ॐ भूर्भुवः स्वः तत्सवितुर्वरेण्यं भर्गो देवस्य धीमहि धियो यो नः प्रचोदयात्।

Kalika

Mathematical Methods (204)

M.Sc. Tech. Mathematics (2nd Sem)

Central University of Rajasthan.

उड़ जायेगा हंस एकदिन अकेला

To find P.I

① $y'' + y = x \Rightarrow (D^2+1)y = x \Rightarrow y = CF + PI$

ⓐ $P.I = \frac{1}{(D^2+1)}(x) = (D^2+1)^{-1}.x = (1 - D^2 + D^4 + \dots)x$

$= x$

ⓑ $\frac{1}{(D^2+a^2)} \sin ax = \frac{-x}{2a} \cos ax$

for polar form $\frac{\partial F}{\partial r} - \frac{d}{d\theta}\left(\frac{\partial F}{\partial r'}\right) = 0$

I II

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Syllabus

UNIT-I Laplace transforms and their properties, laplace transform of derivatives and integrals, Inverse laplace transformation & Convolution theorem. Impulsive function, Application of laplace transform in solving linear differential equations with constant co-efficients and with variable coefficients.

Fourier transforms: Definition & Properties of Fourier sine, cosine and complex transforms. Development of the Fourier integral from the Fourier series - simple Applications. Application to the solution of simple boundary value problems by laplace and Fourier transforms.

UNIT-II (84) Calculus of variations - Functional. Euler's equation and application, Geodesics, Geodesics on a sphere of radius 'a', isoperimetric (94) problems, Variational problem with several variables, approximate sol^n of boundary (114) value problems by Rayleigh - Ritz method, Hamilton's principle, Lagrange equation and applications

UNIT-III (115) Integral equation of the first and second kind of Fredholm and Volterra type, Solutions with separable kernels, characteristic no.s and eigenfunctions. resolvent kernel.

Book: (1) Applied Mathematics for Engineers and Physicists by L.A. Pipe
(2)

Laplace Transform

:- let $F(t)$ be a f^n defined for all the value of t $(t \geq 0)$. then laplace transform of $F(t)$ is a f^n of a new variable and given by -

$$L\{F(t), s\} = L\{F(t)\} = f(s) =$$
$$= F(s) = \int_0^\infty e^{-st} f(t)\,dt \quad , s \geq 0$$

The Laplace transform $F(t)$ is said to exist if the integral (1) converges for some value of s, otherwise it doesn't exist.

Remark:

(1) $\mathcal{L}$ is called Laplace transform operator.

(2) The operator of multiplying $F(t)$ by e^{-st} and integrating b/n the limit 0 to ∞.

Eg. $f(t) = 1$, $L(1) = 1/s$

$$L(e^{at}) = \frac{1}{s-a}$$

$$L(e^{-at}) = \frac{1}{s+a}$$

H.W.

(i) $$L(1) = \int_0^\infty e^{-st}(1)\,dt = \int_0^\infty e^{-st}\,dt$$

$$= \left(\frac{e^{-st}}{-s}\right)_0^\infty = \frac{e^{-\infty}}{-s} - \frac{e^0}{-s}$$

$$= 0 + \frac{1}{s} = \frac{1}{s} \quad \text{Ans}$$

(ii) $$L(e^{at}) = \int_0^\infty e^{-st} e^{at}\,dt$$

$$= \int_0^\infty e^{-(s-a)t}\,dt = \left(\frac{e^{-(s-a)t}}{-(s-a)}\right)_0^\infty$$

$$= \frac{e^{-\infty}}{-(s-a)} - \frac{e^0}{-(s-a)} = 0 + \frac{1}{s-a} = \frac{1}{s-a}$$

(iii) $L(e^{-at}) = \int_0^\infty e^{-st} e^{-at}\,dt$

$$= \int_0^\infty e^{-(s+a)t}\,dt = \left(\frac{e^{-(s+a)}}{-(s+a)}\right)_0^\infty$$

$$= \frac{e^{-\infty}}{-(s+a)} - \frac{e^0}{-(s+a)} = 0 + \frac{1}{s+a}$$

$$= \frac{1}{(s+a)} \quad \text{Ans}$$

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* Properties of Laplace transform:

(1) Linearity Property of Laplace transform:
we know that by defn of L.T —

$$L(f(t)) = \int_0^\infty e^{-st} f(t)\,dt$$

$$L(a_1 f_1(t) + a_2 f_2(t)) = a_1 L(f_1(t)) + a_2 L(f_2(t))$$

where a_1, a_2 are constant.

Proof:- $L(a_1 f_1(t) + a_2 f_2(t)] = \int_0^\infty [a_1 f_1(t) + a_2 f_2(t)] e^{-st}\,dt$

$$= a_1 \int_0^\infty e^{-st} f_1(t)\,dt + a_2 \int_0^\infty e^{-st} f_2(t)\,dt$$

$$= a_1 L[f_1(t)] + a_2 L[f_2(t)]$$

(2) The first shifting theorem or first translation:—

Statement: If $L[f(t)] = f(s)$, then $L[e^{at} f(t)] = f(s-a)$ where a is real or complex const.

Proof: $\because L[f(t)] = f(s) = \int_0^\infty e^{-st} f(t)\,dt$

$$L[e^{at} f(t)] = \int_0^\infty e^{-st} e^{at} f(t)\,dt$$

$$= \int_0^\infty e^{-(s-a)t} f(t)\,dt = \int_0^\infty e^{-rt} f(t)\,dt$$

$$= f(r) = f(s-a). \quad (\text{where } r = s-a)$$

(3) Laplace transform of some Elementary function:—

To find L.T. of function $F(t) = t^n$ where n be any real no. > -1

proof:- Given that $L[F(t)] = f(s)$ — (1)
By definition of laplace transform —

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

$$L[t^n] = \int_0^\infty e^{-st} t^n\,dt \quad (\text{putting } y = st)$$

$y = st$

$\Rightarrow \frac{dy}{dt} = s \Rightarrow dt = \frac{dy}{s}$

Limit: No change

$$L(t^n) = \int_0^\infty e^{-y} \left(\frac{y}{s}\right)^n \frac{dy}{s}$$

$$= \underbrace{\frac{1}{s^{n+1}} \int_0^\infty e^{-y} y^n\,dy}_{\text{Gamma } f^n}$$

$$L(t^n) = \frac{\Gamma(n+1)}{s^{n+1}} = \frac{n!}{s^{n+1}}$$

$$L[e^{at}t^n] = \frac{n!}{(s-a)^{n+1}} \quad \text{(use first shifting theorem)}$$

Since the integral is convergent if $s>a$, and divergent if $s<a$. We must take $s>a$ for the L.T

Problem (1) $f(t) = \sin at$

$$L[\sin at] = \int_0^\infty e^{-st}\sin at\, dt$$

We know that $e^{iat} = \cos at + i\sin at$

& $e^{-iat} = \cos at - i\sin at$

$$\therefore \sin at = \frac{e^{iat} - e^{-iat}}{2i}$$

$$= \int_0^\infty e^{-st}\left(\frac{e^{iat}-e^{-iat}}{2i}\right)dt$$

$$= \frac{1}{2i}\left[\int_0^\infty e^{-(s-ia)t}\,dt - \int_0^\infty e^{-(s+ia)t}\,dt\right]$$

$$= \frac{1}{2i}\left[\frac{-e^{-(s-ia)t}}{(s-ia)} + \frac{e^{-(s+ia)t}}{(s+ia)}\right]_0^\infty$$

H.W

(1) $L[\sin at] = \frac{a}{s^2+a^2}$

(2) $L[\cos at] = \frac{s}{s^2+a^2}$

(3) $L[\sinh at] = \frac{a}{s^2-a^2}$

(4) $L[\cosh at] = \frac{s}{s^2-a^2}$

Sol^n (3) $\sinh at = \frac{e^{at} - e^{-at}}{2}$

$$\therefore L[\sinh at] = \int_0^\infty e^{-st}\left\{\frac{e^{at}-e^{-at}}{2}\right\}dt$$

$$= \frac{1}{2}\int_0^\infty e^{-(s-a)t}\,dt - \frac{1}{2}\int_0^\infty e^{-(s+a)t}\,dt$$

$$= \frac{1}{2}\left[\frac{-e^{-(s-a)}}{(s-a)}\right]_0^\infty - \frac{1}{2}\left[\frac{-e^{-(s+a)}}{(s+a)}\right]_0^\infty$$

$$= \frac{1}{2}\left[0 + \frac{1}{s-a}\right] - \frac{1}{2}\left[0 + \frac{1}{s+a}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] = \frac{1}{2}\,\frac{s+a-s+a}{(s-a)(s+a)}$$

$$= \frac{1}{2}\cdot\frac{2a}{(s-a)(s+a)} = \frac{a}{(s-a)(s+a)}$$

$$= \frac{a}{s^2-a^2}$$

(5) $L\left[\frac{e^{at}-1}{a}\right]$

$$L\left[\frac{e^{at}-1}{a}\right] = \int_0^\infty e^{-st}\left\{\frac{e^{at}-1}{a}\right\}dt$$

$$= \frac{1}{a}\int_0^\infty e^{-(s-a)t}\,dt - \frac{1}{a}\int_0^\infty e^{-st}\,dt$$

$$= \frac{1}{a}\left[\frac{-e^{-(s-a)t}}{(s-a)}\right]_0^\infty - \frac{1}{a}\left[-\frac{e^{-st}}{s}\right]_0^\infty$$

$$= \frac{1}{a}\left[\frac{1}{s-a}\right] - \frac{1}{a}\left[\frac{1}{s}\right]$$

$L\left(\frac{e^{at}}{a}\right) - L\left(\frac{1}{a}\right) = \frac{1}{a}L(e^{at}) - \frac{1}{a}L(1) = \frac{1}{a}\cdot\frac{1}{s-a} - \frac{1}{a}\cdot\frac{1}{s}$

$\cos 2t = \cos^2 t - \sin^2 t = 2\cos^2 t - 1 = 1 - 2\sin^2 t$

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$$= \frac{1}{a}\left[\frac{1}{s-a} - \frac{1}{s}\right]$$

$$= \frac{1}{a}\left[\frac{s-s+a}{s(s-a)}\right] = \frac{1}{s(s-a)} \quad \text{Ans}$$

(6) $L[\cos^2 t]$ (8) $L[\cos^2 t/2]$

(7) $L[\sin^2 t]$ (9) $L[\sin^2 t/2]$

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Integral Transform :—

Suppose that the f^n $K(s,t)$ is a known f^n of two variable s & t. Then integral

$$\int_{-\infty}^{\infty} K(s,t)\, F(t)\, dt \quad \text{—— (1)}$$

the eq^n (1) is convergent then eq^n (1) is called integral transform of the given f^n. It is denoted by

$\bar{F}(s)$ or $T\{f(t)\}$ or $f(s)$

Thus we can write

$$\bar{F}(s) = \int_{-\infty}^{\infty} K(s,t) F(t)\, dt$$

the f^n $K(s,t)$ is known as the Kernal of the transformation, has s as a parameter (real or complex) independent of t.

Remarks: If we choose

$$K(s,t) = \begin{cases} 0 & ; t<0 \\ e^{-st} & ; t>0 \end{cases}$$

then $f(s)$ is known as Laplace transform

Que

Prove that —

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$\sinh at = \frac{e^{at} - e^{-at}}{2}$

(1) $L[e^{at} t^n] = \frac{n!}{(s-a)^{n+1}}$ (Directly By First shifting prop.)

$$L[e^{at}t^n] = \int_0^\infty e^{-st} e^{at} t^n\, dt$$

(2) $L[e^{at}\cosh bt] = \frac{s-a}{(s-a)^2 - b^2}$

(3) $L[e^{at}\sinh bt] = \frac{b}{(s-a)^2 - b^2}$

(4) $L[e^{at}\sin bt] = \frac{b}{(s-a)^2 + b^2}$

(5) $L[e^{at}\cos bt] = \frac{s-a}{(s-a)^2+b^2}$

(Directly may be done also by first shifting property)

(6) if $L[F(t)] = \bar{F}(s)$. Show that

(i) $L[\sinh at\, F(t)] = \frac{1}{2}[\bar{F}(s-a) - \bar{F}(s+a)]$

(ii) $L[\cosh at\, F(t)] = \frac{1}{2}[\bar{F}(s-a) + \bar{F}(s+a)]$

(7) Evaluate —

(i) $\sinh 2t \sin 3t$ Ans: $\frac{12s}{s^4+10s^2+169}$

(ii) $\cosh 3t \cos 2t$ Ans: $\frac{2s(s^2-5)}{s^4-10s^2+169}$

(8) Prove that —

(i) $L[t \sin at] = \frac{2as}{(s^2+a^2)^2}$

(ii) $L[t \cos at] = \frac{s^2-a^2}{(s^2+a^2)^2}$

[5] * **Transform of Periodic function**

If $f(t)$ is a periodic function with period T i.e. $f(t) = f(t+T)$ then prove that $L[f(t)] = \frac{\int_0^T e^{-st} f(t)\, dt}{1 - e^{-sT}}$

By definition —

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^T e^{-st} f(t)\,dt + \int_T^{2T} e^{-st} f(t)\,dt + \int_{2T}^{3T} f(t)\,dt \longrightarrow$$

In the second integral $t = u+T$ in the third integral $t = u+2T$ and so on.

We get —

$$\int_0^T e^{-st} f(t)\,dt + \int_0^T e^{-s(u+T)} f(u+T)\,du + \int_0^T e^{-s(u+2T)} f(u+2T)\,du + \dots$$

$$\Rightarrow \int_0^T e^{-st} f(t)\,dt + e^{-sT}\int_0^T e^{-su} f(u+T)\,du + e^{-2sT}\int_0^T e^{-su} f(u+2T)\,du + \dots$$

$$\Rightarrow \int_0^T e^{-st} f(t)\,dt + e^{-sT}\int_0^T e^{-su} f(u)\,du + e^{-2sT}\int_0^T e^{-su} f(u)\,du \dots$$

$$= \cancel{(1 + e^{-sT} + e^{-2sT} + \dots)}$$

$$= \int_0^T e^{-st} f(t)\,dt + e^{-sT}\int_0^T e^{-st} f(t)\,dt + e^{-2sT}\int_0^T e^{-st} f(t)\,dt + \dots$$

$$= (1 + e^{-sT} + e^{-2sT} + \dots)\int_0^T e^{-st} f(t)\,dt$$

$$= \frac{1}{(1-e^{-sT})}\int_0^t e^{-st} f(t)\,dt$$

$$= \frac{\int_0^T e^{-st} f(t)\,dt}{(1-e^{-sT})}$$

Que (9) Find the Laplace Transform for the given fn

$$f(t) = \begin{cases} \sin\omega t & 0 < t < \pi/\omega \\ 0 & \pi/\omega < t < 2\pi/\omega \end{cases}$$

(Ans: $\frac{\omega e^{-s\pi/\omega} + \omega}{(1-e^{-2\pi s/\omega})(s^2+\omega^2)}$)

Solⁿ:

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^{\pi/\omega} e^{-st} f(t)\,dt + \int_{\pi/\omega}^{2\pi/\omega} e^{-st} f(t)\,dt$$

$$= \int_0^{\pi/\omega} e^{-st}\sin\omega t\,dt + \int_{\pi/\omega} e^{-st}\cdot 0\,dt$$

type-I putting $\sin\omega t = \frac{e^{i\omega t} - e^{-i\omega t}}{2i}$

we have $\frac{1}{2i}\int_0^{\pi/\omega} e^{-st}(e^{i\omega t} - e^{-i\omega t})\,dt$

$$= \frac{1}{2i}\left[\int_0^{\pi/\omega} e^{-(s-i\omega)t}\,dt + \int_0^{\pi/\omega} e^{-(s+i\omega)t}\,dt\right]$$

$$= \frac{1}{2i}\left[\left[\frac{e^{-(s-i\omega)t}}{i\omega - s}\right]_0^{\pi/\omega} - \left[\frac{e^{-(s+i\omega)t}}{s+i\omega}\right]_0^{\pi/\omega}\right]$$

$$= \frac{1}{2i}\left[\frac{e^{-s\pi/\omega}e^{i\pi}}{i\omega - s} - \frac{1}{i\omega - s} - \frac{e^{-s\pi/\omega}e^{-\pi}}{i\omega + s} + \frac{1}{s+i\omega}\right]$$

$$= \frac{1}{2i}\left[\frac{-e^{-s\pi/\omega}}{i\omega - s} + \frac{e^{-s\pi/\omega}}{i\omega + s} + \frac{1}{s+i\omega} - \frac{1}{i\omega - s}\right]$$

$$= \frac{1}{2i}\left[\frac{-i\omega e^{-s\pi/\omega} - s e^{-s\pi/\omega} + i\omega - s - s - i\omega}{-(\omega^2+s^2)}\right]$$

$$= \frac{1}{2i}\left[\frac{-e^{-s\pi/\omega}[s+i\omega] - 2s}{(\omega^2+s^2)}\right]$$

type-II let $I = \int e^{-st}\sin\omega t\,dt$

$$= \sin\omega t\,\frac{e^{-st}}{-s} + \int \frac{\omega}{s}\cos\omega t\, e^{-st}\,dt$$

$$\frac{\omega}{s}\left[\cos\omega t\,\frac{e^{-st}}{-s} - \int \frac{\omega}{s}\sin\omega t\, e^{-st}\right]$$

$I = \frac{\sin\omega t\, e^{-st}}{-s} - \frac{\omega \cos\omega t}{s^2}\cdot e^{-st} - \frac{\omega^2}{s^2}\int e^{-st}\sin\omega t\,dt$

$= -\frac{\sin\omega t\, e^{-st}}{s} - \frac{\omega}{s^2}\cos\omega t\, e^{-st} - \frac{\omega^2}{s^2} I$

$\left(1+\frac{\omega^2}{s^2}\right) I = -\frac{\sin\omega t}{s} e^{-st} - \frac{\omega}{s^2}\cos\omega t\, e^{-st}$

$I = \left(\frac{-s^2}{s^2+\omega^2}\right)\left[\left(\frac{\sin\omega t}{s}e^{-st} + \frac{\omega}{s^2}\cos\omega t\, e^{-st}\right)\right]$

$[I]_0^{\pi/\omega} = \frac{-s^2}{s^2+\omega^2}\left[\frac{-\omega}{s^2} e^{-s\pi/\omega} - \frac{\omega}{s^2}\right]$

$= \frac{s^2\omega}{s^2(s^2+\omega^2)}\left[e^{-s\pi/\omega} + \omega\right]$

$= \frac{\omega}{(s^2+\omega^2)}\left[\omega + e^{-s\pi/\omega}\right]$??

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Transform of some special functions

[6] $J_0(x)$ and $J_1(x)$ where

$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2\cdot4^2} - \frac{x^6}{2^2\cdot4^2\cdot6^2} + \cdots$ — (1)

and $J_0'(x) = -J_1(x)$ and $L[F'(t)] = sf(s) - f(0)$

Bessel function

To prove $L[J_0(x)] = \frac{1}{\sqrt{s^2+1}}$

$L[J_1(x)] = 1 - \frac{s}{\sqrt{s^2-1}}$

Sol^n: — $L[J_0(x)] = L[1] - L\left[\frac{x^2}{2^2}\right] + L\left[\frac{x^4}{2^2\cdot4^2}\right] - L\left[\frac{x^6}{2^2\cdot4^2\cdot6^2}\right] + \cdots$

$= \frac1s - \frac{1}{2^2}\frac{\lfloor 2}{s^3} + \frac{\lfloor 4}{2^2\cdot4^2 s^5} - \frac{\lfloor 6}{2^2\cdot4^2\cdot6^2}\frac{1}{s^7} + \cdots$

$= \frac1s - \frac{1}{2\cdot s^3} + \frac12\cdot\frac{3}{4 s^5} - \frac12\cdot\frac34\cdot\frac56\cdot\frac{1}{s^7} + \cdots$

$= \frac1s\left[1 - \frac{1}{2s^2} + \frac12\cdot\frac{3}{4s^4} - \frac12\cdot\frac34\cdot\frac56\cdot\frac{1}{s^6} + \cdots\right]$

$= \frac1s\left[1+\frac{1}{s^2}\right]^{-1/2} = \frac1s\left[\frac{s^2+1}{s^2}\right]^{-1/2} = \frac1s\left[\frac{s}{\sqrt{s^2+1}}\right] = \frac{1}{\sqrt{s^2+1}}$

$[J_1(x)] = d\left[\frac{2x}{2^2} - \frac{4x^3}{2^2\cdot4^2} + \frac{6x^5}{2^2\cdot4^2\cdot6^2} + \cdots\right]$

$= \frac{x}{2} - \frac{x^3}{2^2\cdot4} + \frac{x^5}{2^2\cdot4^2\cdot6} + \cdots$

$L[J_1(x)] = L\left[\frac x2\right] - L\left[\frac{x^3}{2^2\cdot4}\right] + L\left[\frac{x^5}{2^2\cdot4^2\cdot6}\right] - \cdots$

$= \frac{1}{2s^2} - \frac{3!}{2^2\cdot4\cdot s^4} + \frac{\lfloor 5}{2^2\cdot4^2\cdot6 s^6} - \cdots$

$= \frac{1}{2s^2} - \frac{3}{2\cdot4\cdot s^4} + \frac12\cdot\frac34\cdot\frac{5}{6s^6} + \cdots$

$= \frac{1}{s^2}\left[1 - \frac{1\cdot3}{2\cdot4 s^2} + \frac{1\cdot3\cdot5}{2\cdot4\cdot6 s^4} + \cdots\right]$

$= \frac{1}{s^2}\left[1+\frac{1}{s^2}\right]^{-1/2} = \frac{1}{s^2}\frac{s}{\sqrt{1+s^2}} = \frac{1}{s\sqrt{1+s^2}}$

[7] Change of scale property :—

If $L\{f(t)\} = f(s)$ then —

$L[f(at)] = \frac1a f\left(\frac sa\right)$

Sol^n: $L[f(at)] = \int_0^\infty e^{-st} f(at)\,dt$

let $at = u \Rightarrow t = u/a \Rightarrow dt = du/a$

$\Rightarrow \int_0^\infty e^{-su/a}\frac{f(u)}{a}\,du$

$= \int_0^\infty e^{-ru}\frac{f(u)}{a}\,du$ (where $r = s/a$)

$= \frac1a f(r) = \frac1a f(s/a)$ [$\because r = s/a$]

(8) Prove that $L\left[\frac{\cos\sqrt t}{\sqrt t}\right] = \sqrt{\frac{\pi}{s}}\, e^{-1/4s}$

$\because \frac{\cos\sqrt t}{\sqrt t} = \frac{1}{\sqrt t}\left[1 - \frac{(\sqrt t)^2}{2!} + \frac{(\sqrt t)^4}{4!} - \frac{(\sqrt t)^6}{6!} + \cdots\right.$

H.W.

$$= \left[(t)^{-1/2} - \frac{t^{1/2}}{2!} + \frac{t^{3/2}}{4!} - \frac{t^{5/2}}{6!} + \cdots \right]$$

$$\therefore L\left[\frac{\cos\sqrt{t}}{\sqrt{t}}\right] = \frac{\Gamma 1/2}{s^{1/2}} - \frac{1}{2!}\frac{\Gamma 3/2}{s^{3/2}} + \frac{1}{4!}\frac{\Gamma 5/2}{s^{5/2}} - \frac{1}{6!}\frac{\Gamma 7/2}{s^{7/2}} + \cdots$$

we know that — $\Gamma 1/2 = \sqrt{\pi}$

$$\left(\Gamma 3/2 = \frac{1}{2}\Gamma 1/2 = \frac{1}{2}\sqrt{\pi},\ \Gamma 5/2 = \frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi} \right)$$

$$= \frac{\sqrt{\pi}}{s^{1/2}} - \frac{\sqrt{\pi}/2}{s^{3/2}}\cdot\frac{1}{(2)!} + \frac{3/2\cdot 1/2\sqrt{\pi}}{s^{5/2}}\cdot\frac{1}{4!} - \frac{5/2\cdot 3/2\cdot\frac{1}{2}\sqrt{\pi}}{s^{7/2}}\cdot\frac{1}{6!} + \cdots$$

$$= \frac{\sqrt{\pi}}{s^{1/2}}\left[1 - \frac{1}{2\cdot 2!\, s} + \frac{3\cdot 1}{2\cdot 2\cdot 4!\, s^2} - \frac{5\cdot 3\cdot 1}{s^3\cdot 2\cdot 2\cdot 2\cdot 6!} \cdots\right]$$ *

$$= \sqrt{\frac{\pi}{s}}\left[1 - \frac{1}{(2s)}\cdot\frac{1}{2!} + \frac{3\cdot 1}{(2s)^2\, 4!} - \frac{5\cdot 3\cdot 1}{(2s)^3\, 6!} + \cdots\right]$$

$$= \sqrt{\frac{\pi}{s}}\cdot e^{-1/4s}$$

(9) (c) Find $L\left[\frac{\sin at}{t}\right]$ given that

$$L\left[\frac{\sin t}{t}\right] = \tan^{-1} 1/s.$$

Sol^n Since, $L\left[\frac{\sin t}{t}\right] = \tan^{-1} 1/s$

H.W.

$\therefore$ By the change of scale property that

if $L[F(t)] = f(s)$ then $L[f(at)] = 1/a\, f(s/a)$.

$$\therefore L\left[a\frac{\sin at}{at}\right] = L\left[\frac{\sin at}{t}\right]$$

$$\therefore L\left[\frac{\sin at}{t}\right] =$$

$$a\, L\left[a\frac{\sin at}{at}\right] = a\left[\frac{1}{a} f\left(\frac{s}{a}\right)\right] = a\left[\frac{1}{a}\cdot\tan^{-1}\left(\frac{1}{s/a}\right)\right]$$

$$= \tan^{-1}\left(\frac{a}{s}\right) \quad \text{Ans}$$

self (b) If $L\{F(t)\} = \frac{s^2-s+1}{(2s+1)^2(s-1)}$: ST $L[F(2t)] = \frac{s^2-2s+4}{4(s+1)^2(s-2)}$

By change of scale property, we have

$$L[F(2t)] = \frac{1}{2} f(s/2) = \frac{1}{2}\left[\frac{(s/2)^2 - s/2 + 1}{(2\cdot s/2+1)^2(s/2-1)}\right]$$

$$= \frac{1}{2}\left[\frac{\frac{s^2}{4} - \frac{s}{2} + 1}{(s+1)^2\left(\frac{s-2}{2}\right)}\right]$$

$$= \frac{s^2-2s+4}{4(s+1)^2(s-2)} \quad \text{Ans}$$

$((-x - I(d) / p - 20)$

(10) ## Transform of Error Function:—

$$\operatorname{erf}(\sqrt{x}) = \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{x}} e^{-t^2}\,dt \quad \text{— (ii)}$$

$$\operatorname{erf}(t) = \frac{2}{\sqrt{\pi}}\int_0^t e^{-u^2}\,du$$

$$L[\operatorname{erf}(\sqrt{x})] = \frac{1}{s\sqrt{s+1}}$$

In eq^n(i), [(6)], Applying Laplace transform them we have—

$$L[J_0(x)] = L[1] + \frac{1}{2^2}L[x^2] + \frac{1}{2^2\cdot 4^2}L[x^4] - \frac{1}{2^2\cdot 4^2 6^2}L[x^6] + \cdots$$

$$= \frac{1}{s} - \frac{1}{2^2}\cdot\frac{\Gamma 3}{s^3} + \frac{1}{2^2 4^2}\frac{\Gamma 5}{s^5} - \frac{\Gamma 7}{2^2 4^2 6^2 s^7} + \cdots$$

$$= \frac{1}{s}\left[1 - \frac{1.2}{2^2 s^2} + \frac{3\cdot 1}{2\cdot 4\, s^4} - \frac{1.3.5}{2.4.6}\cdot\frac{1}{s^6} + \cdots\right]$$

$$= \frac{1}{s}\left[1+\frac{1}{s^2}\right]^{-1/2}$$

$$= \frac{1}{s}\left[\frac{s^2+1}{s^2}\right]^{-1/2} = \frac{1}{s}\frac{\sqrt{s^2}}{\sqrt{s^2+1}} = \frac{1}{\sqrt{s^2+1}}$$

Now,

$$J_0'(x) = -J_1(x)$$

$$\Rightarrow -\frac{2x}{2^2} + \frac{4x^3}{2^2 4^2} - \frac{6x^5}{2^2 4^2 6^2} + \dots = -J_1(x)$$

$$\Rightarrow J_1(x) = \frac{x}{2} - \frac{x^3}{2^2\cdot 4} + \frac{x^5}{2^2\cdot 4\cdot 6} - \dots$$

$$L[J_1(x)] = L\left[\frac{x}{2}\right] - \frac{1}{2^2\cdot 4}L[x^3] + \frac{1}{2^2\cdot 4\cdot 6}L[x^5]$$

$\Gamma 2 = 1! = 1$
$\Gamma 4 = 3!$
$\Gamma 6 = 5!$

$$= \frac{1}{2}\cdot\frac{\Gamma 2}{s^2} - \frac{1}{2^2\cdot 4}\frac{\Gamma 4}{s^4} + \frac{1}{2^2\cdot 4\cdot 6}\frac{\Gamma 6}{s^6} - \dots$$

$$= \frac{1}{2\cdot s^2}\left[1 - \frac{\lfloor 3}{2\cdot 4\cdot s^2} + \frac{\lfloor 5}{2\cdot 4\cdot 6\, s^4} - \dots\right]$$

$$= \frac{1}{2s^2}\left[1 - \frac{3\cdot 1}{4\cdot s^2} + \frac{5\cdot 3\cdot 1}{6\, s^4} - \dots\right]$$

In eqn (ii) $\quad erf(\sqrt{x}) = \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{x}}\left[1 - \frac{t^2}{1!} + \frac{t^4}{2!} - \frac{t^6}{3!} + \dots\right]dt$

$$erf[\sqrt{x}] = \frac{2}{\sqrt{\pi}}\left[t - \frac{t^3}{1\cdot 3} + \frac{t^5}{5\cdot 2!} - \frac{1}{7}\cdot\frac{t^7}{3!} + \dots\right]_0^{\sqrt{x}}$$

$$= \frac{2}{\sqrt{\pi}}\left[\sqrt{x} - \frac{x^{3/2}}{3} + \frac{x^{5/2}}{5\cdot 2!} - \frac{x^{7/2}}{7\cdot 3!} + \dots\right]$$

Now, using laplace transform

$$L[erf[\sqrt{x}]] = \frac{2}{\sqrt{\pi}}L\left[\sqrt{x} - \frac{x^{3/2}}{3} + \dots\right]$$

$$L[erf(\sqrt{x})] = \frac{2}{\sqrt{\pi}}\left[\frac{\Gamma 3/2}{s^{3/2}} - \frac{\Gamma 5/2}{3\cdot s^{5/2}} + \frac{\Gamma 7/2}{2\cdot 5\cdot s^{7/2}} - \frac{\Gamma 9/2}{7\cdot s^{9/2}\cdot 3!} + \dots\right]$$

$$= \frac{2}{\sqrt{\pi}\cdot s^{3/2}}\left[\frac{\frac{1}{2}\sqrt{\pi}}{1} - \frac{\frac{3}{2}\frac{1}{2}\sqrt{\pi}}{3\cdot s} + \frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{5\cdot 2!\cdot s^{5/2}} - \dots + \frac{\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{7\cdot 3!\cdot s^{7/2}} + \dots\right]$$

---x---

$$erf(t) = \frac{2}{\sqrt{\pi}}\int_0^t e^{-x^2}dx$$

self

Best pt

$$\therefore erf[\sqrt{t}] = \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}}\left[1 - x^2 + \frac{x^4}{\lfloor 2} - \frac{x^6}{\lfloor 3} + \frac{x^8}{\lfloor 4} - \dots\right]dx$$

$$= \frac{2}{\sqrt{\pi}}\left[x - \frac{x^3}{3} + \frac{x^5}{5\cdot\lfloor 2} - \frac{x^7}{7\cdot\lfloor 3} + \frac{x^9}{9\cdot\lfloor 4} - \dots\right]_0^{\sqrt{t}}$$

$$= \frac{2}{\sqrt{\pi}}\left[\sqrt{t} - \frac{t^{3/2}}{3} + \frac{t^{5/2}}{5\cdot 2!} - \frac{t^{7/2}}{7\cdot 3!} + \dots\right]$$

$$\therefore L[erf(\sqrt{t})] = \frac{2}{\sqrt{\pi}}\left[\frac{\frac{1}{2}\sqrt{\pi}}{s^{3/2}} - \frac{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{3\cdot s^{5/2}} + \frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{5\cdot 2!\, s^{7/2}}\right]$$

$$= \frac{1}{s^{3/2}}\left[1 - \frac{1}{2s} + \frac{3}{(2\cdot s)^2\cdot 2!} - \dots\right]$$

$$= \frac{1}{s^{3/2}}\cdot\left(1+\frac{1}{s}\right)^{-1/2} = \frac{1}{s\sqrt{s}}\frac{\sqrt{s}}{\sqrt{s+1}}$$

$$= \frac{1}{s\sqrt{s+1}}$$

self

Second shifting / translation theorem:

Also known as Heaviside shifting function

If $L[f(t)] = f(s)$ and

$$g(t) = \begin{cases} f(t-a), & t>a \\ 0 & t<a \end{cases}$$

then $L[g(t)] = e^{-as}f(s)$

Proof:- We have

$$L[g(t)] = \int_0^\infty e^{-st} g(t)\,dt = \int_0^a e^{-st} g(t)\,dt + \int_a^\infty e^{-st} g(t)\,dt$$

$$= \int_0^a e^{-st} \cdot 0\,dt + \int_a^\infty e^{-st} g(t)\,dt$$

$$= \int_a^\infty e^{-st} \cdot f(t-a)\,dt$$

$t-a=x$
$dt=dx$
$t=a+x$

$$= \int_0^\infty e^{-s(a+x)} \cdot f(x)\,dt$$

$$= e^{-as} \int_0^\infty e^{-sx} f(x)\,dt = e^{-as} \cdot f(s)$$

$$\Rightarrow \boxed{L[f(t-a)] = e^{-as} f(s)}$$

<u>Piecewise Continuous</u>: A f^n f is called piece-wise continuous on an interval $a \le t \le b$. If the interval (a, b) can be sub-divided into a finite number of intervals in each of which the f^n f is continuous and has finite right and left hand limits.

