Hardworking bits talent ÷ Kalika

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Kalika
Functional Analysis (301)
AND Differential Geometry (304)

MTM 401: Functional Analysis (3) Intro. to Topology & Analysis : G.F. Simmons **LTP: 3+1+0**

Unit-I. Inner product spaces, Normed linear spaces, Banach spaces, Quotient norm spaces, continuous linear transformations, equivalent norms, the Hahn-Banach theorem and its consequences. Conjugate space and separability, second conjugate space, Weak *topology on the conjugate space **(15 Lectures)**

Unit-II. The natural embedding of the normed linear space in its second conjugate space, The open mapping Theorem, The closed graph theorem, The conjugate of an operator, The uniform boundedness principle, Definition and examples of a Hilbert space and simple properties, orthogonal sets and complements **(15 Lectures)**

Unit -III. The projection theorem, separable Hilbert spaces. Bessel's inequality, the conjugate space, Riesz's theorem, The adjoint of an operator, self adjoint operators, Normal and unitary operators, Projections, Eigen values and eigenvectors of on operator on a Hilbert space, The spectral theorem on a finite dimensional Hilbert space **(15 Lectures)**

(4) Functional Analysis : P.K Jain, O.P Ahuja, Kalil Ahmed,

Recommended Reading:

3 ✓ 1. G.F.Simmons: Topology and Modern Analysis, McGraw Hill (1963)
2. G.Bachman and Narici : Functional Analysis, Academic Press 1964
2 ✓ 3. A.E.Taylor : Introduction to Functional analysis, John Wiley and sons (1958)
4. A.L.Brown and Page : Elements of Functional Analysis, Van-Nastrand Reinehold Com
5. B.V. Limaye: Functional Analysis, New age international.
1 ✓ 6. Erwin Kreyszig, Introductory functional analysis with application, Willey.

linear space in its second conjugate space The open mapping theorem, the closed graph theorem, The conjugate of an operator, The uniform boundedness principle Def^n & Example of a Hilbert space and simple properties, orthogonal sets and complements.

UNIT- III

The projection theorem, separable Hilbert spaces. Bessel's inequality, the conjugate space, Riesz's theorem, The Adjoint of an operator, self Adj. operator, Normal and Unitary operators, projections E-values & E-vector of an operator on a Hilbert spaces, The spectral theorem on a f.d Hilbert spaces-

~~$1<p,q<\infty$~~ Some Inequalities

Theorem: (1) Let $1 < p, q < \infty$, be such that $\frac{1}{p}+\frac{1}{q}=1$, then for any non-negative real no.s a, b, we have

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}.$$

Pf: If $b=0$, then the result is trivially true.

Similarly, if $a=0$,

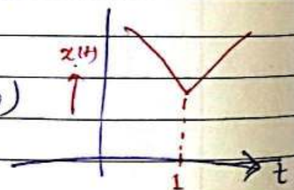
So, consider the case when $a \ne 0, b \ne 0, b>0$.

for any $t \in \mathbb{R}-\{0\}$. Let

$$x(t) = \frac{1}{q} + \frac{1}{p}(t) - (t)^{1/p}$$

then
$$x'(t) = \frac{1}{p} - \frac{1}{p}(t)^{\frac{1}{p}-1}$$
$$= \frac{1}{p}\left(1-(t)^{-1/q}\right) = \frac{1}{p}\left(1-\frac{1}{t^{1/q}}\right)$$
$$= \frac{1}{p}\left(1-t\right)^{-1/q} = \frac{1}{p}\left(1-\frac{1}{t^{1/q}}\right)$$

$$x'(t) = \begin{cases} <0 & \text{for } t<1 \\ >0 & \text{for } t>1 \end{cases}$$



$\therefore x(t) = \min(t)$

$\Rightarrow x(1) \le x(t) \quad \forall \; t (\ne 0)$

$0 \le x(t)$

i.e $0 \le \frac{1}{q} + \frac{1}{p}t - t^{1/p}$

$\therefore t^{1/p} \le \frac{1}{q} + \frac{1}{p}t$

Let $t = \frac{a^p}{b^q}$ (which is NOT zero)

$$\therefore \left(\frac{a^p}{b^q}\right)^{1/p} \le \frac{1}{q} + \frac{1}{p}\left(\frac{a^p}{b^q}\right)$$

$$\therefore \frac{a}{b^{q/p}} \le \frac{1}{q} + \frac{1}{p}\left(\frac{a^p}{b^q}\right)$$

$$\therefore a \le b^{q/p}\left\{\frac{1}{q} + \frac{1}{p}\frac{a^p}{b^q}\right\}$$

$$\therefore ab \le b^{\frac{q}{p}+1}\left\{\frac{1}{q} + \frac{1}{p}\cdot\frac{a^p}{b^q}\right\}$$

$$ab \le \frac{b^q}{q} + \frac{a^p}{p} \qquad \left(\frac{q}{p} + \frac{q}{q} = q\right) ;\ \frac{1}{p}+\frac{1}{q}=1$$

$$\Rightarrow \frac{q}{p} + \frac{q}{q} = q$$

Simple application of $\uparrow$ or $\downarrow$ f^n on the f^n.

$x(t) = \frac{1}{q} + \frac{1}{p}(t) - t^{1/p}$.

* Holder's Inequality (finite form)

* Let $1<p<\infty$. Let q be a conjugates exponent of p (i.e q is such that $\frac{1}{p}+\frac{1}{q}=1$) let a_i, b_i be complex no.s (in particular they can do real numbers)

then —
$$\sum_{i=1}^{n} |a_i b_i| \le \left(\sum_{i=1}^{n} |a_i|^p\right)^{1/p} \left(\sum_{i=1}^{n} |b_i|^q\right)^{1/q} \quad -(1)$$

Hint: Take $\alpha = \left(\sum_{i=1}^{n} |a_i|^p\right)^{1/p}$, $\beta = \left(\sum_{i=1}^{n} |b_i|^q\right)^{1/q}$

take $a = \frac{|a_i|}{\alpha}$, $b = \frac{|b_i|}{\beta}$ is previous inequality.

