

Uniform Continuity, Lipschitz functions and their applications

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I started my summer project under the supervision of Prof. Ajit Iqbal Singh (INSA Honorary Scientist) on 10th June 2014.

Here, I especially focused on the research papers on Visualizing Uniform Continuity by Dwight Paine[8] and K.F. Klopfenstein & John Telste [9]. I consulted books on Metric Spaces, Analysis And Multivariable Calculus [1–7] as suggested by Prof. Ajit Iqbal Singh for understanding these papers. Here, with Prof. Singh's help, I have been able to understand many things in various topics I had never studied, like Lipschitz functions, Euclidean Spaces, Metric Spaces, Functions of Several variables, Application of Real Analysis. And I also got to know a bit about the research going on, nowadays in Mathematics.

We start with definitions and examples of basic concepts, which may be helpful to understand the papers mentioned above.

[1.1]. CONTINUITY^[2-7,10,12]

Continuity at a point

1.1.1 Definition: Let A be a non-empty subset of \mathbb{R} , Let $f: A \rightarrow \mathbb{R}$ be a function, and let $c \in A$. Then we say that f is continuous at c , if given any number $\varepsilon > 0$, $\exists \delta > 0$, such that if x is any point of A satisfying $|x - c| < \delta$, then

$$|f(x) - f(c)| < \varepsilon \quad (1)$$

If f fails to be continuous at c then we say that f is 'discontinuous' at c .

(NOTE: If c is an isolated point of A , then f is continuous at c .)

1.1.2 Example: *The function f given by $f(x)=x^2$, $x \in \mathbb{R}$ is continuous at any $c \in \mathbb{R}$.*

Proof : Let $c \in \mathbb{R}$.

Let $\varepsilon > 0$ be given. We show that $\exists \delta(\varepsilon, c) > 0$ such that

$$|x - c| < \delta \Rightarrow |f(x) - f(c)| = |x^2 - c^2| < \varepsilon$$

Or, equivalently

$$c - \delta < x < c + \delta \Rightarrow c^2 - \varepsilon < x^2 < c^2 + \varepsilon$$

Here $|f(x) - f(c)| = |x^2 - c^2|$

$$= |x - c| |x + c| \quad \text{—————(i)}$$

Now $|x - c| < \delta$

$$\Rightarrow -\delta + c < x < \delta + c$$

$$\Rightarrow 2c - \delta < x + c < 2c + \delta \quad \text{—————(a)}$$

$$\text{Also} \quad -2|c| - \delta \leq 2c - \delta \quad \text{—————(b)}$$

$$\text{and} \quad 2c + \delta \leq 2|c| + \delta \quad \text{—————(c)}$$

Consider any x with $|x - c| < \delta$,

$$\text{Then} \quad |x + c| < 2|c| + \delta \quad (\text{using (b) \& (c) in (a)})$$

$$\text{Therefore} \quad |f(x) - f(c)| = |x - c| |x + c| \quad (\text{from (i)})$$

$$< \delta (2|c| + \delta)$$

$$= 2|c|\delta + \delta^2$$

Now choose a $\delta > 0$ such that $2|c|\delta + \delta^2 < \varepsilon$, for instance,

we may take $\delta = \min\left(1, \frac{\varepsilon}{2|c|+1}\right)$
then $|x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$.

Uniform Continuity

1.1.3 Definition: Let $A \subseteq \mathbb{R}$ and Let $f: A \rightarrow \mathbb{R}$ be defined, then we say that f is **uniformly continuous** on A if for each $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$ such that if $u, x \in A$ are any numbers satisfying $|x - u| < \delta(\varepsilon)$, then we have

$$|f(x) - f(u)| < \varepsilon. \quad (2)$$

On a careful reading of definition, We must try to find how (1.1.1) differs from the other(1.1.2). For f to be continuous on A , we fix both $\varepsilon > 0$ and $u \in A$ before obtaining the value of δ . So the choice of δ might depend on u as well as on ε . Uniform continuity means that for each $\varepsilon > 0$, the value $\delta > 0$ that we obtain can be chosen independently of the point u .

Therefore Uniform continuity of f is a stronger property than the continuity of f on A .

1.1.4 Example: $f(x) = \frac{1}{x}$ for $x > 0$.

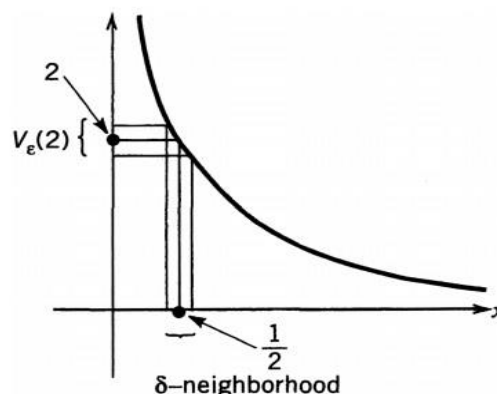
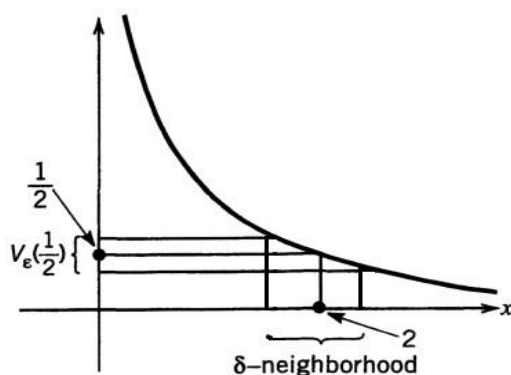