We note that the right and left hand limits of f at t are denoted by

$$\lim_{\varepsilon \to 0} f(t_1 + \varepsilon) = f(t_1 + 0) = f(t^+) \quad (\varepsilon > 0)$$

$$\lim_{\varepsilon \to 0} (f(t_1 - \varepsilon) = f(t_1 - 0) = f(t_1^-) \quad (\varepsilon > 0)$$

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* Laplace transform of the Derivatives

Theorem (11) let $F(t)$ be continuous $\forall t \geq 0$ and be of exponential order as $t \to \infty$. If $\frac{d}{dt}(F(t))$ is a class A. Then laplace transformation of derivative exists. When $s > \eta$ and

$$L\left[\frac{d}{dt} f(t)\right] = s \cdot L[F(t)] - F(0)$$

Que I

$L\left[\sqrt{t} - \frac{1}{\sqrt{t}}\right]^3 =$

let $f(t) = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)^3 = t^{3/2} - 3t^{1/2} + 3t^{-1/2} - t^{-3/2}$

$$\therefore L[F(t)] = \frac{\Gamma 5/2}{s^{5/2}} - \frac{3\,\Gamma 3/2}{s^{3/2}} + \frac{3\,\Gamma 1/2}{s^{1/2}} + \frac{\Gamma 1/2}{s^{-1/2}}$$

$$= \frac{3/2 \cdot 1/2 \sqrt{\pi}}{s^{5/2}} - \frac{3 \cdot 1/2 \cdot \sqrt{\pi}}{s^{3/2}} + \frac{3\sqrt{\pi}}{s^{1/2}} + \frac{-2\sqrt{\pi}}{s^{-1/2}}$$

$$\therefore L[f(t)] = \frac{3\sqrt{\pi}}{4 \cdot s^{5/2}} - \frac{3\sqrt{\pi}}{2\, s^{3/2}} + \frac{3\sqrt{\pi}}{s^{1/2}} + \frac{2\sqrt{\pi}}{s^{-1/2}}$$

$$= \frac{\sqrt{\pi}}{4}\left[\frac{3}{s^{5/2}} - \frac{6}{s^{3/2}} + \frac{12}{s^{1/2}} + \frac{8}{s^{-1/2}}\right]$$

Proof of thm

$$L\left[\frac{d}{dt} F(t)\right] = L(f'(t)) = \int_0^\infty e^{-st} f'(t)\,dt$$

$$= \int_0^\infty e^{-st}\, d(f(t))$$

By Integrating

$$L[F'(t)] = \left(e^{-st} f(t)\right)_0^\infty - \int_0^\infty f(t)\left[e^{-st}(-s)\right] dt$$

$$= -f(0) + s\int_0^\infty e^{-st} f(t)\,dt$$

$$\therefore \boxed{L[F'(t)] = s\bar{f}(s) - f(0)}$$ Hence Proved

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$$L[F'(t)] = L\left[\frac{d}{dt}F(t)\right] = s\,L[F(t)] - F(0)$$

Proof we devide the proof into two parts (two cases)

Case-I let $F(t)$ be continuous $\forall t \geq 0$

Case-II, Let $F'(t)$ be piecewise cts $\forall t \geq 0$

Pf: Case-I. By definition of Laplace Transform

$$L(F(t)) = \int_0^\infty e^{-st} F(t)\,dt$$

Now, $L[F'(t)] = L\left[\frac{d}{dt} F(t)\right]$

$$= \int_0^\infty e^{-st} \frac{d}{dt} F(t)\,dt$$

$$= \left[e^{-st} F(t)\right]_0^\infty + s\int_0^\infty e^{-st} F(t)\,dt$$

$$= \lim_{t\to\infty} e^{-st} F(t) - F(0) + s\int_0^\infty e^{-st} F(t)\,dt \quad \text{—(1)}$$

$\because F(t)$ is exponential order η, we can find the constant $m > 0$. such that

$|f(t)| \leq m e^{\eta t} \quad \forall t \geq 0$ —(2)

$$\therefore |e^{-st} F(t)| \leq e^{-st} |F(t)|$$
$$\leq e^{-st}\, m e^{\eta t}$$
$$\leq m e^{-(s-\eta)t}$$

$\therefore |e^{-st} F(t)| \leq m e^{-(s-\eta)t}$ —(3)

Now, for $s > \eta$, as $t \to \infty$, $m e^{-(s-\eta)t}$

$\therefore$ From eqn (1), we see that $L[F'(t)]$ exist and given by

$$L[F'(t)] = sL[F(t)] - F(0)$$

Pf: Case-II

self

self

Remark.

(I) Suppose, $F(t)$ is not continuous at $t=0$, But limit $t\to 0$ $\lim_{t\to 0} F(t)$, i.e. The right hand limit $F'(0+0)$ exist, then we can prove that —

$$L[F'(t)] = sL[F(t)] - F(0+0)$$

(II) Suppose that $F(t)$, is not continuous at $t=a$, $(0<a<\infty)$, But RHL $F(a+0)$ and LHL $F(a-0)$ exist,

then the Result of the above thm takes the form

$$L[F'(t)] = sL[F(t)] - e^{-as}[F(a+0) - F(a-0)]$$

The quantity $\underline{F(a+0) - F(a-0)}$ is know as the 'Jump at the discontinuity $x=a$'.

III If $F(t)$ is discontinuous at $t=a_1, a_2, a_3, \ldots a_n$, we break up the integral into range from 0 to ∞.

$$L[F'(t)] = \int_0^{a_1} e^{-st} F'(t)\,dt + \int_{a_1}^{a_2} e^{-st} F'(t)\,dt + \cdots + \int_{a_n}^{\infty} e^{-st} F'(t)\,dt$$

Thm: II [1/2] let F & F'' be cts f^n for all $t\geq 0$. & F'' be of exponential order η as $t\to\infty$ if F'' is of class A. then Laplace transformation of $F''(t)$ exist. when $s>\eta$ & given by

$$L[F''(t)] = s^2 L[F(t)] - sF(0) - F'(0).$$

Pf:- For, Proving this theorem, we have two cases.

Case-I: let $F''(t)$ be cts $\forall\ t\geq 0$

Case-II: let $F''(t)$ be piece-wise cts $\forall\ t\geq 0$.

self

Find the L.T of

Problem I: Solve the differential Eqⁿ

$$y'' + 3y' + 2y = 0, \quad y(0) = 1, \; y'(0) = 2$$

Solⁿ: Here L.T. of this transformation is given by —

$$\left(\frac{s+5}{s^2+3s+2}\right)$$

$$L(y'' + 3y' + 2y) = L(0)$$

$$\Rightarrow L(y'') + 3L(y') + 2L(y) = L(0)$$

$$\Rightarrow s^2 L[F(t)] - sF(0) - F'(0) + 3[sL[F(t)] - sF(0)] + 2L[F(t)]$$

$$\Rightarrow L[F(t)][s^2 + 3s + 2] - s(1) - 2 - 3(1) = L(0)$$

$$\Rightarrow L[F(t)](s^2 + 3s + 2) - (s+5) = 0$$

$$\Rightarrow L[F(t)] = \frac{s+5}{s^2+3s+2}$$

*

Theorem-III (13)

Statement: Let $F(t)$ and its derivative $F'(t), F''(t), \ldots, F^{(n-1)}(t)$ be continuous $\forall\, t \geq 0$ and be of exponential order η as $t \to \infty$ and if $F^{(n)}(t)$ is of class A, then Laplace Transform of $F^n(t)$:

$$L[F^n(t)] = s^n L[F(t)] - s^{n-1}F(0) - s^{n-2}F'(0) - \cdots - F^{(n-1)}(0)$$ where

$$F^n(t) = \frac{d^n}{dt^n} F(t)$$

A/B/C Pf - 137

Hint: Method of pf: (1) Mathematical Induction.
(2) Lebniz theorem

Problem (III): If $L[\sin\sqrt{t}] = \frac{\sqrt{\pi}}{2s^{3/2}} e^{-\frac{1}{4s}}$, then, P.T

$$L\left[\frac{\cos\sqrt{t}}{\sqrt{t}}\right] = \sqrt{\frac{\pi}{s}}\, e^{-\frac{1}{4s}}$$

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Theorem-III ~~Laplace transform of the nth order derivative~~ General Case:—

* ~~$y''+3y'+2y=0$; $y(0)=1$,; $y'(0)=2$~~

H.W

$$L[\sin\sqrt{t}] = \frac{\sqrt{\pi}}{2s^{3/2}} e^{-1/4s}$$

$$L\left[\frac{\cos\sqrt{t}}{\sqrt{t}}\right] = ?$$

$$F(t) = \sin\sqrt{t}$$

$$F'(t) = \frac{1}{2}\frac{\cos\sqrt{t}}{\sqrt{t}}$$

$$L[F'(t)] = s.L[F(t)] - F(0)$$

$$L\left[\frac{1}{2}\frac{\cos\sqrt{t}}{\sqrt{t}}\right] = s.\frac{\sqrt{\pi}}{2.s^{3/2}} e^{-1/4s} - 0$$

$$\frac{1}{2}L\left[\frac{\cos\sqrt{t}}{\sqrt{t}}\right] = \frac{1}{2}\sqrt{\frac{\pi}{s}}\, e^{-1/4s}$$

$$L\left[\frac{\cos\sqrt{t}}{\sqrt{t}}\right] = \sqrt{\frac{\pi}{s}}\, e^{-1/4s}$$

—A—

Proof: We shall use the principle of mathematical induction to prove for $n=1$

$$L\left[\frac{d^n}{dt^n}F(t)\right] = s^n L[F(t)] - s^{n-1}F(0) - s^{n-2}F'(0) - \dots - F^{(n-1)}(0) \quad \text{—(1)}$$

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for $n=1$, eq^n (1) reduces to—

$$L[F'(t)] = sL[F(t)] - F(0)$$

Similarly for $n=2$ & $n=3$, then eq^n (1) reduces to

$$L[F''(t)] = s^2 L[F(t)] - sF(0) - F'(0)$$

$$L[F'''(t)] = s^3 L[F(t)] - s^2F(0) - sF'(0) - F''(0)$$

This shows that, the required result is true for $n=1,2,3\dots$

Now, we shall prove that the result is true for $n=k+1$.

Now, putting $n=k$, in (1), we get

$$L[F^{k+1}(t)] = s^{k+1}L[F(t)] - s^kF(0) - s^{k-1}F'(0) - \dots - F^k(0)$$

let $G(t) = F^k(t)$ (k^{th} derivative) $\Rightarrow$ $G'(t) = F^{k+1}(t)$ and $G(0) = F^k(0)$

$$L[G'(t)] = sL[G(t)] - G(0)$$

This shows that this required result is true for $n=k+1$ & $n=k$ hence by the principle of mathematical induction the required

Multiplication by positive integral power of t

Theorem: (15) If $F(t)$ is a f^n of class A and $L[F(t)] = f(s)$, then
$$L[t\,F(t)] = -f'(s) = -\frac{d}{ds} f(s)$$

pf:— let $t F(t) = h(t)$ ⇒
$$\therefore h'(t) = F(t) + t F'(t)$$

since $L(F(t)) = f(s)$
$$\Rightarrow \frac{d}{ds}(f(s)) = \frac{d}{ds}\left(L(F(t))\right) = \int_0^\infty \frac{\partial}{\partial s}\left(e^{-st} F(t)\right) dt$$
$$= \int_0^\infty \left(-t \cdot e^{-st} \cdot F(t)\, dt\right)$$
$$= -\int_0^\infty e^{-st}\left(-t F(t)\right) dt$$
$$= -L[tF(t)]$$
$$\Rightarrow -f'(s) = L[tF(t)]$$ Proved

Remark: Leibnitz Rule for
$$f(s) = \int_a^b F(s,t)\, dt = L[F(t)]$$
$$\Rightarrow \frac{d}{ds} f(s) = \int_a^b \frac{\partial}{\partial s} F(s,t)\, dt$$

Theorem: (16) let $F(t)$ be a function in t is $L[F(t)] = f(s)$; then $L[t^n F(t)] = (-1)^n \frac{d^n}{ds^n} f(s)$ for $n = 1, 2, 3 \ldots$

Proof:→ We prove this theorem by Mathematical Induction—

for $n = 1$,
$$L[t F(t)] = -\frac{d}{ds} f(s)$$

Similarly for $n=2, n=3$, eqn ① reduces to —
$$L[t^2 F(t)] = \frac{d^2}{ds^2} f(s) \text{ and } L[t^3 F(t)] = \frac{-d^3}{ds^3} f(s)$$

Suppose, this result is true for $n = k$.

$$L[t^k F(t)] = (-1)^k \frac{d^k}{ds^k} f(s).$$

Now differentiating both side w.r.t s and apply Lebnitz's rule—
$$\frac{d}{ds}\left[\int_0^\infty e^{-st} t^k F(t)\, dt\right] = (-1)^k \frac{d^{k+1}}{ds^{k+1}} f(s)$$
$$\int_0^\infty \frac{\partial}{\partial s}\left(e^{-st} \cdot t^k F(t)\right) dt = (-1)^k \frac{d^{k+1}}{ds^{k+1}} f(s)$$
$$\Rightarrow \int_0^\infty e^{-st} \cdot (-t)\, t^k F(t)\, dt = (-1)^k \frac{d^{k+1}}{ds^{k+1}} f(s).$$
$$\Rightarrow \int_0^\infty e^{-st} t^{k+1} F(t)\, dt = (-1)^{k+1} \frac{d^{k+1}}{ds^{k+1}} f(s)$$
$$\Rightarrow L[t^{k+1} F(t)] = (-1)^{k+1} \frac{d^{k+1}}{ds^{k+1}} f(s)$$

Hence the result is true for $n = k+1$, so it is true for all n. Hence result is true.
— Proved

H.W

① $L[t \sin at]$ (ii) $L[t \cos at]$

(iii) $L[t^3 \cos at]$ $\left(6(s^4 - 6s^2 + 1)/(s^2+1)^4\right)$

① Let $F(t) = \sin at$ then
$$L[F(t)] = \frac{a}{s^2 + a^2} = f(s)$$
$$\therefore L[t \cdot \sin at] = -f'(s)$$
$$= -\left(\frac{-a \cdot 2s}{(s^2+a^2)^2}\right)$$
$$= \frac{2as}{(s^2+a^2)^2}$$

(ii) $L[t \cos at]$

$\frac{s^2 - a^2}{(s^2+a^2)^2}$

let $F(t) = \cos at$ then
$$L[F(t)] = \frac{s}{s^2 + a^2} = f(s)$$
$$\therefore L[t \cdot \cos at] = -f'(s) = \frac{1(s^2+a^2) - s(2s)}{(s^2+a^2)^2}$$

$$= \frac{1}{s}\left[\ln v - \ln(s+v)\right]_p^\infty$$

$$= \frac{1}{s}\left[-\ln\left|\frac{v+s}{v}\right|\right]_p^\infty = \frac{1}{s}\left[-\ln\left(1+\frac{s}{v}\right)\right]_p^\infty$$

$$= \frac{1}{s}\left[0 - \{-\ln(s+1)\}\right] = \frac{1}{s}\ln(s+1)$$

$$\boxed{L[E(t)] = \frac{1}{s}\log(s+1)}$$

II-Method

$$\because \frac{e^{-u}}{u} = \frac{1}{u}\left[1 - u + \frac{u^2}{2!} - \frac{u^3}{3!} + \frac{u^4}{4!} - \frac{u^5}{5!} + \cdots\right]$$

$$= \frac{1}{u} - 1 + \frac{u}{2!} - \frac{u^2}{3!} + \frac{u^3}{4!} - \frac{u^4}{5!} + \cdots$$

$$\sim \int_1^\infty \frac{e^{-u}}{u}\,du = 1$$

(5) Laguerre Polynomial

$$L_n(t) = \frac{e^{t}}{n!}\frac{d^n}{dt^n}\left(e^{-t}t^n\right), \quad n = 0, 1, 2, \ldots$$

then Prove that

$$L[L_n(t)] = \frac{(s-1)^n}{s^{n+1}}$$

Proof: Here we have

self

$$L_n(t) = \frac{e^{t}}{n!}\frac{d^n}{dt^n}\left(e^{-t}t^n\right)$$

$$\therefore L[L_n(t)] = \int_0^\infty e^{-st}\left[\frac{e^t}{n!}\frac{d^n}{dt^n}(e^{-t}t^n)\right]dt$$

$$= \frac{1}{n!}\int_0^\infty \underbrace{e^{t(1-s)}}_{=u}\,\underbrace{\frac{d^n}{dt^n}(e^{-t}t^n)}_{=v}\,dt$$

$$= \frac{1}{n!}\left[e^{-(s-1)t}\,\frac{d^{n-1}}{dt^{n-1}}(e^{-t}t^n)\right]_0^\infty$$

$$- \frac{1}{n!}\left[(1-s)\int_0^\infty e^{-(s-1)t}\,\frac{d^{n-1}}{dt^{n-1}}(e^{-t}\cdot t^n)\,dt\right]$$

$$\because e^{xt} = \frac{d}{dt}\frac{e^{xt}}{x}, \quad = \int e^{xt}\,dt$$

$$L[L_n(t)] = 0 + \frac{s-1}{n!}\int_0^\infty e^{-(s-1)t}\,\frac{d^{n-1}}{dt^{n-1}}(e^{-t}\cdot t^n)\,dt \quad (s>1)$$

Repeating the Above process (n-times)

$$L[L_n(t)] = \frac{(s-1)^n}{n!}\int_0^\infty e^{-(s-1)t}\cdot(e^{-t}\cdot t^n)\,dt$$

$$= \frac{(s-1)^n}{n!}\int_0^\infty e^{-st+t-t}\cdot t^n\,dt = \frac{(s-1)^n}{n!}\int_0^\infty e^{-st}\cdot t^n\,dt$$

$$= \frac{(s-1)^n}{n!}\,L[t^n]$$

$$= \frac{(s-1)^n}{n!}\,L[t^n]$$

$$L[L_n(t)] = \frac{(s-1)^n}{n!} \times \frac{\Gamma(n+1)}{s^{n+1}} = \frac{(s-1)^n}{s^{n+1}} \quad \Big|\; \because \Gamma(n+1) = \lfloor n$$

(6) UNIT Step function :— (or Heaviside Unit function)

The unit step functions $U(t-a)$ is defined as —

(See next page)

$$U(t-a) = \begin{cases} 0 & \text{when } t<a \\ 1 & \text{when } t\geq a \end{cases}$$

Example (1) Express the following f^n in terms of Unit step f^n and find it L.T

$$f(t) = \begin{cases} 8 & \text{when } t<2 \\ 6 & t\geq 2 \end{cases}$$

$$\frac{8-2e^{-2s}}{s}$$

(2) Prove that $L[J_0(t)] = \frac{1}{\sqrt{s^2+1}}$ and Deduce that

(i) $L[J_0(at)] = \frac{1}{\sqrt{s^2+a^2}}$

(ii) $L[tJ_0(at)] = \frac{s}{(s^2+a^2)^{3/2}}$

(iii) $L[e^{-at}J_0(at)] = \frac{1}{\sqrt{s^2+2as+2a^2}}$

Tue

$$\frac{s^2+a^2-2s^2}{(s^2+a^2)^2} = \frac{a^2-s^2}{(s^2+a^2)^2} = \frac{s^2-a^2}{s^2+a^2}$$

(iii) $\because L[\cos at] = \frac{s}{s^2+a^2} = f(s)$

$\therefore L[t^3 \cos at] = (-1)^3 \frac{d^3}{ds^3} f(s)$

$f(s) = \frac{s}{s^2+a^2}$

$f'(s) = \frac{1.(s^2+a^2) - s.(2s)}{(s^2+a^2)^2}$

$f''(s) = \frac{d}{ds}\left(\frac{s^2+a^2-2s^2}{(s^2+a^2)^2}\right) = \frac{d}{ds}\left(\frac{a^2-s^2}{(s^2+a^2)^2}\right)$

$= \frac{-2s((s^2+a^2)^2) - (a^2-s^2)(2(s^2+a^2).2s)}{(s^2+a^2)^4}$

$= \frac{-2s(s^2+a^2)[s^2+a^2+2(a^2-s^2)]}{(s^2+a^2)^4}$

$= \frac{-2s[3a^2-s^2]}{(s^2+a^2)^3}$

$f'''(s) = \frac{[-6a^2+6s^2](s^2+a^2)^3 - [2s^3-6sa^2][3(s^2+a^2)^2.2s]}{(s^2+a^2)^6}$

Cancelled out (s^2+a^2) from numerator and denominator

$= \frac{6[s^2-a^2](s^2+a^2) - 2s[s^2-3a^2](6s)}{(s^2+a^2)^4}$

$= \frac{6[s^4-a^4-2s^4+6s^2a^2]}{(s^2+a^2)^4}$

$= \frac{-6[s^4+a^4+6s^2a^2]}{(s^2+a^2)^4}$

Laplace transform of

* Some special function :-

(1) The Sine Integral :- The sine integral is defined as

$$S(t) = \int_0^t \frac{\sin u}{u}\, du$$

Self

To prove that $L[S(t)] = \frac{1}{s}\tan^{-1}\left(\frac{1}{s}\right)$

Pf :-

$\because \sin u = u - \frac{u^3}{\lfloor 3} + \frac{u^5}{\lfloor 5} - \frac{u^7}{\lfloor 7} + \cdots$

$\therefore \frac{\sin u}{u} = 1 - \frac{u^2}{\lfloor 3} + \frac{u^4}{\lfloor 5} - \frac{u^6}{\lfloor 7} - \cdots$

$\therefore \int_0^t \frac{\sin u}{u} du = \left[u - \frac{u^3}{3\lfloor 3} + \frac{u^5}{5\lfloor 5} - \frac{u^7}{7\lfloor 7} + \cdots\right]_0^t$

$= \left[t - \frac{t^3}{3\lfloor 3} + \frac{t^5}{5\lfloor 5} - \frac{t^7}{7\lfloor 7} + \cdots - 0\right]$

Self

$\therefore L[S(t)] = L\left[t - \frac{t^3}{3\lfloor 3} + \frac{t^5}{5\lfloor 5} - \frac{t^7}{7\lfloor 7} + \cdots\right]$

By linearity Prop. $= L[t] - \frac{1}{3}L[t^3/\lfloor 3] + \frac{1}{5}L[\frac{t^5}{\lfloor 5}] - \cdots$

$\because \Gamma 4 = \lfloor 3$, $\Gamma 6 = \lfloor 5$, $\Gamma 8 = \lfloor 7$

$= \frac{1}{s^2} - \frac{1}{3\lfloor 3}\frac{\Gamma 4}{s^4} + \frac{1}{5\lfloor 5}\frac{\Gamma 6}{s^6} - \frac{\Gamma 8}{7\lfloor 7 s^8} + \cdots$

$= \frac{1}{s}\left[\frac{1}{s} - \frac{1}{3}\frac{1}{s^3} + \frac{1}{5s^5} - \frac{1}{7s^7} + \cdots\right]$

$= \frac{1}{s}\left[\frac{1}{s} - \frac{1}{3}\left(\frac{1}{s}\right)^3 + \frac{1}{5}\left(\frac{1}{s}\right)^5 - \frac{1}{7}\left(\frac{1}{s}\right)^7 + \cdots\right]$

$= \frac{1}{s}\left[\frac{1}{s} - \frac{1}{3}\left(\frac{1}{s}\right)^3 + \frac{1}{5}\left(\frac{1}{s}\right)^5 - \cdots\right] = \frac{1}{s}\tan^{-1}\frac{1}{s}$

(2) The Cosine integral is denoted by

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$$C(t) = \int_t^\infty \frac{\cos u}{u}\, du$$

To prove ;

$$L[C(t)] = \frac{1}{2s}\log(s^2+1)$$

Self

$\cos x = 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \frac{x^6}{\lfloor 6} + \cdots$

$L[t^n F(t)] = (-1)^n \frac{d^n f(s)}{ds^n}$

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$$\frac{\cos u}{u} = \frac{1}{u}\left[1 - \frac{u^2}{\lfloor 2} + \frac{u^4}{\lfloor 4} + \frac{u^6}{\lfloor 6} + \frac{u^8}{\lfloor 8} - \cdots\right]$$

$$= \left[\frac{1}{u} - \frac{u}{2!} + \frac{u^3}{4!} - \frac{u^5}{6!} + \frac{u^7}{8!} - \cdots\right]$$

self

$$\therefore \int_t^\infty \frac{\cos u}{u}\,du = \left[\log u - \frac{u^2}{2!} + \frac{u^4}{4!} - \frac{u^6}{6!} + \cdots\right] \quad \text{---(1)}$$

$$\because \quad L[t.F'(t)] \cdot - \int_\infty^t \frac{\cos u}{u}\,dt$$

$F'(t) = -\frac{\cos t}{t}$

$tF'(t) = -\cos t$

As we know that $F(t) = \int_t^\infty \frac{\cos u}{u}\,du = -\int_\infty^t \frac{\cos u}{u}\,du$

$$\therefore \quad F'(t) = -\frac{\cos t}{t} \Rightarrow tF'(t) = -\cos t$$

Proof:-

$$\Rightarrow L[tF'(t)] = L[\cos t] = \frac{-s}{s^2+1}$$

$$\Rightarrow -\frac{d}{ds} f'(s) = \frac{-s}{s^2+1} \qquad \left(\because L[tF'(t)] = -\frac{d}{ds}f(s)\right)$$

$L[tF(t)] = -\frac{d}{ds} f(s)$

$$\Rightarrow \cancel{+}\frac{d}{ds} L[F'(t)] = \cancel{+}\frac{s}{s^2+1} \qquad (\because L[F'(t)] = f(s))$$

$L[F(t)]$

$L[F(t)] = f(s)$

$$\Rightarrow \frac{d}{ds}\left[sL[F(t)] - F(0)\right] = \frac{s}{s^2+1}$$

$$= \frac{d}{ds}(sf(s)) = \frac{s}{s^2+1} \qquad (\because F(0) \text{ is constant})$$

$$\Rightarrow sf(s) = \frac{1}{2}\log(s^2+1) + c \qquad \text{(On integrating)} \quad (2)$$

As $s \to 0$

$$\lim_{s\to 0} sf(s) = \lim_{t\to\infty} F(t)$$

[As from (1), $t \to 0 \Rightarrow 0 = 0 + c$, $\Rightarrow c = 0$]

$$\therefore sf(s) = \frac{1}{2}\log(s^2+1)$$

$$\boxed{\therefore f(s) = \frac{1}{2s}\log(s^2+1)}$$

$$\because f(s) = L[F(t)] = L\left[\int_t^\infty \frac{\cos u}{u}\,du\right]$$

$$\Rightarrow \boxed{L[C(t)] = \frac{1}{2s}\left[\log(s^2+1)\right]}$$

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Initial value theorem:-

Let $F(t)$ be continuous for all $t \geq 0$ and exponential order as $t \to \infty$ and $F(t)$ is of class A, then

I. $$\lim_{t\to 0} F(t) = \lim_{s\to\infty} sL[F(t)] = sf(s)$$

$$\lim_{t\to\infty} F(t) = \lim_{s\to 0} sL[F(t)]$$

II. Final value theorem:-

$$\lim_{t\to\infty} F(t) = \lim_{s\to 0} sL[F(t)] = \lim_{s\to 0} sf(s)$$

4 The Exponential Integral:-

The exponential ~~order~~ is defined as

$$E(t) = \int_t^\infty \frac{e^{-u}}{u}\,du$$

then prove that

$$L[E(t)] = \frac{1}{s}\log(1+s)$$

Proof:- (self)

$$L[E(t)] = L\left[\int_t^\infty \frac{e^{-u}}{u}\,du\right] \qquad \text{Let } u = tv \Rightarrow du = t\,dv$$

$$= L\left[\int_1^\infty \frac{e^{-tv}}{tv}\,t\,dv\right]$$

$$= \int_0^\infty e^{-st}\left[\int_1^\infty \frac{e^{-tv}}{v}\,dv\right]dt$$

$$= \int_1^\infty \frac{1}{v}\left(\int_0^\infty e^{-st}.e^{-tv}\,dt\right)dv \qquad \text{[On changing the order of integration]}$$

$$= \int_1^\infty \frac{1}{v}\left[\int_0^\infty e^{-t(s+v)}\,dt\right]dv$$

$$= \int_1^\infty \frac{1}{v}\left[-\frac{e^{-t(s+v)}}{s+v}\right]_0^\infty dv = \int_1^\infty \frac{1}{v}\left[0 - \left(-\frac{1}{s+v}\right)\right]dv$$

$$= \int_1^\infty \frac{1}{s}\left[\frac{1}{v} - \frac{1}{s+v}\right]dv = \frac{1}{s}\int_1^\infty \left[\frac{1}{v} - \frac{1}{s+v}\right]dv$$

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(iv) $\int_0^\infty J_0(t) = 1$

(v) $\int_0^\infty t\, e^{-3t} J_0(4t) = ?$

H.W (6) **Unit step f^n**

We have

$$U(t-a) = \begin{cases} 0 & t < a \\ 1 & t \geq a \end{cases} \quad \text{—(i)}$$

$$\therefore L[U(t-a)] = \int_0^\infty e^{-st}\, U(t-a)\, dt$$

$$= \int_0^a e^{-st} \cdot 0\, dt + \int_a^\infty e^{-st} \cdot 1\, dt$$

$$= 0 + \int_a^\infty e^{-st} dt =$$

$$= \lim_{p \to \infty} \int_a^p e^{-st} dt = \left[\frac{-e^{-st}}{s}\right]_a^p$$

$$= \cancel{\lim_{p\to\infty}\left[\frac{-e^{-s^2}}{s} + \frac{e^{-as}}{s}\right]}$$

$$= \lim_{p \to \infty}\left[\frac{-e^{-sp} + e^{-as}}{s}\right]$$

$$\boxed{L[U(t-a)] = \left[\frac{0 + e^{-as}}{s}\right] = \frac{e^{-as}}{s}}$$

(7) (i) T.S $L[J_0(t)] = \frac{1}{\sqrt{s^2+1}}$

$$\therefore L[J_0(t)] = L\Big[\underbrace{1 - \frac{t^2}{2^2} + \frac{t^4}{2^2 \cdot 4^2} - \frac{t^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots}_{= J_0(t)}\Big]$$

$$= L[1] - \frac{1}{2^2}L[t^2] + \frac{1}{2^2 \cdot 4^2}L[t^4] - \frac{1}{2^2 \cdot 4^2 \cdot 6^2}L[t^6] + \cdots$$

$$= \frac{1}{s} - \frac{1}{2^2}\cdot\frac{L2}{s^3} + \frac{1}{2^2 \cdot 4^2}\cdot\frac{L4}{s^5} - \frac{1}{2^2 \cdot 4^2 \cdot 6^2}\cdot\frac{L6}{s^7} + \cdots$$

$$= \frac{1}{s}\left[1 - \frac{1}{2}\frac{1}{s^2} + \frac{3}{2 \cdot 4}\frac{1}{s^4} - \frac{5 \cdot 3 \cdot 1}{2 \cdot 4 \cdot 6\, s^6} + \cdots\right]$$

$$= \frac{1}{s}\left[\left(1 + \frac{1}{s^2}\right)^{-1/2}\right] = \frac{1}{s}\left[\left(\frac{s^2+1}{s^2}\right)^{-1/2}\right]$$

$$= \frac{1}{s}\sqrt{\frac{s^2}{s^2+1}} = \frac{1}{s}\frac{s}{\sqrt{s^2+1}}$$

$$= \frac{1}{\sqrt{s^2+1}}$$

$$\boxed{L[J_0(t)] = \frac{1}{\sqrt{s^2+1}}}$$

(7)(2) (i) T.S $L[J_0(at)] = \frac{1}{\sqrt{s^2+a^2}}$

$$\cancel{L[J_0(at)] = L\left[1 - \frac{(at)^2}{2^2} + \frac{(at)^4}{2^2 \cdot 4^2} - \frac{(at)^6}{2^2 \cdot 4^2 \cdot 6^2} + \frac{(at)^8}{2^2 4^2 6^2 8^2} - \cdots\right]}$$

$$\cancel{= L[1] - \frac{1}{2^2}L[at^2] + \frac{1}{2^2 \cdot 4^2}L[(at)^4] - \frac{1}{2^2 \cdot 4^2 \cdot 6^2}L[(at)^6] + \cdots}$$

$$= \cancel{\frac{1}{s}}$$

Since $L[F(at)] = \frac{1}{a} f(s/a)$ $\quad (f(s) = L[F(t)])$

$\therefore L[J_0(at)] = \frac{1}{a} f(s/a)$ where $f(s) = \frac{1}{\sqrt{s^2+1}}$

$$\therefore f\left(\frac{s}{a}\right) = \frac{1}{\sqrt{(s/a)^2+1}} = \frac{1}{\sqrt{\frac{s^2+a^2}{a^2}}} = \frac{a}{\sqrt{s^2+a^2}}$$

$$\therefore L[J_0(at)] = \frac{1}{a} f(s/a) = \frac{1}{\sqrt{s^2+a^2}}$$

(7)(2) (ii) As we know that

$$L[t\, F(t)] = -\frac{d}{ds} f(s)$$

where $L[F(t)] = f(s) = \frac{1}{\sqrt{s^2+1}}$

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Similarly

$L[t J_0(at)] = -\frac{d}{ds} f(s) = -\frac{d}{ds} L[J_0(t)]$ $\left(L[F(at)] = \frac{1}{\sqrt{s^2+a^2}} = f(s) \right)$

$-: f(s) = L[J_0(t)]$

$= -\frac{d}{ds}\left(\frac{1}{\sqrt{s^2+a^2}} \right)$

$L[tJ_0(at)] = -\left[\frac{0 - \frac{1}{2} \cdot 2s}{(s^2+a^2)^{3/2}} \right] = \frac{s}{(s^2+a^2)^{3/2}}$

(7)(2)(iii) As $L[F(t)] = f(s)$

we have $L[e^{-at} F(t)] = f(s+a)$

Here we have

$L[F(at)] = f(s) = \frac{1}{\sqrt{s^2+a^2}}$

$\Rightarrow L[F(at)] = f(s+a) = \frac{1}{\sqrt{(s+a)^2+a^2}} = L[J_0(at)]$

$\therefore L[e^{-at} F(at)] = f(s+a)$

i.e

$L[e^{-at} J_0(at)] = \frac{1}{\sqrt{s^2+2as+2a^2}}$

(iv) T.P $\int_0^\infty J_1(t) = 1$

As $L[J_0(t)] = \int_0^\infty e^{-st} J_0(t)\,dt = \frac{1}{\sqrt{s^2+1}}$

$\Rightarrow \lim_{s\to 0} \int_0^\infty e^{-st} J_0(t)\,dt = \lim_{s\to 0} \frac{1}{\sqrt{s^2+1}}$

$\Rightarrow \int_0^\infty J_0(t)\,dt = 1 = \lim_{s\to 0} \frac{1}{\sqrt{s^2+1}}$

(v) T $\int_0^\infty t \cdot e^{-3t} J_0(4t)\,dt$

$= \int_0^\infty e^{-3t} (t \cdot J_0(4t))\,dt$

Here $s = 3$, $a = 4$

As we already proved that

$L[e^{-at} J_0(at)] = \frac{1}{\sqrt{s^2+2as+2a^2}}$

Now putting $a = 4$, $s = 3$

$L[e^{-4t} J_0(4t)] = \frac{1}{\sqrt{9+24+32}} = \frac{1}{\sqrt{65}}$

$= \frac{1}{\sqrt{65}}$ Ans

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* * *

$L[t \cdot J_0(at)] = \frac{s}{(s^2+a^2)^{3/2}}$

$= \frac{3}{(9+16)^{3/2}} = \frac{3}{5\sqrt{5}}$

New Chapter

INVERSE Laplace Transform :-

If $f(s)$ be the laplace transform of a f^n $F(t)$ i.e

$f(s) = L[F(t)]$

then

$F(t)$ is called the Inverse laplace transform of the f^n $f(s)$ and is written as

$F(t) = L^{-1}\{f(s)\}$

where 'L^{-1}' is the inverse laplace transform operators.