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Holder's Inequality

Theorem [2] [3.5.4] (55)

Let $1 < p < \infty$ & q be conjugate exponent of p (i.e $\frac{1}{p}+\frac{1}{q}=1$). Let a_i, b_i be complex no.s (in particular, Real no.s) then

$$\boxed{\sum_{i=1}^{n} |a_i b_i| \le \left(\sum_{i=1}^{n} |a_i|^p\right)^{1/p} \left(\sum_{i=1}^{n} |b_i|^q\right)^{1/q}}$$

Proof: Let $\alpha = \left(\sum_{i=1}^{n} |a_i|^p\right)^{1/p}$, $\beta = \left(\sum_{i=1}^{n} |b_i|^q\right)^{1/q}$

Case-I, If $\alpha = 0$, then—

$$\left(\sum_{i=1}^{n} |a_i|^p\right)^{1/p} = 0 \Rightarrow \sum_{i=0}^{n} |a_i|^p = 0$$

$\Rightarrow |a_i|^p = 0 \quad \forall i = 1, 2, \ldots n$

$\Rightarrow |a_i| = 0 \quad \forall i = 1, 2, \ldots n$

$\Rightarrow a_i = 0 \quad \forall i = 1, 2, \ldots n$

$\therefore$ LHS of (i) = RHS (i) = 0

Similarly the result is true if $\beta = 0$

Case-II, Let $\alpha \ne 0, \beta \ne 0$. Let $a = \frac{|a_i|}{\alpha}$, $b = \frac{|b_i|}{\beta}$

substituting these in the prev. theorem—

$$\frac{|a_i||b_i|}{\alpha \beta} \le \frac{\left(\frac{|a_i|}{\alpha}\right)^p}{p} + \frac{\left(\frac{|b_i|}{\beta}\right)^q}{q}$$

Summing over all $i = 1, 2, \ldots n$, we get

$$\sum_{i=1}^{n} \frac{|a_i||b_i|}{\alpha \beta} \le \sum \frac{\left(\frac{|a_i|}{\alpha}\right)^p}{p} + \sum_{i=1}^{n} \frac{\left(\frac{|b_i|}{\beta}\right)^q}{q}$$

$$\therefore \sum_{i=1}^{n} |a_i b_i| \le \alpha\beta \left[\frac{1}{p} \sum_{i=1}^{n} \frac{|a_i|^p}{\alpha^p} + \frac{1}{q} \sum_{i=1}^{n} \frac{|b_i|^q}{\beta^q}\right]$$

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$$= \alpha\beta \left[\frac{1}{p} \cdot \frac{1}{\alpha^p} \cdot \alpha^p + \frac{1}{q} \cdot \frac{1}{\beta^q} \cdot \beta^q\right]$$

$$= \alpha\beta \left[\frac{1}{p} + \frac{1}{q}\right]$$

$$= \alpha\beta$$

$$\therefore \sum_{i=1}^{n} |a_i b_i| \le \alpha\beta = \left(\sum_{i=1}^{n} |a_i|^p\right)^{1/p} \left(\sum_{i=1}^{n} |b_i|^q\right)^{1/q}$$

Note: If $p = q = 2$, then the Holder's inequality is known as Schwarz's Inequality.

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Minkowski's Inequality

Theorem [3] [3.5.4] (55)

Let $1 < p < \infty$ and let a_i, b_i $(i = 1, 2, \ldots)$ be complex No.s, then

$$\left(\sum_{i=1}^{n} |a_i + b_i|^p\right)^{1/p} \le \left(\sum_{i=1}^{n} |a_i|^p\right)^{1/p} + \left(\sum_{i=1}^{n} |b_i|^p\right)^{1/p}$$

Pf— Let $A = \sum_{i=1}^{n} (|a_i| + |b_i|)^p$. Let q be s.t $\frac{1}{p} + \frac{1}{q} = 1$

Now

$$A = \sum_{i=1}^{n} (|a_i| + |b_i|)(|a_i| + |b_i|)^{p-1}$$

$$= \sum_{i=1}^{n} \left[|a_i| (|a_i| + |b_i|)^{p-1} + |b_i| (|a_i| + |b_i|)^{p-1}\right]$$

$$= \sum_{i=1}^{n} |a_i| (|a_i| + |b_i|)^{p-1} + \sum |b_i| (|a_i| + |b_i|)^{(p-1)}$$

· by Holder's Inequality (Holder's Inequality $\sum_{i=1}^{n} |a_i b_i| \le \left(\sum |a_i|^p\right)^{1/p} \left(\sum |b_i|^q\right)^{1/q}$)

$$\le \left(\sum |a_i|^p\right)^{1/p} \left\{\sum \left((|a_i| + |b_i|)^{p-1}\right)^q\right\}^{1/q} + \left(\sum |b_i|^p\right)^{1/p} \left(\sum \left((|a_i| + |b_i|)^{p-1}\right)^q\right)^{1/q}$$

$\frac{1}{p}+\frac{1}{q}=1 \Rightarrow \frac{1}{q}=\frac{p-1}{p} \Rightarrow q(p-1)=p$

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$$\le \left(\sum|a_i|^p\right)^{1/p}\left(\sum(|a_i|+|b_i|)^{p}\right)^{1/q} + \left(\sum_{i=1}^{n}|b_i|^p\right)^{1/p}\left(\sum(|a_i| + |b_i|)^{p}\right)^{1/q}$$

$$A \le \left\{\left(\sum|a_i|^p\right)^{1/p} + \left(\sum|b_i|^p\right)^{1/p}\right\}\underbrace{\left\{\sum(|a_i|+|b_i|)^p\right\}^{1/q}}_{A^{1/q}}$$

$$\therefore A^{1-1/q} \le \left(\sum|a_i|^p\right)^{1/p}+\left(\sum|b_i|^p\right)^{1/p}$$

$$\therefore A^{1/p} \le \left(\sum|a_i|^p\right)^{1/p}+\left(\sum|b_i|^p\right)^{1/p}$$

Def^n [4]: Let V be a non-empty set and K be the field of reals or complex nos. Then the quadraple $(V, F, +, \cdot)$ is called a linear space or vector space over K. If $+ : V\times V \to V$ and $\cdot : K\times V \to V$ are mappings s.t. $+(x,y) = x+y$ and $\cdot(\alpha, x) = \alpha x$ and satisfying the following conditions.