Fig 1.1.4 (a). $f(x) = \frac{1}{x}$ ($x > 0$)

Fig 1.1.4 (b). $f(x) = \frac{1}{x}$ ($x > 0$)

In fact, there is no way of choosing one value of δ that will “work” for all $u > 0$ for the function $f(x) = \frac{1}{x}$. The situation is exhibited graphically in Figures 1.1.4(a) and 1.1.4(b) where, for a given ε -neighborhood $V_\varepsilon(1/2)$ about $1/2 = f(2)$ and $V_\varepsilon(2)$ about $2 = f(1/2)$, the corresponding maximum values of δ are seen to be considerably different.

Examples:

- Constant function $f(x) = a$ is uniformly continuous on \mathbb{R} .
(Indeed, $|f(x) - f(y)| = 0 < \varepsilon$ for any $\varepsilon > 0$ and $x, y \in \mathbb{R}$.)
- Identity function $f(x) = x$ is uniformly continuous on \mathbb{R} .
(Since $f(x) - f(y) = x - y$, we have $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < \varepsilon$
(taking $\delta = \varepsilon$) and $x, y \in \mathbb{R}$.)

1.1.5 Example: *Let f be the function given by $f(x) = x^2$, $x \in \mathbb{R}$.*

(a). f is not uniformly continuous on \mathbb{R} .

(b). f is uniformly continuous on $[a, b]$ (where $a < b$).

Proof: (a) We prove it,

Let $\varepsilon = 2$, choose an arbitrary $\delta > 0$

let n_δ be a natural number such that $1/n_\delta < \delta$.

Further, let $X_\delta = n_\delta + 1/n_\delta$ & $Y_\delta = n_\delta$,

then

$$\begin{aligned} |X_\delta - Y_\delta| &= |n_\delta + (1/n_\delta) - n_\delta| \\ &= |1/n_\delta| < \delta, \end{aligned}$$

while

$$\begin{aligned} |f(X_\delta) - f(Y_\delta)| &= |(n_\delta + 1/n_\delta)^2 - n_\delta^2| \\ &= |n_\delta^2 + (1/n_\delta^2) + 2 - n_\delta^2| \\ &= 2 + 1/n_\delta^2 > \varepsilon. \end{aligned}$$

So we conclude that f is not uniformly continuous on \mathbb{R} .

(b). Let $-\infty < a < b < \infty$. Then f is uniform continuous on $[a, b]$.

Proof:

Let $x, y \in [a, b]$ i.e $a \leq x \leq b$ and $a \leq y \leq b$.

Then $|x|, |y| \leq \max \{|a|, |b|\} = K$ (say) for all $x, y \in [a, b]$

Let $\varepsilon > 0$ be arbitrary. Now we look for a $\delta(\varepsilon) > 0$ s.t

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| = |x^2 - y^2| < \varepsilon.$$

We have $|f(x) - f(y)| = |x^2 - y^2|$

$$\begin{aligned} &= |x + y||x - y| \\ &\leq (|x| + |y|) |x - y| \\ &\leq 2K |x - y| < \varepsilon \end{aligned}$$

So, as $|x - y| < \frac{\varepsilon}{2K} \Rightarrow |x^2 - y^2| < \varepsilon$.

$\Rightarrow f$ is uniformly continuous.

[1.2]. Illustrating uniform continuity^[1,2,4,6,7,10-12]

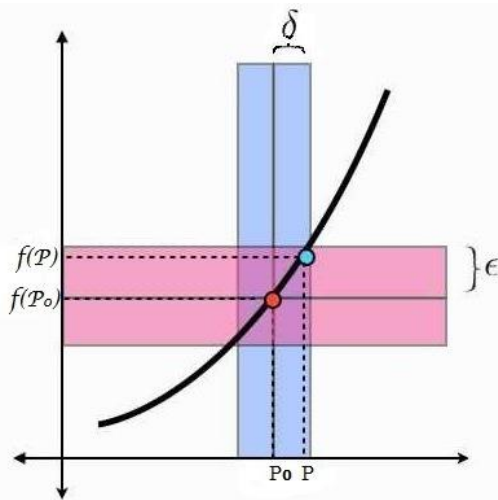


fig.1

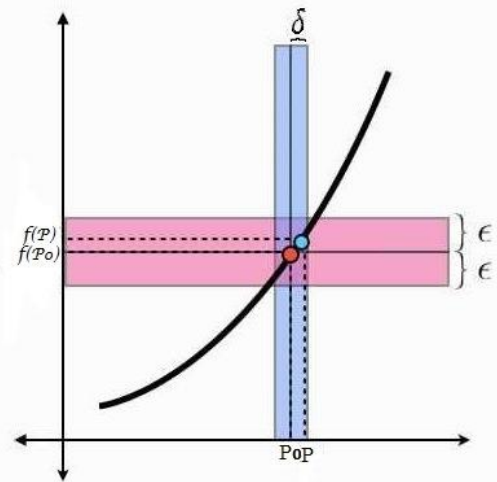


fig.2

A function is continuous...