Formula for Inversion —

$f(s)$	$L^{-1}\{f(s)\} = F(t)$

	$f(s)$	$F(t) = L^{-1}$
(i)	$\frac{1}{s}$	1
(ii)	$\frac{1}{s^{n+1}}$, $n > -1$, $s > 0$	$\frac{t^n}{\Gamma n+1}$
(iii)	$\frac{1}{s^{n+1}}$, n is +ve int. $s > 0$	$\frac{t^n}{n!}$
(iv)	$\frac{1}{s-a}$, $s > a$	e^{at}
(v)	$\frac{1}{s+a}$, $s > a$	e^{-at}
(vi)	$\frac{1}{s^2+a^2}$, $s > 0$	$\frac{\sin at}{a}$
(vii)	$\frac{s}{s^2+a^2}$, $s > 0$	$\cos at$
(viii)	$\frac{1}{s^2-a^2}$, $s > \|a\|$	$\sinh at$
(ix)	$\frac{s}{s^2-a^2}$, $s > \|a\|$	$\cosh at$.

Theorem: (1) If $f_1(s)$ and $f_2(s)$ respectively be the L.T of the f^n $F_1(t)$ and $F_2(t)$ respectively. C_1 & C_2 be two constant, then—

$$L^{-1}\{C_1 f_1(s) + C_2 f_2(s)\} = C_1 L^{-1}\{f_1(s)\} + C_2 L^{-1}\{f_2(s)\}$$

Proof:— Here, we have given that

$$L[F_1(t)] = f_1(s) \ \& \ L[F_2(t)] = f_2(s)$$

∴ By L.H.S —

$$L^{-1}[C_1 L[F_1(t)] + C_2 L[F_2(t)]]$$

$=$

$$= L^{-1}[L[C_1F_1(t)] + L[C_2F(t)]]$$

$$= L^{-1}[L[C_1F_1(t) + C_2F_2(t)]]$$

—x—

We know that the linearity property of L.T

$$L[C_1F_1(t) + C_2F_2(t)] = C_1L[F_1(t)] + C_2L[F_2(t)] \quad —(1)$$

Given that

$$f_1(s) = L[F_1(t)] \ \& \ f_2(s) = L[F_2(t)] \quad —(2)$$

Now from (1) & (2), we have—

$$L[C_1f_1(s) + C_2f_2(s)] = C_1f_1(s) + C_2f_2(s)$$

i.e. $L[C_1F_1(t) + C_2F_2(t)] = C_1f_1(s) + C_2f_2(s)$ —(3)

By def^n of I.L.T —

$$(F(t) = L^{-1}\{f(s)\})$$

$$C_1F_1(t) + C_2F_2(t) = L^{-1}[C_1f_1(s) + C_2f_2(s)]$$

$$\Rightarrow C_1L^{-1}[f_1(s)] + C_2L^{-1}[f_2(s)] = L^{-1}[C_1f_1(s) + C_2f_2(s)]$$

H.W

Example: Evaluate (i) $L^{-1}\left[\frac{3s+7}{s^2-2s-3}\right] = 4e^{3t} - e^{-t}$

(ii) $L^{-1}\left[\frac{s^2}{(s+1)(s+2)(s+3)}\right] = \frac{1}{2}e^{-t} - 4e^{-2t} + \frac{9}{2}e^{-3t}$

Sol^n (i) ∴ $\left[\frac{3s+7}{s^2-2s-3}\right] = \frac{3s+7}{(s+1)(s-3)} = \frac{-1}{(s+1)} + \frac{1}{(s-3)}$

(iii) $L^{-1}\left[\frac{s}{(s^2+a^2)(s^2+b^2)}\right] \rightarrow \frac{\cos bt - \cos at}{a^2-b^2}$

(iv) $L^{-1}\left[\frac{1}{(s+1)(s^2+1)}\right] \rightarrow \frac{1}{2}\left[e^{-t} - \cos t + \sin t\right]$

$$\frac{As+B}{(s^2+a^2)} + \frac{Cs+D}{(s^2+b^2)} = \frac{(As+B)(s^2+b^2) + (Cs+D)(s^2+a^2)}{} = s$$

* I. First Shifting theorem (For ILT)

If $L^{-1}\{f(s)\} = F(t)$ then

$$L^{-1}\{f(s-a)\} = e^{at} L^{-1}\{f(s)\}$$

Proof: Using $s \to s-a$

II. Second shifting theorem (I.L.T)

If $L^{-1}\{f(s)\} = F(t)$ then —

$$L^{-1}\{e^{-as} f(s)\} = G(t)$$

where $G(t) = \begin{cases} F(t-a) & , t>a \\ 0 & , t<a \end{cases}$

Proof: By defn, $L[G(t)] = \int_0^\infty e^{-st} G(t)\,dt$

$$= \int_0^a e^{-st}\, 0\, dt + \int_a^\infty e^{-st} F(t-a)\,dt$$

$$= \cancel{\int_a^\infty e^{-st}\,dt = \left[\frac{e^{-st}}{s}\right]_a^\infty}$$

$$= \int_a^\infty e^{-st} F(t-a)\,dt$$

Let $t-a = x$ | $t=a, x=0$

$\Rightarrow dt = dx$ | $t=\infty, x=\infty$

| $t = a+x$

* ~~Second shifting~~

$$L[G(t)] := \int_0^\infty e^{-s(a+x)} F(x)\,dx$$

$$= \int_0^\infty e^{-sa} e^{-sx} F(x)\,dx$$

$$= e^{-sa} \int_0^\infty e^{-st} F(t)\,dt$$

(By defn of L)

$$= e^{-as} \cdot f(s)$$

$$\therefore G(t) = L^{-1}[e^{-as} f(s)]$$

Remark: Unit step f^n for I.L.T —

$$H(t-a) = \begin{cases} 1 & ; \text{ when } t>a \\ 0 & \quad t<a \end{cases}$$

Change of scale property: —

If $L^{-1}\{f(s)\} = F(t)$, then

$$L^{-1}\{f(as)\} = \frac{1}{a} F\left(\frac{t}{a}\right)$$

Pf: $\because$ By defn of L.T, we know that

$$L\left[\frac{1}{a} F\left(\frac{t}{a}\right)\right] = \int_0^\infty e^{-st} \cdot \frac{1}{a} \cdot F\left(\frac{t}{a}\right) dt$$

Let $t/a = x \Rightarrow dt = a\,dx$

$$= \frac{1}{a} \int_0^\infty e^{-sax} \cdot a F(x)\,dx$$

$$= \int_0^\infty e^{-sax} F(x)\,dx$$

$$= \int_0^\infty e^{-s(at)} F(t)\,dt$$

$$L\left[\frac{1}{a} F\left(\frac{t}{a}\right)\right] = f(as)$$

$$\Rightarrow L^{-1}[f(as)] = \frac{1}{a} F\left(\frac{t}{a}\right)$$

$\to$ Next - Congruent $F(t) * G(t) = \int_0^t$ —

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Def^n: **Convolution:**

let $F(t)$ and $G(t)$ be the two f^n in t, then define the convolution

$$F(t) * G(t) = \int_0^t F(u)\, G(t-u)\, du$$

or $$\int_0^t G(t) F(t-u)\, du$$

This relation $F*G$ is also called as Resultant of $F(t)$ and $G(t)$ or as Falting.

Properties of Convolution:-

(i) Commutative law of convolution

$$F(t) * G(t) = G(t) * F(t).$$

Pf: $\because F(t) * G(t) = \int_0^t F(u) \cdot G(t-u)\, du$

let $t-u = x$ & $u = t-x$ $\quad u=0,\ x=t$

$\Rightarrow du = -dx$

$$\text{so } F(t) * G(t) = \int_t^0 F(t-x) \cdot G(x)\, dx$$

$$= -\int_t^0 G(x) \cdot F(t-x)\, dx$$

$$= \int_0^t G(x) F(t-x)\, dx = G(t) * F(t)$$

(ii) Let $F(t)$, $G(t)$ and $H(t)$ be the f^n in t, then

$$F(t) * [G(t) + H(t)] = F(t) * G(t) + F(t) * H(t)$$

(iii) $F(t) * [G(t) * H(t)] = [F(t) * G(t)] * H(t)$

verify

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(III) $J(t) = G(t) * H(t) = \int_0^t G(u) \cdot H(t-u)\, du$

or $J(t) = G(t) * G(t) = \int_0^t G(t-u) \cdot H(u)\, du$

$\therefore F(t) * [G(t) * H(t)] = F(t) * J(t)$

$\therefore F(t) * J(t) = \int_0^t F(w)\, J(t-w)\, dw$

$$= \int_0^t F(w) \left\{ \int_0^{t-w} G(t-u-w) \cdot H(u)\, du \right\} dw$$

$$= \int_0^t F(w) \int_0^{t-w} G(t-u-w) \cdot H(u)\, du\, dw.$$

$0 < u < t-w$, $0 < w < t$

$$= \int_0^t H(u) \int_0^{t-w} G(t-w-u) \cdot F(w)\, dw\, du$$

$0 < w < t-u$

$$= \int_0^t H(u) \int_0^{t-u} F(w) \cdot G(t-u-w)\, dw\, du$$

$$= H(t) * [F(t) * G(t)]$$

i.e.

$$F(t) * [G(t) * H(t)] = H(t) * [F(t) * G(t)]$$

$$= [F(t) * G(t)] * H(t)$$

Proved

(II) $F(t) * [G(t) + H(t)]$

$$= \int_0^t F(u) [G(t-u) + H(t-u)]\, du$$

$$= \int_0^t F(u) G(t-u)\, du + \int_0^t F(u) \cdot H(t-u)\, du$$

$$= F(t) * G(t) + F(t) * H(t)$$

$$\Rightarrow F(t) * [G(t) + H(t)] = F(t) * G(t) + F(t) * H(t).$$

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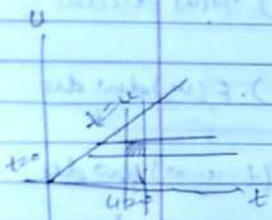
The Convolution theorem:

(st) let $F(t)$ and $G(t)$ be the two f^n in t. then define
$$L^{-1}[f(s)] = F(t) \text{ and } L^{-1}[g(s)] = G(t)$$
Prove that
$$L^{-1}[f(s)\cdot g(s)] = \int_0^t F(u)\cdot G(t-u)\,du = F(t) * G(t)$$

Proof :- let $H(t) = F(t) * G(t)$ ——— (i)

By def $L[H(t)] = \int_0^\infty e^{-st} H(t)\,dt$



$$= \int_0^\infty e^{-st} \{F(t) * G(t)\}\,dt$$
$$= \int_0^\infty e^{-st} \left\{\int_0^t F(u)\,G(t-u)\,du\right\} dt$$
$$= \int_{u=0}^\infty F(u) \left\{\int_{t=u}^\infty e^{-st} G(t-u)\,dt\right\} du$$
$$= \int_{u=0}^\infty e^{-su}\cdot F(u) \left\{\int_{t=u}^\infty e^{-s(t-u)} G(t-u)\,dt\right\} du.$$

Example. Apply the convolution theorem to prove that —
$$\beta(m,n) = \int_0^1 u^{m-1}(1-u)^{n-1}\,du = \frac{\Gamma m\,\Gamma n}{\Gamma m+n}$$
let $F(t) = t^{m-1}$ & $G(t) = t^{n-1}$.
then by using convolution theorem, we have
$$L[\beta(m,n)] = L\left[\int_0^t u^{m-1}(t-u)^{n-1}\,du\right]$$
$$L[F(t) * G(t)] = L[F(t)]\,L[G(t)].$$

Remark: (1) the convolution theorem can be written as —
$$L[F(t) * G(t)] = L[F(t)]\cdot L[G(t)]$$
(2) While using the convolution theorem

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we use one of the following two ways.
$$L^{-1}\{f(s)\cdot g(s)\} = \int_0^t F(u)\,G(t-u)\,du = \int_0^t G(u)\cdot F(t-u)\,du$$
$$= F(t) * G(t).$$
$$f(s)\,g(s) = L[F(t) * G(t)]$$
$$\Rightarrow L[F(t) * G(t)] = L[F(t)]\,L[G(t)]$$

$$L\left[\int_0^t u^{m-1}(t-u)^{n-1}\,du\right] = L[t^{m-1}]\,L[t^{n-1}]$$
$$\int_0^t u^{m-1}(t-u)^{n-1}\,du = L^{-1}\left\{\frac{\Gamma m}{s^m}\cdot\frac{\Gamma n}{s^n}\right\}$$
$$= \Gamma m\,\Gamma n\; L^{-1}\left\{\frac{1}{s^m}\cdot\frac{1}{s^n}\right\}$$
$$= \Gamma m\,\Gamma n\; L^{-1}\left\{\frac{1}{s^{m+n}}\right\}$$
$$= \frac{\Gamma m\,\Gamma n}{1}\cdot\frac{t^{m+n}}{\Gamma m+n}$$
$$\Rightarrow \int_0^t u^{m-1}(t-u)^{n-1}\,du = \frac{\Gamma m\,\Gamma n}{1}\cdot\frac{t^{m+n}}{\Gamma m+n}$$
putting $t=1$.
$$\int_0^1 u^{m-1}\cdot(1-u)^{n-1}\,du = \frac{\Gamma m\,\Gamma n}{\Gamma m+n} \quad (1)$$

* * *

29/1/16 (1) Evaluate the value

(a) $L^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right] = \frac{As}{(s^2+a^2)^2}+$

(b) $L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right]$

using the convolution theorem. Verify that the beta f^n

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$\beta(m+1, n) + \beta(m, n+1) = \beta(m, n)$.

using the convolution theorem.

(2) we know that

$$L[F(t)] \cdot L[G(t)] = L\int_0^t F(u).G(t-u)\,du$$

let $f(s) = \frac{s}{(s^2+a^2)}$ & $g(s) = \frac{s}{(s^2+a^2)}$

$\Rightarrow L^{-1}(f(s)) = \cos at$ & $L^{-1}[g(s)] = \cos at = G(t)$

$\therefore$ ~~$L^{-1}[f(s)\cdot g(s)]$~~

let $F(t) = \cos at = G(t)$

$\therefore$ By convolution

$$L^{-1}[f(s).g(s)] = \int_0^t \cos au.\ \cos a(t-u)\,dt$$

$$= \int_0^t \cos au\,[\cos at.\cos au + \sin at.\sin au]\,du$$

$\cos^2\theta = \frac{\cos 2\theta + 1}{2}$

$$= \int_0^t [\cos at.\cos^2 au + \frac{\sin at}{2}.\sin 2au]\,du$$

$\theta = au$

$$= \cos at \int_0^t \frac{\cos 2\theta + 1}{2}\,du + \frac{\sin at}{2}\int_0^t \sin 2au\,du$$

$$= \frac{\cos at}{2}\left[\frac{\sin 2au}{2a} + u\right]_0^t + \frac{\sin at}{2}\left[\frac{-\cos 2au}{2a}\right]_0^t$$

$$= \frac{\cos at}{2}\left[\frac{\sin 2at}{2a} + t\right] + \frac{\sin at}{2}\left[-\frac{\cos 2at}{2a} + \frac{1}{2a}\right]$$

$$= \frac{\cos at}{2}\left[\frac{2\sin at.\cos at}{2a} + t\right] + \frac{\sin at}{2}\left[\frac{1-\cos^2 at}{2a} + \frac{1}{2a}\right]$$

$$= \frac{\cos^2 at}{2a}\left[2\sin at - \sin^2 at\right] + \frac{t\cos at}{2} + \frac{\sin at}{2} +$$

$f(s) = \frac{1}{s^n} \rightarrow F(t) = \frac{t^{n-1}}{n-1}$

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(3) Problem. Find (i) $L^{-1}\left\{\frac{1}{s(s+1)^3}\right\}$

(ii) $L^{-1}\left\{\frac{1}{(s-2)(s+2)^2}\right\}$

— $f(s) = \frac{1}{s}$ and $g(s) = \frac{1}{(s+1)^3}$

$\therefore L^{-1}\{f(s)\} = 1$ and $L^{-1}\{g(s)\} = G(t)$

$= F(t) = 1$

$G(t) = \frac{1}{2}e^{-t}.t^2$

$\therefore$ By convolution theorem

$G(t) = \frac{1}{2}e^{-t}.t^2$

$$L^{-1}\{f(s)g(s)\} = \int_0^t F(u).G(t-u)\,du$$

$$= \int_0^t 1.\frac{1}{2}e^{-(t-u)}(t-u)^2\,du$$

$$= \frac{1}{2}\int_0^t e^{-(t-u)}.(u^2+t^2-2t.u)\,du$$

let $t-u = p \Rightarrow du = -dp$

at $u=0$, $p=t$

& $u=t$, $p=0$

$$\therefore = -\frac{1}{2}\int_t^0 e^{-p}.p^2\,dp$$

$= 1 - e^{-t}\left[1+t+\frac{t^2}{2}\right]$ Ans

$$\text{or} = \frac{1}{2}\int_0^t e^{-p}p^2\,dp \qquad = \frac{1}{2}.\sqrt{3} = 1$$

$$= \left[\frac{1}{2}\{e^{-p}.p^2\right.$$

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Application of laplace Transform

The laplace application, sol^n of ODE with constant coefficient, sol^n of PDE with constant and without and simulatneous diff eqn. Many more application in a classical Mechanics, Electrical circuit resolving sol^n

Que. Using laplace transform, Determine the sol^n

$$y'' + 3y' + 2y = e^{-t}, \quad y(0) = y'(0) = 0$$

$$L[y'' + 3y' + 2y] = L[e^{-t}]$$

$$\Rightarrow L[y''] + 3L[y'] + 2L[y] = L[e^{-t}]$$

$$\Rightarrow s^2 L(y(t)) - s y(0) - y'(0) + 3[sL(y(t)) - y(0)] + 2L(y(t)) = \frac{1}{(s+1)}$$

As given that $y'(0) = y(0) = 0$

$$\Rightarrow s^2 L(y) + 3s L(y) + 2L(y) = \frac{1}{s+1}$$

$$\Rightarrow (s^2 + 3s + 2) L(y) = \frac{1}{s+1}$$

$$\therefore L(y) = \frac{1}{(s+1)(s^2+3s+2)} = \frac{1}{(s+1)(s+2)(s+1)}$$

$$L[y(t)] = \frac{1}{(s+1)^2(s+2)}$$

$$\therefore y(t) = L^{-1}\left[\frac{1}{(s+1)^2(s+2)}\right]$$

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Let $f(s) = \frac{1}{s+2}$ & $g(s) = \frac{1}{(s+1)^2}$

$\therefore L^{-1}(f(s)) = e^{-2t}$ & $L^{-1}(g(s)) = e^{-t} \cdot t$
$= F(t)$ $\qquad = G(t)$

$\therefore$ By Convolution theorem

$$L^{-1}(f(s) \cdot g(s)) = \int_0^t e^{-2u} \cdot e^{-(t-u)} \cdot (t-u)\, du$$

$$= \int_0^t e^{-2u} \cdot e^{-t+u} (t-u)\, du$$

$$= \int_0^t e^{-t-u} \cdot (t-u)\, du$$

let ~~t~~

$$\left[\frac{e^{-(t+u)}}{-1} (t-u) + (\tfrac{1}{-1}) \int_0^t \frac{e^{-(t+u)}}{-1} \cdot du \right]$$

$$= \left[e^{-(t+u)} (u-t) + \frac{e^{-(t+u)}}{-1} \right]_0^t$$

$$= \left[e^{-2t}(0) + \frac{e^{-2t}}{-1} - \left[e^{-t} \cdot (-t) - \frac{e^{-t}}{-1} \right] \right]$$

$$= \left[+e^{-2t} + t e^{-t} - e^{-t} \right]$$

$$= + e^{-2t} + (t-1) e^{-t}$$

Que Solve the DE

$$(D+1)^2 y = t$$

given that

$$y(0) = -3 \;\; \& \;\; y(1) = -1$$

next → Fourier trans

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$(D+1)^2 y = t$, $y(0) = -3$, $y(1) = -1$

$D^2 y + 2Dy + y = t$

i.e $y'' + 2y' + y = t$

$\therefore L[y'' + 2y' + y] = L(t)$

$\Rightarrow$ By linearity property —

$L[y''] + 2L[y'] + L[y] = L(t)$

$\Rightarrow s^2 L[y] - s y(0) - y'(0) + 2[sL[y] - y(0)] + L(y) = L(t)$

$\Rightarrow L(y)(s^2 + 2s + 1) - s(-3) - y'(0) - 2(-3) = L(t)$

$\Rightarrow L(y)((s+1)^2) + 3s + 6 - y'(0) = L(t) = \frac{1}{s^2}$

let $y'(0) = k$ (any const.)

$L(y)((s+1)^2) = \frac{1}{s^2} - 3s - 6 + k$

$= \left(\frac{1 - 3s^3 - 6s^2 + ks^2}{s^2}\right)$

$L(y) = \left(\frac{1 - 3s^3 - 6s^2 + ks^2}{s^2}\right) \frac{1}{(s+1)^2}$

$= \frac{1}{s^2(s+1)^2} - \frac{s^2(3s + 6 - k)}{s^2(s+1)^2}$

$y = L^{-1}\left[\frac{1}{(s+1)^2 \cdot s^2} - \frac{3s + 6 - k}{(s+1)^2}\right]$

$= L^{-1}\left[\frac{1}{(s+1)^2 s^2}\right] - L^{-1}\left(\frac{3(s+1) + 3 - k}{(s+1)^2}\right)$

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$y = L^{-1}\left[\frac{1}{(s+1)^2 s^2}\right] - L^{-1}\left[\frac{3}{(s+1)} + \frac{(3-k)}{(s+1)^2}\right]$

$= L^{-1}\left[\frac{1}{(s+1)^2 s^2}\right] - 3 L^{-1}\left[\frac{1}{s+1}\right] - (3+k) L^{-1}\left[\frac{1}{(s+1)^2}\right]$

$= e^{-t}$ $\quad = e^{-t} t$

Let $f(s) = \frac{1}{s^2}$, $g(s) = \frac{1}{(s+1)^2}$

$\therefore L^{-1}[f(s)] = t = F(t)$

& $L^{-1}[g(s)] = t \cdot e^{-t} = G(t)$

$\therefore$ By convolution theorem —

$L^{-1}[f(s) g(s)] = \int_0^t F(u) \cdot G(t-u)\, du$

$= \int_0^t (t-u) \cdot u \cdot e^{-u}\, du =$

$= \int_0^t u^2 e^{-u}\, du$

$= \frac{\sqrt{3}}{\ } = \frac{2!}{s^3}$

$I = \left[\frac{e^{-u}}{-1} u^2 + \int e^{-u} 2u\, du\right]_0^t$

$= \left[\frac{u^2 e^{-u}}{-1} + \frac{2 \cdot u e^{-u}}{-1} + \int \frac{e^{-u}}{+1} \cdot 2\, du\right]_0^t$

$= \left[-u^2 e^{-u} - 2u \cdot e^{-u} - 2e^{-u}\right]_0^t$

$= [-t^2 e^{-t} - 2te^{-t} - 2e^{-t} + 2]$

$\therefore y = \frac{2!}{s^3} - 3 \cdot e^{-t} - (3+k) \cdot t e^{-t}$

$\because y(1) = -1$

$\therefore y(1) = \frac{2!}{s^3} - \frac{3}{e} - \frac{(3+k)}{e} = -1$

$\Rightarrow$

$t \int_0^t u e^{-u} du - \int u^2 e^{-u}\, du$

$t\left[[-u \cdot e^{-u}]_0^t + \int e^{-u} du\right] - \int u^2 \left(\frac{e^{-u}}{-1}\right) - \int 2u \cdot \frac{e^{-u}}{-1} du$

$= e^{-1}$

$y = [\ \text{—}\ ] + 3e^{-t} + e^{-t} + (3+k)$

* * *

we know that —

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$$L\left[\int_0^t F(u)\, G(t-u)\, du\right] = L[F(t)]\, L[G(t)] = f(s)\, g(s)$$

(I). Prove that $\int_0^t \sin u \cos(t-u)\, du = \frac{1}{2} t \sin t$ using the convolution theorem.

Proof. Let $F(u) = \sin u$, $G(u) = \cos(t-u)$

given that $F(t) = \sin t$, $G(t) = \cos t$

$$\therefore L[F(t)] = \frac{1}{s^2+1} = f(s)$$

& $L[G(t)] = \frac{s}{s^2+1} = g(s)$

$\therefore$ By Convolution theorem —

$$\therefore L^{-1}\left(f(s)\, g(s) \right) = \int_0^t F(t) \cdot G(t-u)\, du$$

$$\int_0^t F(t) \cdot G(t-u)\, du = L^{-1}\left(\frac{1}{s^2+1} \cdot \frac{s}{s^2+1} \right)$$

$$L^{-1}\left[\frac{1}{s^2+1} \cdot \frac{s}{s^2+1}\right] = \int_0^t \frac{1}{2} \cdot [\sin(u+t-u) + \sin(u-t+u)]\, du$$

$$= \frac{1}{2}\int_0^t [\sin t + \sin(2u-t)]\, du$$

$$= \frac{1}{2} t \sin t + \left[-\frac{\cos(2u-t)}{2}\right]_0^t$$

$$+ \frac{1}{2}\left[-\cos t + \cos t\right]$$

$$= \frac{1}{2} t \sin t$$

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II. If $L^{-1}[f(s)] = F(t)$, then using convolution theorem P.T.

$$L^{-1}\left[\frac{f(s)}{(s-a)^2+b^2}\right] = \frac{1}{b} e^{at} \int_0^t f(z)\, e^{-az} \sin b(t-z)\, dz$$

III. Solve the following simultaneous eqⁿ.

(140) (i) $\frac{dx}{dt} - y + e^t$

(ii) $\frac{dy}{dt} + x = \sin t$. the I.C. $x(0) = 1$ & $y(0) = 0$

(i) $\frac{dx}{dt} + 5x - 2y = t$

(ii) $\frac{dy}{dt} + 2x + y = 0$ when $x = y = 0$ then $t = 0$.

* Dirichlet Conditions

—x—

* Fourier Integral theorem:—

Consider a fⁿ f(x) which satisfies the Dirichlets conditions in every interval $(-c, c)$, so that

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{c} + b_n \sin\frac{n\pi x}{c}\right) \quad \text{—(1)}$$

Note:—

\# Dirichlet Condition

(I). The fⁿ must be periodic.

(II). It must be single value and continuous, except possibly at a finite no. of finite discontinuity.

(III). It must have only a finite no. of maxima and minima within one periodic

(IV) The integral over one period of $|f(x)|$ must be converge.

where $a_0 = \frac{1}{c}\int_{-c}^{c} f(t)\,dt$

$$a_n = \frac{1}{c}\int_{-c}^{c} f(t)\cos\frac{n\pi t}{c}\,dt \quad -(2)$$

and $b_n = \frac{1}{c}\int_{-c}^{c} f(t)\cdot\sin\frac{n\pi t}{c}\,dt$.

Now, using eqn (1) and (2), we have

$$f(x) = \frac{1}{2c}\int_{-c}^{c} f(t)\,dt + \frac{1}{c}\sum_{n=1}^{\infty}\int_{-c}^{c} f(t)\cos\frac{n\pi(t-x)}{c}\,dt \quad -(3)$$

If we assume that $\int_{-\infty}^{\infty}|f(x)|\,dx$ converges, the first term on the right side of eqn (3) approaches 0 as $c\to\infty$. Since

$$\left|\frac{1}{2c}\int_{-c}^{c} f(t)\,dt\right| \le \frac{1}{2c}\int_{-\infty}^{\infty}|f(t)|\,dt$$

the second term —

$$\lim_{c\to\infty}\frac{1}{c}\sum_{n=1}^{\infty}\int_{-\infty}^{\infty} f(t)\,\cancel{dt}\cos\frac{n\pi(t-x)}{c}\,dt$$

Now putting $\delta\lambda = \pi/c$

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$$\lim_{\delta\lambda\to 0}\frac{1}{\pi}\sum_{n=1}^{\infty}\delta\lambda\int_{-\infty}^{\infty} f(t)\cos n\delta\lambda(t-x)\,dt$$

This is of the form $\lim_{\delta\lambda\to 0}\sum_{n=1}^{n} F(n\delta\lambda)$ i.e. $\int_0^{\infty} F(\lambda)\,d\lambda$

Thus as $c\to\infty$, (3) becomes $f(x)$

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\left[\int_{-\infty}^{\infty} f(t)\cos\lambda(t-x)\,dt\right]d\lambda$$

which is known as the Fourier integral of $f(x)$.

Note: Note that, we have given a heristic (involves, discovery and problem sol^n) & demonstration of fourier integral which help is deriving the result eq^n (3). & when f(x) at a pt. discontinuity. we replace $f(x)$ by $\frac{1}{2}[f(x+0)+f(x-0)]$ as in the case of fourier series.

(1) For fourier sine & cosine integral expanding $\cos\lambda(t-x)$ may be written as —

$$f(x) = \frac{1}{\pi}\left[\int_0^{\infty}\cos\lambda x\left(\int_{-\infty}^{\infty} f(t)\cos\lambda t\,dt\right)d\lambda + \int_0^{\infty}\sin\lambda x\left(\int_{-\infty}^{\infty} f(t)\sin\lambda t\,dt\right)d\lambda\right] \quad -(4)$$

(2) If $f(x)$ is an odd f^n, $f(t)\cos\lambda t$ is also an odd f^n while $f(t)\sin\lambda t$ is even then the second term on the right side of eq^n (4) vanishes. and we get

$$f(x) = \frac{2}{\pi}\int_0^{\infty}\sin\lambda x\left[\int_0^{\infty} f(t)\sin\lambda t\,dt\right]d\lambda \quad -(5)$$

$$f(x) = \frac{2}{\pi}\int_0^{\infty}\cos\lambda x\left(\int_0^{\infty} f(t)\cos\lambda t\,dt\right)d\lambda \quad -(6)$$

Here, eq^n (5) & (6) are known as fourier sine integral & fourier cosine integral resp.

* A f^n f(x) defined in the interval $(0,\infty)$ is expressed either as a Fourier sine integral or as a Fourier cosine integral.

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merely looking upon it as an odd or even f^n in $(-\infty, \infty)$ on the lines of half range Fourier series

(I) For complex form of Fourier Integral eq^n (3) can be written as

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\cos\lambda(t-x)\,dt\,d\lambda \quad (7)$$

because $\cos\lambda(t-x)$ is an even f^n λ, Also $\sin\lambda(t-x)$ is an odd f^n of λ.

~~Now eq^n (3) + i eq^n~~ (3), we have —

$$0 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\sin\lambda(t-x)\,dt\,d\lambda \quad (8)$$

Now, eq^n (7) + i eq^n (8) on both sides (side by side)

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\left[\cos\lambda(t-x) + i\sin\lambda(t-x)\right]dt\,d\lambda$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\,e^{i\lambda(t-x)}\,dt\,d\lambda \quad (9)$$

which is ~~known as~~ complex form of Fourier Integral.

(II) For Fourier integral represent of a f^n :—

Example (1) Express $f(x) = \begin{cases} 1 & \text{for } 0 \le x \le \pi \\ 0 & \text{for } x > \pi \end{cases}$

By a fourier sine of integral and hence evaluate $\int_0^{\infty} \frac{(1-\cos\pi\lambda)}{\lambda}\sin\lambda x\,d\lambda$

* * *

Sol^n: The fourier sine integral for

$$f(x) = \frac{2}{\pi}\int_0^{\infty}\sin(\lambda x)\,d\lambda\int_0^{\infty} f(t)\sin(\lambda t)\,dt$$

$\because f(t) = 1$

$$= \frac{2}{\pi}\int_0^{\infty}\sin(\lambda x)\left[\int_0^{\pi}\sin\lambda t\cdot 1\cdot dt\right]d\lambda$$

$$= \frac{2}{\pi}\int_0^{\infty}\sin\lambda x\left[\frac{-\cos\lambda t}{\lambda}\right]_0^{\pi}d\lambda$$

$$= \frac{2}{\pi}\int_0^{\infty}\frac{\sin\lambda x}{\lambda}\left[-1+\cos\lambda\pi\right]d\lambda$$

$$\therefore f(x) = \frac{2}{\pi}\int_0^{\infty}\frac{(1-\cos\lambda\pi)}{\lambda}\sin\lambda x\,d\lambda$$

$$\Rightarrow \int_0^{\infty}\left(\frac{1-\cos\lambda\pi}{\lambda}\right)\sin\lambda x\,d\lambda = \frac{\pi}{2}f(x)$$

$$= \frac{\pi}{2}\begin{cases} 1 & 0 \le x < \pi \\ 0 & x > \pi \end{cases}$$

$$= \begin{cases} \pi/2 & 0 \le x < \pi \\ 0 & x > \pi \end{cases}$$

at $x = \pi$, which is a pt. of discontinuity of $f(x)$, the value of the above integral

$$= \frac{\pi}{2}\left[\frac{f(\pi+0)+f(\pi-0)}{2}\right] = \frac{\pi}{4} \quad \text{Ans}$$

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For Fourier Integral Representation of a f^n :—

using (4), of a function $f(x)$, may be represented by a Fourier integral as

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\cos\lambda x\left(\int_{-\infty}^{\infty} f(t)\cos\lambda t\,dt\right)d\lambda + \int_0^{\infty}\sin\lambda x\left(\int_{-\infty}^{\infty} f(t)\sin\lambda t\,dt\right)d\lambda$$

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$$= \frac{1}{\pi}\int_0^\infty A(\lambda)\cos\lambda x\, d\lambda + \frac{1}{\pi}\int_0^\infty B(\lambda)\sin\lambda x\, d\lambda$$

$$= \frac{1}{\pi}\left[\int_0^\infty A(\lambda)\cos\lambda x\, d\lambda + \int_0^\infty B(\lambda)\sin\lambda x\, d\lambda\right]$$

where

$$A(\lambda) = \int_{-\infty}^{\infty} f(t)\cos\lambda t\, dt = 2\int_0^\infty f(t)\cos\lambda t\, dt$$

$$B(\lambda) = \int_{-\infty}^{\infty} f(t)\sin\lambda t\, dt = 2\int_0^\infty f(t)\sin\lambda t\, dt$$

if $f(x)$ is an odd f^n then eq^n (5) is

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin\lambda x \int_0^\infty f(t)\sin\lambda t\, dt\, d\lambda$$

$$f(x) = \frac{1}{\pi}\int_0^\infty B(\lambda)\sin\lambda x\, d\lambda$$

$$f(x) = \frac{1}{\pi}\int_0^\infty A(\lambda)\sin\lambda x\, d\lambda$$

Eq^n (9)

$$\boxed{f(x) = \frac{1}{2\pi}\int_0^\infty \int_{-\infty}^{\infty} f(t)\cdot e^{i\lambda(t-x)}\, dt\, d\lambda} \quad \text{eq}^n (9)$$

Now putting $\lambda = s$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\cdot e^{is(t-x)}\, dt\, ds$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\cdot e^{ist}\cdot e^{-isx}\, ds$$

$$F(s) = \int_{-\infty}^{\infty} f(t)\, e^{ist}\, dt \quad (10)$$

form

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds \quad (11)$$

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The $F(s)$ is defined by

$$\cancel{F(s) = \int_{-\infty}^{\infty} f(t)\cdot e^{ist}\, dt}$$

~~Also the~~ is called Fourier trans-form.