(1). $x+y = y+x \quad \forall x,y \in V$ (Commutative prop.)

(2). $(x+y)+z = x+(y+z) \quad \forall x,y,z \in V$ — (Associativity)

(3). $\exists$ a unique $\theta \in V$ s.t. $x+\theta = \theta+x = x$
(θ is called zero vector or Null vector)

(4). $\forall x \in V$, $\exists\, w \in V$: $x+w = w+x = \theta$
we denote w by $-x$.

(5). $(\alpha\beta)x = \alpha(\beta x) \quad \forall x \in V, \alpha,\beta \in F$

(6). $\alpha(x+y) = \alpha x+\alpha y \quad \forall x,y \in V, \alpha \in F$

(7) $(\alpha+\beta)x = \alpha x+\beta x \quad \forall \alpha,\beta \in F, x \in V$

(8). $1.x = x \quad \forall x \in V$.

The elements of the linear space are called vectors and V is called real (or complex)

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linear space a/c as F is real or complex.

* Eg. $(\mathbb{R}, \mathbb{R}, +, \cdot)$ is a vector space

W.O.T.F are v.s —

(i) $(\mathbb{C}, \mathbb{R}, +, \cdot)$

× (ii) $(\mathbb{R}, \mathbb{C}, +, \cdot)$

(iii) $(\mathbb{C}, \mathbb{C}, +, \cdot)$

$(3+4i, 2)$ should be vector $\mathbb{R}$.
$(3+4i)(2)$ is NOT Real.

Eg. Let $\mathbb{R}^n$ be the set of all ordered n-tuple of Real no.s. For $x = (x_1, \dots x_n)$
$y = (y_1, \dots y_n)$

define $x+y = (x_1+y_1, \dots x_n+y_n)$
$\alpha x = (\alpha x_1, \alpha x_2, \dots \alpha x_n)$.
Then $(\mathbb{R}^n, \mathbb{F}, +, \cdot)$ is a vector space over F.

* * *

Ex. — [P.18] [6] Let $1 \le p < \infty$, let ℓ^p denote the set of all seq.s x s.t.
Let $\ell^p = \left\{x = \{\xi_n\}_{n=1}^{\infty} \;\middle|\; \sum_{n=1}^{\infty} |\xi_n|^p < \infty, \text{ where } \xi_n \in K\ (\mathbb{R} \text{ or } \mathbb{C}), n=1,2\dots\right\}$.

If $x = \{\xi_n\}_{n=1}^{\infty}$, $y = \{\eta_n\}_{n=1}^{\infty}$ and $\alpha \in F$.

Define $x+y = \{\xi_n+\eta_n\}_{n=1}^{\infty}$

$\alpha x = \{\alpha \xi_n\}_{n=1}^{\infty}$

* * *

~~Holder's Inequality~~

Hint: $\sum|\xi_n|^p < \infty$, $\sum|\eta_n|^p < \infty$

$\therefore \left(\sum|\xi_n|^p\right)^{1/p} + \left(\sum|\eta_n|^p\right)^{1/p} < \infty$

$\therefore \left(\sum|\xi_n+\eta_n|^p\right)^{1/p} \le (\infty)$ TS mult

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$\therefore \sum |\xi_n + \eta_n|^p < \infty$

$\therefore x+y \in l^p$ } ⟹ Now it is easy to verify that l^p is a l.s / V.S

Def^n: 5] Let X be a vector space over a field F, where $F = \mathbb{R}$ or $\mathbb{C}$. A Norm on X is a real valued f^n whose value at x denoted by $\|x\|$ satisfies the following conditions—

(i) $\|x\| > 0$ if $x \neq 0$

(ii) $\|\alpha x\| = |\alpha| \|x\|$

(iii) $\|x+y\| \leq \|x\| + \|y\|$

* W.O.T.F. is true—

✗ $\|\alpha x\| = |\alpha| |x|$

✗ $|\alpha x| = \|\alpha\| \|x\|$

✗ $\|\alpha x\| = \|\alpha\| |x|$

✗ $|\alpha x| = \|\alpha\| |x|$

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Monday 25/7/16

Definition: Let X be a vector space (V.S) over a field F, where F is $\mathbb{C}$ or $\mathbb{R}$. A Norm X is a real valued f^n whose value at x denoted by $\|x\|$ satisfies the following conditions.

(i) $\|x\| > 0$ if $x \neq 0$

(ii) $\|\alpha x\| = |\alpha| \|x\|$ $\qquad \| \ \| : X \to \mathbb{R}$

(iii) $\|x+y\| \leq \|x\| + \|y\|$

NB: $\|0\| = \|0 \cdot x\| = |0| \|x\| = 0$ (vector, scalar, vector, scalar, scalar)

N.L.S → Then $(X, F, +, \cdot, \| \ \|)$ is called a normed linear space.

Ex: Let X be a n.l.s. Define $d(x,y) = \|x-y\|$. Then ST (X,d) is a metric space.