If for every point in domain, we can make the images of that point and another point arbitrary close enough to our given point.

We want the distance between the images to be less than ε (for “error”). To accomplish this, it is enough that the distance between the points is less than δ (for “distance”).

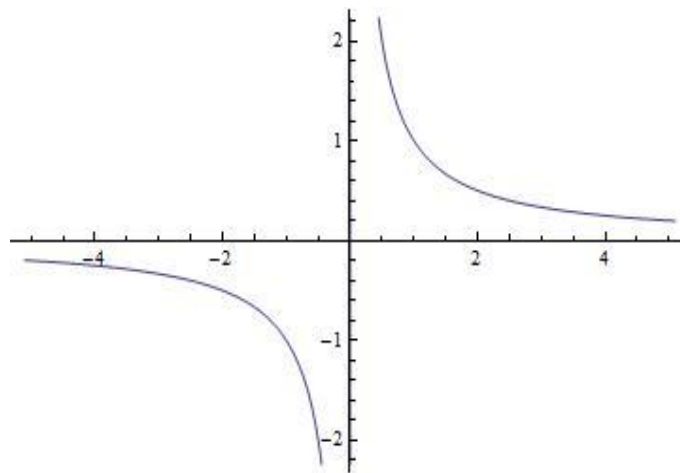


Fig.3 $f(x) = 1/x$

Now coming up with an example of a function that is continuous but not uniformly continuous.

- For this consider $f(x) = 1/x$ (fig.3) over $(0,2)$.

Second Proof:

Clearly $1/x$ is continuous over $(0, 2)$ as it is the quotient of two polynomial and the denominator doesn't vanish here.

Now if the function was uniformly continuous here, then

$\forall \varepsilon > 0, \exists \delta > 0$ such that whenever $|x - y| < \delta$,

$$\Rightarrow |f(x) - f(y)| < \varepsilon.$$

Now taking $\varepsilon = 1$, then for $\delta > 0$, we take $x = \min(\delta, 1)$ and $y := x/2$

$$\Rightarrow |x - y| = x/2 < \delta,$$

but nevertheless

$$\begin{aligned} |f(x) - f(y)| &= |1/x - 1/y| = |1/x - 2/x| \\ &= |1/x| \geq 1 = \varepsilon \end{aligned}$$

So f is not uniformly continuous here.

For uniform continuity, there has to be one single δ that works for a fixed given ε . In [fig.4](#) that is not possible.

Because, if the ε -interval slides up the positive y -axis, the corresponding δ must get smaller and smaller. There is no single δ that will work for any possible location of ε -interval on the y -axis.

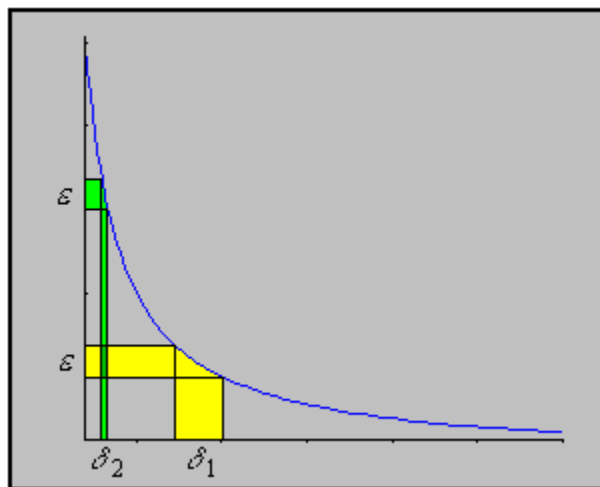


Fig.4

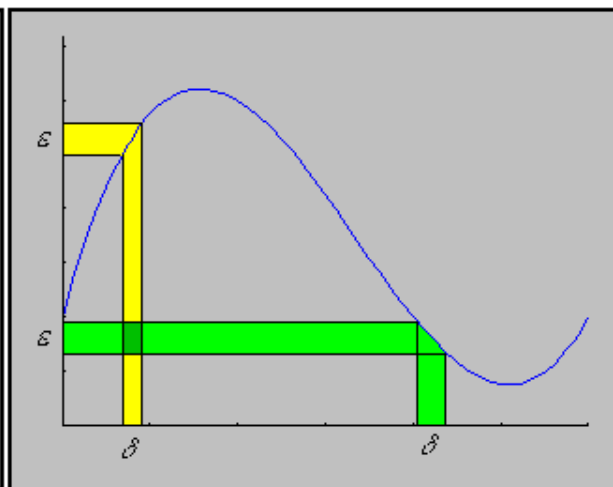


Fig.5

In [fig.5](#) one can see that regardless of where I (interval) places the ε -interval on the y -axis, it is possible to find one single δ that will work for each of those locations of ε . That is to say, there is one δ that will work uniformly for

all locations of ε (of course choosing smaller ε means that I am also allowed to pick another smaller δ that will work again uniformly for all ε -locations).