Also the the $f(x)$ as given by eq^n (11) is called Invers fourier transform of $F(s)$.

For Fourier Sine & Cosine transform

Using eq^n (5),

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin\lambda x\left(\int_0^\infty f(t)\sin\lambda t\, dt\right)d\lambda$$

$F_s(s)$ for sine

define the f^n $F_s(s) = \int_0^\infty f(t)\sin st\, dt$

$$\boxed{f(x) = \frac{2}{\pi}\int_0^\infty F_s(s)\,\sin sx\, ds}$$

And for Cosine, using (6) —

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos\lambda x\left(\int_0^\infty f(t)\cos\lambda t\, dt\right)d\lambda$$

define the f^n $F_c(s) = \int_0^\infty f(t)\cos st\, dt$

$$\boxed{f(x) = \frac{2}{\pi}\int_0^\infty F_c(s)\cdot\cos sx\, ds}$$

Example: (I) Find the Fourier transform of $f(x)$

$$f(x) = \begin{cases} 1 - x^2 & |x| \le 1 \\ 0 & |x| > 1 \end{cases}$$

$\frac{3\pi}{16}$ Ans

Hence evaluate $\int_0^\infty \left(\frac{x\cos x - \sin x}{x^2}\right)\cos\frac{x}{2}\, dx$

*** Next - modulation

8/2/16 No class due to Meeting with VC (missed sindex)

Tuesday

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Example II: ~~Division~~ In last example, Evaluate the following

$$\int_0^\infty \left(\frac{x\cos x - \sin x}{x^3}\right) \sin\frac{x}{2}\, dx$$

Sol^n: Here, $f(s) = \int_{-\infty}^{\infty} f(x) e^{isx} dx$ = fourier transform

$$f(s) = \int_{-\infty}^{-1} 0\, dx + \int_{-1}^{1} (1-x^2) e^{isx} dx + \int_1^\infty 0\, dx$$

$$= \int_{-1}^{1} e^{isx} dx - \int_{-1}^{1} x^2 e^{isx} dx$$

$$= \left[\frac{e^{is} - e^{-is}}{is}\right] - \left[\frac{x^2 e^{isx}}{is} - \int \frac{e^{isx}}{is} 2x\, dx\right]$$

$$= \left[\frac{e^{is} - e^{-is}}{is}\right] - \left[\frac{x^2 e^{isx}}{is} - \frac{e^{isx}}{(is)^2} \cdot 2x + \int \frac{e^{isx}}{(is)^2} \cdot 2\, dx\right]_{-1}^{1}$$

$$= \left[\frac{e^{is} - e^{-is}}{is}\right] - \left[\frac{x^2 e^{isx}}{is} - \frac{e^{isx}}{(is)^2} \cdot 2x + 2\frac{e^{isx}}{(is)^3}\right]_{-1}^{1}$$

$$= \cancel{\left[\frac{e^{is} - e^{-is}}{is}\right]} - \left[\cancel{\frac{e^{is} - e^{-is}}{is}} - \frac{2(e^{is} + e^{-is})}{(is)^2} + \frac{2(e^{is} - e^{-is})}{(is)^3}\right]$$

$$= \frac{2 \cdot 2(e^{is} + e^{-is})}{2(is)^2} - \frac{2(e^{is} - e^{-is})}{(is)^3}$$

$$= \frac{-4}{s^2} \cos s + \frac{4}{s^3} \sin s$$

$$\therefore f(s) = \frac{4}{s^3} \sin s - \frac{4}{s^2} \cos s$$

Second part

Now evaluating $f(x)$:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(s) e^{-isx} ds$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} -\frac{4}{s^3} (s\cos s - \sin s)\, e^{-isx} ds$$

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$= \quad (\because e^{i\theta} = \cos\theta + i\sin\theta)$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{-4}{s^3} (s\cos s - \sin s)(\cos sx - i\sin sx)\, ds$$

$$= \frac{1}{2\pi} \int \frac{-4}{s^3} \left[(s\cos s \cdot \cos sx - \sin s \cdot \cos sx) - i(\sin sx \cdot s\cos s - \sin s \cdot \sin sx)\right] ds$$

Now, Here leaving term with i, that is we take only Real part —

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{-4}{s^3} [s\cos s \cos sx - \sin s \cdot \cos sx]\, ds$$

or $$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{-4}{s^3} [s\cos s - \sin s] \cos sx\, ds$$

putting $x = 1/2$, we have

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{-4}{s^3} [s\cos s - \sin s] \cos\frac{s}{2}\, ds$$

$$\frac{3}{4} = f(s) = f(x) = \frac{-4}{\pi} \int_0^\infty \left(\frac{s\cos s - \sin s}{s^3}\right) \cdot \cos\frac{s}{2}\, ds$$

$$\frac{3}{4} \times \frac{\pi}{-4} = \int_0^\infty \frac{s\cos s - \sin s}{s^3} \cos\frac{s}{2}\, ds$$

i.e. $$\int_0^\infty \frac{x\cos x - \sin x}{x^3} \cos\frac{x}{2}\, dx = \frac{3\pi}{-16} \quad \text{Ans}$$

HW

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* Properties of Fourier Transform

self (I) **Linearity Property :**

If $f(s)$ & $g(s)$ are fourier transform of $f(x)$ & $g(x)$ respectively, then—

$$F[af(x) + bg(x)] = af(s) + bg(s)$$

i.e $F[af(x) + bg(x)] = aF[f(x)] + bF[g(x)]$
$= a\,f(s) + b\,g(s)$

where a, b are constants

Pf: We know that —

$$f(s) = F[f(x)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x).e^{+isx}\,dx$$

$$g(s) = F[g(x)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(x).e^{isx}\,dx$$

$$\therefore F[af(x)+bg(x)] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(af(x)+bg(x)\right)e^{isx}\,dx$$

$$= \frac{1}{2\pi}\left[\int_{-\infty}^{\infty} af(x).e^{isx}dx + \int_{-\infty}^{\infty} bg(x)e^{isx}dx\right]$$

$$= \frac{a}{2\pi}\int_{-\infty}^{\infty} f(x).e^{isx}dx + \frac{b}{2\pi}\int_{-\infty}^{\infty} g(x).e^{isx}\,dx$$

$$= a.f(s) + b.g(s)$$

$$\Rightarrow F[af(x)+bg(x)] = a.f(s) + bg(s)$$

Proved

self (II) **Change of Scale Property :—**

If $f(s)$ is the fourier transform of

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$f(x)$, then the fourier transform of

$$f(ax) = \frac{1}{a}f\left(\frac{s}{a}\right), \quad a \neq 0$$

i.e $F\{f(ax)\} = \frac{1}{a}\cdot f\left(\frac{s}{a}\right) - f(ax)\}$

Pf: As, we know that —

$$F(s) = F(f(x)) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x).e^{isx}\,dx$$

$$\therefore F(f(ax)) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(ax).\,e^{isax}\,dx$$

Now putting $ax = t \Rightarrow dx = \frac{dt}{a}$

$$F = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t).\,e^{ist/a}\frac{dt}{a}$$

$$= \frac{1}{a}\cdot\underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} f(t).\,e^{is(t/a)}\,dt}_{= f(s/a)}$$

$$= \frac{1}{a}\cdot f(s/a)$$

$$\Rightarrow \boxed{F\{f(ax)\} = \frac{1}{a}f\left(\frac{s}{a}\right)}, \quad a \neq 0$$

self 3. **Shifting property :**

If $f(s)$ is the fourier transform of $f(x)$, then the fourier transform of $f(x-a) = e^{isa}.f(s)$

i.e $\boxed{F\{f(x-a)\} = e^{isa}\cdot f(s)}$

Imp

As we know that —

$$F\{f(x)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x).\,e^{isx}dx = f(s)$$

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Now

$$F\{f(x-a)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x-a)\, e^{isx}\, dx$$

let $x-a=t \Rightarrow dx = dt$

$\Rightarrow x = t+a$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\cdot e^{is(t+a)}\, dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\, e^{ist}\cdot e^{isa}\, dt$$

$$= e^{isa}\cdot \underbrace{\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\cdot e^{ist}\, dt\right]}_{=f(s)}$$

$$\boxed{F\{f(x-a)\} = e^{isa}\cdot f(s)}$$ proved

Note: If $F_s(s)$ and $F_c(s)$ are the fourier transform of the Fourier sine & Fourier Cosine transform of $f(x)$ respectively, then,

$$F_s\{f(ax)\} = \frac{1}{a} F_s\left(\frac{s}{a}\right) \quad \text{and}$$

$$F_c\{f(ax)\} = \frac{1}{a}\cdot F_c\left(\frac{s}{a}\right)$$

(4) Modulation theorem:
If $f(s)$ is the Complex fourier transform of $f(x)$ then the fourier transform of $f(x)\cos ax = \frac{1}{2}\left[f(s-a) + f(s+a)\right]$

$$\boxed{F\{f(x)\cos ax\} = \frac{1}{2}\left[f(s-a) + f(s+a)\right]}$$

self proff

we know that
$$F\{f(x)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\cdot e^{isx}\, dx$$

$\because \cos ax = \dfrac{e^{iax} + e^{-iax}}{2}$

$$\therefore F\{f(x)\cos ax\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\cdot\left(\frac{e^{iax}+e^{-iax}}{2}\right)\cdot e^{isx}\, dx$$

$$= \frac{1}{2\pi}\left[\int_{-\infty}^{\infty} f(x)\cdot \frac{e^{iax}}{2}\cdot e^{isx}\, dx + \int_{-\infty}^{\infty} f(x)\cdot\frac{e^{-iax}}{2}\cdot e^{isx}\, dx\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{2}\int_{-\infty}^{\infty} f(x)\cdot e^{i(a+s)x}\, dx + \int_{-\infty}^{\infty} f(x)\cdot \frac{e^{i(s-a)x}}{2}\, dx\right]$$

$$= \frac{1}{2}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\cdot e^{ik_1x}\, dx + \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\cdot e^{ik_2x}\, dx\right]$$

where $k_1 = s+a$, $k_2 = s-a$

$$= \frac{1}{2}\left[f(k_1) + f(k_2)\right]$$

$$= \frac{1}{2}\left[f(s+a) + f(s-a)\right]$$

$$\Rightarrow \boxed{F\{f(x)\cos ax\} = \frac{1}{2}\left[f(s+a) + f(s-a)\right]}$$

Note: (I) $F_s(s)$ & $F_c(s)$ are Fourier sine & cosine transform of $f(x)$

$$F_s\{f(x)\sin ax\} = \tfrac{1}{2}\left[F_s(s+a) + F_s(s-a)\right]$$
$$F_c\{f(x)\sin ax\} = \tfrac{1}{2}\left[F_c(s+a) + F_c(s-a)\right]$$
$$F_s\{f(x)\cos ax\} = \tfrac{1}{2}\left[F_s(s+a) + F_s(s-a)\right]$$
$$F_c\{f(x)\cos ax\} = \tfrac{1}{2}\left[F_c(s+a) + F_c(s-a)\right]$$

Example: Find the fourier sine transform & cosine transform of x^{n-1}, $n>0$.

As we know that

$$f_s(\alpha) = F_S(f(x)) = \int_0^\infty f(x)\, \sin st\, dt \quad —(a)$$

$$= \int_0^\infty x^{n-1} \sin st\, dt \quad —(2)$$

$$f_c(x) = F_c(f(x)) = \int_0^\infty x^{n-1} \cdot \cos st\, dt \quad —(b)$$

$$\therefore \Gamma n = \int_0^\infty e^{-x} x^{n-1}\, dx \quad —(1)$$

$$\left(\because \sin st = \frac{e^{ist} - e^{-ist}}{2i} \ \& \ \cos st = \frac{e^{ist} + e^{-ist}}{2} \right)$$

In (1), putting

$dx = is\, dy \Rightarrow x = isy \ \& \ i = e^{i\pi/2}$

$$\Gamma n = \int_0^\infty e^{-isy} (isy)^{n-1} \cdot is\, dy$$

$$= \int_0^\infty e^{-isy} y^{n-1} \cdot (is)^n\, dy$$

$$= (is)^n \int_0^\infty e^{-isy} \cdot y^{n-1}\, dy$$

$$\frac{(-1)^n \frac{1}{s^n}}{i^n} \quad \Gamma n$$

$$= i s^n \int_0^\infty e^{-isy} (iy)^{n-1}\, dy$$

2

Example: Solve the integral eqn

$$\int_0^\infty f(\theta) \cdot \cos \alpha\theta \cdot d\theta = \begin{cases} 1-\alpha & 0 \le \alpha \le 1 \\ 0 & \alpha > 1 \end{cases}$$

Evaluate $\int_0^\infty \frac{\sin^2 t}{t^2}\, dt = \pi/2$

$$F_c(\alpha) = \int_0^\infty f(\theta) \cdot \cos \alpha\theta\, d\theta = \cancel{\int_0^\infty f(\theta) \cdot \frac{e^{i\alpha\theta} + e^{-is\theta}}{2} d\theta}$$

By the Inversion formula, we have

$$f(\theta) = \frac{2}{\pi} \int_0^\infty F_c(\alpha) \cdot \cos \alpha\theta \cdot d\alpha$$

$$= \frac{2}{\pi} \int_0^1 (1-\alpha) \cos \alpha\theta \cdot d\alpha + \int_1^\infty 0 \cdot \cos \alpha\theta\, d\alpha$$

$$= \frac{2}{\pi} \int_0^1 \cos \alpha\theta \cdot d\alpha - \frac{2}{\pi} \int_0^1 \alpha \cos \alpha\theta\, d\alpha$$

$$= \frac{2}{\pi} \left[\left[\frac{\sin \alpha\theta}{\alpha}\right]_0^1 - \left[\alpha \cdot \frac{\sin \alpha\theta}{\alpha} - \int \frac{\sin \alpha\theta\, d\alpha}{\alpha} \right]_0^1 \right]$$

$$= \frac{2}{\pi} \left[\sin\theta - \left[\alpha \sin \alpha\theta + \frac{\cos \alpha\theta}{\alpha^2} \right]_0^1 \right]$$

$$= \frac{2}{\pi} \left[\cancel{\sin\theta} - [\cancel{\sin\theta} + \cos\theta - 1] \right]$$

$$= \frac{2}{\pi} \left[1 - \cos\theta \right] = \frac{2}{\pi} \left[2\left(1 - \cos^2\frac{\theta}{2}\right) \right]$$

$$= \frac{4}{\pi} \sin^2 \theta/2$$

$$\Rightarrow \int_0^\infty$$

* * *

After 1st Internal — Monday

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Internal

(4) Given that: $\frac{\partial y}{\partial t} = 3\frac{\partial^2 y}{\partial x^2}$ —— (1)

where $y(\pi/2, t) = 0$, $\left(\frac{\partial y}{\partial x}\right)_{x=0} = 0$, &

$y(x,0) = 30\cos 5x$

In eqn (1), taking L.T on the both sides, we have

$s\bar{y}(x,s) - y(x,s) = 3\frac{d^2\bar{y}}{dx^2}$

$\frac{s}{3}\bar{y}(x,s) - \frac{30\cos 5x}{3} = \frac{d^2\bar{y}}{dx^2}$

$\left(D^2 - \frac{s}{3}\right)\bar{y}(x,s) = -10\cos 5x$

$f(D)\,\bar{y}(x,s) = X$ —— (2)

$\cos x = 1, \quad x = 0$ &

$\cos x = \pi, \quad x \neq 0$

$\bar{y}(x,s) = c_1 e^{\sqrt{s/3}\,x} + c_2 e^{-\sqrt{s/3}\,x} + \frac{30\cos 5x}{75+s}$

$y(x,t) = +30e^{-75t}\cos 5x$.

(2) $y(x) = 2 - \int y(t)\cdot e^{x-t}\,dt$

$L[y(x)] = 2L[1] - L\left[\int_0^x y(t)\cdot e^{x-t}\,dt\right]$

$F(x) = y(x)$

$G(x) = e^{x}$

$= \frac{2}{s} - L[F(x)]\;L[G(x)]$

$= \frac{2}{s} -$

* * *

Tuesday

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Convolution:

The convolution of two function $f(x)$ & $g(x)$ over the interval $(-\infty, \infty)$ is defined as $f*g = \int_{-\infty}^{\infty} f(u)g(x-u)\,du$

$f*g = \int_{-\infty}^{\infty} f(u)g(x-u)\,du = h(x)$

$\Rightarrow \boxed{f*g = h(x)}$

Theorem: Convolution theorem for fourier transform

The fourier transform of the convolution of $f(x)$ and $g(x)$ is the product of their Fourier Transform.

$F[f(x)*g(x)] = F[f(x)]\cdot F[g(x)]$

Proof: We know that Fourier transform

$F[f(x)] = \int_{-\infty}^{\infty} f(x)e^{isx}\,dx$

also let fourier transfer of $g(x)$ be similar

so

$F[f(x)*g(x)] = \int_{-\infty}^{\infty} \cancel{(f(x)*g(x))e^{isx}}\,dx$

$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(u)*g(x-u)\,du\right] e^{isx}\,dx$

$= \int_{-\infty}^{\infty} f(u)\left\{\int_{-\infty}^{\infty} g(x-u)e^{isx}\,dx\right\}du$

(∵ changing the order of integration)

$= \int_{-\infty}^{\infty} f(u)\left\{\int_{-\infty}^{\infty} e^{is(x-u)}\cdot g(x-u)\,dx\right\} e^{isu}\,du$

$= \int_{-\infty}^{\infty} f(u)\cdot e^{isu}\,du * \int_{-\infty}^{\infty} e^{is(x-u)}g(x-u)\,dx$

tuesday

$$\therefore F[f(x) * g(x)] = F[f(x)] * F[g(x)]$$

Parseval's Identity for Fourier transform

If the fourier transform of $f(x)$ and $g(x)$ are respectively $F(s)$ and $G(s)$ then

(i) $$\frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)\,\bar{G}(s)\,ds = \int_{-\infty}^{\infty} f(x)\,\bar{g}(x)\,dx$$

(ii) $$\frac{1}{2\pi}\int_{-\infty}^{\infty} [F(s)]^2\,ds = \int_{-\infty}^{\infty} |f(x)|^2\,dx$$

where bar implies that complex conjugate.

Verify (i) Proof.

Proof: We have $$\int_{-\infty}^{\infty} f(x)\,\bar{g}(x)\,dx = \int_{-\infty}^{\infty} f(x)\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{G}(s)\,e^{isx}\,ds\right\}dx$$

We know that

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)\,e^{-isx}\,ds$$

$$\overline{f(x)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{F}(s)\,e^{isx}\,ds$$

$$\therefore \int_{-\infty}^{\infty} f(x)\cdot\bar{g}(x)\,dx = \int f(x)\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{G}(s)\,e^{isx}\,ds\right\}dx$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{G}(s)\left\{\int_{-\infty}^{\infty} f(x)\,e^{isx}\,dx\right\}ds$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{G}(s)\cdot F(s)\,ds$$

$$\int_{-\infty}^{\infty} f(x)\,\overline{g(x)}\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)\,\bar{G}(s)\,ds$$

We know that $|z|^2 = z\bar{z}$

Proof (ii)

$$\therefore f(x)\cdot\overline{f(x)} = |f(x)|^2$$

$$\therefore \int_{-\infty}^{\infty} |f(x)|^2\,dx = \int_{-\infty}^{\infty} f(x)\cdot\bar{f}(x)\,dx$$

$$\because f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)\cdot e^{-isx}\,dx$$

$$\therefore \int_{-\infty}^{\infty} f(x)\,\bar{f}(x)\,dx = \int_{-\infty}^{\infty} f(x)\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{F}(s)\,e^{isx}\,ds\right\}dx$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{F}(s)\underbrace{\left\{\int_{-\infty}^{\infty} f(x)\,e^{isx}\,dx\right\}}\,ds$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{F}(s)\cdot F(s)\,ds$$

$\because \bar{F}(s)\cdot F(s) = |F(s)|^2$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} [F(s)]^2\,ds$$

$$\Rightarrow \int_{-\infty}^{\infty} |f(x)|^2\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} [F(s)]^2\,ds$$

NB: Note that; The following Parseval's identity for Fourier Cosine & Sine transform

(i) $$\frac{2}{\pi}\int_{0}^{\infty} F_c(s)\,G_c(s)\,ds = \int_{0}^{\infty} f(x)\,g(x)\,dx$$

(ii) $$\frac{2}{\pi}\int_{0}^{\infty} F_s(s)\,G_s(s)\,ds = \int_{0}^{\infty} f(x)\,g(x)\,dx$$

(iii) $$\frac{2}{\pi}\int_{0}^{\infty} [F_c(s)]^2\,ds = \int_{0}^{\infty} \{f(x)\}^2\,dx$$

(iv) $$\frac{2}{\pi}\int_{0}^{\infty} [F_s(s)]^2\,ds = \int_{0}^{\infty} \{f(x)\}^2\,dx$$

$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$

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Fourier transform of the derivatives of function. The Fourier transform

$F(u(x,t))$ is define as

$$F[u(x,t)] = \int_{-\infty}^{\infty} u e^{isx} dx \quad \text{--- (1)}$$

Now taking the derivative from $\frac{\partial^2 u}{\partial x^2}$

Sol^n Replacing u by $\frac{\partial^2 u}{\partial x^2}$ in (1) we have

$$F\left[\frac{\partial^2 u(x,t)}{\partial x^2}\right] = \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial x^2} \cdot e^{isx} dx$$

How we ILATE

$\frac{\partial u}{\partial x} \to 0$ as $x \to \infty$

$$= \left[e^{isx} \frac{\partial u}{\partial x} - \int \frac{\partial u}{\partial x} \cdot e^{isx} \cdot is\, dx\right]_{-\infty}^{\infty}$$

$$= \left[\underbrace{e^{isx}\frac{du}{dx}}_{=0} - is\left\{u e^{isx} - \int u \cdot e^{isx} is\, dx\right\}\right]_{-\infty}^{\infty}$$

on applying general rule of integration

Here, as $x \to \infty$ then $\frac{\partial u}{\partial x} \to 0$, then

$$F\left[\frac{\partial^2 u}{\partial x^2}\right] = -s^2 F$$

Similarly, in the case of fourier sine and Cosine transform, we have

$$F_s\left[\frac{\partial^2 u}{\partial x^2}\right] = s(u)_{x=0} - s^2 F_s(u)$$

$$F_c\left[\frac{\partial^2 u}{\partial x^2}\right] = -\left(\frac{\partial u}{\partial x}\right)_{x=0} - s^2 F_c[u]$$

In general, the Fourier transform of the n^{th} derivative of $f(x)$ is given by —

$$F\left[\frac{d^n f(x)}{dx^n}\right] = (-is)^n F[f(x)]$$

* * *

Friday

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Application of Fourier Transform

Application of fourier transform for BVP

In one-dim BVP, the pde can be easily transform into ODE by applying a suitable transform. The required sol^n is obtained by solving the eq^n and inverting by the complex inversion formula or any other method.

(1) If in a problem $u(x,t)_{x=0}$ or $u(0,t)$ is given, then we use infinite sine transform to remove $\frac{\partial^2 u}{\partial x^2}$ from the DE.

In case $\left[\frac{\partial(u(x,t))}{dx}\right]_{x=0}$ is given then we employ ~~within~~ fourier cosine transform to remove $\frac{\partial^2 u}{\partial x^2}$

(2) If in a problem $u(0,t)$ & $u(l,t)$ are given, then we use finite sine transform to remove $\frac{\partial^2 u}{\partial x^2}$ from the DE.

In case $\left(\frac{\partial u}{\partial x}\right)_{x=0}$ and $\left(\frac{\partial u}{\partial x}\right)_{x=l}$ are given, we employ finite fourier cosine transform to remove $\frac{\partial^2 u}{\partial x^2}$.

(1) Solve, $\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}$ if $u(0,t) = 0$, $u(x,0) = e^{-x}$ $(x>0)$, $u(x,t)$ is b.dd where $x>0$, $t>0$

Sol^n Given that —

$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}, \quad x>0, t>0 \quad \text{--- (1)}$$

B.C $u(0,t) = 0$, $u(x)$ is b.dd — (2)

I.C $u(x,0) = e^{-x}$, $x>0$ — (3)

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$\because u(0,t)$ is given then we take Fourier sine transform —

$$\int_0^\infty \frac{\partial u}{\partial t}\sin px\,dx = 2\int_0^\infty \frac{\partial^2 u}{\partial x^2}\sin px\,dx$$

$$\frac{d}{dt}\int_0^\infty u\sin px\,dx = 2\left[\frac{\partial u}{\partial x}\sin px\Big|_0^\infty - \int_0^\infty \frac{\partial u}{\partial x}p\cos px\,dx\right]$$

$$\left(\bar{u}_s = \int_0^\infty u\sin px\,dx\right)$$

$$\frac{d\bar{u}_s}{dt} = 2\left[0 - \int_0^\infty \frac{\partial u}{\partial x}p\cos px\,dx\right]$$

$$\frac{d\bar{u}_s}{dt} = -2p\left[u(x,t)\cos px\Big|_0^\infty - \int_0^\infty u(x,t)\{(-p)\cdot\sin px\}\,dx\right]$$

$$\frac{d\bar{u}_s}{dt} = -2p\times p\int_0^\infty u(x,t)\sin px\,dx.$$

$$\frac{d\bar{u}_s}{dt} = -2p^2\bar{u}_s$$

Integrating on the both sides, we have —

$$\log\left(\frac{\bar{u}_s}{c}\right) = -2p^2t$$

$$\boxed{\bar{u}_s = ce^{-2p^2t}}$$

In eqn (3). $\int_0^\infty u(x,0)\sin px\,dx = \int_0^\infty e^{-x}\sin px\,dx$

$$\bar{u}_s(x,0) = \int_0^\infty e^{-x}\sin px\,dx \quad \times$$

$$\bar{u}_s(p,0) = \int_0^\infty e^{-x}\sin px\,dx$$

$$\bar{u}_s(p,0) = \frac{p}{1+p^2} \quad —(5)$$

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In eqn (5), putting $c = p/1+p^2$

then,

$$\bar{u}_s(p,t) = \frac{p}{(1+p^2)}e^{-2p^2t}$$

$$u(x,t) = \frac{2}{\pi}\int_0^\infty \left(\frac{p}{1+p^2}\right)e^{-2p^2t}\sin px\,dp$$

Example:- 1 Solve the eqn

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad (x>0,\ u>0)$$

Subject to the condition

(i) $u=0$, when $x=0,\ t>0$

(ii) $u = \begin{cases}1 & 0<x \\ 0 & x\geq 1\end{cases}$, when $t=0$

(iii) $u(x,t)$ is bdd

Monday 29-2-16

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Soln:- Here $u(0,t)=0$

so, we take Fourier Sine transform of both sides of the given eqn.

$$\int_0^\infty \frac{\partial u}{\partial t}\sin px\,dx = \int_0^\infty \frac{\partial^2 u}{\partial x^2}\sin px\,dx$$

$$\Rightarrow \frac{\partial}{\partial t}\int_0^\infty u\sin px\,dx = \left[\frac{\partial u}{\partial x}\sin px\right]_0^\infty - \int_0^\infty p\frac{\partial u}{\partial x}\cos px\,dx$$

Here $\bar{u}_s = \int_0^\infty u\sin px\,dx$

$$\Rightarrow \frac{\partial \bar{u}_s}{\partial t} = \left[0 - \int_0^\infty p\frac{\partial u}{\partial x}\cos px\,dx\right]$$

$$= -p\left[u(x,t)\cos px + p\int_0^\infty u(x,t)\sin px\,dx\right]$$

$$= -p\left[0 + p\int_0^\infty u(x,t)\sin px\,dx\right]$$

$$= -p^2\bar{u}_s$$

$\Rightarrow \frac{\partial \bar{U}_s}{\partial t} = -p^2 \bar{U}_s$

$\Rightarrow \frac{\partial \bar{U}_s}{\bar{U}_s} = -p^2 \partial t$

$\Rightarrow \boxed{\bar{U}_s = c\, e^{-p^2 t}}$

By (ii) $\int_0^\infty u(x,0)\sin px\,dx = \int_0^1 1.\sin px\,dx + \int_1^\infty 0.\sin px\,dx$

$= \int_0^1 \sin px\,dx$

$= \left[-\frac{\cos px}{p}\right]_0^1 = \left[-\frac{\cos x}{p} + \frac{1}{p}\right]$

$\bar{u}(p,0) = \frac{1}{p} - \frac{\cos p}{p}$

$\Rightarrow \bar{u}(p,0) = c.e^{-p^2.0} = c$

$\Rightarrow c = \frac{1}{p} - \frac{\cos p}{p}$

$\Rightarrow \bar{u}(p,t) = c.e^{-p^2 t} = \left(\frac{1}{p} - \frac{\cos p}{p}\right) e^{-p^2 t}$

Now, inversion formula —

$u(x,t) = \frac{2}{\pi}\int_0^\infty \left(\frac{1}{p} - \frac{\cos p}{p}\right) e^{-p^2 t} \sin px\,dp$

Example-2 Solve the

$\frac{\partial \theta}{\partial t} = c^2 \frac{\partial^2 \theta}{\partial x^2} \quad (t>0)$ — (1)

If the initial temperature of an infinite bar is given by

$\theta(x) = \begin{cases} \theta_0 & \text{for } |x|<a \\ 0 & \text{for } |x|>a \end{cases}$ — (2)

Determine temp. at any pt. x & at any instant time t, $\theta(x,t)$

Here $\bar{\theta} = \int \theta \sin px\,dx$

So, $\therefore \frac{d\bar{\theta}}{dt} = -c^2 p^2 \bar{\theta} \Rightarrow \boxed{\bar{\theta} = d\, e^{-c^2 p^2 t}}$

Here

$\theta(x,0) = \begin{cases} \theta_0 & |x|<a \\ 0 & |x|>a \end{cases}$ — (2)

~~So, we take fourier sine transform~~ —

Now,

$\int_0^\infty u(x,0)\, e^{i x p}\,dx = \int_{-a}^{a} \theta_0 . e^{ixp}\,dx + \int_a^\infty 0.\, e^{ixp}\,dx$

$= \int_{-a}^{a} \theta_0 . e^{ix\phi}\,dx + \int_{-\infty}^{-a} 0.$

$= \theta_0 \left[\frac{e^{ixp}}{ip}\right]_{-a}^{a} = \theta_0 \left[\frac{e^{iap} - e^{-iap}}{ip}\right]$

$\bar{\theta}(p) = \frac{2\theta_0 \sin ap}{p} - \cancel{\frac{\theta_0}{p}[\cos ap + \cos ap]}$

$\bar{\theta}(p,0) = \frac{\theta_0}{p} - \frac{\theta_0 \cos ap}{p}$

$\Rightarrow \bar{\theta}(p,0) = d.e^{-c^2 p^2 .0} = d$

$\Rightarrow d = \frac{\theta_0}{p} - \frac{\theta_0 \cos ap}{p}$

$\therefore \bar{\theta}(p,t) = \left(\frac{\theta_0}{p} - \frac{\theta_0 \cos ap}{p}\right) e^{-c^2 p^2 t^2}$

$\therefore$ By inversion formula —

$\theta(x,t) = \frac{2}{\pi}\int \Big($

$$\bar{\theta} = \frac{2\theta_0 \cdot \sin ap}{p} \cdot e^{-c^2p^2t}$$

Now, taking its inverse Fourier transf

We get —

$$\theta = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\,\theta_0 \sin ap}{p} \cdot e^{-c^2p^2t} \cdot e^{ixp}\,dp$$

$$= \frac{\theta_0}{\pi}\int_{-\infty}^{\infty} \frac{\sin ap}{p} \cdot e^{-c^2p^2t}\,(\cos px - i\sin px)\,dp$$

$$= \frac{\theta_0}{\pi}\int_{-\infty}^{\infty} \frac{(\sin ap\cos px - i\sin ap \cdot \sin px)}{p}\, e^{-c^2p^2t}\,dp$$

$$= \frac{\theta_0}{\pi}\int_{-\infty}^{\infty} e^{-c^2p^2t} \cdot \frac{\sin ap\cos px}{p}\,dp \; -$$

$$\frac{\theta_0}{\pi}\, i\int_{-\infty}^{\infty} e^{-c^2p^2t}\,\frac{\sin ap \cdot \sin px}{p}\,dp$$

$$\theta = \frac{\theta_0}{2}\left[\operatorname{erf}\left(\frac{a+x}{2c\sqrt{t}}\right) + \operatorname{erf}\left(\frac{a-x}{2c\sqrt{t}}\right)\right]$$

Example-3

An infinite string is initially at rest and that the initial displacement is $f(x)$ $(-\infty < x < \infty)$. Determine the displacement $y(x,t)$ of the string.

Example-4 Solve the PDE

$$U_t(x,t) = U_{xx}(x,t) \Rightarrow \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

then $u(0,t) = u(2^2,t) = 0$ and

$u(x,0) = 2x$ where

$0 < x < 2^2$, $t > 0$.

