GATE-2016 | 21 Marks

Page No. 120 Date 10 01 18

Prob. & Stat

p.d.f = Probability density f^n

(a) p.m.f = probability mass f^n.

r.v = random variable.

(1) (1M) Let the pdf of a r.v X be $f(x)$,

$$f(x) = \begin{cases} x & : 0 \le x < 1/2 \\ c(2x-1)^2 & ; 1/2 < x \le 1 \\ 0 & : \text{elsewhere} \end{cases}$$

Then the value of $C =$ ______

Sol^n As we know that

$$\boxed{\int p.d.f = 1}$$

$$\Rightarrow \int_0^{1/2} x\,dx + \int_{1/2}^{1} c(2x-1)^2 dx = 1$$

$$\Rightarrow \left[\frac{x^2}{2}\right]_0^{1/2} + c\left[\frac{(2x-1)^3}{3\times 2}\right]_{1/2}^{1} = 1$$

$$\Rightarrow \frac{1}{8} + c\left(\frac{1}{6} - 0\right) = 1$$

$$\Rightarrow c = \frac{24-3}{4} = \frac{21}{4}$$

$$= \frac{21}{4} = 5.25$$

(2) Let $x_1, x_2, x_3, \ldots x_n$ be a random sample from the following p.d.f for $0 < \mu < \infty$, $0 < \alpha < 1$:

$$f(x;\mu,\alpha) = \begin{cases} \frac{1}{\Gamma(\alpha)}(x-\mu)^{\alpha-1} e^{-(x-\mu)}; & x > \mu \\ 0 & : ow \end{cases}$$

Here α & μ are unknown parameters.

By P. Kalika

Page No. 121 Date

W.o.t.f statements is true —

(a). Max. likehood estimator of only μ exists.

(b). " " " " α exists.

(c). " " " of both μ & α exist.

(d). " " " of neither α & μ exists.

NB: Question based on MLE every time asked in GATE/NET.

Sol^n

(3). Suppose X & Y are two r.v.s s.t $aX+bY$ is Normal r.v. $\forall a, b \in \mathbb{R}$. W.o.t.f. Statements ALWAYS hold TRUE. ?

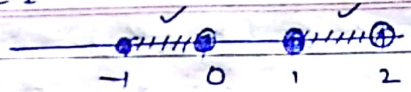
P: X is a std. normal r.v.

Q: The conditional distribution of $X|Y$ (X given Y) is Normal.

R: The cond^n dist^n of $X|(X+Y)$ is normal.

S: $X-Y$ has mean 0.

i.e $|[x]| = 1$



& $\because$ $\underbrace{x < 0 \;\&\; |[x]| = 0}$ $\Rightarrow -1 \le x < 0$

$\therefore P[x < 0 \cap |[x]| = 1] = P(-1 < x < 0)$

let $P(x \le x) = \Phi(x)$

$\Rightarrow P(x < 0, |[x]| = 1) = P(-1 < x < 0)$
$= P(x < 0) - P(x < -1)$
$= \Phi(0) - \Phi(-1)$
$= \Phi(0) - (1 - \Phi(1))$
$= \Phi(0) + \Phi(1) - 1$ — ⓐ

$\because$ In std. normal distⁿ $(\sigma = 1, \mu = 0)$
& $P(x \le 0) = \frac{1}{2} = P(x \ge 0)$

$\Rightarrow \Phi(0) = 1/2 \Rightarrow$ ⓐ $\Rightarrow$ $= \frac{1}{2} - 1 + \Phi(1)$
$= \Phi(1) - 1/2$ — ⓑ

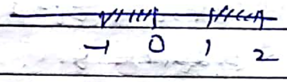
$\left\{ \because N(\sigma, \mu^2) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \Rightarrow N(1, 0) = \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} \right\}$

Since Normal distⁿ is symmetric
$\Rightarrow \Phi(x) = 1 - \Phi(-x)$, $\Phi(\,) =$ c.d.f

