The recent most exciting news in number theory is striking result by Yitang Zhang on gaps in primes announced in April 2013. The purpose of this project is to give an idea of this.

Gaps in Prime Numbers

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1. Short Introduction

This introduction is based on material from books and websites like [1-7], [16], [17] [18] & [19].

(1.1) . *Definitions:*

- ***Prime Number*** : A prime number is an integer n > 1, whose only positive divisors are 1 and n.
  First few primes are 2,3,5,7,11,13,17…etc.
- An integer greater than 1 which is not prime is said to be **composite.**

- ***Twin Primes*** : A pair of prime numbers that differ by two (successive odd numbers that are both prime numbers).
  We give a few twin prime pairs: 3 and 5, 5 and 7, 11 and 13, 17 and 19, 101 and 103,…

- ***Cousin primes*** : Cousin primes are prime pairs that differ by four.
  We note a few cousin primes: (3, 7), (7,11), (13,17),….. (883,887),…

- ***Sexy primes*** : Sexy primes are prime pairs that differ from each other by six.
  Here are some sexy primes (5,11), (11,17), (17,23),…. (991,997),…

(1.2). *Twin Primes Conjecture:*

The Twin Primes Conjecture states that there are infinitely many primes p such that $p+2$ is also a prime.

2. Progress on the Twin Primes Conjecture

This is based on [1], [5], [6], [7], [17], [18] & [19].

(2.1)(i) We first proves that there are infinitely many primes:

We assume for a contradiction that there are only finitely many primes, let's say k of them. Then we can make a list of all the primes.

$$p_1 , p_2 , p_3 , p_4 \ldots\ldots p_k$$

Now we can define a number M by multiplying all of these numbers together and adding 1. That is, we have:

$$M = p_1p_2p_3p_4\ldots.p_k + 1$$

However, we know that M must be divisible by some prime number, since every number has at least one prime factor. (Of course, the number M might itself be prime, but that's ok, since every number is a factor of itself.). It cannot be any of the primes on our list, since M leaves a remainder of 1 when divided by any of those primes.

So finally, conclusion is that there must be a prime number that is not on the list. This contradicts our assumption that our list was complete. It follows that no finite list can possibly contain all the primes, and therefore there are infinitely many prime numbers.

(ii) Thus, there is an infinite sequence of all primes, (p_n) with $p_1 = 2 < p_2 = 3 < p_3 < p_4 \ldots$

At times, we will write p_n as p(n).

(iii) Primes up to 3000 have been given in Table 4.1.

(2.2). *Gaps in primes*

$$6! = 6\times5\times4\times3\times2\times1 = 720.$$

Note that 6! is a multiple of the numbers 2, 3, 4, 5 and 6. I can use this fact to produce a sequence of five consecutive non-prime numbers, namely 722, 723, 724, 725 and 726. 722 is a multiple of 2, 723 is a multiple of 3, 724 is a multiple of 4, 725 is a multiple of 5 and 726 is a multiple of 6. If you want n consecutive non–prime numbers, just consider the sequence:

$$(n+1)!+2,\ (n+1)!+3,\ (n+1)!+4,\ldots\ldots\ldots,\ (n+1)!+n,\ (n+1)!+(n+1)$$

The first number is divisible by 2, the next is divisible by 3, the next is divisible by 4, and so on until the last number, which is a multiple of n+1.

Therefore, there are infinitely many prime numbers, but also arbitrarily long gaps between primes.

(2.3). *Prime number theorem*

This tells us that for a real number x, the number of primes less than or equal to *x* is closely approximated by $\frac{x}{\log x}$.
This is an important topic in itself, which we will not discuss in this project.

(2.4). Yitang Zhang paper "Bounded gaps between primes", On April 17, 2013, this paper arrived in the inbox of Annals of Mathematics.