The basic difference between uniform continuity and continuity is that –

- In uniform continuity for a given ε one can take a single δ which works $\forall x_0 \in X$ but for ordinary continuity each $x_0 \in X$ might use a different δ .

Thus every uniformly continuous function is continuous but not conversely.

Cauchy Sequence

A sequence (x_n) of real numbers is said to be a Cauchy Sequence if for every $\varepsilon > 0 \exists$ a natural number $H(\varepsilon) > 0$ such that \forall natural numbers $n, m \geq H(\varepsilon)$, the terms x_n, x_m satisfy

$$|x_n - x_m| < \varepsilon.$$

1.2.1 Theorem^{[3][4]}: *If $f: A \rightarrow \mathbb{R}$ is uniformly continuous on a subset A of \mathbb{R} and if (x_n) is a Cauchy sequence in A , then $(f(x_n))$ is a Cauchy sequence in \mathbb{R} .*

Proof :

Let (x_n) be a Cauchy sequence in A , and let $\varepsilon > 0$ be given.

Since f is uniformly continuous on A , then there exists $\delta > 0$ such that if $x, u \in A$ and satisfy $|x - u| < \delta$, then

$$|f(x) - f(u)| < \varepsilon. \quad \text{_____} (3)$$

Since (x_n) is a Cauchy sequence, there exists $H(\delta)$ such that $\forall n, m > H(\delta)$

$$\text{We have } |x_n - x_m| < \delta.$$

From (3),

$$\forall n, m > H(\delta), \text{ we have } |f(x_n) - f(x_m)| < \varepsilon.$$

Therefore the sequence $(f(x_n))$ is a Cauchy sequence.

1.2.2 EXAMPLES (a). $f(x) := 1/x, \quad x \in (0, 1).$

Let $x_n = 1/n$, for $n \in \mathbb{N}$.

Then x_n is obviously a Cauchy sequence in $(0, 1)$.

Since

$$f(x_n) := 1/(1/n) = n \quad \text{for } n \in \mathbb{N}.$$

Therefore $(f(x_n))$ is not a Cauchy sequence.

Therefore f cannot be uniformly continuous on $(0, 1)$ by **Theorem 1.2.1**.

$$(b). f(x) := 1/x^2, \quad x \in (0,1).$$

Let $x_n = 1/n$, for $n \in \mathbb{N}$.

Then x_n is obviously a Cauchy sequence in $(0,1)$.

Now

$$f(x_n) := 1/(1/n)^2 = n^2 \quad \text{for } n \in \mathbb{N}.$$

Therefore $(f(x_n))$ is not a Cauchy sequence.

Therefore f cannot be uniformly continuous on $(0,1)$ by **Theorem 1.2.1**.

[1.3]. Euclidean Space and Distance

The Euclidean distance (or Euclidean metric) is the “ordinary” distance between two points that one would measure with a rule and is given by Pythagorean formula.

Euclidean Spaces: In mathematics, Euclidean space is the Euclidean plane and 3D space of Euclidean geometry.

Euclidean spaces^[2] in \mathbb{R}^n

A point in 2-dimensional spaces is an ordered pair of real numbers (x_1, x_2) . Similarly a point in 3-dimensional space is an ordered triple of real numbers (x_1, x_2, x_3) . It is just easy to consider an ordered n -tuple of real numbers (x_1, x_2, \dots, x_n) and to refer to this as a point in n -dimensional space.

1.3.1 Definition: *Let $n > 0$ be an integer. An ordered set of n real numbers (x_1, x_2, \dots, x_n) is called an n -dimensional point or a vector with n -components.*

For example,

$$x = (x_1, x_2, \dots, x_n) \text{ or } y = (y_1, y_2, \dots, y_n)$$

the number x_k is called the n^{th} coordinate of the point x or the n^{th} component of the vector x . The set of all n -dimensional points is called n -dimensional Euclidean space or simply n -space and denoted by \mathbb{R}^n .

Norm or length $\|x\| = \sqrt{x \cdot x}$

$$= \sqrt{\sum_{k=1}^n (x_k)^2}$$

For $x = (x_1, x_2, x_3, \dots, x_n)$ and $y = (y_1, y_2, y_3, \dots, y_n)$.

$$\|x - y\| = \left(\sum_{i=1}^n (x_i - y_i)^2 \right)^{\frac{1}{2}}.$$

The $d(x, y) = \|x - y\|$ is called the distance between x and y .

And Euclidean distance satisfies:

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + \dots + (x_n - y_n)^2}$$

1.3.2 Definition: Let $E \subseteq \mathbb{R}^m$ and let $f: E \rightarrow \mathbb{R}^n$ be any function. Then we say that f is **uniformly continuous** on E if for a given $\epsilon > 0$, $\exists \delta(\epsilon) > 0$ such that for any $x, y \in E$,

$$\|f(x) - f(y)\|_{\mathbb{R}^n} < \epsilon \quad \text{whenever } \|x - y\|_{\mathbb{R}^m} < \delta(\epsilon).$$

[1.4] LIPSCHITZ FUNCTIONS ^[1,4-7,10,11]

What is a Lipschitz function ?