$\Rightarrow P[\,|[x]| = 1\,] = P[0 < x < 2]$

P. KALIKA

$\left\{ \begin{array}{l} \because |[x]| = 1 \Rightarrow \\ \text{\& } \cancel{\text{Now}}\; N(\sigma, \mu^2) \text{ is symmetric} \Rightarrow (-1, 0) = (0, 1) \end{array} \right.$



$\therefore P[\,|[x]| = 1\,] = P(0 < x < 2)$
$= \Phi(2) - \Phi(0)$
$= \Phi(2) - 1/2$ — ⓒ

$\therefore$ by ⓑ & ⓒ

$P[x < 0 \,|\, |[x]| = 1] = \dfrac{\Phi(1) - 1/2}{\Phi(2) - 1/2}$

ⓑ (2M) Let $x_1, x_2, \ldots x_n$ be r. sample from the pdf

$f(x) = \begin{cases} \theta\alpha e^{-\alpha x} + (1-\theta) 2\alpha e^{-2\alpha x}, & x > 0 \\ 0 & \text{o.w} \end{cases}$

where $\alpha > 0$, $0 \le \theta \le 1$ are parameters.
Consider the following testing problems —
$H_0: \theta = 1, \alpha = 1$ vs $H_1: \theta = 0, \alpha = 2$.

W.o.t.f. statements is true?

ⓐ Uniformly most powerfull test d.n.e
ⓑ Uniformly most powerfull test is of the form $\sum_{i=1}^{n} x_i > c$, for some $0 < c < \infty$
ⓒ Uniformly most Powerfull test is of the form $\sum_{i=1}^{n} x_i < c$, for some $0 < c < \infty$
ⓓ Uniformly Most Powerfull test is of the form $c_1 < \sum_{i=1}^{n} x_i < c_2$, for some $0 < c_1 < c_2 < \infty$.

GATE-16 || Prob. & Stat

P. KALIKA

Sol^n

Here $\theta_0 = 1,\ \theta_1 = 0,\ \alpha_0 = 1,\ \alpha_1 = 2.$

~~$f_0(x) = \theta$~~

Now $f_0(x) = \theta_0\alpha_0 e^{-\alpha_0 x} + 2(1-\theta_0)\alpha_0 e^{-2\alpha_0 x}$

$= 1e^{-x} + 0$

$= e^{-x}$

and $f_1(x) = \theta_1\alpha_1 e^{-\alpha_1 x} + (1-\theta_1)2\alpha_1 e^{-2\alpha_1 x}$

$= 0 + 4e^{-4x}$

$= 4e^{-4x}$

Joint pdf of $x_1 \ldots x_n$ is $f_0(x)$ under $H_0 = f_0(x) = \prod_{i=1}^{n} e^{-x_i}$

$= e^{-\sum_{i=1}^{n} x_i}$

&

$f_1(x)$ under $H_1 = f_1(x) = 4e^{-4\sum_{i=1}^{n} x_i}$

and the Best Critical region of $(U\alpha) \cap \beta$ test is —

$$\frac{f_1(x)}{f_0(x)} > k$$

$$\Rightarrow \frac{4e^{-4\sum_{i=1}^{n} x_i}}{e^{-\sum_{i=1}^{n} x_i}} = 4e^{-4\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} x_i} > k$$

$$\Rightarrow e^{-3\sum_{i=1}^{n} x_i} > k/4$$

Now take Log both side —

$$-3\sum_{i=1}^{n} x_i > \log(k) - \log(4)$$

$$\Rightarrow \sum_{i=1}^{n} x_i < \underbrace{-\frac{1}{3}\log 4/k}_{= c \text{ (say)}}$$

$$\Rightarrow \boxed{\sum_{i=1}^{n} x_i < c}$$

So, if $\sum_{i=1}^{n} x_i < c$ is critical region

$\Rightarrow 0 < c < \infty$

So the test ~~reg~~ rejects H_0.

Hence

Uniformly most powerfull test is of the form $\sum_{i=1}^{n} x_i < c$; for some $0 < c < \infty$.