In fact Zhang shows that, there is some number N smaller than 70 million such that there are infinitely many pairs of primes that differ by N. No matter how far you go into the deserts of the truly gargantuan prime numbers. No matter how sparse the primes become – you will keep finding prime pairs that differ by less than 70 million.

The main result of Zhang[11] is

$$\liminf_{n\to\infty} (p_{n+1} - p_n) < 7 \times 10^7$$

where p_n is n-th prime.

(2.5). Zhang's result builds on the paper by Goldston, Pintz And Yildrim[9] & [10].
D. A. Goldston, J. Pintz & C. Y. Yıldırım established[18] that there is a constant c < 1 and infinitely many primes p such that $0 < (p(n+1) - p(n)) < c \log p(n)$, c can be chosen to be arbitrarily small

$$\liminf_{n\to\infty}\left(\frac{p(n+1) - p(n)}{\log p(n)}\right) = 0$$

(2.6). *This motivated me to study the behaviour of prime numbers with different gaps. I obtained tables for gaps 4, 6, 8 & 10, these are given in Tables 4.2 to 4.6. Primes up to 3000 are given in Tables 4.1.*
I used software [12], [13], [14] & [15] for this purpose. A comparison chart [4.8] for different gaps was prepared to study the pattern.

3. Elementary Partition theory

We follow [2], [3] & [18] for this section.

(3.1). Partition of a non−negative integer n is a representation of n as a sum of positive integers, called summands or parts of the partition.

(3.2). Examples

(a) Partitions of 4.

4 = 4	or (reverse order)	1 + 1 + 1 + 1
3 + 1		1 + 1 + 2
2 + 2		1 + 2 + 1
2 + 1 + 1		1 + 3
1 + 3		2 + 1 + 1
1 + 2 + 1		2 + 2
1 + 1 + 2		3 + 1
1 + 1 + 1 + 1		4

(b) Partitions of 5.

5 = 5	or (reverse order)	1 + 1 + 1 + 1 + 1
4 + 1		2 + 1 + 1 + 1
3 + 2		2 + 2 + 1
3 + 1 + 1		3 + 1 + 1
2 + 2 + 1		3 + 2
2 + 1 + 1 + 1		4 + 1
1 + 1 + 1 + 1 + 1		5

(3.3). For this project, I use only even partitions i.e, partitions of even integers in terms of even summands, because gaps between any two odd prime numbers are always even. For instance, even partitions of 4 are 4 & 2+2 and of 6 are 6, 4+2, 2+4 & 2+2+2. For our purpose, the even partition 4+2 & 2+4 will be treated as different.

4. Tables

We begin with table of prime numbers up to 3000 in (4.1). It is based on [12], [14] & [19]. It is divided into three parts.

(4.1)(i) *Table of prime numbers (1–1000)*

Interval →	001 – 100	101 – 200	201 – 300	301 – 400	401 – 500	501 – 600	601 – 700	701 – 800	801 – 900	901 – 1000
Prime numbers	2	101	211	307	401	503	601	701	809	907
	3	103	223	311	409	509	607	709	811	911
	5	107	227	313	419	521	613	719	821	919
	7	109	229	317	421	523	617	727	823	929
	11	113	233	331	431	541	619	733	827	937
	13	127	239	337	433	547	631	739	829	941
	17	131	241	347	439	557	641	743	839	947
	19	137	251	349	443	563	643	751	853	953
	23	139	257	353	449	569	647	757	857	967
	29	149	263	359	457	571	653	761	859	971
	31	151	269	367	461	577	659	769	863	977
	37	157	271	373	463	587	661	773	877	983
	41	163	277	379	467	593	673	787	881	991
	43	167	281	383	479	599	677	797	883	997
	47	173	283	389	487		683		887	
	53	179	293	397	491		691			
	59	181			499					
	61	191								
	67	193								
	71	197								
	73	199								
	79									
	83									
	89									
	97									
Total	**25**	**21**	**16**	**16**	**17**	**14**	**16**	**14**	**15**	**14**
Total prime numbers in the interval 0001–1000 = 168										