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Proof: $d(x,y) = \|x-y\| = \begin{cases} > 0 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$

$d(x,y) = \|x-y\| = \|y-x\| = d(y,x)$

and $d(x,z) = \|x-z\|$

$= \|(x-y) + (y-z)\|$

$\leq \|x-y\| + \|y-z\|$

$\|x-z\| \leq \|x-y\| + \|y-z\|$

Ex. [6] Let $(X_1, \| \ \|_1)$ and $(X_2, \| \ \|_2)$ be two normed linear spaces over F. On $X_1 \times X_2$, define addition and scalar multiplication as follows—

$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$

$\alpha(x_1, y_1) = (\alpha x_1, \alpha y_1)$

Clearly, then $X_1 \times X_2$ is a vector space.

Now, for $v = (v_1, w_1) \in X_1 \times X_2$

define (i) $\|v\| = \|v_1\|_1 + \|w_1\|_2$

Then—

$(X_1 \times X_2, \| \ \|)$ is a n.l.s.

define (ii) $\|v\| = \max\{\|v_1\|_1, \|w_1\|_2\}$. Then Show that $(X_1 \times X_2, \| \ \|)$ is a n.l.s.

NB. We can have more than one norm on a given vector space.

Def^n: A sequence $\{x_n\}$ in a metric space is said to be Cauchy sequence if $\forall \varepsilon > 0, \exists n_0(\varepsilon)$ such that

$d(x_n, x_m) < \varepsilon \ \forall n, m \geq n_0(\varepsilon)$.

Def^n: A metric space (X,d) is said to be complete iff every Cauchy seq. in X converges in X.

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Def^n: A complete normed linear space is called a Banach space.
(2.1.7/61)

Ex. [7] The space $\mathbb{R}^n$ and $\mathbb{C}^n$ are Banach spaces with the norm defined by
$$\|x\| = \left(\sum_{i=1}^{n}|x_i|^2\right)^{1/2} \quad \text{where } x = (x_1, \dots x_n)$$
In general, $\|x\| = \left(\sum_{i=1}^{n}|x_i|^p\right)^{1/p}$
(we denote this norm by $\| \ \|_2$)

Ex. [8] In the vector space $\mathbb{R}^n$, define —
(2.2.3/63)
$$\|x\| = \left(\sum_{i=1}^{n}|x_i|^p\right)^{1/p}$$
Then also $(\mathbb{R}^n, \| \ \|)$ is a Banach space.
(we denote this norm by $\| \ \|_p$)

Ex. [9] In a normed linear space,
(2.1.4/64)
$$\big|\|x\| - \|y\|\big| \le \|x - y\| \quad \forall x, y \in X$$
$$\because \|x\| = \|x - y + y\| \le \|x - y\| + \|y\|$$
$$\Rightarrow \|x\| - \|y\| \le \|x - y\|$$
Interchanging x & y,
$$\|y\| - \|x\| \le \|y - x\|$$
$\therefore$ from both above, we have
$$\big|\|x\| - \|y\|\big| \le \|x - y\|$$

Theorem: [10] The $\|\cdot\|$ is a cts mapping from X into $\mathbb{R}$
(2.1.5/64)
[i.e $f: X \to \mathbb{R}$ defined by $f(x) = \|x\|$ is a cts mapping] i.e to prove $x_n \to x$
$\Rightarrow f(x_n) \to f(x)$ as $n \to \infty$
i.e to prove ~~$\|x_n - x\| \to 0$~~
$x_n \to x \Rightarrow \|x_n\| \to \|x\|$ as $n \to \infty$

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Consider
$$\big|\|x_n\| - \|x\|\big| \le \|x_n - x\| \xrightarrow{} \|0\| = 0$$
$\Rightarrow \|x_n\| - \|x\| \to 0$ as $n \to \infty$
i.e $\|x_n\| \to \|x\|$ as $n \to \infty$
$\therefore \|\cdot\|$ is continuous

other method
To show that f is cts at x_0
i.e given $\varepsilon > 0$, $\exists\ \delta > 0$.
$|x - x_0| < \delta \ni |f(x) - f(x_0)| < \varepsilon$
i.e
$\|x_n - x\| < \delta \ni (\|x\| - \|x_n\|) < \varepsilon$
$$\big|\|x\| - \|x_0\|\big| \le \|x - x_0\| < \delta = \varepsilon$$

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wed 27/7/16

Exercise: [11] Let X be a n.l.s over a field K. Let $\{x_n\}, \{y_n\}$ be a seq. in X with $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$. Let $\alpha_n \in K$
(2.1.6/65)
with $\lim \alpha_n = \alpha$, then —

(i) $\lim_{n\to\infty}(x_n + y_n) = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n$ Proof:

(ii) $\lim_{n\to\infty}(\alpha_n x_n) = \alpha\ \lim_{n\to\infty}(x_n)$

Def^n: A seq. $\{x_k\}$ in a normed space X is s.t.b **summable** to s, if the sequence
(2.1.9/66)
$s_n \left(= \sum_{k=1}^{n} x_k\right)$ of partial sums of the series $\sum_{k=1}^{\infty} x_k$ converges to s in X
i.e $\|s_n - s\| \to 0$ as $n \to \infty$ or $\left\|\sum_{k=1}^{n} x_k - s\right\| \to 0$

NB We write $s = \sum_{k=1}^{\infty} x_k$

Def^n: A sequence $\{x_n\}$ in a normed linear space X is s.t.b absolutely summable if $\sum_{k=1}^{\infty}\|x_k\| <$

NB: For a sequence of real nos and also for complex nos we know that absolute summability implies summability. But in a normed linear space this need not be true.

However in a Banach space (i.e. a normed linear space which is complete), this result is true.

Def^n: A n.l.s X is said to be complete if every Cauchy sequence in X converges in X.
(2013/03 (v))

Theorem: A n.l.s X is a Banach space iff every absolutely summable sequence in X is summable in X.
[12]
(2010/67)

NB: Then any n.l.s which is NOT a Banach space will always have at least one sequence which is absolutely summable but NOT summable.