If a uniformly continuous function is given on a set that is not a closed bounded interval, then it is sometimes difficult to establish its uniform continuity. However, there is a condition that frequently occurs that is sufficient to guarantee uniform continuity (*It is named after Rudolf Lipschitz (1832–1903) who was a student of Dirichlet and who worked extensively in differential equations and Riemannian geometry*).

1.4.1 Definition: Let $\mathcal{A} \subseteq \mathbb{R}$ & Let $f: \mathcal{A} \rightarrow \mathbb{R}$. If there exists a constant $K > 0$ such that

$$|f(x) - f(u)| \leq K|x - u| \quad (4)$$

for all $x, u \in \mathcal{A}$, then f is said to be a **Lipschitz function** (or to satisfy a Lipschitz condition) on \mathcal{A} .

The condition (4) that a function $f: I \rightarrow \mathbb{R}$ on an interval I is a Lipschitz function can be interpreted geometrically as follows. If we write the condition as

$$\left| \frac{f(v) - f(u)}{v - u} \right| \leq \mathcal{K}, \quad v, u \in I, v \neq u,$$

then the quantity inside the absolute values is the slope of the line segment joining the points $(v, f(v))$ and $(u, f(u))$. Thus a function f satisfies a **Lipschitz condition** if and only if the slopes of all the line segments joining two points on the graph of $y = f(x)$ over I are bounded by some number \mathcal{K} .

LIPSCHITZ CONTINUITY

Our purpose is to summarize the idea of Lipschitz continuity and, in particular, to make clear the distinctions between the notion of *Lipschitz continuity* of a function at a point and the notion of a *Lipschitz function* (as defined in the textbook^{[1] [6]}).

1.4.2 Definition: A function f from $S \subset \mathbb{R}^n$ into \mathbb{R}^m is *Lipschitz continuous at* $x \in S$ if there is a constant $C > 0$, $\exists \alpha > 0$ such that for all $x, y \in S$,

$$\|f(x) - f(y)\| \leq C\|x - y\| \quad (5)$$

whenever $\|x - y\| < \alpha$.

Note that *Lipschitz continuity* at a point depends only on the behavior of the function near that point.

For f to be Lipschitz continuous at x , an inequality (5) must hold for all y sufficiently near x , but it is not necessary that (5) holds if y is not near x . Also, f may be *Lipschitz continuous* at other points, but different values of C may be required for inequality (5) to hold near those points.

Remarks: A Lipschitz function is Lipschitz continuous but conversely need not be true. For example (Example 1.1.4), we can see that $f(x) = \frac{1}{x}$ for $x > 0$ is Lipschitz continuous at each $x > 0$, but there is no single C for which (5) holds for all $x > 0$.

1.4.3 Theorem: If $f: \mathcal{A} \rightarrow \mathbb{R}$ is a Lipschitz function, then f is uniformly continuous on \mathcal{A} .

Proof: Let f be a Lipschitz function. Then condition (5) is satisfied, so for

given $\varepsilon > 0$, we can take $\delta = \frac{\varepsilon}{K}$. If $x, u \in \mathcal{A}$ satisfy $|x - u| < \delta$, then

$$|f(x) - f(u)| \leq K|x - u|$$

$$|f(x) - f(u)| < K\delta$$

$$|f(x) - f(u)| < K \cdot \frac{\varepsilon}{K} = \varepsilon.$$

Therefore f is uniformly continuous on A .

For example,

1.4.4 EXAMPLE: *The function $f(x)=x^2$ is Lipschitz on any bounded interval $[a,b]$ where $a < b$.*

i.e the function $f(x)=x^2$ is Lipschitz continuous at each $x \in [a, b]$.

Proof:

For any $x, y \in [a, b]$, we obtain

$$\begin{aligned} |f(x) - f(y)| &= |x^2 - y^2| \\ &= |(x+y)(x-y)| \\ &= |x+y| |x-y| \\ &\leq (|x| + |y|) |x-y| \\ &\leq 2 \cdot \max(|a|, |b|) |x-y| \\ &\leq C |x-y| \quad (\text{taking } C = 2 \cdot \max(|a|, |b|)). \end{aligned}$$

[1.5]. VISUALIZING UNIFORM CONTINUITY[8] [D'WIGHT PAINE, Wells College]

A continuous function defined on an interval has a connected graph, and although this fact is seldom used in proofs it is helpful in thinking about continuity.

Is there likewise a helpful way to visualize uniform continuity ?

One may feel that the graph of a uniform continuous function must not become too steep; but as R.C. Buck point out [1,p.68], this is too strong a condition. For example, the square root function is uniformly continuous even though it has arbitrarily steep chords near the origin. In the theorem below,(given by D. Paine), however he finds a similar property, easy to visualize, which does characterize the graph of a uniformly continuous function on an interval. The theorem says, essentially that “UNIFORM CONTINUITY ON AN INTERVAL MEANS THAT STEEP CHORDS ARE SHORT”.