Table 4.1 (i)

(4.1)(ii) *Table of prime numbers (1001–2000)*

Interval →	1001 – 1100	1101 – 1200	1201 – 1300	1301 – 1400	1401 – 1500	1501 – 1600	1601 – 1700	1701 – 1800	1801 – 1900	1901 – 2000
Prime numbers	1009	1103	1201	1301	1409	1511	1601	1709	1801	1901
	1013	1109	1213	1303	1423	1523	1607	1721	1811	1907
	1019	1117	1217	1307	1427	1531	1609	1723	1823	1913
	1021	1123	1223	1319	1429	1543	1613	1733	1831	1931
	1031	1129	1229	1321	1433	1549	1619	1741	1847	1933
	1033	1151	1231	1327	1439	1553	1621	1747	1861	1949
	1039	1153	1237	1361	1447	1559	1627	1753	1867	1951
	1049	1163	1249	1367	1451	1567	1637	1759	1871	1973
	1051	1171	1259	1373	1453	1571	1657	1777	1873	1979
	1061	1181	1277	1381	1459	1579	1663	1783	1877	1987
	1063	1187	1279	1399	1471	1583	1667	1787	1879	1993
	1069	1193	1283		1481	1597	1669	1789	1889	1997
	1087		1289		1483		1693			1999
	1091		1291		1487		1697			
	1093		1297		1489		1699			
	1097				1493					
					1499					
Total	**16**	**12**	**15**	**11**	**17**	**12**	**15**	**12**	**12**	**13**
Total prime numbers in the interval 1001–2000 = **135**										

Table (4.1) (ii)

(4.1)(iii) *Table of prime numbers (2001–3000)*

Interval →	2001 – 2100	2101 – 2200	2201 – 2300	2301 – 2400	2401 – 2500	2501 – 2600	2601 – 2700	2701 – 2800	2801 – 2900	2901 – 3000
Prime numbers	2003	2111	2203	2309	2411	2503	2609	2707	2801	2903
	2011	2113	2207	2311	2417	2521	2617	2711	2803	2909
	2017	2129	2213	2333	2423	2531	2621	2713	2819	2917
	2027	2131	2221	2339	2437	2539	2633	2719	2833	2927
	2029	2137	2237	2341	2441	2543	2647	2729	2837	2939
	2039	2141	2239	2347	2447	2549	2657	2731	2843	2953
	2053	2143	2243	2351	2459	2551	2659	2741	2851	2957
	2063	2153	2251	2357	2467	2557	2663	2749	2857	2963
	2069	2161	2267	2371	2473	2579	2671	2753	2861	2969
	2081	2179	2269	2377	2477	2591	2677	2767	2879	2971
	2083		2273	2381		2593	2683	2777	2887	2999
	2087		2281	2383			2687	2789	2897	
	2089		2287	2389			2689	2791		
	2099		2293	2393			2693	2797		
			2297	2399			2699			
Total	**14**	**10**	**15**	**15**	**10**	**11**	**15**	**14**	**12**	**11**
Total prime numbers in the interval 2001–3000 = **127**										

Table (4.1) (iii)

Sources are Wikipedia and use of software Prime Number Generator.

(4.2). *Prime numbers with gap 2 (Twin Primes) in 1-1000*

Interval	Twin prime pairs	No. of twin prime pairs	No. of primes
001-100	(3,5); (5,7); (11,13); (17,19); (29,31); (41,43); (59,61); (71,73)	8	25
101-200	(101,103); (107,109); (137,139); (149,151); (179,181); (191,193); (197,199)	7	21
201-300	(227,229); (239,241); (269,271); (281,283)	4	16
301-400	(311,313); (347,349)	2	16
401-500	(419,421); (431,433); (461,463)	3	17
501-600	(521,523); (569,571); (599,601)	3	14
601-700	(617,619); (641,643); (659,661)	3	16
701-800	–	0	14
801-900	(809,811)(821,823); (827,829); (857,859); (881,883)	5	15
901-1000	–	0	14
	Total =	**35**	**168**

Table (4.2) Twin primes

Remarks:

(i) It is not known whether the set of twin prime numbers ends or not.