Note:

(a) Both P & Q (c) Both Q & S
(b) Both Q & R (d) Both P & S.

Sol^n Hint: direct application of result.
As we know that, if X & Y are independent r.v.. The $aX+bY$ is normal distributed iff X & Y are both normal distributed.

(P) Here—
As $X \sim N(0,\mu^2)$ but NOT given $X \sim N(1,0)$

(ii) ∵ If $X \sim N(0,\mu^2)$, $Y \sim N(0,\mu^2)$ then Conditional dist^n of $X|Y \sim N(0,\mu^2)$.
& $X|(X+Y) \sim N(0,\mu^2)$.

(iv) $(X-Y) \sim N(0,\mu^2)$. But mean $(X-Y)$ may or may NOT be zero. (i.e. $E(X-Y) = \neq 0$)
⇒ S is NOT always true.

(4). (2M) Let X be a r.v with the following c.d.f —

$$F(x) = \begin{cases} 0 & , x < 0 \\ x^2 & , 0 \le x < 1/2 \\ 3/4 & , 1/2 \le x < 1 \\ 1 & x \ge 1 \end{cases}$$

Then $P(1/4 < X < 1)$ is equal to —

Sol^n ∵ By c.d.f $P(a < x < b) = F(b) - F(a)$

$\therefore P(1/4 < X < 1) = F(1) - F(1/4)$

P. KALIKA

$= 3/4 - \left(\frac{1}{4}\right)^2 = \cancel{\frac{15}{16}} = \cancel{\frac{15}{16}}$

$= \frac{12-1}{16} = \frac{11}{16} = 0.6875$

(5) (2M), Let $x_1, x_2, x_3, \dots$ be a seq. of i.i.d. r.v with mean 1. If N is a geo. r.v with the p.m.f $P(N=k) = \frac{1}{2^k}, k=1,2,3,\dots$ & it is independent of x_i's, then — $E(X_1+X_2+\dots+X_N)$ is equal to 2

Sol^n ∵ for p.m.f

$$E(N) = \sum_{n=1}^{\infty} n \cdot P(N=n)$$

$$= \sum_{n=1}^{\infty} n \cdot \frac{1}{2^n}$$

$$= 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{2^2} + 3 \cdot \frac{1}{2^3} + 4 \frac{1}{2^4} + \dots$$

$$= \frac{1}{2}\left(1 + 2 \cdot \frac{1}{2} + 3 \cdot \frac{1}{2^2} + 4 \times \frac{1}{2^3} + \dots\right)$$

$$= \frac{1}{2}\left[\left(1-\frac{1}{2}\right)^{-2}\right] = \frac{1}{2} \times \left(\frac{1}{2}\right)^{-2} = 2$$

$(\because (1-\theta)^{-2} = 1 + 2\theta + 3\theta^2 + 4\theta^3 + \dots)$

Now

$\because E(X_1+X_2+\dots+X_N) = E\left(E(X_1+X_2+\dots+X_N) \mid N\right)$

$\{$ As $E(X) = E(E(X|Y))\}$

for this E, N is r.v & x_i is fixed

for this E, X is r.v & N is fixed

($\because$ x_i are i.i.d)

$= E\left(E(X_1) + E(X_2) + \dots + E(X_N) \mid N \right)$

r.v. fixed

$= E(N \cdot E(X_1))$

$= E(X_1) \cdot E(N) \quad = E(N)$

$= 1 \cdot 2 \quad = 2$

(given $E(X_i) = 1$)

(6) (2M) Let X_1 be an exp. r.v. with mean 1 & X_2 a gamma r.v. with mean 2 & variance = 2. If X_1 & X_2 are indep^n. distributed, then $P(X_1 < X_2)$ is equal to ——— $= 3/4$