So $\frac{1}{2}$ (4) self

Since $u(0,t) = 0$ is given

we take finite fourier sine transf

$$\int_0^4 \frac{\partial u}{\partial t}\sin\left(\frac{n\pi}{4}\right)x\,dx = \int_0^4 \frac{\partial^2 u}{\partial x^2}\sin\frac{n\pi}{4}\,dx$$

$$\frac{d\,\bar{U}_s}{dt} = -s^2 F_s\left(\frac{\partial^2 u}{\partial x^2}\right) = -s^2\,\bar{U}_s$$

$$\Rightarrow \frac{d\,\bar{U}_s}{\bar{U}_s} = -s^2\,dt = -\left(\frac{n\pi}{4}\right)^2 dt$$

$$\Rightarrow \boxed{\bar{U}_s = c\,e^{-\frac{n^2\pi^2}{16}t}} \text{ —— (ii)}$$

Putting $t = 0$,

$$\bar{U}(s,0) = c$$

Now taking finite fourier sine tr on second term

$$\bar{U}_s(s,0) = \int_0^4 u(x,0)\sin\frac{n\pi x}{4}\,dx$$

$$= \int_0^4 2x\sin\frac{n\pi x}{4}\,dx$$

$$= \frac{-32}{n\pi}\cos n\pi$$

$$\Rightarrow c = -\frac{32}{n\pi}\cos n\pi$$

$\therefore$ by (ii),

$$\bar{U}_s(x,t) = -\frac{32}{n\pi}\cos n\pi \cdot e^{-\frac{n^2\pi^2}{16}t}$$

taking fourier inverse transfor

$$U_s(x,t) = \int_0^4 -\frac{32}{n\pi}\cos n\pi\; e^{-\frac{n^2\pi^2}{16}t} \cdot \sin\frac{n\pi x}{4}\,dx$$

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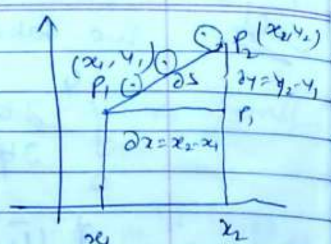
Review

By pythagoras formula



$$\partial s = \sqrt{(\partial x)^2 + (\partial y)^2}$$

$$\partial s = \partial x \sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2}$$

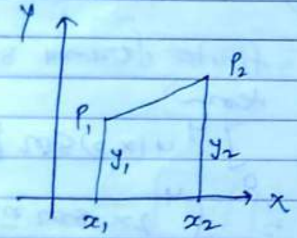
$$= \partial x \sqrt{1 + (y')^2}$$

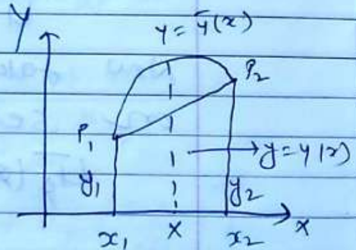
$$\therefore \int_{x_1}^{x_2} \sqrt{1+(y')^2}\, dx \Leftarrow \int_{x_1}^{x_2} ds$$

* —

(99) EULER's Equation or Euler's – Lagrange Eqn: —

(Go details p-183)





Let us examine the

$$I[y(x)] = \int_{x_1}^{x_2} F(x, y, y')\, dx \quad \text{—(1)}$$

For an ~~example~~ extreme value, the Boundary point of the curve $y(x_1) = y_1$ & $y(x_2) = y_2$ —(2)

We assume that the given function — $F(x, y, y')$ (Max or min).

let $y = y(x)$ be the curve which (Max or min). the eqn (1) then assume that $y = \bar{y}(x)$ is an curve close to $y = y(x)$ st both $y(x)$ & $\bar{y}(x)$ are included in a one-parameter of curve

$$Y(x, \beta) = y(x) + \beta\{\bar{y}(x) - y(x)\} \quad \text{—(3)}$$

For $\beta = 0$, then —
$Y = y(x)$

For $\beta = 1$
$Y = \bar{y}(x)$

Now, $\bar{y}(x) - y(x) = \delta y$ —(4)

Where, δy is known as the variation of the function $y(x)$ then eqn (3) reduces to

$$Y(x, \beta) = y(x) + \beta\, \delta y \quad \text{—(5)}$$

The variation δy is a function of x; In eqn (4), we have —

$$\delta y' = \bar{y}'(x) - y'(x) = (\delta y)'$$

In eqn (5), $\frac{\partial Y}{\partial \beta} = \delta y$

Similarly $\frac{\partial Y'}{\partial \beta} = \delta y'$ } —(6)

Now eqn (1) & (5) reduces of β, say

$$\phi(\beta) = \int_{x_1}^{x_2} F[x, Y(x,\beta), Y'(x,\beta)]\, dx \quad \text{—(7)}$$

For the extremum of the function, $\phi(\beta)$ for $\beta = 0$

then $\left(\frac{\partial \phi}{\partial \beta}\right)_{\beta=0} = 0$ — (next page)

$$\left(\frac{\partial \phi}{\partial \beta}\right)_{\beta=0} = 0 = \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial x}\cdot\frac{\partial x}{\partial \beta} + \frac{\partial F}{\partial y}\cdot\frac{\partial y}{\partial \beta} + \frac{\partial F}{\partial y'}\cdot\frac{\partial y'}{\partial \beta}\right] dx$$

$$\Rightarrow \int_{x_1}^{x_2} \left[0 + \frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y' \right] dx = 0$$

$$\Rightarrow \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y}\right)\delta y\, dx + \int_{x_1}^{x_2}\left(\frac{\partial F}{\partial y'}\right)\delta y'\, dx = 0$$

$$\Rightarrow \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y}\right)\delta y\, dx + \left[\left(\frac{\partial F}{\partial y'}\right)\delta y\right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\cdot \delta y\, dx = 0$$

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Note: —

$\because$ $y = y(x)$ & $y = \bar{y}(x)$ both passes through $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$

$\Rightarrow y(x_1) = \bar{y}(x_1)$ and $y(x_2) = \bar{y}(x_2)$ using eq^n (4)

$$\delta y\Big|_{x=x_1} = \bar{y}(x_1) - y(x_1) = 0 \quad \&$$

$$\delta y\Big|_{x=x_2} = \bar{y}(x_2) - y(x_2) = 0$$

$$\therefore \left[\frac{\partial F}{\partial y'}\delta y\right]_{x_1}^{x_2} = 0$$

$$\int_{x_1}^{x_2}\left(\frac{\partial F}{\partial y} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y'}\right)\right)\delta y\, dx = 0 \quad \text{—(8)}$$

Now the first term (factor)

$\left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right] = 0$ on the extremizing curve $y = y(x)$ is a given cts f^n, while the second factor $\delta y = 0$ because

of the arbitrary choice of the curve $y = \bar{y}(x)$ $\{\delta y$ at $x = x_1$ & $x = x_2\}$

Hence, the function extremizing f^n $y = y(x)$ satisfy the differentiate eq^n

$$\frac{\partial F}{\partial y} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y'}\right) = 0 \qquad x_1 \le x \le x_2$$

Which is known as Euler's Equation for functional for eq^n (1) and also known as Euler's-Lagrange's eq^n.

* * *

Tuesday 8|3|16

Example — Find the extremal for the functional

(i) $\int_0^{\pi/2} [y'^2 - y^2 + 2xy]\, dx$ with $y(0) = 0$, & $y(\pi/2) = 0$ — P. 147, Similar (i)

(ii) $\int_0^{\pi/2} [y^2 - y'^2 - 2y\sin x]\, dx$ with $y(0) = 0$ & $y(\pi/2) = 1$ — (148) $y = \sin x - \frac{x\cos x}{2}$

(iii) $\int_0^1 (xy + y^2 - 2y^2y')\, dx$ with $y(0) = 1$ & $y(1) = 2$ — Not

Sol^n: (1) We know that

$$I[y(x)] = \int_{x_1}^{x_2} F[x, y, y']\, dx$$

$$F[x, y, y'] = y'^2 - y^2 + 2xy$$

Using Euler's Equation —

$$\frac{\partial F}{\partial y} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y'}\right) = 0$$

$$2x - 2y - \frac{d}{dx}(2y') = 0 \Rightarrow y'' = -y + x$$

or $(D^2+1)y = 0$

y'^2 $2 \cdot y' \cdot y''$ (2)

y'' $2y''$

$\Rightarrow \cancel{2y' \cdot y'' - \frac{\partial}{\partial x}(}$

$\Rightarrow \cancel{(2y' \cdot y'' - 2y + 2x) - \frac{\partial}{\partial x}(2y') = 0}$

$-2y + 2x - 2y'' = 0$

$\Rightarrow y'' + y = x$

$(D^2+1)y = x \Rightarrow$

Case-I, $x = 0 \Rightarrow D = \pm i$

$CF = c_1 \cos x + c_2 \sin x$

Case-II, $x \neq 0$

$PI = \frac{1}{(D^2+1)} x = \frac{1}{(D^2+1)} x = x$

$= (1+D^2)^{-1} x = (1 + D^2 + D^4 - \dots) x = x$

$\boxed{y = x - \pi/_2 \sin x.}$

$y = CF + PI$

$y = c_1 \cos x + c_2 \sin x + x$

$y(0) = c_1 \times \cos 0 + c_2 \times \sin 0 + 0$

$= c_1 + 0 = c_1 = 0$

$y(\pi/_2) = c_1 \cdot 0 + c_2 \cdot 1 + \pi/_2$

$\Rightarrow c_2 = -\pi/_2$

$\therefore \boxed{y = x - \frac{\pi}{2} \sin x}$

* Some Important Cases in Euler's-Lagrange:-

Case-I If F is independent of y' so that $\frac{\partial F}{\partial y'} = 0$. Hence the Euler's eq^n

$\frac{\partial F}{\partial y} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y'}\right) = 0$ reduces to

$\frac{\partial F}{\partial y} = 0$ which is a first sol^n.

Case-II, The f^n F is L.D on y' s.t

$F(x, y, y') = M(x,y) + N(x,y)y'$

then $\frac{\partial F}{\partial y} = \frac{\partial M}{\partial y} + \frac{\partial N}{\partial y} y'$

$\frac{\partial F}{\partial y'} = N(x,y)$, $\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = \frac{d}{dx} N(x,y)$

Hence the Euler eq^n

$\frac{\partial F}{\partial y} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y'}\right) = 0$

$\frac{\partial M}{\partial y} + \frac{\partial N}{\partial y} y' - \frac{d}{dx} N(x,y) = 0$

$\frac{\partial M}{\partial y} + \frac{\partial N}{\partial y} y' - \left(\frac{\partial N}{\partial x} + \frac{\partial N}{\partial y}\frac{dy}{dx}\right) = 0$

$\frac{\partial M}{\partial y} + \frac{\partial N}{\partial y} y' - \frac{\partial N}{\partial x} - \frac{\partial N}{\partial y} y' = 0$

$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 0$

Hence the expression $Mdx + Ndy$ is an exact DE's, then the functional

$I[y(x)] = \int_{x_1}^{x_2} F(x, y, y') dx$

$= \int_{x_1}^{x_2} (Mdx + Ndy)$

Case-III, The functional F is independent of x and y i.e F is dependent only on y'.

Second form

$$\frac{\partial F}{\partial y} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y'}\right) = 0 \quad \text{— (1)}$$

Second form

so that $F = F(y')$

$$\Rightarrow \frac{dF}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} + \frac{\partial F}{\partial y'}\cdot\frac{dy'}{dx} \qquad (\because F = F(x,y,y'))$$

$$\Rightarrow \frac{dF}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}y' + \frac{\partial F}{\partial y'}\cdot y'' \quad \text{— (2)}$$

$$\frac{d}{dx}\left(y'\frac{\partial F}{\partial y'}\right) = y'\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{\partial F}{\partial y'}y'' \quad \text{— (3)}$$

Now by eqn (2) & (3) —

$$\frac{dF}{dx} - \frac{d}{dx}\left(y'\frac{\partial F}{\partial y'}\right) = \frac{\partial F}{\partial x} + y'\left\{\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right\}$$

$$\boxed{\frac{d}{dx}\left(F - y'\frac{dF}{\partial y'}\right) - \frac{\partial F}{\partial x} = 0}$$

Third form

Since, $\frac{\partial F}{\partial y'}$ is function of x, y & y'', we have

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y'}\right) + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial y'}\right)\frac{dy}{dx} + \frac{\partial}{\partial y'}\left(\frac{\partial F}{\partial y'}\right)\cdot\frac{dy'}{dx}$$

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = \frac{\partial^2 F}{\partial x\partial y} + y'\frac{\partial^2 F}{\partial y\partial y'} + y''\frac{\partial^2 F}{\partial y'^2} \quad \text{— (4)}$$

using eqn (4) from eqn (1), reduces to —

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

$$\boxed{\frac{\partial F}{\partial y} - \frac{\partial^2 F}{\partial x\partial y'} - y'\frac{\partial^2 F}{\partial y\partial y'} - y''\frac{\partial^2 F}{\partial y'^2} = 0} \quad \text{— (5)}$$

The above expression is third form of Euler's eqn.

In the case of III, $\frac{\partial F}{\partial y} = 0$, $\frac{\partial F}{\partial x} = 0$

$\frac{\partial^2 F}{\partial x\partial y'} = 0$, $\frac{\partial^2 F}{\partial y\partial y'} = 0$

then eq (5) reduces

$$-y''\frac{\partial^2 F}{\partial y'^2} = 0 \Rightarrow y'' = 0$$

$$\Rightarrow \boxed{y = c_1 + c_2 x}$$

(P-9)

Example: 1 Prove that, the shortest distance between the two points in a plane is st. line

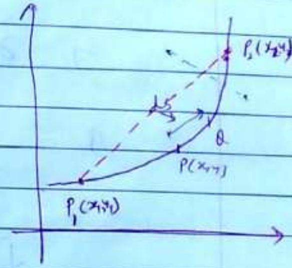
Soln:-

f^n

$$(y - y_1) = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$\Rightarrow y - y_1 = (x - x_1)\frac{dy}{dx}$$

$$= (x - x_1)ds$$



Soln:-

let $P_1(x_1, y_1)$ & $P_2(x_2, y_2)$ be two pts given on a plane. P_1 & P_2.

let $P(x, y)$ be any point on curve joing

let the arc $P_1P_2 = S$, then arc length

$$P_1P_2 = \int_{x_1}^{x_2} ds$$

$\therefore ds = \sqrt{dx^2 + dy^2}$

$$= \int_{x_1}^{x_2}\sqrt{1 + (y')^2}\,dx$$

$$P_1P_2 = \int_{x_1}^{x_2} f'(y)\,dx, \quad f(y) = \sqrt{1 + y'^2}$$

The condition is that the curve b/n P_1 to P_2 be the shortest path.

$\therefore$ By Euler's formula

$$\frac{\partial f}{\partial y} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y'}\right) = 0$$

$$\left(\because f \text{ is independent of } y \Rightarrow \partial f/\partial y = 0\right)$$

$$\Rightarrow -\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y'}\right) = 0$$

self

$$\Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y'}\right) = 0$$

on integrating both side, we have

$$\frac{\partial f}{\partial y'} = c$$

$$\Rightarrow \frac{\partial\left(\sqrt{1+(y')^2}\right)}{\partial y'} = c$$

$$\Rightarrow \frac{1}{2}\left\{\frac{1}{\sqrt{1+(y')^2}}\right\}\cdot 2y' = c$$

$$\Rightarrow \frac{y'}{\sqrt{1+(y')^2}} = c$$

$$\Rightarrow (y')^2 = c^2(1+(y')^2)$$

$$\text{or } (y')^2(1-c^2) = c^2$$

$$\Rightarrow y' = \frac{c}{\sqrt{1-c^2}} = a \quad (a \text{ is constant})$$

$$\Rightarrow y' = \frac{dy}{dx} = a$$

$$\Rightarrow dy = a\,dx$$

$$\Rightarrow \boxed{y = ax + b}, \quad b \text{ is constant}$$

$\Rightarrow$ which is st. line.

self

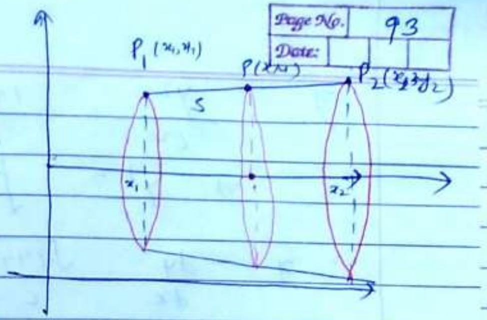
(2) Show that the area of the surface of revolution of a curve $y = y(x)$ is $2\pi\int_{x_1}^{x_2} y\sqrt{1+y'^2}\,dx$.

Hence, ST for this to be minimum, the curve must be catenary.

Sol: Let $P_1(x_1,y_1)$ & $P_2(x_2,y_2)$ be the extremities of any curve, now rotate the curve about x-axis.

self

Let $P(x,y)$ be any point on the curve. ∴ then the surface area of revolution is given by—



$$S = \int 2\pi y\,ds$$

$$= \int 2\pi y \frac{ds}{dx}\,dx$$

$$= 2\pi\int_{x_1}^{x_2} y\sqrt{1+y'^2}\,dx$$

$$\left[\because \frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,.dx = \sqrt{1+y'^2}\,.dx\right.$$

Proved

Part-II, This has to be minimum

Since $f = y\sqrt{1+y'^2}$ is independent of x. ∴ By Euler's Equation —

$$\frac{\partial f}{\partial y} - \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y'}\right) = 0$$

$$\frac{\partial}{\partial y}\left(f - y'\frac{\partial f}{\partial y'}\right) = 0$$

on integrating both side, we have—

$$f - y'\frac{\partial f}{\partial y'} = c$$

$$\Rightarrow y\sqrt{1+y'^2} - y'\frac{\partial}{\partial y'}\left(y\sqrt{1+y'^2}\right) = c$$

$$\Rightarrow y\sqrt{1+y'^2} - y'\left\{y\cdot\frac{1}{2}\frac{2y'}{\sqrt{1+y'^2}}\right\} = c$$

$$\Rightarrow y\sqrt{1+y'^2} - \frac{yy'^2}{\sqrt{1+y'^2}} = c$$

$$\Rightarrow y\left\{\frac{1+y'^2-y'^2}{\sqrt{1+y'^2}}\right\} = c$$

$$\Rightarrow \frac{y}{\sqrt{1+y'^2}} = c$$

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$$\Rightarrow \frac{y^2}{c^2} = 1 + y'^2$$

$$\Rightarrow y' = \sqrt{-1 + \frac{y^2}{c^2}} = \frac{\sqrt{-c^2+y^2}}{c}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sqrt{-c^2+y^2}}{c} \Rightarrow \int \frac{dy}{\sqrt{-c^2+y^2}} = \int \frac{dx}{c}$$

Now integrating both side, we have

$$\Rightarrow \int \frac{dy}{\sqrt{-c^2+y^2}} = \int \frac{dx}{c}$$

$$\Rightarrow \cosh^{-1}\left(\frac{y}{c}\right) = \frac{x}{c} + a = \frac{x+ac}{c}$$

$$\boxed{\therefore y = c\cosh\left(\frac{x+ac}{c}\right)}$$

Which is the eq^n of the curve of which is catenary.

* * *

Holi Vacation
19-3-16 to - 27-3-16

AFTER II - INTERNAL Exam

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28/3/16
No class due to Absent of Sir

29/3/16
Paper discussion

$$= \int_{\theta_1}^{\theta_2} (r^2 + r'^2)^{1/2} d\theta, \quad r' = \frac{dr}{d\theta}$$

$$F(\theta, r, r') = (r^2 + r'^2)^{1/2}$$

$$\frac{\partial F}{\partial r} - \frac{d}{d\theta}\left(\frac{\partial F}{\partial r_1}\right) = 0 \quad \text{---(1)}$$

$$r^3 + 2rr'^2 - r^2 r'' = 0$$

$$rr'' - 2r'^2 - r^2 = 0 \quad \text{---(2)}$$

putting $p = r' = \frac{dr}{d\theta}$

$$\frac{dp}{dr} = \frac{2p}{r} + \frac{r}{p}$$

putting $v = p/r$

$$\Rightarrow p = \cancel{rv}\; rv$$

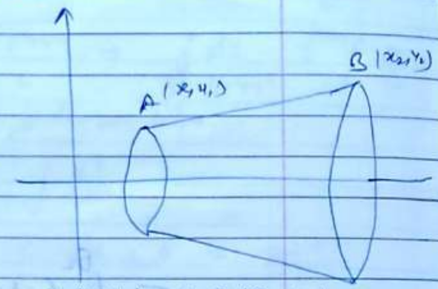
$$\left(\frac{v}{1+v^2}\right) dv = \frac{dr}{r}$$

$$\Rightarrow v = \frac{\sqrt{r^2-a^2}}{a}$$

$$p = \frac{r\sqrt{r^2-a^2}}{a}$$

Exercise: Find the curve passes through the pts (x_1, y_1) & (x_2, y_2) which rotated about the x-axis gives a minimum surface area.

Sol:



A curve joining $A(x_1, y_1)$ & $B(x_2, y_2)$ is revolved about x-axis. the area of surface of revolution

$$S[y(x)] = \int_{x_1}^{x_2} 2\pi y \, ds$$

$$= 2\pi \int_{x_1}^{x_2} y\sqrt{1+y'^2}\, dx$$

where $F[x, y, y'] = y\sqrt{1+y'^2}$

Using Euler's-Lagrange Eq^n where x absent case-(v).

$$\frac{d}{dx}\left(F - y'\frac{\partial F}{\partial y'}\right) = \frac{\partial F}{\partial x} = 0$$

$$\Rightarrow F - y'\frac{\partial F}{\partial y'} = c_1$$

$$\Rightarrow \cancel{y = c_1\sqrt{1+y'^2}}$$

$$\Rightarrow y\sqrt{1+y'^2} - y'\left(\frac{y}{2}(1+y'^2)^{-1/2} \cdot 2y'\right) = c_1$$

$$\Rightarrow y(1+y'^2) - y'^2 y = c_1\sqrt{1+y'^2}$$

$$\Rightarrow y(1+y'^2 - y'^2) = c_1\sqrt{1+y'^2}$$

$$\Rightarrow y = c_1\sqrt{1+y'^2} \Rightarrow y' = \sqrt{\frac{y^2}{c_1^2} - 1}$$

$$\Rightarrow c_1 = \frac{y}{\sqrt{1+y'^2}}$$

② Find the curve connecting the pts A & B which is traversed by a particle sliding from A to B in the shortest (time) (Friction & resistance of the medium are ignored) → (Brachistochrone)

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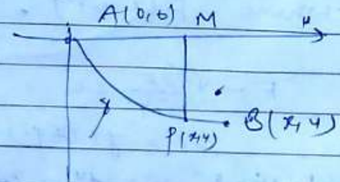

Let A(0,0) be the origin, we minimize the time of descent from A to B at any time t using the principle of conservation of Energy.

We have workdone in moving particle from A to B is equal to KE of P − PE of P. $(W = KE - PE)$

$$\Rightarrow mgy = \frac{1}{2}mv^2 - 0$$

$$v = \sqrt{2gy} \qquad \left(v = \frac{ds}{dt}\right)$$

$$\Rightarrow dt = \frac{1}{\sqrt{2gy}}\, ds$$

the time spent in moving from A to B

$$t[y(x)] = \int_A^B \frac{ds}{\sqrt{2gy}}$$

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$$t[y(x)] = \int_0^{x_1} \frac{\sqrt{1+y'^2}}{\sqrt{2gy}}\,dx$$

where $F(x,y,y') = \frac{\sqrt{1+y'^2}}{\sqrt{2gy}}$

$F(x,y,y')$ is independent of x, then Euler Eq^n

$$\frac{d}{dx}\left[F - y'\frac{\partial F}{\partial y'}\right] - \frac{\partial F}{\partial x} = 0$$

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$$t[y(x)] = \int_A^B \frac{ds}{\sqrt{2gy}} = \int_0^{x_1} \frac{\sqrt{1+y'^2}}{\sqrt{2gy}}\,dx$$

where $F(x,y,y') = \sqrt{\frac{1+y'^2}{y}}$

$$F - y'\frac{\partial F}{\partial y'} = c$$

$$c_1 = y(1+y'^2)$$

putting $y' = \frac{dy}{dx} = \cot\theta$

$$y = \frac{c_1}{2}(1-\cos 2\theta) \quad \text{—— } (*)$$

$$x = c_1\left(\theta - \frac{\sin 2\theta}{2}\right) + c_2$$

$$\Rightarrow (x-c_2) = \frac{c_1}{2}(2\theta - \sin 2\theta)$$

$\sin^2\theta = \frac{1-\cos 2\theta}{2}$

putting $\varphi = 2\theta$, $\theta = \frac{\varphi}{2}$

Now by $(*)$ and $(**)$, we have

Reduce $x - c_2 = a(\varphi - \sin\varphi)$

$y = a(1-\cos\varphi)$

Now putting $A(0,0)$, where $x=0, y=0$ then $c_2 = ?$, $\varphi = ?$

Hence, we get the eq^n of cycloids in the std. form —

$$\left.\begin{aligned} x &= a(\varphi - \sin\varphi) \\ y &= a(1-\cos\varphi) \end{aligned}\right\} \text{ putting } y' = \frac{dy}{dx} = \cot\theta$$

where a is the radius of a rolling circle

$(D^2+1)y = 0$

$D = \pm i$

$\therefore y = (CF + PI)$

$CF = c_1\cos x + c_2 \sin x$

$PI = (D^2+1)^{-1} 0 = 0$

$\therefore y = c_1 \cos x + c_2 \sin x$

$\because y(0) = 0 \Rightarrow c_1 = 0$

$y = c_1(\cos x - 1) + c_2 \sin x$

Iso-perimetric Problems

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(JSU)

In certain problem: It is necessary to make a given integral

$$I = \int_{x_1}^{x_2} f(x,y,y')\,dx \quad \text{—— (1)}$$

max. or min. while keeping other integral

$$J = \int_{x_1}^{x_2} g(x,y,y')\,dx \quad \text{—— (2)}$$

In such problems involves one or more constraints condition. Just as $J = a$, constant perimeter and surface area. We obtain refer to the problems of this type as Isoperimetric Problems.

Such problems are generally solved by the method of Lagrange Multipliers.

To exterimize Eq^n (1), in multiply Eq^n (2) by λ and adding to Eq^n (1), where λ is the Lagrange Multipliers.

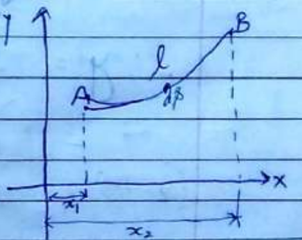
Hence the necessary condition for this integral $\int_{x_1}^{x_2} H\,dx$ to be an extremum is

$$\frac{\partial H}{\partial y} - \frac{d}{dx}\left(\frac{\partial H}{\partial y'}\right) = 0 \text{ where } H = f + \lambda g.$$

The value of the two constants of integration & the parameter of λ.

Example: (Self) Find the plane curve of fixed perimeter and Maximum Area.

Sol^n Let l be the fixed perimeter of a plane curve b/n the points A & B with x_1 & x_2



$$l = \int_{x_1}^{x_2} \sqrt{1+y'^2}\,dx \quad \text{—— (I)}$$

A = the area between the curve and x-axis.

M C | @ Guwahati

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$$A = \int_{x_1}^{x_2} y\,dx \quad \text{--- (2)}$$

Here, we have to maximise area, so max. (2) s.t.o (without) (1).

Now, $H = f + \lambda g = y + \lambda\sqrt{1+y'^2}$

Using Euler-Lagrange,

$$\frac{\partial H}{\partial y} - \frac{d}{dx}\left(\frac{\partial H}{\partial y'}\right) = 0 \qquad 1 - \frac{d}{dx}\left(\frac{\lambda y'}{\sqrt{1+y'^2}}\right) = 0$$

$$x - \frac{\lambda y'}{\sqrt{1+y'^2}} = a \Rightarrow (x-a) = \frac{\lambda y'}{\sqrt{1+y'^2}}$$

$$\frac{\partial H}{\partial y} - \frac{d}{dx}\left(\frac{\partial H}{\partial y'}\right) = 1 - \frac{d}{dx}\left(\frac{\lambda y'}{\sqrt{1+y'^2}}\right) = 0$$

$$\Rightarrow \frac{d}{dx}\left(\frac{\lambda y'}{\sqrt{1+y'^2}}\right) = 1$$

$$\Rightarrow \frac{\lambda y'}{\sqrt{1+y'^2}} = x + a$$

$$\Rightarrow \cancel{(\lambda y')^2} = \frac{(x-a)^2}{(\sqrt{1+y'^2})}$$

$$\frac{\lambda^2}{(x-a)^2} = 1 + \frac{1}{y'^2}$$

$$\Rightarrow (x-a)^2 = \frac{(\lambda y')^2}{(1+y'^2)} \Rightarrow \frac{1+y'^2}{y'^2} = \frac{\lambda^2}{(x-a)^2}$$

$$\Rightarrow (y')^2\left((x-a)^2 - \lambda^2\right) = -(x-a)^2$$

$$\Rightarrow \frac{1}{y'^2} = \frac{\lambda^2}{(x-a)^2} - 1$$

$$\Rightarrow y' = \sqrt{\frac{(x-a)^2}{\lambda^2 - (x-a)^2}}$$

$$y' = \frac{(x-a)}{\sqrt{\lambda^2-(x-a)^2}} \qquad \lambda^2-(x-a)^2 = t,\ -2(x-a)dx = dt$$

$$\therefore y' = \frac{(x-a)}{\sqrt{\lambda^2-(x-a)^2}} = \frac{(x-a)}{\sqrt{t}}\cdot\frac{dt}{-2(x-a)}$$

$$y = \sqrt{\lambda^2-(x-a)^2} + b$$

$$(y-b)^2 = \lambda^2 - (x-a)^2$$

$$= \frac{(t)^{-\frac{1}{2}+1}}{\left(-\frac{1}{2}+1\right)}\cdot\frac{1}{-2}$$

$$\Rightarrow (y-b)^2 + (x-a)^2 = \lambda^2$$

which is circle

$$= \sqrt{t}$$

$\sqrt{x^2+y^2} = z$

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$$= -\sqrt{t}$$

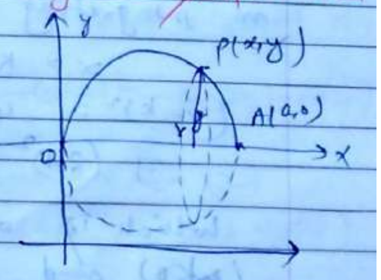
$$\Rightarrow y = -\sqrt{\lambda^2-(x-a)^2} + b$$

$$\Rightarrow (y-b)^2 = \lambda^2 - (x-a)^2$$

$$\Rightarrow \lambda^2 = (x-a)^2 + (y-b)^2$$

which represents a circle.

Example: Prove that the sphere is the solid of revolution which for a given surface area has max. volume



Sol^n: Consider the arc OPA of the curve which rotate about the x-axis, then the surface area

$$S = \int_0^a 2\pi y\,ds$$

$$S = 2\pi\int_0^a y\,ds = 2\pi\int_0^a \underbrace{y\sqrt{1+y'^2}}_{=f}\,dx$$

And the volume of the ~~surface~~ solid, so formed –

$$V = \int_0^a \pi y^2 dx \quad (\text{where } f \text{ is independent of } x)$$

Then $H = \pi y^2 + \lambda \times 2\pi y\sqrt{1+y'^2}$

Using Euler's Equation (Case: when the given H is independent of x).

(we use $\frac{d}{dx}\left(F - y'\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial x} = 0$ — P-90)

$$\pi y^2 + 2\pi\lambda y\sqrt{1+y'^2} - y'\left(2\pi y\lambda\cdot\frac{y'}{\sqrt{1+y'^2}}\right) = C$$

$$H - y'\frac{\partial H}{\partial y'} = C$$

$$\Rightarrow \pi y^2 + 2\pi\lambda y\left(\frac{1+y'^2-y'^2}{\sqrt{1+y'^2}}\right) = C$$

$$\Rightarrow \pi y^2 + \frac{2\pi\lambda y}{\sqrt{1+y'^2}} = C$$

$$\pi y^2 + \lambda 2\pi y' y\left(\frac{1}{2}\cdot\frac{2y'}{\sqrt{1+y'^2}}\right) = C$$

$$\Rightarrow \cancel{\pi y^2 + \frac{2\pi\lambda y'^2 y}{\sqrt{1+y'^2}}} = C$$

$$\pi y^2 + \frac{2\pi\lambda y}{\sqrt{1+y'^2}} = c \quad \text{—(1)}$$

Since the curve passing through O and A for which $y=0$, then Eqn (1), $c=0$

$\therefore \pi y^2 + \frac{2\pi\lambda y}{\sqrt{1+y'^2}} = 0 \Rightarrow y + \frac{\lambda 2}{\sqrt{1+y'^2}} = 0$

$\frac{y\,dy}{\sqrt{4\lambda^2-y^2}} = dx$

$\Rightarrow y' = \frac{dy}{dx} = \frac{\sqrt{4\lambda^2-y^2}}{y}$

on integrating both sides, we have —

$x = k - \sqrt{4\lambda^2-y^2}$

$(x-k)^2 = 4\lambda^2-y^2$

$\Rightarrow (2\lambda)^2 = (x-k)^2 + y^2$

which is a circle with centre $(k,0)$ and radius 2λ.