(ii) Largest known twin prime pair is

$$(3,756,801,695,685 \times 2^{666,669} - 1,\ 3,756,801,695,685 \times 2^{666,669} + 1)$$

One can see [20].

(4.3). *Prime numbers with gap 4 (Cousin Primes) in 1–1000*

Partition	Interval	Prime pairs with gap 4	No. of Prime pairs with gap 4	Total Prime numbers
2+2	001-100	(**3**,5,**7**)	1	25
	101-1000	–	0	143
4	1-100	(7,11); (13,17); (19,23); (37,41); (43,47); (67,71); (79,83); (97,101)	8	25
	101-200	(103,107); (109,113); (127,131); (163,167); (193,197)	5	21
	201-300	(223,227); (229,233); (277,281)	3	16
	301-400	(307,311); (313,317); (344,353); (379,383); (397,401)	5	16
	401-500	(443,449); (457,461); (463,467); (487,491); (499,503)	5	17
	501-600	–	0	14
	601-700	(613,617); (643,647); (673,677)	3	16
	701-800	(739,743); (757,761); (769,773)	3	14
	801-900	(823,827); (853,857); (859,863); (877,881); (883,887)	5	15
	901-1000	(907,911); (937,941); (967,971)	3	14
	Total		**41**	168

Table (4.3) Cousin primes

Remark:

The only prime less than 1000 belonging to two pairs of cousin primes is 7.

(4.4). Prime numbers with gap 6 (Sexy Primes) in 1- 1000

Interval	Partition	Prime pairs	Total
0001-1000	2+2+2	–	0
	2+4	(5,7,11); (11,13,17); (17,19,23); (41,43,47); (101,103,107); (107,109,113), (191,193,197); (227,229,233); (311,313,317); (347,349,353); (461,463,467) (641,643,647); (821,823,827); (857,859,863); (881,883,887)	15
	4+2	(7,11,13); (13,17,19); (37,41,43); (67,71,73); (97,101,103); (103,107,109), (193,197,199); (223,227,229); (277,281,283); (307,311,313); (457,461,463), (613,617,619); (823,827,829); (853,857,859); (877,881,883)	15
	6	(23,29); (31,37); (47,53); (53,59); (61,67); (73,79); (83,89); (131,137); (151,157); (157,163); (167,173); (173,179); (233,239); (251,257); (257,263); (263,269); (271,277); (331,337); (353,359); (367,373); (373,379); (383,389); (433,439); (443,449); (503,509); (541,547); (557,563); (563,569); (571,577); (587,593); (593,599); (601,607); (607,613); (647,653); (653,659); (677,683); (727,733); (733,739); (751,757); (941,947); (947,953); (971,977); (977,983); (991,997)	44
		Total =	**74**

Table (4.4) Sexy primes

Remark:

The only prime numbers up to 1000 belonging to two pairs of sexy primes are 11, 13, 17, 53, 103, 107, 157, 173, 257, 263, 373, 563, 593, 607, 653, 733, 947 & 977.

(4.5). Prime numbers with gap 8 in 1- 1000

Interval 0001-1000	Partition	Prime pairs	Total
	2+2+2+2	–	0
	2+2+4	(3,5,7,11)	1
	2+4+2	(5,7,11,13); (11,13,17,19); (101,103,107,109); (191,193,197,199); (821,823,827,829)	5
	2+6	(29,31,37); (59,61,67); (131,137,139); (149,151,157); (269,271,277); (431,433,439); (569,571,577); (599,601,607)	8
	4+2+2	–	0
	4+4	–	0
	6+2	(23,29,31); (53,59,61); (71,73,79); (173,179,181); (233,239,241); (263,269,271), (563,569,571); (593,599,601); (653,659,661)	9
	8	(89,97); (359,367); (389,397); (401,409); (449,457); (479,487); (491,499); (683,691); (701,709); (719,727); (743,751); (761,769); (911,919); (929,937); (983,991)	15
		Total =	38

Table (4.5)

Remark:

The only prime numbers less than 1000 belonging to two pairs of primes with gap 8 is 11.