THEOREM:1 *For a real function f defined on a real interval, the following two conditions are equivalent:*

$$(i). \forall \varepsilon > 0, \exists \delta > 0 \text{ s.t } |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon;$$

$$(ii). \forall \varepsilon > 0, \exists N > 0 \text{ s.t } \left| \frac{f(x)-f(y)}{x-y} \right| > N \Rightarrow |f(x) - f(y)| < \varepsilon.$$

Proof:

I. Assume (i) i.e $\forall \varepsilon > 0 \exists \delta > 0$ such that $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$

Let $N = 2\varepsilon/\delta$.

We shall prove the contrapositive of (ii). Note that if $|f(x) - f(y)| \geq \varepsilon$ it is in fact an integral multiple $k\eta$ of some number $\eta \in [\varepsilon, 2\varepsilon]$.

We can assume without loss of generality that $f(x) < f(y)$. If $x < y$, the intermediate value theorem ensures that there are numbers $x = x_0 < \dots < x_k = y$ such that $f(x_i) = f(x) + i\eta$. If $x > y$ then the numbers can be chosen to satisfy $x = x_0 > \dots > x_k = y$.

$$\begin{aligned} \text{In either case, each } |f(x_i) - f(x_{i-1})| &= |f(x) + i\eta - (f(x) + (i-1)\eta)| \\ &= |i\eta - (i-1)\eta| \\ &= \eta \geq \varepsilon \end{aligned}$$

So by our hypothesis each $|x_i - x_{i-1}| \geq \delta$.

Consequently $|x - y| \geq k\delta$ and

$$\left| \frac{f(x)-f(y)}{x-y} \right| \leq \frac{k\eta}{k\delta} = \frac{\eta}{\delta} \leq \frac{2\varepsilon}{\delta} = N.$$

II. Assume (ii), and let $\delta = \frac{\varepsilon}{N}$. To prove contrapositive of (i), we observe that if $|f(x) - f(y)| \geq \varepsilon$, then

$$\begin{aligned} |x - y| &= \frac{|x - y|}{|f(x) - f(y)|} \cdot |f(x) - f(y)| \\ &\geq \frac{1}{N} \cdot \varepsilon = \delta \end{aligned}$$

D. Paine mentioned that, if the domain is not an interval, (ii) still implies (i) but the two conditions need not be equivalent.

For example, any function restricted to the integers satisfies the definition of uniform continuity automatically, because distinct points in the domain are never less than one unit apart and we may take $\delta = 1$ for any $\varepsilon > 0$. But it may still (like the function n^2) have arbitrarily steep and long chords.

Nevertheless, no matter what the domain, the two conditions are sure to be equivalent if the range is bounded, for then N may be taken to be the diameter of the range divided by δ . And of course the same is true if the domain is bounded because then the range is bounded too.

PAPER-II

[1.6]. VISUALIZING UNIFORM CONTINUITY OF FUNCTIONS OF SEVERAL VARIABLES[9]

[K.F. Klopfenstein & John Telste]

In the paper of K.F. Klopfenstein and John Telste [VISUALAZING UNIFORM CONTINUITY OF FUNCTIONS OF SEVERAL VARIABLES], they say that

1.6.1 Definition: A function f from a subset D of E^n into E^m has the steep chords are shorts (abbreviated SCS) property when for every $\varepsilon > 0$ there exists $M > 0$ such that if

$$\|f(x) - f(y)\| > M\|x - y\|, \text{ then } \|(x, f(x)) - (y, f(y))\| < \varepsilon$$

Here norm symbols denote Euclidean distance in the appropriate Euclidean spaces.

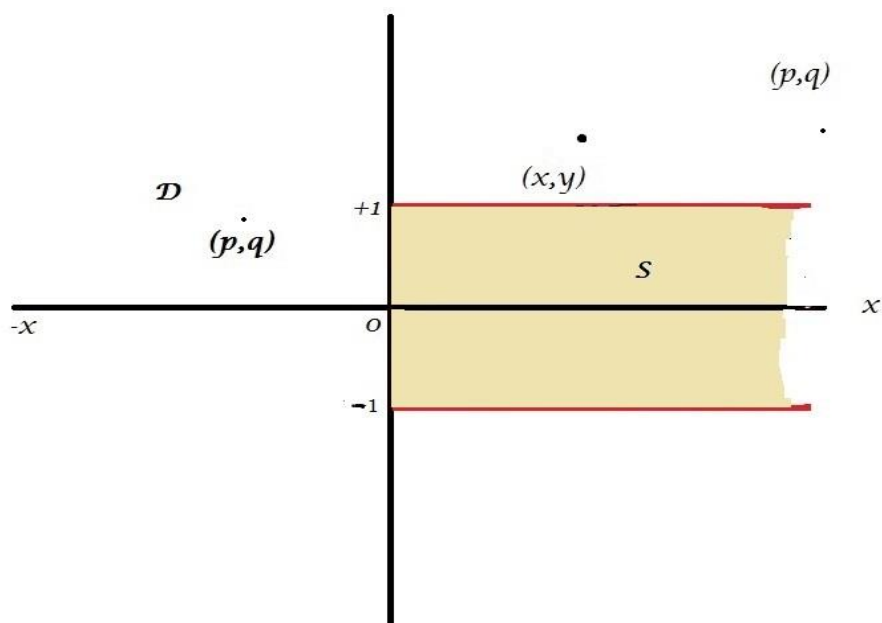
Is this easily visualized characterization of uniform continuity valid for function between Euclidean spaces ?