(on putting $x=0, y=0$, we have $k=2\lambda$)

Example Show that the functional

$$I[x(t),y(t)] = \int_0^{\pi/2} \left[2xy + \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right] dt$$

s.t. $x(0)=0$, $x(\pi/2)=-1$, $y(0)=0$, $y(\pi/2)=1$

is stationary for $x=-\sin t$, $y=\sin t$

pf:- Given that $F(x,y,y') = [2xy + (x')^2 + (y')^2]$

using Euler's Eqn,

(a) $\frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial x'}\right) = 0$ and

(b) $\frac{\partial F}{\partial y} - \frac{d}{dt}\left(\frac{\partial F}{\partial y'}\right) = 0$

(Here we solve for x & y w.r.t. t)

(a) $\Rightarrow 2y - \frac{d}{dt}[2x'] = 0$

$x'' \Rightarrow \frac{d^2x}{dt^2} = y$ —(1)

lly, $2x - \frac{d}{dt}(2y') = 0 \Rightarrow \frac{d^2y}{dt^2} = x$ —(2)

Now, solve the Eqn (1) & (2), then we have —

$\frac{d^2}{dt^2}\left(\frac{d^2x}{dt^2}\right) = x \Rightarrow \frac{d^4x}{dt^4} - x = 0$

$\Rightarrow (D^4-1)x = 0$

$\Rightarrow (D^2+1)(D^2-1)(x) = 0 \Rightarrow D = \pm i, \pm 1$

$\Rightarrow$ G.F. is,

$x = c_1e^t + c_2e^{-t} + c_3\cos t + c_4\sin t$ —(3)

$y = x'' = c_1e^t + c_2e^{-t} - c_3\cos t - c_4\sin t$ —(4)

In Eqn (4),

$y(0)=0 \Rightarrow c_1+c_2-c_3=0$ —(7)

$y(\pi/2)=1 \Rightarrow c_1e^{\pi/2}+c_2e^{-\pi/2} - c_4 = 1$ —(8)

In Eqn (3),

$x(0)=0$ then $c_1+c_2+c_3=0$ —(5)

$x(\pi/2)=-1 \Rightarrow c_1e^{\pi/2}+c_2e^{-\pi/2}+c_4 = -1$ —(6)

then $x = -\sin t$

$y = \sin t$

Example: Find the Extremal of the functional

$$I[x(t),y(t)] = \int_0^{\pi/4} [x'y' + 2x^2 + 2y^2]\,dt$$

where $x' = dx/dt$, $y' = dy/dt$ subject to the IC at $t=0$, then $x=y=0$. $t=\pi$[illegible] then $x=y=1$

Ans $x = y = \frac{\sinh 2t}{\sin \pi/2}$

Sol $F = x'y' + 2x^2 + 2y^2$

$\frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial x'}\right) = 0$

$\Rightarrow 4x - \frac{d}{dt}y' = 0 \Rightarrow \frac{d^2y}{dt^2} = 4x$ —(1)

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$$\Rightarrow \frac{\partial F}{\partial y} - \frac{d}{dt}\left(\frac{\partial F}{\partial y'}\right) = 0$$

$$\Rightarrow 4y - \frac{d}{dt}x' = 0 \Rightarrow \frac{d^2x}{dt^2} = 4y \Rightarrow y = \frac{1}{4}x'' \quad \text{(ii)}$$

$$\Rightarrow \frac{d^2}{dt^2}\left(\frac{1}{4}\frac{d^2x}{dt^2}\right) = 4x \quad (\text{using (ii) in (i)})$$

$$\Rightarrow \frac{1}{4}\frac{d^4x}{dt^4} = 4x \Rightarrow (D^4 - 16)x = 0$$

$$\Rightarrow (D^2+4)(D^2-4) = 0$$

$$\Rightarrow D = \pm 2i, \pm 2$$

$$\therefore x = c_1e^{2t} + c_2e^{-2t} + c_3\cos 2t + c_4\sin 2t$$

$$y = \frac{1}{4}x'' = \frac{1}{4}\cdot 4\left[c_1e^{2t} + c_2e^{-2t} - c_3\cos 2t - c_4\sin 2t\right]$$

Now by, $x(0) = 0$, $c_1 + c_2 + c_3 = 0$

$x(\pi/4) = 1$, $c_1e^{\pi/4} + c_2e^{-\pi/4} + c_4 = 1$

$y(0) = 0$, $c_1 + c_2 - c_3 = 0$

$y(\pi/4) = 1$, $c_1e^{\pi/4} + c_2e^{-\pi/4} - c_4 = 1$

$\Rightarrow c_1 + c_2 = 0$ (a) — $x(\pi/4) = 1 = c_1e^{\pi/2} + c_2e^{-\pi/2} + c_4$

$c_1e^{\pi/2} + c_2e^{-\pi/4} = 2$ — $x(0) = 0 = c_1 + c_2 + c_3$

$x = y = \frac{\sinh 2t}{\sin \pi/2}$

& $y(0) = c_1 + c_2 - c_3 = 0$

(b) — $y(\pi/4) = c_1e^{\pi/2} + c_2e^{-\pi/2} - c_4 = 0$

$$a + b = 1 = 2c_4 \Rightarrow c_4 = 1/2$$

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* Geodesic :—

A Geodesic on a surface is a curve along which the distance between two pt. on the surface is a minimum.

Example: Find the Geodesic of a Right Circular cylinder of Radius a.

Sol^n:— In cylinder polar co-ordinate (ρ, ϕ, z), $x = \rho\cos\phi$, $y = \rho\sin\phi$, $z = z$

The element of arc ds on the surface of the circular cylinder of Radius a is given then

$$(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2$$

$$= (d\rho)^2 + \rho^2(d\phi)^2 + (dz)^2$$

$$(ds)^2 = 0 + \rho^2(d\phi)^2 + (dz)^2 \quad \left(\because \rho = a,\ d\rho = 0\right)$$

$$= (d\phi)^2\left\{\rho^2 + \left(\frac{dz}{d\phi}\right)^2\right\}$$

$$ds = \sqrt{a^2 + \left(\frac{dz}{d\phi}\right)^2}\, d\phi$$

then minimum surface occupied

$$S = \int_{\phi_1}^{\phi_2} \sqrt{a^2 + (z')^2}\, d\phi \quad \text{— (i)}$$

for the Geodesics, given cylinder,

$$\frac{\partial F}{\partial z} - \frac{d}{d\phi}\left(\frac{\partial F}{\partial z'}\right) = 0$$

$$\Rightarrow -\frac{d}{d\phi}\cdot\frac{2z'}{\sqrt{a^2 + (z')^2}} = 0$$

$$\Rightarrow \frac{z'}{\sqrt{a^2 + z'^2}} = c$$

$$\Rightarrow z = \left(\frac{ac}{1-c}\right)\phi + c_2$$

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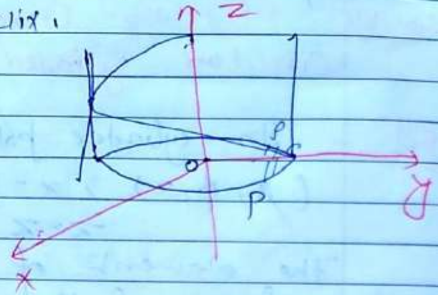
The required Geodesics are Circular Helix. This curve is drawn on a circular cutting its generators at a Constant angle and it is known as a Circular Helix.



Home Work:

(Q) (1) Prove that the Geodesics on a plane is straight line.

(2) Find a function $y(x)$ for which $\int_0^1 (x^2+y'^2)\,dx$ is stationary. Given that $\int_0^1 y^2 dx = 2$, $y(0)=0$, $y(1)=0$

(3) Find the extremals of the isoperimetric problem $v[y(x)] = \int_{x_0}^{x_1} y'^2 dx$, given that $\int_{x_0}^{x_1} y\,dx = c$, a Constant.

$x_0=0, x_1=2$
$c=1, y(0)=0$
$y(2)=1$

(4) Find the Externals of the function $I = \int_0^{\pi} [y'^2 - y^2]\,dx$ under the conditions $y(0)=0$, $y(\pi)=1$ and subject to the constraints $\int_0^{\pi} y\,dx = 1$

(5) Determine the curve of length which passes through the points (0,0)

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and (1,0) and for which the area between the Curve and the x-axis is a maximum.

(6) Find the curve of fixed length l that joining the point $P_1(x_1,0)$ and $P_2(x_2,0)$ lies above the ~~axis~~ x-axis and encloses the maximum area between itself and the x-axis.

(7) Show that the curve c of given length l which minimize the curved surface area of solid generated by the revolution of c about x-axis is a Catenary.

(Imp)

Tuesday 5/4/16

Sol^n

(1) goto P- 91

(2) $I = \int_0^1 (x^2+y'^2)\,dx$

Here $F(xy) = x^2+y'^2$

Now using Euler's Eq^n —

$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$

$\Rightarrow 0 - \frac{d}{dx}(2y') = 0$

$\Rightarrow 2y' = c$

$\Rightarrow y' = c/2 = d$

$\Rightarrow \boxed{y = dx + k} \Rightarrow y^2 = (dx+k)^2$

Wrong Method

→ II-Method

⇒ we have given that $I = \int_0^1 (x^2+y'^2)dx$ & $\int_0^1 y^2 dx = 2$ i.e. $f = x^2+y'^2$, $g = y^2$

$\therefore H = f + \lambda g = x^2 + y'^2 + \lambda y^2$

∴ By Euler's Eqⁿ

$\frac{\partial H}{\partial y} - \frac{d}{dx}\left(\frac{\partial H}{\partial y'}\right) = 0 \Rightarrow 2y\lambda + \frac{d}{dx}(2y') = 0$

$\Rightarrow \frac{d}{dx}(2y') = 2\lambda y$

$\Rightarrow y' = \lambda x + c$

$\Rightarrow \cancel{y = \lambda x^2 + cx + d}$

or $y'' = \lambda$

$(D^2 - \lambda)y = 0$

$\Rightarrow \cancel{D = \pm\sqrt{\lambda}}$

$D = \pm\sqrt{\lambda}$

$D^2 - \lambda = 0$

$D^2 = \lambda$

$D = \pm\sqrt{\lambda}$

$\boxed{y = c_1 e^{\sqrt{\lambda}t} + c_2 e^{-\sqrt{\lambda}t}}$

Now using given BC

$y(0) = 0 = c_1 + c_2$

$y(1) = 0 = c_1 e^{\sqrt{\lambda}} + c_2 e^{-\sqrt{\lambda}}$

$c_1(e^{\sqrt{\lambda}} - e^{-\sqrt{\lambda}}) = 0$

$\Rightarrow c_1 = 0 = c_2$

if $f = x^2 - y'^2$

then $H = x^2 - y'^2 + \lambda y^2$

∴ By Euler's Eqⁿ $\left[\frac{\partial H}{\partial y} - \frac{d}{dx}\left(\frac{\partial}{\partial y'}\right)\right] = 0$

$2\lambda y - \frac{d}{dx}(-2y') = 0$

$\Rightarrow y'' + \lambda y = 0$

or $(D^2 + \lambda) = 0 \Rightarrow D = \pm\sqrt{\lambda}\, i$

$\therefore y = c_1 \cos\sqrt{\lambda}t + c_2 \sin\sqrt{\lambda}t$

$y(0) = 0 = c_1 \Rightarrow c_1 = 0$

$y(1) = 0 = c_2 \sin\sqrt{\lambda}t$

$\Rightarrow \sin\sqrt{\lambda} = \sin n\pi$

$\Rightarrow \sqrt{\lambda} = n\pi$

$\therefore y = c_2 \sin n\pi t$

∴ By condiⁿ

$\int_0^1 c_2^2 \sin^2 n\pi t\, dt = 2$

$\Rightarrow c_2^2 \int_0^1 \frac{1 - \cos 2n\pi t}{2}\, dt = 2$

$\Rightarrow c_2^2\left[\frac{1}{2}t - \frac{\sin 2n\pi t \cdot 2}{2n\pi}\right]_0^1 = 4$

$\Rightarrow c_2^2\left\{1 - \frac{\sin 2n\pi}{2n\pi}\right\} = 4$

$c_2 = 2$

(3) given that if $v(y(x)) = \int_{x_0}^{x_1} y'^2\, dx$ and $\int_{x_0}^{x_1} y\, dx = c$

Here take $f = y'^2$ & $g = y$

Let $H = f + \lambda g = y'^2 + \lambda y$

Now using Euler's Eqⁿ

$\frac{\partial H}{\partial y} - \frac{d}{dx}\left(\frac{\partial H}{\partial y'}\right) = 0$

$\Rightarrow \lambda - \frac{d}{dx}(2y') = 0$

$\Rightarrow \frac{d}{dx}(2y') = \lambda$

$\Rightarrow 2y'' = \lambda \Rightarrow 2D^2 y = \lambda$

$\cancel{D = \pm\sqrt{\lambda/2}}$

$\cancel{y = c_1 e^{\sqrt{\lambda/2}\,x} + c_2 e^{-\sqrt{\lambda/2}\,x}}$

~~Now using given condition~~

$D^2 y = \lambda/2$

on integrating both side twice

$$\boxed{y = \frac{\lambda}{4}x^2 + cx + d} \quad (1)$$

Note Suppose that in this Ques, it is given as

$I = \int_0^2 y'^2\, dx$ & $J = \int_0^2 y\, dx = 1$

BC, $y(0) = 0$, $y(2) = 1$

$y(0) = 0 = d$

& $y(2) = 1 = \lambda + 2c \Rightarrow c = (1-\lambda)/2$

∴ (1) becomes

$$\boxed{y = \frac{\lambda}{4}x^2 + \frac{(1-\lambda)}{2}x}$$

(4) given that $I = \int_0^\pi (y'^2 - y^2)\, dx$

and $J(y(x)) = \int_0^\pi y\, dx = 1$ ——(*)

Here we have

$f = (y'^2 - y^2)$ & $g = y$

Let $F = f + \lambda g$

$F = (y'^2 - y^2) + \lambda y$

Now using Euler's theorem

$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$

$\Rightarrow (-2y + \lambda) - \frac{d}{dx}(2y') = 0$

$\frac{d}{dx}(2y') = (-2y + \lambda)$

$$\boxed{y'' = \frac{(-2y + \lambda)}{2}}$$

On integrating twice,

$$\boxed{y = \frac{(-2y + \lambda)}{2}\frac{x^2}{2} + cx + d} \quad (1)$$

Now using the condition

$y(0) = 0 = d \Rightarrow d = 0$

& $y(\pi) = 1 = \frac{(1-\lambda)}{4}\pi^2 + c\pi$

$\Rightarrow c = \frac{4 - (1-\lambda)\pi^2}{4\pi}$

∴ By (1), we have

$$\boxed{y = \frac{(1-\lambda)}{4}x^2 + \frac{4 - (1-\lambda)\pi^2}{4\pi}x} \quad (2)$$

Now substituting these values in (*), we have

$\int_0^\pi \left(\frac{(1-\lambda)}{4}x^2 + \frac{4 - (1-\lambda)\pi^2 x}{4\pi}\right) dx = 1$

$= \left[\frac{(1-\lambda)}{4}\frac{x^3}{3} + \frac{4 - (1-\lambda)\pi^2}{4\pi}\cdot\frac{x^2}{2}\right]_0^\pi$

$= \frac{(1-\lambda)}{12}\pi^3 + \frac{(4 - (1-\lambda)\pi^2)\pi^2}{8\pi} = 1$

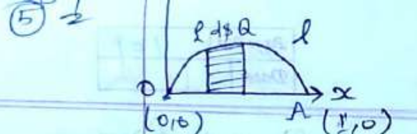
$\Rightarrow \frac{(1-\lambda)\pi^3}{4}\left[\frac{1}{3} - \frac{1}{2}\right] + \frac{1}{2}\pi = 1$

$\Rightarrow (1-\lambda) = \frac{(2-\pi)}{-2} \times \frac{4 \times -6 \times}{\pi^3}$

$\Rightarrow \lambda = 1 + \frac{(2-\pi) \times 12}{\pi^3}$

Putting λ in (2), we get desired result.

(5) $\frac{1}{2}$



Here length of curve is give (i.e. perimeter) $= \ell$ which passes through O and A. Let S be the area enclosed by the curve & x-axis. Then

$$S = \int_0^1 y\,dx = f$$

subject to constraints

$$\int_0^1 ds = \ell$$

i.e. $\int_0^1 \underbrace{\sqrt{1+y'^2}}_{=g}\,dx = \ell$

& BCs are $y(0)=y(1)=0$

Let $F = f + \lambda g$

$F = y + \lambda\sqrt{1+y'^2}$

Now using Euler's Eqn —

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

$$\Rightarrow 1 - \frac{d}{dx}\left(\frac{\lambda y'}{\sqrt{1+y'^2}}\right) = 0$$

$$\Rightarrow \frac{d}{dx}\left(\frac{\lambda y'}{\sqrt{1+y'^2}}\right) = 1$$

Now integrating both side —

$$\frac{\lambda y'}{\sqrt{1+y'^2}} = x + c$$

$$\frac{\lambda^2 y'^2}{(1+y'^2)} = (x+c)^2$$

$$\frac{1+y'^2}{y'^2} = \frac{\lambda^2}{(x+c)^2}$$

$$\Rightarrow \frac{1}{y'^2} + 1 = \frac{\lambda^2}{(x+c)^2}$$

$$\Rightarrow \frac{1}{y'^2} = \frac{\lambda^2 - (x+c)^2}{(x+c)^2}$$

$$\Rightarrow y' = \frac{x+c}{\sqrt{\lambda^2 - (x+c)^2}}$$

on integrating both side, we get

$$y = \int \frac{(x+c)\,dx}{\sqrt{\lambda^2-(x+c)^2}}$$

let $\lambda^2 - (x+c)^2 = t^2$

$\Rightarrow -2(x+c)dx = 2t\,dt$

$\therefore y = \int \frac{-t\,dt}{t} = -\int dt = -t$

$$\Rightarrow \boxed{y = -\sqrt{\lambda^2-(x+c)^2} + d}$$

~~Now Putting BC~~

~~$y(0) = -\sqrt{\lambda^2 - c^2} + d = 0$~~

~~$y(1) = -\sqrt{\lambda^2-(c+1)^2} + d = 0$~~

~~$\sqrt{\lambda^2-(c+1)^2} = \sqrt{\lambda^2 - c^2}$~~

~~$\Rightarrow \lambda^2-(c+1)^2 = \lambda^2 - c^2$~~

~~$c+1 = c$~~

$$(y-d)^2 = (\lambda^2 - (x+c)^2)$$

$$(y-d)^2 + (x+c)^2 = \lambda^2$$

which is equation of circle with radius λ.

~~II Method~~ Now using BC —

$y(0) = 0 \Rightarrow (-d)^2 + (c)^2 = \lambda^2$

$\Rightarrow c^2 = \sqrt{\lambda^2 - d^2}$

$y(1) = 0 \Rightarrow (c+1)^2 = \sqrt{\lambda^2 - d^2}$

(2) Find the functional for the B.V.P defined by $\frac{d^2y}{dx^2} = f(x)$, $y(a) = y(b) = 0$

$\delta y = ? \quad = y_2 - y_1$

we have

$$\int_a^b \underset{\text{I}}{f(x)}\,\underset{\text{II}}{\delta y}\,dx = \int_a^b \left(\frac{d^2y}{dx^2}\right)\delta y\,dx$$

Now ~~differentiating~~ integrating by part

$$= \left[\frac{dy}{dx}\delta y\right]_a^b - \int_a^b \frac{dy}{dx}\cdot\frac{d}{dx}(\delta y)\,dx$$

$\delta \equiv \partial$

$$\int_a^b f(x)\delta y\,dx = 0 - \int_a^b \frac{dy}{dx}\,\delta\left(\frac{dy}{dx}\right)dx$$

$\frac{d}{dx}(\delta y) = \delta\left(\frac{dy}{dx}\right)$

$\delta y(a) = \delta y(b) = y(a) = y(b) = 0$

$$= -\frac{1}{2}\int_a^b \delta\left(\frac{dy}{dx}\right)^2 dx$$

$$\Rightarrow \delta\int_a^b f(x)\,dx + \frac{1}{2}\int_a^b \delta\left(\frac{dy}{dx}\right)^2 = 0$$

$\Rightarrow$ ~~$\delta\{f\} + \int^b$~~

$$\Rightarrow I(v) = \int_a^b \left\{ f v + \frac{1}{2}\left(\frac{dv}{dx}\right)^2 \right\} dx$$

It follows that a unique soln of the problem exist at a min. value of the integral (1). Therefore, the integral in eqn (1) represents the required f^al of the problem.

In similar way, functional of other BVP & Initial BVP can be derived.

Tuesday

(sol^n back side)

(i) $\frac{d^2y}{dx^2} = f(x)$, $y(a) = y(b) = 0$

$$I(v) = \int_a^b v(2f - v'')\,dx$$

(ii) $\frac{d^2y}{dx^2} + ky = x^2$, $0 < x < 1$, $y(0) = 0$,

$\left(\frac{dy}{dx}\right)_{x=1} = 1$, $I(v) = \frac{1}{2}\int_a^b \left[\left(\frac{dv}{dx}\right)^2 - kv^2 + 2vx^2\right]dx - v(1)$.

(iii) $x^2 y'' + 2xy' = f(x)$. $y(a) = y(b) = 0$.

$$I(v) = \int_a^b v\left(2f - \frac{d}{dx}(x^2 y')\right)dx$$

(iv) $\nabla^2 u = 0$, $u = 0$ on the Boundary C to R

$$I(v) = \iint_R \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right]dx\,dy$$

(v) $\nabla^2 u = -f$, $u = 0$, on the Boundary C to R. $I(u) = \iint_R \frac{1}{2}\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right\} - uf\,dx\,dy$

(vi) $EI\frac{d^4y}{dx^4} + ky = f(x)$. $0 < x < l$

$y = 0 = \frac{d^2y}{dx^2}$ at $x = 0$, l

$$I(v) = \frac{1}{2}\int_0^l \left[EI\left(\frac{d^2v}{dx^2}\right)^2 + kv^2 - 2vf\right]dx$$

Base function :-

Suppose that to approximate a real valued f^n $f(x)$ over a finite interval $[a,b]$. A usual approach is to define $[a,b]$ into a no. of

Subintervals $[x_i, x_{i+1}]$, $i = 0, 1, 2, \ldots n-1$. Where $x_0 = a$ & $x_n = b$ & to interpolate linearity bet^n the value of $f(x)$ at the end points of each sub-intervals. Along $[x_i, x_{i+1}]$ the linear approximation f^n is given by

$$L_i(x) = \frac{1}{h_i}\left[(x_{i+1} - x)f_i + (x - x_i)f_{i+1}\right] \quad \text{--- (1)}$$

where $h_i = x_{i+1} - x_i$, from this to construct the piecewise linear interpolating f^n over $[x_0, x_n]$.

by the formula —

$$p(x) = \sum_{i=0}^{n} \phi_i(x) f_i \quad \text{--- (2)}$$

where
$$\phi_0(x) = \begin{cases} \frac{(x_1 - x)}{h_0}, & x_0 \le x \le x_1 \\ 0, & x_1 \le x \le x_n \end{cases}$$

$$\phi_i(x) = \begin{cases} \frac{(x - x_{i-1})}{h_{i-1}}, & x_{i-1} \le x \le x_i \\ \frac{(x_{i+1} - x)}{h_i}, & x_i \le x \le x_{i+1} \\ 0, & x > x_{i+1} \end{cases}$$

$$\phi_n(x) = \begin{cases} 0 & ; x_0 \le x \le x_{n-1} \\ \frac{x - x_{n-1}}{h_{n-1}}, & x_{n-1} \le x \le x_n \end{cases}$$

The f^n $\phi_i(x)$, $i = 1, 2, 3, \ldots n$ are called Base function or Shape function. It is easily seen that the Base f^n $\phi_i(x)$ are identically zero except for the range $[x_{i-1}, x_{i+1}]$ with $\phi_i(x_i) = 1$.

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Tuesday

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Rayleigh-Ritz Method

Consider second order BVP defined by

$$y'' + p(x)y + q(x) = 0, \quad y(a) = y(b) = 0$$ — (1) — (2)

$$I[v] = \int_a^b \left[\left(\frac{dv}{dx}\right)^2 - pv^2 - 2qv\right] dx \quad \text{— (3)}$$

If $y(x)$, the sol^n eq^n (1) is substituting in eq^n (3), then the Integral I will be min.

Now let $v(x) = \sum_{i=1}^{n} \alpha_i \phi_i(x)$ be an approximate sol^n.

$$I(\alpha_1, \alpha_2, \ldots \alpha_n) = \int_a^b \left\{\left[\frac{d}{dx}\sum_{i=1}^{n} \alpha_i \phi_i\right]^2 - p\left[\sum_{i=1}^{n} \alpha_i \phi_i(x)\right]\right.$$
$$\left. - 2q\sum_{i=1}^{n} \alpha_i \phi_i(x)\right\} dx \quad \text{— (4)}$$

The base f^n $\phi_i(x)$ are L.I & satisfy the B.C in eq(2)

$$\phi_i(a) = 0 \quad \& \quad \phi_i(b) = 0. \quad \text{— (5)}$$

For minimum, we have $\frac{\partial I}{\partial \alpha_1} \cdot \delta\alpha_1 + \frac{\partial I_1}{\partial \alpha_2} \delta\alpha_2$

$$+ \cdots + \frac{\partial I}{\partial \alpha_n} \cdot \delta\alpha_n = 0$$

Since the $\delta\alpha_i$ are arbitrary eq^n (6)

$$\frac{\partial I}{\partial \alpha_i} = 0, \quad i = 1, 2, 3, \ldots n \quad \text{— (7)}$$

If I is a quadratic of y & $\frac{dy}{dx}$, the eq^n (7) will be linear in α_i & can be solved easily.

* * *

UNIT – III

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* Integral Equation:—

A Eq^n in which an unknown function Appears under one, two or more integral signs is called integral Equation.

If the derivative & the integral of the unknown f^n is in the Eq^n then this type of Eq^n is called Integro-Diff Eq^n.

Types of Linear integral Eq^n:–

$$\boxed{\alpha(x)u(x) + F(x) + \lambda\int_D K(x,t)u(t)dt = 0} \quad \text{— (1)}$$

An integral Eq^n is called linear if only linear operator (linear operations) are perform upon the unknown function.

Thus an Eq^n of the form (1) is known as the Linear Integral Equation.

where $u(x)$ is the unknown function, $\alpha(x)$, $F(x)$ and $K(x,t)$ are known function. λ is a n-z real or complex parameter.

The integration extends over the domain D of the variable t.

Linear Integral Eq^{n}s are classifieds into two types.

→ (I) Volterra linear Integral Eq^n

The eq^n of the form

$$\alpha(x)u(x) = F(x) + \lambda\int_a^x K(x,t)u(t)dt$$

is a Volterra linear integral Eq^n. The eq^n in which one of the limit is a variable, is called Volterra linear integral Eq^n.

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Particular cases:

(i) When $\alpha(x) = 0$, Then the eqⁿ involves unknown fⁿ is only under the integral sign and such Eqⁿ is known as Volterra linear integral of the first kind.

$$F(x) = \lambda \int_a^x K(x,t)\, u(t)\, dt, \quad a > -\infty$$

(ii) When $\alpha(x) = 1$, Then the eqⁿ involves the unknown fⁿ is only under the integral sign & such Eqⁿ is known as Volterra linear integral of the second type

$$u(x) = F(x) + \lambda \int_a^x K(x,t)\, u(t)\, dt$$

(iii) When $\alpha(x) = 1$, $F(x) = 0$

$$u(x) = \lambda \int_a^x k(x,t)\, u(t)\, dt$$

is known as Volterra Homogeneous linear integral Eqⁿ.

Example: If $f(x) = \frac{1}{\pi\sqrt{x}}$ is a solⁿ of $\int_0^x \frac{y(s)}{\sqrt{x-s}}\, ds = 1$

Let $y(x) = f(x)$

$\Rightarrow y(s) = f(s)$

$$\therefore \int_0^x \frac{f(s)}{\sqrt{x-s}}\, ds = \frac{1}{\pi}\int_0^x \frac{1}{\sqrt{s^2 - sx}}\, ds$$

Let

$$I = \int_0^x \frac{1}{\pi\sqrt{s}} \frac{1}{\sqrt{x-s}}\, ds$$

$$= \frac{1}{\pi}\int_0^x \frac{1}{\sqrt{xs - s^2}}\, ds = \frac{1}{\pi}\int_0^x \frac{ds}{\sqrt{\left(\frac{x}{2}\right)^2 - \left(s - \frac{x}{2}\right)^2}}$$

UNIT-III

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$$= \frac{1}{\pi}\left[\sin^{-1}\left(\frac{s - x/2}{x/2}\right)\right]_0^x$$

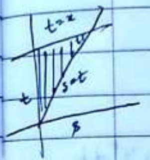
$$= \frac{1}{\pi}\left[\sin^{-1}(1) - \sin^{-1}(-1)\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = 1$$

Example: Solve the $y'' + 2y' + y = x^2$ with $y(0) = 1$, $y'(0) = 0$

Let $y'' = \phi(x)$ ← Integrating both

$$\int_0^x d\left(\frac{dy}{dx}\right) = \int_0^x \phi(s)\, ds$$



$$\frac{dy}{dx} - 0 = \int_0^x \phi(s)\, ds$$

$$y(x) - y(0) = \int_0^x \int_0^t \phi(s)\, ds\, dt = \int_0^x \int_s^x \phi(s)\, dt\, ds$$

$$\Rightarrow y(x) = 1 + \int_0^x (x-s)\, \phi(s)\, ds$$

100

$$\boxed{\phi(x) = 1 - x^2 - \int_0^x (2 + x - s)\, \phi(s)\, ds}$$

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Transformation of DE into Integral Eqn.

Example: Consider the IVP consisting, 2nd order Linear DE

$$\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2 y(x) = F(x) \quad \text{--- (1)}$$

with $y(0) = y_0$ & $y'(0) = y_1$

Soln: ~~define~~ $\cancel{\frac{d^2y}{dx^2} = \phi(x)}$

Now eqn (1) can be written as

$$y'' = F(x) - a_1 y' + a_2 y$$

$$\int_0^x \frac{d^2y}{dx^2}\,dt = \int_0^x F(t)\,dt - \int_0^x a_1 y'\,dt - \int_0^x a_2 y\,dt$$

$$\frac{dy}{dx} - y_1 = \int_0^x F(t)\,dt - \int_0^x a_2(t)\cdot y(t)\,dt - \left\{ \left[a_1(t) y(t)\right]_0^x - \int_0^x \frac{d}{dt}\left(a_1(t)\, y(t)\right) dt \right\}$$

Unable to understand

$$y(x) = y_0 + \left[a_1(0) y_0 + y_1\right]x + \int_0^x \left\{ \frac{d}{dt} a_1(t)' - a_2(t) \right\}(x-t) - a_1(t) \Big\} y(t)\,dt + \int_0^x (x-t) F(t)\,dt$$

Now $y(x) = G(x) + \lambda \int_0^x k(x,t)\, y(t)\,dt$

where $G(x) = y_0 + \left[a_1(0) y_0 + y_1\right] x + \int_0^x (x-t) F(t)\,dt$

$$k(x,t) = \left\{ \frac{d}{dt} a_1(t) - a_2(t) \right\}(x-t) - a_1(t) \Big\}$$

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→ II - Method: $u(x) = f(x) + \lambda \int^x k(x,t)\, u(t)\,dt$, $\lambda = -1$

$f(x) = F(x) - a_1 y_1 - a_2 y_0 - a_2 y_1 x$

$k(x,t) = (a_1 + a_2(x-t))$

Let $y''(x) = u(x)$

$$\int_0^x y''(t)\,dt = \int_0^x u(t)\,dt$$

$$y'(x) - y'(0) = \int_0^x u(t)\,dt$$

$$y'(x) = y_1 + \int_0^x u(t)\,dt$$

Now again integrating both side

$$y(x) - y(0) = y_1 x + \int_0^x \left(\int_t^x (u(t))\,dt \right) dt$$

$$y(x) = y_0 + y_1 x + \int_0^x (x-t)\, u(t)\,dt$$

Now putting these values in eqn (1) —

$$y'' + a_1 y' + a_2 y = F(x)$$

$$u + a_1\left(y_1(0) + \int_0^x u(t)\,dt \right) + a_2\left[y_0 + y_1 x + \int_0^x (x-t)\, u(t)\,dt \right] = F(x)$$

$$u(x) = F(x) - y_0 a_2(x) - y_1(0)\, a_1(x) - a_2 y_1 x - \lambda \int_0^x (a_1 + a_2(x-t))\, u(t)\,dt$$

Example: Show that f^n $u(x) = xe^x$ is a soln of the linear integral Eqn.

$$u(x) = \sin x + 2\int_0^x \cos(x-t)\cdot u(t)\,dt$$

Soln: take R.H.S

given $u(x) = xe^x$ or $u(t) = te^t$

$$\sin x + 2\int_0^x \cos(x-t)\cdot t e^t\,dt$$

$$= \sin x + 2\int_0^x \left(\cos x \cdot \cos t + \sin x \sin t\right) t e^t\,dt$$

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$= \sin x + 2\cos x\left[\int_0^x t e^t \cos t\,dt\right] + 2\sin x\left[\int_0^x t e^t \sin t\,dt\right]$

formula

We know

let $I_1 = \int e^t \cos t\, dt = \frac{e^t}{2}(\sin t + \cos t)$

$I_2 = \int e^t \sin t\, dt = \frac{e^t}{2}(\sin t - \cos t)$

self

(simplify by self)

$= \sin x + 2\cos x\left[x \cdot \left\{\frac{e^x}{2}(\sin x + \cos x)\right\} - \int \frac{e^t}{2}(\sin t + \cos t)\,dt\right] + 2\sin x\left[x \cdot \frac{e^x}{2}(\sin x - \cos x) - \int \frac{e^t}{2}(\sin t - \cos t)\,dt\right]$

$= \sin x + \cos x\left[x e^x(\sin x + \cos x)\right] + \sin x\left[x e^x(\sin x - \cos x)\right]$

$\cos x\left[-\int e^t \sin t\,dt - \int e^t \cos t\,dt\right] + \sin x\left[-\int e^t \sin t\,dt + \int e^t \cos t\,dt\right]$

$= \sin x + x e^x\left[\cancel{\cos x \cdot \sin x} + \underbrace{\cos^2 x + \sin^2 x}_{=1} - \cancel{\sin x \cos x}\right]$

$- \int e^t \sin t\,dt\,(\cos x + \sin x) - \int e^t \cos t\,dt\,(\cos x - \sin x)$

$= \sin x + x e^x - \left[\left(\frac{e^x}{2}(\sin x - \cos x) + \frac{1}{2}\right)(\sin x + \cos x)\right.$

$\left. + \left(\frac{e^x}{2}(\sin x + \cos x) + \frac{1}{2}\right)(\cos x - \sin x)\right]$

$= \sin x + x e^x -$

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Example

Sol: Obtain Fredholm integration of second kind corresponding to the BVP

$\frac{d^2u}{dx^2} + \lambda u = x$, $u(0) = 0$, $u'(1) = 0$

Sol: let $\frac{d^2u}{dx^2} = \phi(x)$

$\int_0^x \frac{d^2u}{dx^2}\,dt = \int_0^x \phi(x)\,dt$

$\Rightarrow u'(x) - u'(0) = \int_0^x \phi(x)\,dt$

$\Rightarrow u'(x) = u'(0) + \int_0^x \phi(x)\,dt$

Given that $u'' = x - \lambda u$ with $u(0) = 0$, $u'(1) = 0$

Integrating w.r.t from 0 to 1

$\int_0^x u''(t)\,dt = \int_0^x (t - \lambda u(t))\,dt$

$u'(x) - u'(0) = \frac{x^2}{2} - \lambda\int_0^x u(t)\,dt$ — (1)

Again integrating both side

$u(x) - u(0) = Ax + \frac{x^3}{6} - \lambda\int_0^x (x-t)\,u(t)\,dt$ — (11)

where $u'(0) = A$

Now putting $x = 1$ in eqn (1)

$A = -\frac{1}{2} + \lambda\int_0^1 u(t)\,dt$ — (3)

Now putting A in (2)

$u(x) = \left[-\frac{1}{2} + \lambda\int_0^1 u(t)\,dt\right]x + \frac{x^3}{6} - \lambda\int_0^x (x-t)\,u(t)\,dt$

$= -\frac{x}{2} + \lambda\int_0^1 x\,u(t)\,dt + \frac{x^3}{6} - \lambda\int_0^x (x-t)\,u(t)\,dt$

$= -\frac{x}{2} + \frac{x^3}{6} + \lambda\left[\int_0^x x\,u(t)\,dt + \int_x^1 x\,u(t)\,dt\right] - \lambda\int_0^x (x-t)\,u(t)\,dt$

Solution of Volterra's integral Eqⁿ of the IInd Kind by the method of successive approximation

The Volterra's integral Eqⁿ of second Kind

$$U(x) = F(x) + \lambda \int_0^x K(x,t)\, u(t)\, dt \quad \text{--- (1)}$$

Where $K(x,t)$ is a cts fⁿ $0 \le x \le a$, $0 \le t \le a$ and $F(x)$ is cts for $0 \le x \le a$.