Sol^n

$\because$ $E(X) =$ mean $= 1/\lambda$ for exp. r.v

$\Rightarrow \frac{1}{\lambda} = 1 \Rightarrow \lambda = 1 \Rightarrow$

$\Rightarrow f(x) = \lambda e^{-\lambda x} = e^{-x}$

Next

Mean for Gamma distr^n $G(\lambda, n)$

$E(X) = \frac{n}{\lambda} = 2$ & $\frac{n}{\lambda^2} = 2$

$\Rightarrow n = 2\lambda$, $n = 2\lambda^2$

$\Rightarrow 2\lambda - 2\lambda^2 = 0 \Rightarrow \lambda = 0, 1$

$\lambda = 0 \leftarrow$ NOT possible

$\Rightarrow \lambda = 1$

$\Rightarrow n = 2$

$\therefore G(1, 2) = \frac{\lambda^n e^{-\lambda x_2} x^{n-1}}{\Gamma n} = \frac{e^{-x_2} x_2}{\Gamma 2} = x e^{-x_2}$

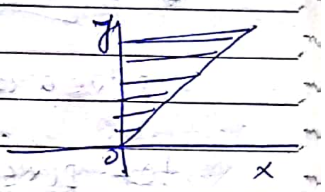
Let $X_2 = Y$

$= y e^{-y}$ for r.v Y.

$\therefore P(X_1 < X_2)$

$= P(X < Y) = \iint (e^{-x})(y e^{-y})\, dx\, dy$

$= \int_0^{\infty}\int_0^{y} e^{-x} y e^{-y}\, dx\, dy$



$= \int_0^{\infty} y e^{-y} \left[-e^{-x} \right]_0^{y} dy$

$= -\int_0^{\infty} y e^{-y} \left[e^{-y} - 1 \right] dy$

$= \int_0^{\infty} \left[y e^{-y} - y e^{-2y} \right] dy$

$= \Gamma 2 - \frac{\Gamma 2}{2^2} = 1 - \frac{1}{4} = \frac{3}{4}$

(7) (2M) Let X be a std. normal random variable. Then $P(X < 0 \mid |[X]| = 1)$ is equal to ———

(a) $\frac{\Phi(1) - 1/2}{\Phi(2) - 1/2}$ (b) $\frac{\Phi(1) + 1/2}{\Phi(2) + 1/2}$

(c) $\frac{\Phi(1) - 1/2}{\Phi(2) + 1/2}$ (d) $\frac{\Phi(1) + 1}{\Phi(2) + 1}$

Sol^n

$\because$ Cond^n probability is —

$P(X < 0 \mid |[X]| = 1) = \frac{P(X < 0, |[X]| = 1)}{P(|[X]| = 1)}$

$\because |[X]| = 1$ only if $x \in [1, 2)$

or $x \in [-1, 0)$

GATE-2016

P. KALIKA

∵ p.d.f for Poisson distⁿ is—

$$f(x) = \begin{cases} \frac{e^{-\theta}\theta^x}{x!} & ; x = 1,2,\ldots \\ 0 & ; o.w \end{cases}$$

$$f(x,\theta) = \prod_{i=1}^{n} \frac{e^{-\theta}\theta^{x_i}}{x_i!}, \quad x = 1,2,\ldots$$

$$f(x,\theta) = \frac{(e^{-\theta})^n \cdot \theta^{\sum_{i=1}^{n} x_i}}{\prod x_i!}, \quad x = 1,2,\ldots$$

now take log both sides—
As log. fⁿ is also gives the max. likehood—
(As, ∵ log x is a cts, strictly ↑ fⁿ over the range of the likehood, the value which max. the likehood will also max. its log).

$$\Rightarrow \log f(x,\theta) = \log\left[e^{-n\theta} \cdot \theta^{\sum_{i=1}^{n} x_i}\right] + \log\left(\frac{1}{\prod x_i}\right)$$

$$= -n\theta + \sum_{i=1}^{n} x_i \log\theta + \log \frac{1}{\prod_{i=1}^{n} x_i!}$$

now diff w.r.t θ for max.—

$$\frac{\partial(\log f(x,\theta))}{\partial\theta} = -n + \frac{\sum_{i=1}^{n} x_i}{\theta} = 0 \ (\text{put} = 0)$$