Proof:- Let X be a Banach space. Let $\{x_n\}$ be an absolutely summable

$$\Rightarrow \sum_{n=1}^{\infty} \|x_n\| = M < \infty$$

$\forall\ \epsilon > 0$, $\exists\ N_\epsilon$ such that $\sum_{n=N_\epsilon}^{\infty} \|x_n\| < \epsilon$.

Let $s_n = \sum_{k=1}^{n} x_k$ ~~then~~ then

$$\|s_n - s_m\| = \|x_{m+1} + \dots + x_n\| \quad \text{(WLOG we take } n > m)$$

$$\le \sum_{k=m+1}^{n} \|x_k\|$$

$$\le \sum_{k=N_\epsilon}^{n} \|x_k\| < \epsilon \quad \forall\ n > m > N_\epsilon.$$

N_ϵ m n

Thus $\{s_n\}$ is a Cauchy sequence in X. But by hypothesis, X is complete and hence $\{s_n\}$ must converge to s say in X.

$$\Rightarrow \sum_{k=1}^{\infty} x_k = s.$$

$\therefore \{x_n\}$ is summable.

(Conversely)

Suppose that every absolutely summable sequence is summable in X.

Claim: X is complete.

Let $\{x_n\}$ be a Cauchy sequence in X. Then for each k, $\exists$ an integer n_k s.t $\|x_n - x_m\| < \frac{1}{2^k}$ $\forall\ n, m \ge n_k$.

We choose n_k so that $n_{k+1} > n_k$.

Define $y_k = x_{n_{k+1}} - x_{n_k}$ then $\|x_{n_{k+1}} - x_{n_k}\| < \frac{1}{2^k}$

i.e. $\|y_k\| < 1/2^k$.

$$\therefore \sum_{k=0}^{\infty} \|y_k\| = \|y_0\| + \|y_1\| + \dots + \|y_k\| \to$$

$$< \|y_0\| + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^k} \to$$

$$\le \|y_0\| + 1/2 + 1/2^2 + \dots + \dots$$

$$\le \|y_0\| + 1$$

$< \infty$

$\therefore \{y_k\}$ is ~~summable~~ absolutely summable.

$\therefore$ By hypothesis, $\{y_k\}$ is summable.

$$\therefore \sum_{k=1}^{\infty} y_k = x \quad \text{say}$$

$\therefore$ the sequence of partial sums $\{t_n\}$ converges to x. (where $t_n = y_0 + y_1 + \dots + y_n$ $= x_{n+1} - x_0$)

$\therefore x_{n+1} - x_{n_0} \to x$ as $n \to \infty$

$\therefore x_{n+1} \to x + x_{n_0}$ as $n \to \infty$

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$\therefore \{x_n\}$ is a convergent sequence

$\therefore X$ is complete

$\therefore X \sim$ ~~B~~ Banach space

Example: The linear space $\mathbb{K}^n$ is a Banach space with the Norm Given by —

(i) $\|x\|_1 = \sum_{i=1}^{n} |x_i|$, where $x = (x_1, x_2, \ldots x_n)$

(ii) $\|x\|_\infty = \max \{ |x_i| : 1 \le i \le n \}$

Note: $\|\cdot\|_1$ is called ℓ^1 - norm.

$\|\cdot\|_\infty$ is called sub-Norm.

Example: In $\mathbb{R}^5$, if $x = (-3, 4, 7, 5, 2)$, $y = (-8, 3, 2, 4, 7)$

$\|x\|_\infty = 7$, $\|y\|_\infty = 8$

Def^n: let $1 \le p \le \infty$ be a real no. Define $\|\cdot\|_p : \mathbb{K}^n \to \mathbb{R}$ by $\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$ where $x = (x_1, x_2, \ldots, x_n)$, then $(\mathbb{K}^n, \|\cdot\|_p)$ is a n.l.s and is denoted by $\ell^p(n)$.

Theorem: The space $\ell^p(n)$ is a Banach space

[13] Pf. Clearly $\ell^p(n)$ is a n.l.S.

Hint (i). $\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \ge 0$ and

$= 0$ iff $\left(\sum |x_i|^p\right)^{1/p} = 0$

iff $\sum |x_i|^p = 0$

iff $|x_i|^p = 0 \ \forall i$

iff $|x_i| = 0$

iff $x_i = 0$

iff $x = 0$

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(ii) $\|\alpha x\|_p = \left(\sum |\alpha x_i|^p\right)^{1/p}$

$= \left(\sum |\alpha|^p |x_i|^p\right)^{1/p}$

$= |\alpha| \left(\sum |x_i|^p\right)^{1/p} = |\alpha| \|x\|_p$.

for triangular inequality. use Minkowski inequality.

We know that $\ell^p(n)$ is complete.

let $\{x_m\}$ be a ~~ele~~ Cauchy sequence in $\ell^p(n)$.

where $x_m = (x_1^m, x_2^m, \ldots x_n^m) \in \mathbb{K}^n$,

* * *

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$\|\cdot\|_p : \mathbb{K}^n \to \mathbb{R}$, $\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$

T.P $\ell^p(n)$ is a B.S

Clearly $\ell^p(n)$ is normed linear space.

claim: $\ell^p(n)$ is Banach space.

let $\{x_m\}$ be a cauchy seq^n in $\ell^p(n)$.

where $x_m = (x_1^m, x_2^m, \ldots x_n^m)$ where $x_i^m \in \mathbb{K}$

Then for every $\varepsilon > 0$, $\exists$ a positive integer n_0 for all $m, k \ge n_0$

$\|x_m - x_k\|_p < \varepsilon$ i.e $\left(\sum_{i=1}^{n} |x_i^{(m)} - x_i^{(k)}|^p\right)^{1/p} < \varepsilon$

i.e $\sum_{i=1}^{n} |x_i^m - x_i^k|^p < \varepsilon^p \quad \forall m, k \ge n_0$ —(*)

$\Rightarrow |x_i^m - x_i^k|^p < \varepsilon^p$

$|x_i^m - x_i^k| < \varepsilon \quad \forall m, k \ge n_0$.