The solⁿ of the integral Eqⁿ (1) in the form of an infinite Power series is λ

$$u(x) = u_0(x) + \lambda u_1(x) + \lambda^2 u_2(x) + \lambda^3 u_3(x) + \cdots + \lambda^n u_n(x) \quad \text{--- (2)}$$

Substituting this series in eqⁿ (1), we have

$$u_0(x) + \lambda u_1(x) + \lambda^2 u_2(x) + \cdots + \lambda^n u_n(x)$$
$$= F(x) + \lambda \int_0^x K(x,t) \{ u_0(t) + \lambda u_1(t) + \lambda^2 u_2(t) + \cdots + \lambda^n u_{n-1}(t) \} dt \quad \text{--- (3)}$$

Equating the coefficient of equal power of λ, we have

$$u_0(x) = F(x)$$
$$u_1(x) = \int_0^x K(x,t)\, u_0(t)\, dt$$
$$\boxed{u_1(x) = \int_0^x K(x,t)\, F(t)\, dt}$$

$$u_2(x) = \int_0^x K(x,t)\, u_1(t)\, dt$$
$$= \int_0^x K(x,t) \left[\int_0^t K(t,t_1)\, F(t_1)\, dt_1 \right] dt \quad \text{--- (4)}$$

The relation Eqⁿ (4) yields a method for a successive determination of the fⁿ $u_n(x)$.

It may be show that the series eqⁿ (2) is convergent uniformly in x and λ for any λ and $x \in [0,a]$ under assumption with respect $F(x)$ & $K(x,t)$, its sum is unique solⁿ of Eqⁿ (1).

$$u_1(x) = \int_0^x K(x,t)\, F(t)\, dt$$

$$u_2(x) = \int_0^x K(x,t) \left[\int_0^t K(t,t_1)\, F(t_1) \right] dt$$

By interchanging the order of integration, we have

$$u_2(x) = \int_0^x F(t_1)\, dt_1 \int_{t_1}^x K(x,t)\, K(t,t_1)\, dt$$

$$u_2(x) = \int_0^x K_2(x,t)\, F(t_1)\, dt, \quad \text{--- (5)}$$

where $K_2(x,t) = \int_{t_1}^x K(x,t)\, K(t,t_1)\, dt$

Similarly in general

$$U_n(x) = \int_0^x K_n(x,t)\, F(t)\, dt \quad \text{--- (6)}$$

The fⁿ $K_n(x,t)$ is called Iterated Kernel.

By Eq^n (6), we have

$$U_1(x) = \int_0^x K_1(x,t)\, F(t)\, dt$$

$$U_1(x) = \int_0^x K(x,t)\, F(t)\, dt$$

Hence $K_1(x,t) = K(x,t)$

And for determination $K_2(x,t)$, $K_3(x,t)$ .. etc, we have Recursion formula

$$K_{n+1}(x,t) = \int_t^x K(x,z)\, K_n(z,t)\, dz \quad \text{—(7)}$$

$n = 1, 2, \dots$

using eq^n (6) & (7), the relation eq^n (2) may be written as

$$U(x) = F(x) + \sum \lambda^m \int_0^x K_m(x,t) F(t)\, dt$$

The f^n $R(x,t,\lambda)$ defined as

$$R(x,t,\lambda) = K_1(x,t) + \lambda K_2(x,t) + \lambda^2 K_3(x,t) + \dots + \lambda^n K_{n+1}(x,t) + \dots$$

$$= \sum_{m=0}^{\infty} \lambda^m K_{m+1}(x,t)$$

is called the Resolvent kernal. (or Reciprocal kernel) for the integral.

eq^n (1)

$$\boxed{U(x) = F(x) + \lambda \int_0^x R(x,t,\lambda)\, F(t)\, dt}$$

Example.

Prove that

$$R(x,t,\lambda) = K(x,t) + \lambda \int_t^x K(x,z)\, R(z,t,\lambda)\, dz$$

Where $R(x,t,\lambda)$ be the Resolvent kernel of a Volterra integral Eq^n

$$U(x) = F(x) + \lambda \int_0^x K(x,t)\, U(t)\, dt$$

Proof:- We have $R(x,t,\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x,t)$ —(1)

Where iterated kernel are given by

$$K_1(x,t) = K(x,t)$$

$$K_m(x,t) = \int_t^x K(x,z)\, K_{m-1}(z,t)\, dz$$

$$R(x,t,\lambda) = K_1(x,t) + \sum_{m=2}^{\infty} \lambda^{m-1} K_m(x,t)$$

$$= K_1(x,t) + \sum_{m=2}^{\infty} \lambda^{m-1} \int_t^x K(x,z)\, K_{m-1}(z,t)\, dz$$

$$= K(x,t) + \sum_{n=1}^{\infty} \lambda^n \int_t^x K(x,z)\, K_n(z,t)\, dz$$

(putting $n = m-1$)

$$R(x,t,\lambda) = K(x,t) + \lambda \sum_{n=1}^{\infty} \lambda^{n-1} \int_t^x K_n(z,t)\, K(x,z)\, dz$$

$$= K(x,t) + \lambda \int_t^x \left\{ \sum_{n=1}^{\infty} \lambda^{n-1} K_n(z,t) \right\} K(x,z)\, dz$$

then

$$R(x,t,\lambda) = K(x,t) + \lambda \int_t^x K(x,z)\, R(z,t,\lambda)\, dz$$

Example: Find the Resolvent kernel of the volterra integral Eq^n with $K(x,t) = 1$.

Sol^n Given that $K(x,t) = 1$

we know that $K_1(x,t) = K(x,t) = 1$

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$$K_2(x,t) = \int_t^x K(x,z)\, K_1(z,t)\, dz$$

$$= \int_t^x 1.1\, dz = (x-t)$$

$$K_3(x,t) = \int_t^x K(x,z)\, K_2(z,t)\, dz$$

$$= \int_t^x 1.(z-t)\, dz = \frac{(x-t)^2}{2!}$$

Similarly for

K_4, K_5, K_6 ----

$$K_m(x,t) = \int_t^x K(x,z) K_{m-1}(z,t)\, dz$$

$$= \int_t^x 1.\ \frac{(z-t)^{m-2}}{(m-2)!}\, dz$$

$$= \frac{(x-t)^{m-1}}{(m-1)!}$$

Hence the Resolvent kernel

$$R(x,t,\lambda) = \sum_{m=0}^{\infty} \lambda^m K_{m+1}(x,t)$$

$$= \sum_{m=0}^{\infty} \lambda^m \frac{(x-t)^m}{m!}$$

$$= 1 + \frac{\lambda(x-t)}{1!} + \frac{\lambda^2 (x-t)^2}{2!} + \frac{\lambda^3 (x-t)^3}{3!} + \cdots$$

$$= e^{\lambda(x-t)}$$

Note –

(1) The VIE of 2^{nd} kind

$$u(x) = F(x) + \lambda_0 \int^x K(x,t)\, u(t)\, dt$$

(i) $u(x) = F(x) + \lambda \int_0^x R(x, t, \lambda)\, F(t)\, dt$

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Where $R(x,t,\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x,t)$

$K_m(x,t)$ is the m^{th} iteration

taking $K_1(x,t) = K(x,t)$

(ii) $u(x) = \lim_{n \to \infty} u_n(x)$

Example : (i) $K(x,t) = \dfrac{2+\cos x}{2+\cos t}$ (Similar probs. in exam)

(ii) $K(x,t) = e^{x-t}$

(iii) $K(x,t) = e^{x^2-t^2}$

(iv) $K(x,t) = a^{x-t}$ $\quad a > 0$

* Solution of fredholm integral Eq^n by the method of successive approximation ——

The Fredholm integral Eq^n of 2^{nd}

$$u(x) = F(x) + \lambda \int_a^b K(x,t)\, u(t)\, dt \quad \text{. where}$$

$K(x,t) \neq 0$ is real & cts in the Rectangle R, $a \le x \le b$ & $a \le t \le b$. $F(x) \neq 0$ (2)

has the sol^n $u(x) = F(x) + \lambda \int_a^b R(x, t, \lambda)\, F(t)\, dt$

where $R(x,t,\lambda) = \sum_{n=0}^{\infty} \lambda^n K_{n+1}(x,t)$ is called

Resolvent-Kernel

Let the sol^n of Eq^n (2) in the form of a Power series in λ

$$u(x) = u_0(x) + \lambda u_1(x) + \lambda^2 u_2(x) + \cdots + \lambda^n u_n(x)$$ —— (3)

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$u_0(x) = F(x)$

$u_1(x) = \int_a^b k(x,t)\, u_0(t)\, dt$

$= \int_a^b k(x,t)\, F(t)\, dt$

$u_2(x) = \int_a^b k(x,t)\, u_1(t)\, dt$

Similarly $u_n(x) = \int_a^b k(x,t)\, u_{n-1}(t)\, dt$

If we define the iterated kernel as

$K_1(x,t) = K(x,t)$

$K_2(x,t) = \int_a^b K(x,t_1)\, K_1(t_1,t)\, dt_1$

$K_2(x,t) = \int_a^b K(x,t_1)\, K(t_1,t)\, dt_1$

$K_3(x,t) = \int_a^b K(x,t_1)\, K_2(t_1,t)\, dt_1$.

$= \int_a^b \int_a^b K(x,t_1)\, K(t_1,t_2)\, K(t_2,t)\, dt_1\, dt_2$

So on

Hence, in the general

$K_n(x,t) = \int_a^b \cdots \int_a^b K(x,t_1)\, K(t_1,t_2) \cdots K(t_{n-2},t_{n-1})\, K(t_{n-1},t)\, dt_1\, dt_2 \cdots dt_{n-1}$

It follows at once that iterated kernel satisfy $K_n(x,t) = \int_a^b K_r(x,z)\, K_{n-r}(z,t)\, dz$

where $r < n$ ——— (4)

Now consider the series

$R(x,t,\lambda) = K_1(x,t) + \lambda K_2(x,t) + \cdots + \lambda^{n-1} K_n(x,t) + \cdots$

$= K(x,t) + \lambda K_2(x,t) + \cdots$

$= \sum_{n=0}^{\infty} \lambda^n K_{n+1}(x,t)$ ——— (5)

This series is called the Neumann series of the Kernel $K(x,t)$.

Suppose that $F(x)$ & $K(x,t)$ are cts function in the range of

$|F(x)| \le m, \quad a \le x \le b$

$|K(x,t)| \le M \quad (a \le x \le b,\ a \le t \le b)$

~~This series~~ where m, M are positive constant. Now $K_1(x,t) = K(x,t) \le M$

$K_2(x,t) = \int_a^b K(x,t_1)\, K(t_1,t)\, dt_1$

$\le \int_a^b M.M\, dt_1 = M^2 (b-a)$

$K_3(x,t) = \int_a^b K(x,t_1)\, K_2(t_1,t_2)\, dt_1$.

$K_3(x,t) \le M^3 (b-a)^2$

In general $K_{n+1}(x,t) \le M^{n+1} (b-a)^n$ ——— (6)

Since $\left| \sum_{n=0}^{\infty} \lambda^n K_{n+1}(x,t) \right| \le \sum_{n=0}^{\infty} |\lambda|^n M^{n+1} (b-a)^n$

where $|\lambda| < \frac{1}{M(b-a)}$

Hence the solⁿ of eqⁿ (2).

$u(x) = F(x) + \lambda \int_a^b R(x,t,\lambda)\, F(t)\, dt$

where $R(x,t,\lambda) = \sum_{n=0}^{\infty} \lambda^n K_{n+1}(x,t)$

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Find the Resolvent kernel of the following

(i) $K(x,t) = (1+x)(1-t)$ with $a=-1,\ b=0$

(ii) $K(x,t) = e^{(x+t)}$, with $a=0,\ b=1$

Sol^n

(1) We have $K_1(x,t) = K(x,t) = (1+x)(1-z)$

$$K_2 = \int_{-1}^{0} \underbrace{(1+x)(1-z)}\ \underbrace{(1+z)(1-t)}dz$$

$$= (1+x)(1-t)\int_{-1}^{0}(1-z^2)\,dz$$

$$= (1+x)(1-t)\left[z - \frac{z^3}{3}\right]_{-1}^{0}$$

$$= (1+x)(1-t)\cdot\left(\frac{2}{3}\right)$$

$$K_3 = \left(\frac{2}{3}\right)^2 (1+x)(1-t)$$

$$K_n(x,t) = \int_{-1}^{0} K(x,z)\,K_{n-1}(z,t)\,dz$$

$$= \left(\frac{2}{3}\right)^{n-1}(1+x)(1-t)$$

$$R(x,t,\lambda) = \sum_{n=0}^{\infty}\lambda^n K_{n+1}(x,t)$$

$$= \sum_{n=0}^{\infty}\lambda^n\cdot\left(\frac{2}{3}\right)^n(1+x)(1-t)$$

$$= (1+x)(1-t)\left[1 + \left(\frac{2}{3}\right)\lambda + (2/3)^2\lambda^2 + \dots\right]$$

$$= \frac{(1+x)(1-t)\cdot 3}{(3-2\lambda)} \quad \text{when } |\lambda| < 3/2$$

Solve the Integral Eq^n

$$u(x) = 1 + \lambda\int_0^1 (1-3xt)\,u(t)\,dt$$

Evaluate the Resolvent kernel.

Sol^n: We have $K_1(x,t) = K(x,t) = 1-3xt$

$$K_2(x,t) = \int_0^1 K(x,z)\,K_1(z,t)\,dz$$

$$= \int_0^1 (1-3xz)(1-3zt)\,dz$$

$$= \int_0^1 \left[1 - 3z(x+t) + 9xz^2t\right]dz$$

$$= \left[1 - \frac{3}{2}(x+t) + 3xt\right]$$

$$K_3(x,t) = \int_0^1 K(x,z)\,K_2(z,t)\,dz$$

$$= \int_0^1 (1-3xz)\left[1 - \frac{3}{2}(z+t) + 3zt\right]dz$$

$$= \int_0^1 \left[\left(1-\frac{3}{2}t\right) - 3z\left(\frac{1}{2} - t + x - \frac{3}{2}xt\right) + 9xz^2\left(\frac{1}{2} - t\right)\right]dz$$

$$= \frac{1}{4} - \frac{3}{4}xt = \frac{1}{4}(1-3xt)$$

$$\boxed{K_3(x,t) = \frac{1}{4}K_1(x,t)}$$

$$K_4(x,t) = \int_0^1 K(x,z)\,K_3(z,t)\,dz$$

$$= \frac{1}{4}\int_0^1 (1-3xz)(1-3zt)\,dz$$

$$= \frac{1}{4}\int_0^1 \left[1 - 3z(x+t) + 9xz^2t\right]dz$$

$$= \frac{1}{4}K_2(x,t)$$

Similarly

$$K_5(x,t) = \left(\frac{1}{4}\right)^2 K_1(x,t)$$

$$K_6(x,t) = \left(\frac{1}{4}\right)^2 K_2(x,t)$$

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Now the Resolvent Kernal

$$R(x,t,\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x,t)$$

Course finished
R. Kalika
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SCFF SECTION

REVISION

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Rayleigh Ritz Method for Boundary Value Problem

Consider the second order LODE —

$$A(x)\frac{d^2y}{dx^2} + B(x)\frac{dy}{dx} + C(x)y = f(x) \quad \text{—(1)}$$

with BC $y(x_1) = y_1$ & $y(x_2) = y_2$

then we can construct an integrand in the functional (*) for which Euler's Equation satisfies

(*) = $\boxed{\text{Consider a functional along with non-homog. fixed BVP as } I = \int_{x_1}^{x_2} F(x,y,y')\,dx \text{ where } \frac{dy}{dx} = y',\ y(x_1)=y_1,\ \&\ y(x_2)=y_2}$

Functional for DE (1)

$$F = p(x)y'^2 + q(x)y^2 + r(x)y$$

By Euler's Eq^n. $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$

$$\Rightarrow q(x).2y + r(x) - \frac{d}{dx}\left[2y'.p(x)\right] = 0$$

$y' = \frac{dy}{dx}$

$$\Rightarrow 2p(x).y'' - 2q(x)y - r(x) + 2y'.p'(x) = 0$$

$$\Rightarrow 2\left[p(x)y'' + p'(x)y' - q(x)y\right] = r(x)$$

$$\text{or} \quad p(x)y'' + p'(x)y' - q(x)y = \frac{r(x)}{2} \quad \text{—(2)}$$

$$\text{or} \quad y'' + \frac{p'(x)}{p(x)}y' - \frac{q(x)}{p(x)}y = \frac{r(x)}{2p(x)}$$

$$\cancel{IF = e^{\int \frac{p'(x)}{p(x)}dx} = e^{\log(p(x))} = p(x)}$$

Now multiply eq^n (1) by arbitrary f^n $h(x)$ then Compare with (2), we get —

$p(x) = A(x)\,h(x)$	$q(x) = -C(x).h(x)$
$p'(x) = B(x)\,h(x)$	$r(x) = 2f(x).h(x)$

Example: Solve the BVP — (By Ritz Method)

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 2x^2 \quad \text{—(1)}$$

Subject to the conditions $y(0)=0$ & $y(1)=1$

Sol^n (1) $\Rightarrow$ $y'' + \frac{1}{x}y' - \frac{y}{x^2} = 2$

∴ one of its sol^n is given by $\quad IF = e^{\int \frac{1}{x}dx} = x$

$y = x$

$y' = 1$

Solving by Ritz Method

The corresponding functional is given by

(2) — $I = \int_0^1 (py'^2 + qy^2 + ry)\,dx$, $y(0)=0$, $y(1)=1$

Now comparing (1) with

$a(x)\frac{d^2y}{dx^2} + b(x)\frac{dy}{dx} + c(x)y = f(x)$

we have $a(x) = x^2$, $b(x) = x$, $c(x) = -1$, $f(x) = 2x^2$

Now using

$p(x) = a(x)\,h(x)$

$h(x) = \frac{p(x)}{a(x)} = \frac{x}{x^2} = 1/x$

$p'(x) = b(x)\,h(x)$

$q(x) = -c(x)\,h(x)$

$r(x) = 2f(x)\,h(x)$

$p(x) = x$

$q(x) = 1\cdot h(x) = 1/x$

$r(x) = 4x^2\cdot h(x) = 4x$

here $p(x) = e^{\int \frac{b(x)}{a(x)}dx} = e^{\int \frac{1}{x}dx} = x$

∴ (2) Becomes

$$I = \int_0^1 \left(xy'^2 + \frac{y^2}{x} + 4xy\right)dx, \quad y(0)=0,\ y(1)=1$$

let $\phi_1(x) = x(1-x)$

$\phi_2(x) = x$ be two sol^n of (1)

then approximate sol^n is given by —

$y = u_0(x) + c_1(x-0)(x-1)$, $u_0(0)=0, u_0(1)=1$

(1) S.T. $\phi(x) = xe^x$ is a sol^n of the Volterra integral eq^n

$$\phi(x) = \sin x + 2\int_0^x \cos(x-t)\,\phi(t)\,dt$$

Sol^n taking RHS $\phi(t) = te^t$

$\sin x + 2\int_0^x (\cos x\cdot\cos t + \sin x\cdot\sin t)(t\cdot e^t)\,dt$

let $x - t = -p \Rightarrow t = x+p$

$\Rightarrow dt = +dp$

$\therefore \sin x + 2\int_{-x}^{0} \cos p\,(x+p)\cdot e^{(x+p)}\,dp$

$= \sin x + 2\int_{-x}^{0} \cos(-p)\,(x+p)\cdot e^x\cdot e^{p}\,dp$

$\because \cos(-p) = \cos p$

$= \sin x + 2\left[e^x \int_{-x}^{0} (x+p)\cos(p)\,e^{p}\,dp\right]$

$= \sin x + 2e^x\left[x\int_{-x}^{0} \cos p\cdot e^p\,dp + \int_{-x}^{0} pe^p\cos p\,dp\right]$

Formula: $\int e^t \sin t\,dt = \frac{e^t}{2}(\sin t - \cos t)$

$\int e^t\cos t\,dt = \frac{e^t}{2}(\sin t + \cos t)$

$= \sin x + 2e^x\left\{x\cdot\frac{e^p}{2}(\sin p + \cos p) + p\cdot\frac{e^p}{2}(\sin p + \cos p) - \int \frac{e^p}{2}(\sin p + \cos p)\,dp\right\}$

$= \sin x + e^x\left[xe^p(\sin p + \cos p) + pe^p(\sin p + \cos p) - \left\{\frac{e^p}{2}(\sin p - \cos p) - \frac{e^p}{2}(\sin p + \cos p)\right\}_{-x}^{0}\right.$

$= \sin x + e^x\left[(\sin p + \cos p)\left(xe^p + pe^p + \frac{e^p}{2}\right) - \frac{e^p}{2}(\sin p - \cos p)\right]_{-x}^{0}$

$= \sin x + e^x \left[1(x+\tfrac{1}{2}) + \tfrac{1}{2}(1) - \left[(\cos x - \sin x) \cancel{x e^{-x}} - x\cancel{e^{-x}} + \frac{e^{-x}}{2} \right] + \frac{e^{-x}}{2}(\sin x + \cos x) \right]$

$= \sin x + e^x \left[x + 1 - \frac{e^{-x}}{2}\cos x + \frac{e^{-x}}{2}\sin x - \frac{e^{-x}}{2}\sin x - \frac{e^{-x}}{2}\cos x \right]$

$= \sin x + e^x \left[x + 1 - e^{-x}\cos x \right]$

$= \sin x + xe^x + e^x - \cos x$

Pending

Laplace transform of the n^{th} order derivative of $f(t)$

thm: derivative of $f(t)$

Statement: ① Let $f(t)$ & its derivatives $f'(t), f''(t), \ldots f^{n-1}(t)$ are cts for $t \geq 0$ & of exponential order as $t \to \infty$. and if $f^n(t)$ is of class A & if $L[f(t)] = f(s)$ then Laplace transform of $f^n(t)$ exists when $s > a$. & given by ——

$L\{f^n(t)\} = s^n L[f(t)] - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - sf^{n-2}(0) - f^{n-1}(0)$

form pr 29

Pf: We have for $f'(t)$

$L[f'(t)] = s[(f(t)] - f(0)$ — ①

then for $f''(t)$, we have

$L[f''(t)] = s^2 L[f(t)] - sf(0) - f'(0)$

i.e $L[f''(t)] = sL[f'(t)] - f'(0)$

$= s[s[f(t)] - f(0)] - f'(0)$

$= s^2 L[f(t)] - sf(0) - f'(0)$

for $f'''(t)$,

$L[f'''(t)] = sL[f''(t)] - f''(0)$

$= s[s^2 L[f(t)] - sf(0) - f'(0)] - f''(0)$

$= s^3 f(s) - s^2 f(0) - sf'(0) - f''(0)$

proceeding in similar way, we have

$L\{f^n(t)\} = s^n f(s) - \sum_{r=0}^{n-1} s^{n-1-r} f^r(0)$

Proved

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(2) cosine integral

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$$C(t) = \int_t^\infty \frac{\cos u}{u}\,du, \quad L[C(t)] = ??$$

$$\Rightarrow C'(t) = -\frac{\cos t}{t}$$

$$\Rightarrow -tC'(t) = \cos t$$

$$\therefore -L[tC'(t)] = L[\cos t] = \frac{s}{s^2+1}$$

$$L[tC'(t)] = \frac{-s}{s^2+1} \quad \text{--- (1)}$$

Now

$$\because L[tf(t)] = -\frac{d}{ds}f(s)$$

where $L[f(t)] = f(s)$

$$= -\frac{d}{ds}L[f(t)]$$

$$\therefore L[tC'(t)] = -\frac{d}{ds}L[C'(t)]$$

$$= -\frac{d}{ds}\left[sL[C(t)] - C(0) \right]$$

by (1)

$$\frac{-s}{s^2+1} = \frac{d}{ds}\left(sL[C(t)] - C[0] \right)$$

$$= \frac{d}{ds}\left(sL[C(t)] \right) - 0$$

Because $C[0]$ so $\frac{d}{ds}C(0) = 0$

$$\Rightarrow \frac{d}{ds}[sL[C(t)]] = \frac{s}{s^2+1}$$

Now integrating both side, we get

$$sL[C(t)] = \frac{1}{2}\int \frac{2s}{s^2+1}\,ds$$

$$= \frac{1}{2}\log(s^2+1) + c \quad \text{--- (1)}$$

$$\boxed{L[C(t)] = \frac{1}{2s}\log(s^2+1)}$$ Proved

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in (1) by final value theorem

$$\lim_{s\to 0} sf(s) = \lim_{t\to\infty} f(t) = 0$$

$$\therefore c = 0$$

$$\therefore f(s) = -1$$

$$\boxed{L[C(t)] = \frac{1}{2s}\log(s^2+1)}$$

(3) To find inverse laplace transform by convolution

(1) Use convolution to evaluate

$$L^{-1}\left[\frac{s}{(s^2+4)^2}\right] = \left[\frac{1}{(s^2+4)}\cdot\frac{s}{(s^2+4)}\right]$$

let $f(s) = \frac{2}{2(s^2+4)}$, $g(s) = \frac{s}{(s^2+4)}$

$$\therefore L^{-1}[f(s)] = \frac{1}{2}\sin 2t$$

$$L^{-1}[g(s)] = \cos 2t$$

$\therefore$ A/c to Convolution theorem

$$L^{-1}[f(s)\,g(s)] = \int_0^t f(u)\,g(t-u)\,du$$

$$= \frac{1}{2}\int_0^t \sin 2u \cdot \cos 2(t-u)\,du$$

$$= \frac{1}{2}\int_0^t \frac{1}{2}\left[\sin(2u+2t-2u) + \sin(2u-2t+2u)\right]du$$

$$= \frac{1}{4}\left[\int_0^t \sin 2t\,du + \int_0^t \sin(4u-2t)\,du\right]$$

$$= \frac{1}{4}\left[u\sin 2t + \cancel{\cos}(4u-2t)/4\right]_0^t$$

$$\frac{1}{4}t\sin 2t = \frac{1}{4}\left[t\sin 2t + \cancel{\cos}(2t) + \cancel{\cos}(2t)\right]$$

(2) $L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right]$

let $g(s) = \frac{1}{(s^2+a^2)} = f(s)$

$g(t) = L^{-1}[g(s)] = \frac{1}{a}\sin at = L^{-1}[f(s)] = f(t)$

∴ By convolution theorem

$L^{-1}[f(s)\,g(s)] = \int_0^t \frac{1}{a}\sin au \,\frac{1}{a}\sin a(t-u)$

$= \frac{1}{a^2}\left[\int_0^t \sin^2 at\, dt\right] = \frac{1}{a^2}\int_0^t \frac{(1-\cos 2at)}{2}$

(3) $\frac{dx}{dt} - y = e^t$ — (1) $\quad (x(0)=1,\ y(0)=0$

$L[x'(t)] - L[y] = L[e^t]$

$\Rightarrow sL[x(t)] - x(0) - L(y) = \frac{1}{s-1}$

$sL[x(t)] - 1 - L[y] = \frac{1}{s-1}$ — (1) $\times s$

& $\frac{dy}{dt} + x = \sin t$ $\quad \rightarrow s\bar{x} - \bar{y} = \frac{1}{s-1} + 1 = \frac{s}{s-1}$

$L[y'(t)] + L[x] = L[\sin t]$

$\Rightarrow sL[y(t)] - y(0) + L[x] = \frac{1}{s^2+1}$

$\Rightarrow sL[y(t)] + L[x] = \frac{1}{s^2+1}$ — (11) $\times s$ $\quad L[y] = \bar{y}$

Sub (i) (ii) ...,

$L(x)\,(s^2+1) - s = \frac{s}{s-1} + \frac{1}{s^2+1}$

$L(x) =$

$s\bar{x} - s\bar{y} = \frac{s^2}{s+1}$

$\bar{x} + s\bar{y} = \frac{1}{s+1}$

$\bar{x}(s^2+1) = \frac{s^2}{s+1} + \frac{1}{s^2+1}$

$\bar{x} = \frac{s^2}{(s+1)(s^2+1)} + \frac{1}{(s^2+1)^2}$

$\bar{y} = \frac{s}{(s^2+1)^2} - \frac{s}{(s-1)(s^2+1)}$

[from Here, we solve $\bar{x}$ & $\bar{y}$ by taking Inverse Laplace trans. and get x & y, that is the soln of this que.]

(4) Solve the DE using Laplace transf. method.

$\frac{\partial y}{\partial t} = 3\frac{\partial^2 y}{\partial x^2}$ where $y(\frac{\pi}{2}, t) = 0$, $\left(\frac{\partial y}{\partial x}\right)_{x=0} = 0$, $y(x,0) = 30\cos 5x$

Soln: given Equation is

$\frac{\partial y}{\partial t} = 3\frac{\partial^2 y}{\partial x^2}$

taking Laplace transfor. on both side, we

$L[y'(t)] = 3L[y''(t)]$

$\Rightarrow$ ~~$s\bar{y} - y(0) = 3[s^2\bar{y} - sy(0) - y'(0)]$~~

$\Rightarrow s\bar{y} - y(x,0) = L\left[3\frac{\partial^2 y}{\partial x^2}\right] = 3\frac{d^2}{dx^2}L(y)$

$s\bar{y} - y(x,0) = 3\frac{d^2\bar{y}}{dx^2}$

$\Rightarrow 3\frac{d^2\bar{y}}{dx^2} - s\bar{y} = -y(x,0) = -30\cos 5x$

$\Rightarrow (3D^2 - s)\bar{y} = -30\cos 5x$

$(D^2 - 8/3)\bar{y} = -10\cos 5x$

Soln is given as
$y = CF + PI$

$m^2 - 8/3 = 0 \Rightarrow m = \pm\sqrt{8/3}$

∴ $C.F = c_1 e^{x\sqrt{8/3}} + c_2 e^{-x\sqrt{8/3}}$

$P.I = \frac{(-10\cos 5x)}{(D^2 - 8/3)} = \frac{30\cos 5x}{75+\phi}$

$\bar{y} = c_1 e^{x\sqrt{8/3}} + c_2 e^{-x\sqrt{8/3}} + \frac{30\cos 5x}{75+\phi}$ — (i)

Now

$\frac{\partial y}{\partial x}\Big|_{x=0} = 0$ — (ii)

$L\left[\frac{\partial y}{\partial x}\right] = L[0] \Rightarrow \frac{\partial \bar{y}}{\partial x} = 0$

Again, $y(\pi/2, t) = 0 \Rightarrow L[y(\pi/2, t)] = 0$

$\Rightarrow \bar{y}(\pi/2, \phi) = 0$ — (iii)

diff (i) w.r.t x,

$\frac{dy}{dx} = c_1\sqrt{8/3}\, e^{x\sqrt{8/3}} - c_2\sqrt{8/3}\, e^{-x\sqrt{8/3}} - \frac{150\sin 5x}{(75+\phi)}$

$= \sqrt{8/3}\left[c_1 e^{x\sqrt{8/3}} - c_2 e^{-x\sqrt{8/3}}\right] - \frac{150\sin 5x}{75+\phi}$

putting $x = 0 \Rightarrow \frac{d\bar{y}}{dx} = 0$ (by (ii))

$0 = \sqrt{8/3}\,[c_1 - c_2] - 0 \Rightarrow c_1 = c_2$

∴ by (i)

$\bar{y} = c_1\left[e^{x\sqrt{8/3}} + e^{-x\sqrt{8/3}}\right] - \frac{150\sin 5x}{75+\phi}$

Now subject to the cond.

$\bar{y}(\pi/2, \phi) = 0$ (by iii)

by (i) $0 = c_1\left[e^{\frac{\pi}{2}\sqrt{8/3}} + e^{-\frac{\pi}{2}\sqrt{8/3}}\right] + \frac{30\cos(5\pi/2)}{75+\phi}$ $=0$

∴ $c_1[\quad] = 0$

$\Rightarrow c_1 = 0$

$L[y] = \bar{y} = \frac{30\cos 5x}{75+\phi}$ (by (i))

taking inverse laplace

$y = L^{-1}\left[\frac{30\cos 5x}{75+\phi}\right] = 30\cos 5x \cdot e^{-75t}$

II – UNIT (Also p-84)

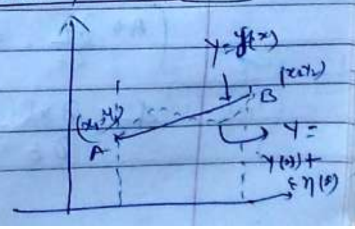
(1) Euler's Equation.

The necessary condition for $\int_{x_1}^{x_2} f(x,y,y')\,dx$ to be an extremum is that

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

which is called Euler's Equation.

Pf. Let $y = y(x)$ be a curve joining the pts (x_1, y_1) & (x_2, y_2).



$$I = \int_{x_1}^{x_2} f(x, y, y')\,dx$$

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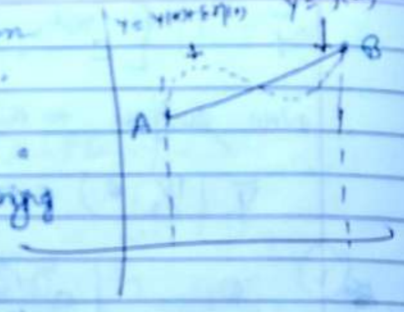
which makes I an stationary or extremum.