(As for critical pts, we put $f'(x) = 0$)

$$\Rightarrow n\hat{\theta} = \sum_{i=1}^{n} x_i \Rightarrow \hat{\theta} = \frac{\sum x_i}{n}$$

$$\Rightarrow \hat{\theta} = \bar{x}$$

∴ Max. likehood eastimator for θ

$$= \frac{x_1 + x_2 + x_3 + x_4 + x_5 + x_6}{6}$$

$$= \frac{1+1+1+2+2+2}{6} = \frac{9}{6} = 1.5$$

* * *

Note: $\boxed{\int_0^n [x]\,dx = \frac{n(n-1)}{2}}$

GATE-2016

Page No. 130 Date 05/02/18

P. KALIKA

(9) (2M) Let $x_1, x_2, x_3 \ldots$ be a seq. of iid. $N(\mu, 1)$ r.v.s. Then, $\lim_{n\to\infty} \frac{\sqrt{\pi}}{2n} \sum_{i=1}^{n} E(|x_i - \mu|)$ is equal to $1/\sqrt{2}$

Sol^n: $\because E(|x_i - \mu|) = \int_{-\infty}^{\infty} |x-\mu| f(x)\, dx$

[formula for $E(x) = \int x (\text{p.d.f})\, dx$]

$$= \int_{-\infty}^{\infty} \frac{|x-\mu|}{\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2}} dx$$

(By putting $\sigma = 1$ in $N(\mu, \sigma^2)$)

$$= -\int_{-\infty}^{\mu} \frac{(x-\mu)}{\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2}} dx + \int_{\mu}^{\infty} \frac{(x-\mu)}{\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2}} dx$$

$$= I_1 + I_2$$

I_1: let $\mu - x = y \Rightarrow -dx = dy$ in I_1

& as $x \to \mu \Rightarrow y \to 0$

$x \to -\infty \Rightarrow y \to +\infty$

I_2: let $-\mu + x = z \Rightarrow +dx = dz$ in I_2

& as $x \to \mu \Rightarrow z = 0$

$x \to \infty \Rightarrow z \to \infty$

$$\therefore I_1 + I_2 = \frac{1}{\sqrt{2\pi}} \left[\int_0^{\infty} y e^{-y^2/2} dy + \int_0^{\infty} z e^{-z^2/2} dz \right]$$

Page No. 137 Date

$$= \frac{2}{\sqrt{2\pi}} \left[\int_0^{\infty} y e^{-y^2/2} dy \right]$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} y e^{-y^2/2} dy$$

let $y^2/2 = t \Rightarrow y\,dy = dt$

$$\therefore I_1 + I_2 = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-t} dt$$

$$= \sqrt{\frac{2}{\pi}} [\Gamma 1] = \sqrt{2/\pi}$$

which is free from x.

$$\Rightarrow E[|x_i - \mu|] = \sqrt{2/\pi} \quad \forall i = 1, 2, \ldots n$$

$$\therefore \sum_{i=1}^{n} E[|x_i - \mu|] = \sum_{i=1}^{n} \sqrt{2/\pi} = n \cdot \sqrt{2/\pi}$$

Hence, $\lim_{n\to\infty} \frac{\sqrt{\pi}}{2n} \sum_{i=1}^{n} E[|x_i - \mu|] = \frac{\sqrt{\pi}}{2n} \cdot n \sqrt{\frac{2}{\pi}}$

$$= \frac{1}{\sqrt{2}}$$

Repeated

(10) (2M) Let $x_1 = x_2 = x_3 = 1$, $x_4 = x_5 = x_6 = 2$ be a r. sample from a poisson r.v with mean θ, where $\theta \in \{1, 2\}$. Then, the max. likelihood estimator of θ is equal to $1/2 = 0.5$

Sol: given $x_i \sim P_i(\theta)$.