$\Rightarrow \{x_i^m\}_{m=1}^{\infty}$ is a cauchy sequence in $\mathbb{K}$ (i.e $\mathbb{R}$ or $\mathbb{C}$)

which is complete & hence the seq. must be converge to say x_i.

$\Rightarrow \ell^p(n)$ is complete

$\Rightarrow \ell^p(n)$ is Banach space Proved

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Define : $x = (x_1, x_2, \dots x_n)$

Then clearly $x \in l^p(n)$ and further

$\|x_m - x\|_p$

$\|x_m - x\|_p = \left[\sum_{i=1}^{n} |x_i^{m} - x_i|^p \right]^{1/p}$

$= \left[\lim_{k\to\infty} \sum_{i=1}^{n} |x_i^m - x_i^k|^p \right]^{1/p}$

$= \lim_{k\to\infty} \left(\sum_{i=1}^{n} |x_i^m - x_i^k|^p \right)^{1/p}$ (see *)

$< \varepsilon$

$\therefore \{x_m\}$ is converge to x.

Note :

The condition $1 \le p < \infty$ is essential for $l^p(n)$ to be a normed linear space.

Ex. $l^p(n)$ $1 \le p < \infty$
(i) B.S only
(ii) n.l.s only
(iii) Both B.S & n.l.s ✓
(iv) None

$0 < p < \infty$
(i) B.S only
(ii) n.l.s only ✗
(iii) Both B.S & n.l.s
(iv) None

Ex. [15] Consider the space $l^p(2)$, where $0 < p < 1$. Then $x = (1,0)$, $y = (0,1)$. Then clearly $x, y \in l^p(2)$.

$\|x\|_p = \left[\sum_{i=1}^{2} |x_i|^p \right]^{1/p} = (|x_1|^p + |x_2|^p)^{1/p}$
$= (1^p + 0^p)^{1/p} = 1$

$x + y = (1,1)$

$\therefore \|x+y\|_p = (1^p + 1^p)^{1/p} = (2)^{1/p}$, then clearly

$\therefore \|x+y\|_p > \|x\|_p + \|y\|_p$

$2^{1/p} > 2$ (if $0 < p < 1$)

and so $l^p(n)$ for $0 < p < 1$ is not n.l.s.

N.B : $l^p(n)$, $1 \le p < \infty$ is a complete metric space with the metric d defined by

$d(x,y) = \|x - y\|_p^p$, $x, y \in l^p(n)$.

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Ex : [16] Let p be a real no. s.t $1 \le p < \infty$ & let l^p denote the set of all sequence $x = \{x_i\}_{i=1}^{\infty}$ in $\mathbb{R}$ s.t
$\sum_{i=1}^{\infty} |x_i|^p < \infty$.

Define $x + y = \{x_i + y_i\}_{i=1}^{\infty}$

$\alpha x = \{\alpha x_i\}_{i=1}^{\infty}$

(verify) Then $(l^p, +, \cdot)$ is a vector space.

Define $\|x\|_p = \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{1/p}$

(verify) Then l^p is a Banach Space.

Ex. [17] Let l^∞ be the set of all b.dd seq. in $\mathbb{R}$. Define the usual $+, \cdot$. Then $(l^\infty, +, \cdot)$ is a V.S. Define

$\|x\|_\infty = \sup_{1 \le i < \infty} |x_i|$ $\forall x = \{x_i\} \in l^\infty$

* * *

Mon 01/08/16

Example : [18] (The function space $C[a,b]$.)

Banach Space is Complete normed Space

Let $C[a,b] = \{ f_x \mid f : [a,b] \to K \}$, such that f is cts on $[a,b]$.

Define addition & scalar multiplication on $C[a,b]$ by $(f+g)(t) = f(t) + g(t)$ $\forall f, g \in C[a,b]$
$(\alpha f)(t) = \alpha f(t)$. $\forall f \in C[a,b]$.

(verify) Then clearly $C[a,b]$ is a V.S.

Define $\|\cdot\|_\infty : C[a,b] \to \mathbb{R}$ by

$\|x\|_\infty = \max_{t \in [a,b]} |x(t)|$

(verify) Then $(C[a,b], +, \cdot, \|\ \|_\infty)$ is a Banach space.

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Clearly $(C[a,b], +, \cdot, \|\ \|_\infty)$ is a n.l.s.

Claim: $C[a,b]$ is complete

Let $\{x_n\}$ be a Cauchy sequence in $C[a,b]$.

$\Rightarrow$ for each $\epsilon > 0$, $\exists$ a positive integer N : $\forall$ $n, m \geq N$.

$$\|x_n - x_m\|_\infty = \max_{t \in [a,b]} |x_n(t) - x_m(t)| < \frac{\epsilon}{2} \quad (*)$$

$\therefore$ for any fixed, $t_0 \in [a,b]$, we get —

$|x_n(t_0) - x_m(t_0)| < \epsilon \quad \forall\ n, m \geq N$

$\Rightarrow \{x_n(t_0)\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{K}$ which is complete and hence this sequence must converge to say $x(t_0)$.

But $t_0 \in [a,b]$ was arbitrary

$\therefore$ The sequence $x_n \to x$ pointwise.

We claim $x_n \to x$ uniformly

$\Rightarrow x$ is cts. $\exists$ $x \in C[a,b]$

And so $C[a,b]$ would be complete and hence Banach space.

from $(*)$,

$|x_n(t) - x_m(t)| < \epsilon/2 \quad \forall\ n, m \geq N, t \in [a,b]$.

letting $m \to \infty$

$|x_n(t) - x(t)| \leq \epsilon/2 \quad \forall\ n, m \geq N$

$\forall\ t \in [a,b]$.

$\Rightarrow \{x_n\}$ Converges to x unif. on $[a,b]$.

$\therefore x$ must be cts on $[a,b]$.