Let $y = y(x) + \epsilon\eta(x)$ be a neighbouring curve joining these points.
where ϵ is small and $\eta(x)$ is an arbitrary differentiable f^n of x satisfying $\eta(x_1) = 0$ & $\eta(x_2) = 0$ at A & B respectively.

then the value along $y = y(x) + \epsilon\eta(x)$ is

$$I(\epsilon) = \int_{x_1}^{x_2} f[x, y(x) + \epsilon\eta(x), y'(x) + \epsilon\eta'(x)]\,dx \quad \text{—(1)}$$

this being a f^n of ϵ is maximum or minimum for $\epsilon = 0$, when

$$\frac{dI(\epsilon)}{d\epsilon} = 0 \quad \text{at } \epsilon = 0 \quad \text{—(2)}$$

Now differentiating $I(\epsilon)$ under the integral sign by leibnitz's rule, we have

$$\frac{dI(\epsilon)}{d\epsilon} = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial x}\frac{\partial x}{\partial \epsilon} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \epsilon} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial \epsilon}\right] dx \quad \text{—(3)}$$

Since ϵ is independent of x.

$$\therefore \frac{\partial x}{\partial \epsilon} = 0 \quad \text{Also } \frac{\partial y}{\partial \epsilon} = \eta(x)$$

$$\left(\text{As } \frac{\partial y}{\partial \epsilon} = \frac{\partial y(x)}{\partial \epsilon} + \frac{\partial[\epsilon\eta(x)]}{\partial \epsilon} \Rightarrow \frac{\partial y}{\partial \epsilon} = \epsilon\eta(x)\right)$$

& $\frac{\partial y'}{\partial \epsilon} = \eta'(x)$ with $y = y(x) + \epsilon\eta(x)$

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Substituting these values in eq^n (3), we get—

$$\frac{dI(\epsilon)}{d\epsilon} = \int_{x_1}^{x_2} \left[0 + \frac{\partial f}{\partial y}\eta(x) + \frac{\partial f}{\partial y'}\eta'(x)\right] dx$$

$$= \int_{x_1}^{x_2} \frac{\partial f}{\partial y}\eta(x)\,dx + \left[\left|\frac{\partial f}{\partial y'}\eta(x)\right|_{x_1}^{x_2} - \int \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\eta(x)\,dx\right]$$

$$= \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y}\eta(x)\right] dx + 0 - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\eta(x)\,dx$$

$$\left[\because \eta(x_1) = 0 \ \& \ \eta(x_2) = 0 \text{ at A \& B}\right]$$

$$= \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]\eta(x)\,dx \quad \text{—(4)}$$

Now from Eq^n (2) & (4), we get—

$$\boxed{\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0} \quad \text{—(5)}$$

which is is desired Euler's Eq^n.

Second form of Euler's Eq^n

II. Another form of Euler's Equation

the Euler's Equation is

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \qquad (\text{by } (5))$$

$$\Rightarrow y'\frac{\partial f}{\partial y} - y'\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \qquad \left(\text{multiply by } y'\right)$$

$$\Rightarrow y'\frac{\partial f}{\partial y} - \left[\frac{d}{dx}\left(y'\frac{\partial f}{\partial y'}\right) - \left(\frac{\partial f}{\partial y'}\frac{dy'}{dx}\right)\right] = 0$$

$$\Rightarrow \frac{\partial f}{\partial x} + y'\frac{\partial f}{\partial y} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} - \frac{d}{dx}\left(y'\frac{\partial f}{\partial y'}\right) = \frac{\partial f}{\partial x}$$

$$\Rightarrow \frac{df}{dx}$$

$f(x, y, y')$ -

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx}$$

$$\Rightarrow \frac{df}{dx} - \frac{d}{dx}\left(y'\frac{\partial f}{\partial y'}\right) = \frac{\partial f}{\partial x}$$

$$\Rightarrow \frac{d}{dx}\left[f - y'\frac{\partial f}{\partial y'}\right] = \frac{\partial f}{\partial x}$$

$$\Rightarrow \frac{d}{dx}\left[f - y'\frac{\partial f}{\partial y'}\right] - \frac{\partial f}{\partial x} = 0 \quad -(6)$$

which second form of Euler's Equation

III. <u>Particular Cases of Euler's Eqn</u>

The Euler's Equation is

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \quad -(1)$$

Every soln of Euler's Eqn which satisfy the BC is called a stationary fn or ~~extremal~~ external of the problem

(a) If f is independent of y,

then $\frac{\partial f}{\partial y} = 0$, so by (1),

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \Rightarrow \frac{\partial f}{\partial y'} = c$$

(b) If f is independent of y'.

then $\frac{\partial f}{\partial y'} = 0$, then by eqn (1).

$$\frac{\partial f}{\partial y} = 0$$

which gives soln directly.

(iii) If ~~f~~ is independent of x.

for this we use second form

then $\frac{\partial f}{\partial x} = 0$, then Eqn (6) gives

$$\frac{d}{dx}\left[f - y'\frac{\partial f}{\partial y'}\right] = 0$$

$$\Rightarrow \boxed{f - y'\frac{\partial f}{\partial y'} = c}$$

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(2) Find the extremals of the following functions

(i) $\int_{x_0}^{x_1} (x+y')y'\,dx$

Soln Here $f(x,y,y') = (x+y')y'$

which is independent of y.

So $\frac{\partial f}{\partial y} = 0$ $\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0\right)$

$$0 - \frac{d}{dx}(x + 2y') = 0$$

$$\Rightarrow x + 2y' = c \Rightarrow y' = \frac{c-x}{2}$$

on integrating

$$\Rightarrow y = \frac{c}{2}x - \frac{x^2}{4} + d$$

(ii) $\int_{x_0}^{x_1} (y^2 + y'^2 - 2y\sin x)\,dx$

Here $f = y^2 + y'^2 - 2y\sin x$

Euler's Equation $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \quad -(1)$

$$(2y - 2\sin x) - \frac{d}{dx}(2y') = 0$$

$$\Rightarrow 2\frac{d}{dx}(y') = 2y - 2\sin x$$

$$\Rightarrow y'' - y = -2\sin x$$

$$(D^2 - 1) = -2\sin x$$

Soln $y = CF + PI$

$\therefore D^2 - 1 = 0 \Rightarrow D = \pm 1$

$CF = c_1 e^{x} + c_2 e^{-x}$

& $P.I = \dfrac{(-\sin x)}{(D^2 - 1)} = \dfrac{1}{2}\sin x$

$(1-D^2)^{-1}(\sin x) = (1 + D^2 + D^4 + D^6 + \cdots)\sin x$

$= \sin x$

$\because d(\sin x) = \cos x$ | $d[-\sin x] = -\cos x$

$d[\cos x] = -\sin x$ | $d[-\cos x] = \sin x$

$\therefore P.I = \frac{\sin x + \sin x + \sin x - (-\sin x) + \cdots}{2}$

$\therefore y = c_1 e^{x} + c_2 e^{-x} + \frac{1}{2}\sin x$

(c) $\displaystyle\int_{x_0}^{x_1} \frac{1+y^2}{y'^2}\,dx$, $F(x, y, y') = \dfrac{1+y^2}{y'^2}$

I-Method $\Rightarrow$ second form of Euler's eqn

$\dfrac{d}{dx}\left(f - y'\dfrac{\partial f}{\partial y'}\right) - \dfrac{\partial f}{\partial x} = 0 \quad \left(\because \dfrac{\partial f}{\partial x} = 0\right)$

$\Rightarrow \dfrac{1+y^2}{y'^2} - y'\left((1+y^2)\cdot(-2\,y'^{-3})\right) = c$

$\Rightarrow$ ~~$\frac{1+y^2}{y'^2} + \frac{1+y^2}{...}$~~

$\dfrac{1+y^2}{y'^2} + \dfrac{y'(1+y^2)}{y'^3} = c$

$\dfrac{(1+y^2)}{y'^2} = c \Rightarrow y' = \dfrac{\sqrt{1+y^2}}{\sqrt{c}}$

$\Rightarrow \displaystyle\int \frac{dy}{\sqrt{1+y^2}} = \frac{1}{\sqrt{c}}\int dx = \frac{x}{\sqrt{c}} + d$

$\Rightarrow \sinh^{-1}(y) = \dfrac{x}{\sqrt{c}} + d$

$y = \sinh^{-1}\left(\dfrac{x}{\sqrt{c}} + d\right)$

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(d) $\displaystyle\int_0^{\pi/2} [y^2 - y'^2 - 2y\sin x]\,dx$, $y(0) = 0$, $y(\pi/2) = 1$

$f = y^2 - y'^2 - 2y\sin x$

$\therefore$ By Euler's equation $\dfrac{\partial f}{\partial y} - \dfrac{d}{dx}\left(\dfrac{\partial f}{\partial y'}\right) = 0$

$(2y - 2\sin x) - \dfrac{d}{dx}(-2y') = 0$

$\Rightarrow \boxed{y'' + y = \sin x} \Rightarrow (D^2+1)y = \sin x$

$\therefore y = CF + PI$ $\qquad (D^2+1) = 0$

$\Rightarrow D = \pm i$

CF, $y = c_1\cos x + c_2 \sin x$

P.I $= \dfrac{1}{(D^2+1)}\sin x = -\dfrac{x}{2}\cos x$

$\therefore y = c_1\cos x + c_2\sin x + \frac{1}{2}\cancel{\sin x} - \frac{x}{2}\cos x$

$y(0) = 0 = c_1 \Rightarrow c_1 = 0$

$y(\pi/2) = 1 = c_2 + 1/2 \Rightarrow c_2 = 1/2$

$\boxed{y = \sin x - \frac{x}{2}\cos x}$

(e) $\displaystyle\int_0^1 (xy + y^2 - 2y^2 y')\,dx$; $y(0) = 1$; $y(1) = 2$

$f(x, y, y') = xy + y^2 - 2y^2y'$

$\dfrac{\partial f}{\partial y} - \dfrac{d}{dx}\left(\dfrac{\partial f}{\partial y'}\right) = 0$

$\Rightarrow (x + 2y - 4yy') - \dfrac{d}{dx}(-2y^2) = 0$

$\Rightarrow x + 2y - \cancel{4yy'} + \cancel{2\cdot 2y\cdot y'} = 0$

$\Rightarrow$ ~~$4yy'$~~ $y = -x/2$

$y(0) = 1 \Rightarrow 1 = -0$

Since B.C not satisfied.

Hence extremum can't be achieved.

III. 3rd form of Euler's eqn

(90) Since $\frac{\partial F}{\partial y'}$ is a fn of x, y, y', we have

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y'}\right) + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial y'}\right)\frac{dy}{dx} + \frac{\partial}{\partial y'}\left(\frac{\partial F}{\partial y'}\right)\frac{dy'}{dx}$$

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = \frac{\partial^2 F}{\partial x \partial y'} + y'\cdot\frac{\partial^2 F}{\partial y \partial y'} + y''\frac{\partial^2 F}{\partial y'^2} \quad \text{--- (1)}$$

Since Euler's Equation is —

$$\frac{\partial F}{\partial y} - \underbrace{\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)}_{} = 0$$

$$\Rightarrow \frac{\partial F}{\partial y} - \frac{\partial^2 F}{\partial x \partial y'} - y'\frac{\partial^2 F}{\partial y \partial y'} - y''\frac{\partial^2 F}{\partial y'^2} = 0$$

Which is third form of Euler's eqn.

(3) If (r_1, θ_1) & (r_2, θ_2) are two pts in the plane then the length of arc of the curve joining these pts is given by

$$I[r(\theta)] = \int_{\theta_1}^{\theta_2} (r^2 + r'^2)^{1/2}\, d\theta, \quad r' = dr/d\theta$$

by minimizing the above integral obtain the eqn of st. line in the polar Co-ordinates

Soln Comparing the given functional with $\int_{\theta_1}^{\theta_2} F(\theta, r, r')\, d\theta$, we get

$$F(\theta, r, r') = (r^2 + r'^2)^{1/2}$$

∴ By Euler's Eqn

$$\frac{\partial F}{\partial r} - \frac{d}{d\theta}\left(\frac{\partial F}{\partial r'}\right) = 0 \quad \text{--- (1)}$$

r, θ $\frac{d}{dx}\left(\frac{x^2+5}{x^3+3}\right)$

$$\frac{\partial F}{\partial r} = \frac{1}{2}\cdot\frac{2\cdot r}{\sqrt{r^2 + r'^2}}$$

$$\& \frac{d}{d\theta}\left(\frac{\partial F}{\partial r'}\right) = \frac{r'}{\sqrt{r^2 + r'^2}}$$

∴ By (1)

$$\frac{r}{\sqrt{r^2 + r'^2}} - \frac{d}{d\theta}\left(\frac{r'}{\sqrt{r^2 + r'^2}}\right) = 0$$

$$\frac{r}{\sqrt{r^2 + r'^2}} - \frac{r''\left(\sqrt{r^2 + r'^2}\right) - r'\frac{1}{2}\frac{1}{\sqrt{r^2 + r'^2}}\cdot(2rr' + 2r'r'')}{(r^2 + r'^2)}$$

$$\frac{r}{\sqrt{r^2 + r'^2}} - \frac{r''(r^2 + r'^2) - r'(rr' + r'r'')}{(r'^2 + r^2)\left(\sqrt{r^2 + r'^2}\right)} = 0$$

$$\frac{r^3 + r'^2 r - [r^2 r'' + r'^2 r'' - rr'^2 - r'^2 r'']}{(r^2 + r'^2)} = 0$$

$$\Rightarrow r^3 + rr'^2 - r^2 r'' \cancel{- r'^2 r''} + rr'^2 + \cancel{r'^2 r''} = 0$$

$$\Rightarrow r^3 + 2rr'^2 - r^2 r'' = 0$$

$$\Rightarrow r^2 + 2r'^2 - rr'' = 0$$

Let $p = r' \Rightarrow r'' = \frac{d^2 r}{d\theta^2} = \frac{d}{d\theta}\left(\frac{dr}{d\theta}\right) = \frac{dp}{d\theta} = \frac{dp}{dr}\frac{dr}{d\theta} = p\frac{dp}{dr}$

$\Rightarrow$ ~~$p^2 + 2p^2 - rp = 0$~~

$$\Rightarrow r^2 + 2p^2 - rp\frac{dp}{dr} = 0$$

$$\frac{dp}{dr} = \frac{r^2 + 2p^2}{rp} = \frac{r}{p} + 2\,p/r \quad \text{--- (5)}$$

Let $p/r = x \Rightarrow p = rx$

$\Rightarrow$ ~~$\frac{dp}{dr} = dx$~~

$$\Rightarrow \frac{dp}{dr} = x + r\frac{dx}{dr}$$

∴ (5) $\Rightarrow$

$$x + r\frac{dx}{dr} = 2x + 1/x \Rightarrow r\frac{dx}{dr} = x + 1/x$$

$$\Rightarrow \frac{dr}{r} = \frac{x}{x^2+1}dx \Rightarrow \cancel{\log r = \frac{(x^2+1) - x \cdot 2x}{(x^2+1)^2}}$$

$$\Rightarrow \cancel{\log r = \left(\frac{1-x^2}{x^2+1}\right)} \Rightarrow \log r = \frac{1}{2}\log(x^2+1) + \log c$$

$\frac{r^2}{c^2} = (x^2+1) \Rightarrow \boxed{r = c\sqrt{x^2+1}} = c\sqrt{p^2/r^2+1}$

$x = \frac{\sqrt{r^2-c^2}}{c}$

$\frac{p}{r} = \frac{\sqrt{r^2-c^2}}{c}$

$p = \frac{r\sqrt{r^2-c^2}}{c} \Rightarrow \frac{dr}{d\theta} = \frac{r\sqrt{r^2-c^2}}{c}$

$r = \frac{c^2\sqrt{p^2+1}}{r} \Rightarrow p = \sqrt{\left(\frac{r^2}{c^2}\right)^2 - 1}$

$\Rightarrow p = \frac{\sqrt{r^4-c^4}}{c^2}$

$\frac{dr}{d\theta} = \frac{\sqrt{r^4-c^4}}{c^2}$

$\Rightarrow \frac{dr}{\sqrt{r^4-c^4}} = \frac{d\theta}{c^2} \Rightarrow \log\left|\frac{r^2}{c^2}\right|$

$\Rightarrow \frac{dr}{r\sqrt{r^2-c^2}} = \frac{1}{c}\,d\theta$

$\Rightarrow \frac{\theta}{c} + d = \frac{1}{c}\sec^{-1}(r/c)$

$\Rightarrow \sec(\theta+d) = r/c$

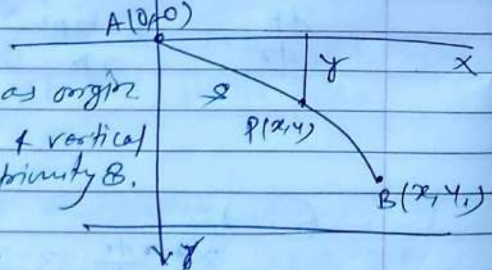
$\Rightarrow \boxed{r = c\sec(\theta+d)}$

or $c = r\cos(\theta+d)$

where c & d arbitrary constants

Q. (P-174) Find the curve connecting the pts A & B which is traversed by a particle sliding from A to B in the shortest time.

Sol^n: taking the fixed pt. A as origin & horizontal & vertical of other extremity B.



At time t,
let v be the velocity in the position P. and actual dista from origin is s.
∴ By Energy eq^n

Change in KE = Workdone in moving particle from A to B,

$W = KE + PE \Rightarrow mgy = \frac{1}{2}mv^2 + 0$

$\Rightarrow v = \sqrt{2gy}$ $\left(\because dv = \frac{ds}{dt}\right)$

$\Rightarrow dt = \frac{1}{\sqrt{2gy}}\,ds$

Thus time taken by the particle from A to B is —

$$t[y(x)] = \int_0^{x_1}\frac{ds}{\sqrt{2gy}} = \frac{1}{\sqrt{2g}}\int_0^{x_1}\frac{\sqrt{1+y'^2}\,dx}{\sqrt{y}}$$

$$= \int_0^{a} f(x,y,y')\,dx \text{ where}$$

$$f = \frac{1}{\sqrt{2g}}\cdot\frac{\sqrt{1+y'^2}}{\sqrt{y}}$$

Since f is independent of $x \Rightarrow \frac{\partial F}{\partial x} = 0$

∴ By Second form of Euler's eq

$$f - y'\frac{\partial f}{\partial y'} = c$$

$$\therefore \sqrt{\frac{1+y'^2}{2gy}} - y'\left(\frac{1}{\sqrt{2g}\sqrt{y}}\cdot\frac{1}{2}\frac{2y'}{\sqrt{1+y'^2}}\right) = c$$

$$\Rightarrow \frac{1}{\sqrt{2g}\sqrt{y}}\left(\frac{1+y'^2-y'^2}{\sqrt{1+y'^2}}\right) = c$$

$$\Rightarrow \frac{1}{\sqrt{y}\sqrt{1+y'^2}} = c\cdot\sqrt{2g}$$

$$\Rightarrow y(1+y'^2) = \frac{1}{c^2 2g} = 2k, \quad k = \text{const}$$

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$\Rightarrow y(1+y'^2) = 2k$

$\left[\because \text{ then we have } \frac{dy}{dx} = y' = \tan\theta\right]$

$\Rightarrow y(1+\tan^2\theta) = 2k$

$\Rightarrow y \sec^2\theta = 2k$

$\Rightarrow y = 2k.\cos^2\theta = k(1+\cos 2\theta)$

Now $\because \frac{dy}{dx} = \tan\theta \Rightarrow \frac{dx}{dy} = \cot\theta$

$\Rightarrow dx = \cot\theta . dy$

$= \cot\theta . k . -\sin 2\theta . 2$

$= -2k\cot\theta (2\sin\theta\cos\theta)\, d\theta$

$= -2k . 2\cos^2\theta\, d\theta$

$= -2k(1+\cos 2\theta)\, d\theta$

$\therefore x = -2k\left[\theta + \frac{\sin 2\theta}{2}\right] + c$

$= c - k(2\theta + \sin 2\theta)$

(Here we find value of x & y in this problem)

(9) Isoperimetric Problem

It is necessary to make a given integral $I = \int_{x_1}^{x_2} f(x, y, y')\,dx$ subject to the constraint $J = \int_{x_1}^{x_2} I(x, y, y')\,dx = \text{constant}$

Method: Such problem involve one or more constraints conditions. This type of problems are called Isoperimetric Problem.

Rayleigh-Ritz method

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(1) Consider the two-pt. BVP —

$y'' + x = 0, \quad 0 < x < 1, \quad y(0) = y(1) = 0$

Sol^n $\left[v(x) = \frac{1}{6}x(1-x^2)\right]$

(2) Sol^n the BVP, defined below —

$y'' + y = -x, \quad 0 < x < 1$ — (1)

with $y(0) = 0 = y(1)$

Sol^n To find the approximate sol^n by Rayleigh Ritz method, we take functional

$I(v) = \int_0^1 (vv'' + v^2 + 2vx)\,dx$ — (2)

let an approximate sol^n is given by $v(x) = \sum_{i=1}^{n} \alpha_i \phi_i(x)$ — (3)

where base f^n ϕ_i satisfies BC. $\phi_i(0) = 0 = \phi_i(1) \quad \forall\, i$

Now substituting (3) in (2), we have

$I(v) = \int_0^1 \left[\sum_{i=1}^{n} \alpha_i \phi_i(x)\right]\left[\sum_{i=1}^{n} \alpha_i \phi_i''(x) + \sum_{i=1}^{n} \alpha_i \phi_i(x) + 2.x\right] dx$ — (4)

let $P_i = \int_0^1 x . \phi_i(x)\,dx$ — (5)

$\gamma_{ij} = \int_0^1 \phi_i(x) . \phi_j(x)\,dx$ — (6)

$\beta_{ij} = \int_0^1 \phi_i(x)\phi_j''(x)\,dx = -\int_0^1 \phi_i'(x)\phi_j'(x)$ — (7)

Now, putting Eq^n (4), we get

$$I(v) = \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_j \alpha_i q_{ij} + \sum\sum_{i,j=1}^{n} \alpha_i \alpha_j r_{ij} + 2\sum_{i=1}^{n} \alpha_i p_i$$

for minimum $\frac{\partial I}{\partial \alpha_i} = 0$, which gives

$$2\sum_{j=1}^{n} \alpha_j q_{ij} + 2\sum_{j=1}^{n} \alpha_j r_{ij} + 2 p_i = 0$$

$$\Rightarrow \sum_{j=1}^{n} \alpha_j (q_{ij} + r_{ij}) + p_i = 0 \quad \text{—(8)}$$

to obtain approximate sol^n. Here, we take $n=2$,

then by eq^n (8).

$\cancel{\alpha_1 (q_{11} + r_{11}) + p_1 = 0}$

(9)
$$\begin{cases} \alpha_1 (q_{11} + r_{11}) + \alpha_2 (q_{12} + r_{12}) = -p_1 \\ \alpha_1 (q_{21} + r_{21}) + \alpha_2 (q_{22} + r_{22}) = -p_2 \end{cases}$$

(Here $q_{21} = q_{12}$ by symmetry)

let $\phi_1(x) = x(1-x)$ & $\phi_2(x) = x^2(1-x)$

be ~~an ap~~, then we h-

$$p_1 = \int_0^1 x\,\phi(x) = \int_0^1 x^2(1-x)\,dx$$

$$= \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \left[\frac{1}{3} - \frac{1}{4}\right] = \frac{1}{12}$$

$$p_2 = \int_0^1 x\,x^2(1-x)\,dx = \left[\frac{x^4}{4} - \frac{x^5}{5}\right]_0^1 = \frac{1}{20}$$

$$q_{11} = -\int_0^1 \phi_1'(x)\,\phi_1'(x) = -\int_0^1 (1-2x)^2\,dx$$

$$= -\left[\frac{(1-2x)^3}{3(-2)}\right]_0^1 = \frac{1}{6}[-1-1]$$

$$= -\frac{1}{3}$$

$$q_{21} = q_{12} = -\int_0^1 \phi_1'(x)\,\phi_2'(x)\,dx$$

$$= -\int_0^1 (1-2x)(2x-3x^2)\,dx$$

$$= -\int_0^1 (2x - 4x^2 - 3x^2 + 6x^3)\,dx$$

$$= -\left[x^2 - \frac{7x^3}{3} + \frac{3}{2}x^4\right]_0^1 = -\left(1 - \frac{7}{3} + \frac{3}{2}\right)$$

$$= -\left[\frac{6-14+9}{6}\right] = -\frac{1}{6}$$

$$q_{22} = -\int_0^1 \phi_2'^2(x)\,dx = -\int_0^1 (2x-3x^2)^2\,dx$$

$$\cancel{= -\left[\frac{(2x-3x^2)^3}{3(2-6x)}\right]_0^1 = -\frac{1}{6}\left[\frac{-1}{-2} - 0\right] = -\frac{1}{12}}$$

$$= -\int_0^1 \left[4x^2 + 9x^4 - 12x^3\right]dx$$

$$= -\left[\frac{4}{3}x^3 + \frac{9}{5}x^5 - 3x^4\right]_0^1 = \left(\frac{4}{3} + \frac{9}{5} - 3\right)$$

$$= \frac{20+27-45}{15} = -\frac{2}{15}$$

$r_{11} = 1/30$

$r_{12} = 1/60$

$r_{22} = 1/105$

$$q_{11} + r_{11} = -\frac{1}{3} + \frac{1}{30} = \frac{-9}{30} = \frac{-3}{10}$$

$$q_{12} + r_{12} = \frac{1}{60} - \frac{1}{6} = \frac{-9}{60} = \frac{-3}{20}$$

putting in (9), we get

some eq^n in for of α_1 & α_2.

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$$12\alpha_1 + 9\alpha_2 = 5 \times \tfrac{4}{9}$$
$$63\alpha_1 + 52\alpha_2 = 21 \times \tfrac{1}{7}$$
$$0 + \left(\frac{9}{2} + \frac{52}{7}\right)\alpha = \frac{5}{2} + 3 = \frac{11}{2}$$
$$\alpha_2 = \frac{11}{2} \times \frac{14}{(63+104)} = \frac{7\times 11}{167}$$
$$= 167$$

10-4

(5) Solve $y'' - x = 0$, $y(0) = 0$ — (1)

$y'(1) = 1/2$

by Ritz method

functional for this problem is given by

$$I(v) = \int_0^1 (v'^2 + 2vx)\, dx + v(1) \quad \text{— (2)}$$

let

$$v = \alpha_1 \phi_1(x) + \alpha_2 \phi_2(x)$$

$$\boxed{v = \alpha_1 x + \alpha_2 x^2} \quad \text{— (3)}$$

be an approximate solⁿ so that $v(x)$ satisfies the B.C

$v(0) = 0$

& $v'(x) = \alpha_1 + 2\alpha_2 x \Rightarrow v'(1) = 1/2 = \alpha_1 + \alpha_2$

Now putting in eqⁿ (2), we have

$$I(v) = \int_0^1 \left[(\alpha_1 + 2\alpha_2 x)^2 + 2(\alpha_1 x + \alpha_2 x^2) x\right] dx + v(1)$$

$$v(1) = \alpha_1 + \alpha_2$$

Now for min

$$\frac{\partial I}{\partial \alpha_1} = 0 = \int_0^1 \left[2(\alpha_1 + 2\alpha_2 x)\cdot 1 + 2x^2\right] + 1$$

& $$\frac{\partial I}{\partial \alpha_2} = 0 = \int_0^1 \left[2(\alpha_1 + 2\alpha_2 x)\cdot 2x + 2x(x^2)\right] + 1$$

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$$\Rightarrow \left[2\alpha_1 x + 4\alpha_2 \frac{x^2}{2} + \tfrac{2}{3}x^3\right]_0^1 + 1/2 = 0$$

$$\Rightarrow 2(\alpha_1 + \alpha_2) + 2/3 + 1 = 0 \Rightarrow \alpha_1 + \alpha_2 = -5/3 \times 2$$

& $$\left[4\alpha_1 \frac{x^2}{2} + 8\alpha_2 \frac{x^3}{3} + 2\frac{x^4}{4}\right]_0^1 + 1 = 0$$

$$\Rightarrow 2\alpha_1 + \frac{8\alpha_2}{3} + \frac{1}{2} + 1 = 0$$

$$\Rightarrow 2\left(\alpha_1 + \tfrac{4}{3}\alpha_2\right) = -3/2 = -3/4$$

$$\Rightarrow \alpha_1 = -13/12,\ \alpha_2 = 1/4$$

∴ Approximate solⁿ is given by

$$\boxed{y^{(x)}(x) = -\frac{13}{12}x + \frac{1}{4}x^2}$$

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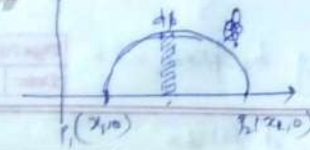
Soln



(7) (107)

Here length of plane curve is given $= l$ (i.e perimeter) which passing through $(x_1,0)$ & $(x_2,0)$. let S be the area enclosed by the plane curve & x-axis — then

$$S = \int_{x_1}^{x_2} y\,dx$$

subject to the constraints —

$$\int_{x_1}^{x_2} ds = l$$

i.e $\int_{x_1}^{x_2} (1+y'^2)^{1/2}\,dx = l$

Let $F = f + \lambda g = y + \lambda(1+y'^2)^{1/2}$

Now applying Euler's eqn — $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$

$$1 - \frac{d}{dx}\left(\frac{\lambda y'}{\sqrt{1+y'^2}}\right) = 0$$

Now integrating both side, we get —

$$x - \frac{\lambda y'}{\sqrt{1+y'^2}} = c \Rightarrow \frac{\lambda y'}{\sqrt{1+y'^2}} = x - c$$

$$\Rightarrow \frac{(\lambda y')^2}{(1+y'^2)} = (x-c)^2 \Rightarrow \frac{1}{y'^2} + 1 = \frac{\lambda^2}{(x-c)^2}$$

$$\Rightarrow y'^2 = \frac{(x-c)^2}{\lambda^2 - (x-c)^2} \Rightarrow y' = \frac{(x-c)}{\sqrt{\lambda^2 - (x-c)^2}}$$

$\Rightarrow$ Now integrating both sides, we get —

$$y = d - \sqrt{\lambda^2 - (x-c)^2} \Rightarrow (y-d)^2 = \lambda^2 - (x-c)^2$$

$$\Rightarrow (x-c)^2 + (y-d)^2 = \lambda^2$$

which is the eqn of circle with Radius λ.

(112) (i) $\frac{d^2y}{dx^2} = f(x)$, $y(a) = y(b) = 0$

Soln: We have $\delta \int_a^b f y\,x = \int_a^b f\,\delta y\,dx$

$$= \int_a^b \frac{d^2y}{dx^2}\cdot \delta y\,dx$$

I. Geodesics

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(1) To find the geodesics on a surface is a variational problem involving the conditional extremum —

(a). Geodesics on a plane are st. line

(b). Geodesics on a sphere of fixed radius are its great circle.

(c) Geodesics on a circular cylinder of fixed radius are circular helix.

II. ISOPERIMETRIC PROBLEMS

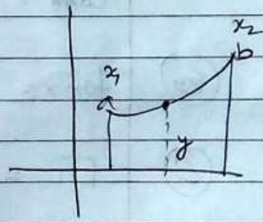
Isoperimetric Problems includes finding extremals of

$$I[y(x)] = \int_{x_1}^{x_2} F(x, y, y')\,dx$$

when another functional

$$J[y(x)] = \int_{x_1}^{x_2} G(x, y, y')\,dx = \text{constant}$$

Such problems are generally solved by method of Lagrange Multipliers.



(1) length of curve $= l$

$$l = \int_{x_1}^{x_2} ds = \int_{x_1}^{x_2} \sqrt{1+y'^2}\,dx$$

A = Area b/n the curve & x-axis

$$= \int_{x_1}^{x_2} y\,dx$$

S = surface area when rotating about x-axis

$$= \int_{x_1}^{x_2} 2\pi y\,ds = 2\pi \int_{x_1}^{x_2} y\sqrt{1+y'^2}\,dx$$

V = vol. of the solid surface

$$= \int_{x_1}^{x_2} \pi y^2\,dx$$

Formula

$(1-x)^{-1} = 1+x+x^2+x^3+\dots$ | $(1+x)^{-1} = 1-x+x^2-x^3+x^4-x^5+\dots$

(1) $\frac{1/x}{1} - \frac{(1/x)^3}{3} + \frac{(1/x)^5}{5} - \frac{(1/x)^7}{7} + \dots = \tan^{-1}(1/x)$

ie $\frac{p}{1} - \frac{p^3}{3} + \frac{p^5}{5} - \frac{p^7}{7} + \dots = \tan^{-1}(p)$

(2) $J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 6^2} + \dots$

(3) $\left(1+\frac{1}{s^2}\right)^{-1/2} = 1 - \frac{1}{2}\left(\frac{1}{s^2}\right) + \frac{1\cdot 3}{2\cdot 4}\left(\frac{1}{s^4}\right) - \frac{1.3.5}{2.4.6}\left(\frac{1}{s^6}\right) + \dots$

(4) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$

(5) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$

(8) $\tan^{-1} x = \frac{x}{1} - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$

(6) $\Gamma n = \int_0^\infty e^{-x} x^{n-1}\, dx$

Laplace transform of special function

(1) $H(t-a) = e^{-as}/s$ $(= e^{-as} f(s))$

(2) $Exp(t) = \log(s+1)/s$

(3) Sine, $Si(t) = (1/p) \cdot \tan^{-1}(1/p)$

(4) Cosine, $Ci(t) = (1/2p) \cdot \log(s^2+1)$

(5) $\operatorname{erf}\sqrt{t}$, $1/(s\sqrt{s+1})$

(6) $\sin\sqrt{t}$, $(\sqrt{\pi}/2s^{3/2})(e^{-1/4s})$

(7) $\frac{\cos\sqrt{t}}{\sqrt{t}}$, $\sqrt{\pi/s} \cdot e^{-1/s}$

(8) $J_0(at)$, $1/(\sqrt{s^2+a^2})$

Inverse formula - 44

	$F(t)$	$f(s) = L[F(t)]$
1.	1	$1/s$
2.	t	$1/s^2$
3.	t^n	$\frac{n!}{s^{n+1}} = \frac{\Gamma n+1}{s^{n+1}}$, $s>0$
4.	e^{at}	$\frac{1}{s-a}$, $s>a$
5.	$t^n e^{at}$	$\frac{1}{(s-a)^{n+1}}$
6.	$\sin at = \frac{a}{s^2+a^2}$,	$\cos at = \frac{s}{s^2+a^2}$ $s>0$
7.	$\sinh at = \frac{a}{s^2-a^2}$,	$\cosh at = \frac{s}{s^2-a^2}$ $s>\lvert a\rvert$
8.	$e^{bt} \sin at = \frac{a}{(s-b)^2+a^2}$	
9.	$e^{bt}\cos at = \frac{s-b}{(s-b)^2+a^2}$	

(1) $\boxed{\frac{1}{D^2+a^2} \sin ax = \frac{-x}{2a}\cos ax}$ for PI of non-Hom Eqn

(10) $L[f(at)] = \frac{1}{a} f(s/a)$

(11) $L[e^{at} f(t)] = f(s-a)$ (shifting)

(12) $L[f(t)] = \frac{\int_0^T e^{-st} f(t)\,dt}{1-e^{-sT}}$, when $f(t) = f(t+T)$ periodic function

(13) $L[f'(t)] = s f(s) - f(0)$

(14) $L[f^n(t)] = s^n L[f(t)] - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) - \dots - f^{n-1}(0)$

(15) $L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} f(s)$

$[L[t f(t)] = -\frac{d}{ds} f(s)]$