To show it we consider the following example.

1.6.2 Example: Let \mathcal{D} be the complement in \mathbb{R}^2 of the semi-infinite strip $x \geq 0$ and $-1 \leq y \leq 1$. Define f on \mathcal{D} by

$$f(x, y) = \begin{cases} x & \text{for } x \geq 0 \text{ and } y > 1 \\ -x & \text{for } x \geq 0 \text{ and } y < -1 \\ 0 & \text{for } x < 0 \end{cases}$$

Solution:



The function f is uniformly continuous on \mathcal{D} , if for given $\varepsilon > 0$, $\exists \delta(\varepsilon) > 0$ such that

$$\begin{aligned} \|(x, y) - (p, q)\| &< \delta \\ \Rightarrow |f(x, y) - f(p, q)| &< \varepsilon \end{aligned}$$

Choose $\delta = \min\{\varepsilon, 2\} > 0$

let $P = (x, y)$ & and $Q = (p, q)$

Here

$$|f(x, y) - f(p, q)| = |x - p| < \delta \leq \varepsilon \quad (\text{whenever } x, p \geq 0 \text{ \& } y, q > 1), \text{ and}$$

$$\begin{aligned} |f(x, y) - f(p, q)| &= |-x - (-p)| && (\text{whenever } x, p \geq 0 \text{ \& } y, q \leq -1) \\ &= |-x + p| \\ &= |-(x - p)| \\ &= |x - p| < \delta \leq \varepsilon \end{aligned}$$

And $|f(x, y) - f(p, q)| = |0 - 0| < \delta \leq \varepsilon$ (whenever $x < 0$ and $p < 0$).

So f is uniformly continuous on D .

In this note K.F. KLOPFENSTEIN AND JOHN TESTLE determine conditions under which the SCS property is equivalent to uniform continuity for functions from a subset D of E^n into E^m .

When the implication in the definition of the SCS property is replaced by its contrapositive, it is easy to see that if a function f has the SCS property, then it also has the property that for every $\varepsilon > 0$ there exists $M > 0$ such that if

$$\|x - y\| > \varepsilon, \text{ then } \|f(x) - f(y)\| < M\|x - y\|$$

We shall refer to this property as the Lipschitz condition in the large, abbreviated LCL.

THEOREM 1: Let f be a function from a subset D of E^n into E^m . The function f has the SCS property if and only if f is uniformly continuous and satisfies the LCL.

Proof:

Let $f : D \subseteq E^n \rightarrow E^m$ be defined and satisfies SCS Property.

Then for all $\varepsilon > 0$, there exists $N > 0$ such that if

$$\|f(x) - f(y)\|_{R^m} > N \|x - y\|_{R^n} \text{ then}$$

$$\|(x, f(x)) - (y, f(y))\|_{R^{n+m}} < \varepsilon.$$

This gives

$$\frac{\|f(x) - f(y)\|_{R^m}}{\|x - y\|_{R^n}} > N \Rightarrow$$

$$\|x - y\|_{R^n} < \varepsilon \quad \text{and} \quad \|f(x) - f(y)\|_{R^m} < \varepsilon.$$

We now show that f has SCS property $\Rightarrow f$ is uniformly continuous.

i.e Let f satisfy SCS. We have to show,

$\forall \varepsilon > 0$, there exists $\delta > 0$ such that

$$\|x - y\| < \delta \Rightarrow \|f(x) - f(y)\| < \varepsilon$$

Or equivalently,

$$\|f(x) - f(y)\| \geq \varepsilon \Rightarrow \|x - y\| \geq \delta \tag{1}$$

Let $\delta = \varepsilon/N > 0$. Consider x, y with $\|f(x) - f(y)\|_{R^m} \geq \varepsilon$.

$$\text{Then } \|(x, f(x)) - (y, f(y))\|_{R^{n+m}} \geq \varepsilon$$

$$\text{which, in turn, gives } \frac{\|f(x) - f(y)\|}{\|x - y\|} \leq N.$$

$$\begin{aligned} \text{So, } \|x - y\| &= \frac{\|x - y\|}{\|f(x) - f(y)\|} \|f(x) - f(y)\| \\ &\geq \frac{1}{N} \cdot \varepsilon = \delta > 0. \end{aligned}$$

This proves (1).

\Leftarrow (ii).

Consider an f which is uniformly continuous and satisfies LCL property.