[19] Example of a Normed linear space which is NOT a Banach space.

(2.10.23) Let $C[-1,1]$ be the set of all cts f^n on $[-1,1]$. Define $\|x\|_1 = \int_{-1}^{1} |x(t)|\, dt$

where the integration is the usual Riemann

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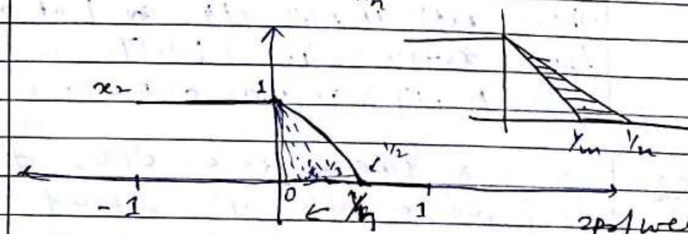
integral. Then $(C[-1,1], +, \cdot, \|\ \|_1)$ is a n.l.s but not B.s.

Clearly $(C[-1,1], +, \cdot, \|\cdot\|)$ is a n.l.s (verify)

We now show that $C[-1,1]$ is NOT complete.

Let $\{x_n\}$ be a sequence in $C[-1,1]$ definitely.

$$x_n(t) = \begin{cases} 1 & -1 \leq t \leq 0 \\ 1 - nt & 0 < t \leq 1/n \\ 0 & 1/n < t \leq 1 \end{cases}$$



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* * *

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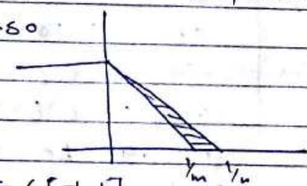
Thu 11th 4/8/16

[20] Geometrically $\|x_n - x_m\|$ represent — the shaded area, so

Clearly $\{x_n\}$ is a Cauchy sequence.



Suppose $x_n \to x \in C[-1,1]$

Then $\|x_n - x\| = \int_{-1}^{1} |x_n(t) - x(t)|\, dt$

$$= \int_{-1}^{0} |x_n(t) - x(t)|\, dt + \int_{0}^{1/n} |x_n(t) - x(t)|\, dt + \int_{1/n}^{1} |x_n(t) - x(t)|\, dt \quad \text{— (1)}$$

As $\|x_n - x\| \to 0$ each term on the RHS of (1) must tend to zero as $n \to \infty$.

In particular $\lim \int_{1/n}^{1} |x(t)|\, dt$

$(C[-1,1], +, \cdot, \|\ \|)$ is n.l.s.

$$= \left(\int_{-1}^{0} |1 - x(t)|\, dt + \int_{0}^{1/n} |x_n(t) - x(t)|\, dt + \int_{1/n}^{1} |x(t)|\, dt \right)$$

$$\int_{-1}^{0} |1 - x(t)|\, dt = 0$$

$$\Rightarrow x(t) = 1 \quad \forall\ t \in [-1, 0]$$

Next $x(t) = \begin{cases} 1 & t \le 0 \\ 0 & t \ge 1 \end{cases}$ $\int_{0}^{1} |x(t)|\, dt = 0 \Rightarrow |x(t)| = 0$ in $[0,1]$

Then $x(t)$ is NOT cts as [at 0).

Thus, $x_n \not\to x$ in $C[-1,1]$

$\therefore C[-1,1]$ is NOT a B.S.

Ex. [21] (2.2.11/85) The real linear space $C'[0,1]$ of all ctsly differentiable f^n defined on $[0,1]$ with the norm given by

$$\|x\|_\infty = \sup_{t \in [0,1]} |x(t)|$$

is NOT complete n.l.s.

sol^n If f is cts $\Rightarrow$ f' exist & cts.

H.W $\rightarrow$ Clearly $C'[0,1]$ is a n.l.s.

Claim: $C'[0,1]$ is NOT complete.

Let $\{x_n\}$ be a cauchy sequence in $C'[0,1]$ given by $x_n(t) = \sqrt{t^2 + 1/n}$ then

$\{x_n\}$ converges pt-wise to the f^n x. given by $x(t) = |t|$, $t \in [0,1]$

H.W $\{x_n\}$ defined is indeed a Cauchy seq.

Now $\|x_n - x\|_\infty = \sup_{t \in [0,1]} |x_n(t) - x(t)|$

$$= \sup_{t \in [0,1]} \left| \sqrt{t^2 + 1/n} - |t| \right|$$

$$\text{or } \sup_{t \in [0,1]} \left[\sqrt{t^2 + 1/n} - t \right] \quad (\because t \in [0,1])$$

$= \frac{1}{\sqrt{n}} \to 0$ as $n \to \infty$ $t = 0$.

Thus $x_n \to x$ on $[0,1]$, But x is not diff at & hence $x \notin C'[0,1]$.

$\therefore C[0,1]$ is Not a B.S.

Note:- A n.l.s can always be made into a metric space. But the converse is NOT true.

i.e $\exists$ example of metric space which can never be made into a n.l.s.

Eg. [22] (2.2.12/86) Let W be the linear space of all seq. in K with the usual pt-wise add^n & scalar multiplication.

Define $d(x,y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|\alpha_i - \beta_i|}{1 + |\alpha_i - \beta_i|}$

where $x = \{\alpha_i\}_{i=1}^{\infty}$

$y = \{\beta_i\}_{i=1}^{\infty}$

H.W $\Rightarrow$ then clearly d is metric space

[23.] Suppose there is a norm on W, then we should have for any λ.

$$d(\lambda x, \lambda y) = \|\lambda x - \lambda y\| = |\lambda| \|x - y\|$$
$$= |\lambda|\, d(x,y)$$

* * *

TUE 10/08/16 Eg. of a linear space which is a metric space but can't be made into n.l.s.

(2.2.12/86) (W) Let W be the linear space of all sequences in K with the usual pt-wise addition and scalar multiplication.