(4.6). Prime numbers with gap 10 in 1- 1000

Interval 1-1000	Partition	Primes	Total
	2+2+2+2+2	–	0
	2+2+6	–	0
	2+4+4	–	0
	2+6+2	–	0
	2+8	–	0
	4+2+4	(7,11,13,17); (13,17,19,23); (37,41,43,47); (97,101,103,107); (103,107,109,113); (223,227,229,233); (307,311,313,317); (457,461,463,467); (853,857,859,863); (877,881,883,887)	10
	4+4+2	–	0
	6+2+2	–	0
	8+2	–	0
	10	(139,149); (181,191); (241,251); (283,293); (337,347); (409,419); (421,431); (547,557); (577,587); (631,641); (691,701); (709,719); (787,797); (811,821); (829,839); (919,929)	16
		Total =	**26**

Table (4.6)

Remark:

There are no primes less than 1000 belonging to two pairs of primes with gap 10.

(4.7). Method

- I have prepared and checked all the data by using the following software:
  - ✓ Mathematica
  - ✓ Microsoft Mathematics
  - ✓ Prime No. generator
  - ✓ wxMaxima

- **Microsoft Mathematics:**

  Command: The *isPrime* function tells whether a number is prime. The *nextPrime* and *prevPrime* functions return, respectively, the first prime greater than a given number and the first prime less than a given number.

  Command: *factor(n)*, n is an integer. The prime factors of the integer or expression. If an integer is returned unfactored, it is prime.

- **Prime No. generator:**

  Simply, enter the number and generate the prime numbers

  Mode A: if you want to find a certain number of primes, starting with a specific number.

  Mode B: if you want to find all primes between two specific numbers.

- **wxMaxima**

  Command : ***primep(n)***

  Primality test: If ***primep(n)*** returns *false*, then ***n*** is a composite number and if it returns *true*, ***n*** is a prime number.

  Command : ***prev_prime(n)***

  Returns the greatest prime smaller than *n.*

  Command : ***next_prime(n)***

  Returns the smallest prime bigger than *n.*

(4.8). Comparison chart of pairs of Primes with different gaps

Interval ↓	G_2	G_4		G_6			G_8					G_{10}		Total
		{2+2}	{4}	{2+4}	{4+2}	{6}	A	B	C	D	E	F	G	
1-100	8	1	8	4	5	7	1	2	2	3	1	4	0	25
101-200	7	0	5	3	2	5	0	2	2	1	0	1	2	21
201-300	4	0	3	1	2	5	0	0	1	2	0	1	2	16
301-400	2	0	5	2	1	5	0	0	0	0	2	1	1	16
401-500	3	0	5	1	1	2	0	0	1	0	4	1	2	17
501-600	3	0	0	0	0	7	0	0	2	2	0	0	2	14
601-700	3	0	3	1	1	5	0	0	0	1	1	0	2	16
701-800	0	0	3	0	0	3	0	1	0	0	4	0	2	14
801-900	5	0	5	3	3	0	0	0	0	0	0	2	2	15
901-1000	0	0	3	0	0	5	0	0	0	0	3	0	1	14
Subtotal	35	1	40	**15**	**15**	**44**	**1**	**5**	**8**	**9**	**15**	**10**	**16**	168
Total	35	41		74			38					26		168

Table (4.8)

Notation:

G_n = Number of prime pairs with gap n

A = 2 + 2 + 4,

B = 2 + 4 + 2,

C = 2 + 6,

D = 6 + 2

E = 8

F = 4 + 2 + 4

G = 10

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