To show that f has SCS property,

let $\varepsilon > 0$. There exists a $\delta > 0$ such that

$$\varepsilon > \varepsilon/2 > \delta$$

then

$$\|x - y\| < \delta \Rightarrow \|f(x) - f(y)\| < \varepsilon/2$$

Now, LCL property \Rightarrow

there exists $N > 0$ such that $\|x - y\| > \delta/2 \Rightarrow$

$$\|f(x) - f(y)\| < N \|x - y\|$$

or equivalently,

$$\|f(x) - f(y)\| \geq N \|x - y\|$$

$$\Rightarrow \|x - y\| \leq \delta/2 < \delta < \varepsilon/2.$$

$$\begin{aligned} \text{So } \|(x, f(x)) - (y, f(y))\| &\leq \|x - y\| + \|f(x) - f(y)\| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon \end{aligned}$$

Let $x = (x_1, x_2, x_3, \dots, x_n)$

$$y = (y_1, y_2, y_3, \dots, y_n)$$

then $f(x) = (f_1(x), f_2(x), f_3(x), \dots, f_n(x))$

$$f(y) = (f_1(y), f_2(y), f_3(y), \dots, f_n(y)).$$

$$\begin{aligned}
\therefore \quad \|(x, f(x)) - (y, f(y))\|_{R^{n+m}} &= \sqrt{\sum_{i=1}^n (x_i - y_i)^2 + \sum_{j=1}^m (f_j(x) - f_j(y))^2} \\
&\leq \sqrt{\sum_{i=1}^n (x_i - y_i)^2} + \sqrt{\sum_{j=1}^m (f_j(x) - f_j(y))^2} \\
&\leq \|x - y\|_{R^n} + \|f(x) - f(y)\|_{R^m} \\
&< \varepsilon/2 + \varepsilon/2 = \varepsilon
\end{aligned}$$

So f has SCS property.

THEOREM 2: Let D be a subset of E^n which has the property that for every $\eta > 0$ there is a constant K such that for every pair of points x and y in D , if $\|x - y\| > \eta$ then there is a rectifiable curve $\gamma: [0, 1] \rightarrow D$ such that $\gamma(0) = x$, $\gamma(1) = y$, and the length of γ is less than $K \|x - y\|$. A function f from D into E^m is uniformly continuous on D if and only if f has the SCS property.

{Here $\gamma : [0,1] \rightarrow D$ be a rectifiable curve (A **rectifiable curve** is a curve with finite length) from x to y of length less than $K\|x - y\|$ }

Proof:

Let $f: D \subseteq E^n \rightarrow E^m$ be given where D is as in the statement of the theorem.

From THEOREM:1 we have, f is uniformly continuous if f has the SCS property.

So thanks to THEOREM:1, we need only show that a function which is uniformly continuous and whose domain D satisfies the hypotheses of theorem satisfies the LCL.

i.e f has uniformly on D + hypothesis of the theorem $\Rightarrow f$ has LCL.

Let $\eta > 0$ be given, $K(\text{constant}) > 0$, and $x, y \in D$, such that $\|x - y\| > \eta$,

then $\exists \gamma : [0,1] \rightarrow D$ s.t

$$\gamma(0) = x, \gamma(1) = y \text{ and } |\gamma| = L \leq \|x - y\|.$$

Now for every $\varepsilon > 0$ (take $\varepsilon=1$), $\exists \delta > 0$ then

$$\|x - y\| < \delta \Rightarrow \|f(x) - f(y)\| < 1 (= \varepsilon)$$

Because f is uniformly continuous.

by the intermediate value theorem applied to the function which associates with each s in $[0,1]$

$\psi : [0, 1] \rightarrow [0, L]$ continuous (uniformly continuous), as γ is rectifiable curve.

$\psi(s) = \text{arc length of } \gamma$.

the interval $[0,1]$ can be partitioned into k intervals by points

$$0 \leq t_0 < t_1 < t_2 \dots \leq t_k = 1$$

in such a way that

$$\|\gamma(t_i) - \gamma(t_{i-1})\| < 1 \quad \text{for } i=1,2,3\dots k,$$

where 'k' is the such least integer greater than $\frac{1}{\delta} \cdot |\gamma| = \frac{1}{\delta} \cdot L = L/\delta$

Since $\|x - y\| > \eta$, then

$$\begin{aligned} \|f(x) - f(y)\| &= \|f(\gamma(t_0)) - f(\gamma(t_k))\| \\ &= \|f(\gamma(t_0)) - f(\gamma(t_1)) + f(\gamma(t_1)) - f(\gamma(t_2)) + \dots + f(\gamma(t_k))\| \\ &\leq \sum_{i=1}^k \|f(\gamma(t_i)) - f(\gamma(t_{i-1}))\| \\ &< k < 1 + L/\delta \\ &< K/\delta \|x - y\| + 1 \quad (\text{K is positive constant}) \\ &< \frac{K \|x - y\|}{\delta} + \frac{\|x - y\|}{\eta} \\ &< \left(\frac{K}{\delta} + \frac{1}{\eta}\right) \|x - y\| \end{aligned}$$

It follows that f satisfies the LCL, and the theorem is proved.

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