Define $d(x,y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|\alpha_i - \beta_i|}{1 + |\alpha_i - \beta_i|}$

where $x = \{\alpha_i\}_{i=1}^{\infty}$, $y = \{\beta_i\}_{i=1}^{\infty}$

Then clearly d is metric on W.

Suppose there is a norm on W. Then for any λ, we should have –

$\|\lambda(x-y)\| = |\lambda| \|x-y\|$

$\|\lambda x - \lambda y\|$

R.H.S. $|\lambda| \, d(x,y)$

$$= |\lambda| \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|\alpha_i - \beta_i|}{1+|\alpha_i - \beta_i|} \quad \text{—} \ (1)$$

L.H.S. $= \|\lambda x - \lambda y\|$

$$= \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|\lambda\alpha_i - \lambda\beta_i|}{1+|\lambda\alpha_i - \lambda\beta_i|}$$

$$= \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|\lambda| \, |\alpha_i - \beta_i|}{1+|\lambda| \, |\alpha_i - \beta_i|} \quad \text{—} \ (2)$$

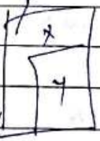
Clearly $(1) \neq (2)$ for $|\lambda| \neq 1, 0$

and hence the $\|\cdot\|$ conditions are NOT satisfied.

$\therefore (X, \|\cdot\|)$ is not a n.l.s

[24] **SUBSPACES – 2.3 – 89**

Def^n: [2.3.1 | 89] Let $(X, \|\cdot\|)$ be a n.l.s. Let $\phi \neq Y \subseteq X$. Then Y is s.t.b a subspace of X if –



(i). Y is a linear subspace of X.

(ii). Y is equipped with a norm $\|\cdot\|_Y$ induced by the norm $\|\ \|_X$ on X.

i.e $\|t\|_Y = \|t\|_X \quad \forall t \in Y$

Note: [25] Every subspace of a normed lin. space is a n.l.s.

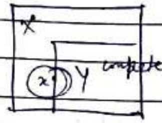
Def^n: [2.3.2 | 89] A subspace Y of a n.l.s X is called a closed subspace of X if Y is a closed set in X, considered as a metric space.

Def: [2.3.3 | 89] Let X be a B.S. A subspace Y of X considered as a normed linear space is called a subspace of a Banach space X.

[2.3.6 | 90] Theorem [26]: Let Y be a subspace of a n.l.s X. If Y be a complete. Then Y is closed.

Proof: Let x be a limit pt. of Y.

(To prove: $x \in Y$).



$\Rightarrow B(x) \cap Y - \{x\} \neq \phi$

Thus $B(x, 1/n) \cap Y - \{x\} \neq \phi$ say

$x_n \in B(x, 1/n) \cap Y - \{x\}$

Thus $\{x_n\}$ is a ~~subspace~~ sequence converging to x.

As each $x_n \in Y$ and Y is complete and hence $\lim_{n\to\infty} x_n \in Y$ is $x \in Y$.

Thus every limit pt. of Y is in Y.

$\therefore Y$ is closed.

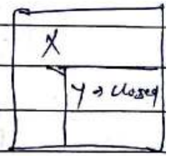
[2.3.7 | 90] Theorem [27]: Let Y be a subspace of a B.S X. Let Y be a closed. Then Y is complete.

Pf: Let $\{x_n\}$ be a cauchy sequence in Y.

$\Rightarrow \{x_n\}$ is cauchy sequence in X.

$\Rightarrow x_n \to x$ for some $x \in X$.

$\because X$ is complete.



$\therefore$ every Ball $B(x)$ contains a pt. x_n of Y other than x.

$\Rightarrow x$ is limit point of Y.

But Y is closed

$\therefore x \in Y$.

Thus every Cauchy seq. in Y converges in Y.

$\therefore Y$ is complete.

$\therefore Y$ is a Banach space.

Cor. Let Y be a subspace of a B.S. X. Then Y is complete iff Y is closed.

(23rd)

* * *

11-8-16

Quotient space [2.4] - 92

[28] Let V be a vector space and W be a subspace of V. For an elt. $x \in V$, define

$$x+W = \{x+w \mid w \in W\}$$

Then $x+W$ is called a coset of x w.r.t. W.

Note (i). Any two cosets are identical or disjoint.

(ii) The collection of all cosets form a partition of V.

(iii) Define $V/W = \{v+W \mid v \in V\}$

define linear operators on V/W by

$(x+W)+(y+W) = (x+y)+W$

& $\alpha(x+W) = \alpha x + W$

Then $(V/W, +, \cdot)$ is a vector space and is called Quotient space of V by W.

Note: The additive identity of V/W is $0+W \equiv W$ and $-(x+W) = -x+W$

We shall now construct new normed lin. sp. from the given n.l.s.

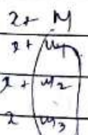
Thm: Let X be a n.l.s over a field K.

[29] [2.4.1] [92] Let M be a closed subspace of X. Defined

$$\|\cdot\|_q : X/M \to \mathbb{R}$$

by $\|x+M\|_q = \inf\{\|x+m\| : m \in M\}$

Then $(X/M, +, \cdot, \|\cdot\|_q)$ is n.l.s.



Further, if X is a B.S then X/M is also a Banach space.

Proof: We first show that $(X/M, \|\cdot\|_q)$ is a n.l.s.

(i) $\|x+M\|_q \geq 0$ is obvious

$\left(\|x+M\|_q = \inf\{\underbrace{\|x+m\|}_{\geq 0} : m \in M\}\right)$

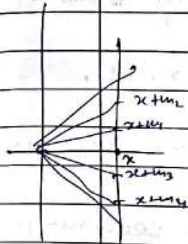
$\|x+M\|_q = 0$ iff $x+M = M$ $\quad \therefore \inf \geq 0$

(ii) Let $x+M = M$ (i.e let $x+M$ be the zero vector of X/M)

Then $\|x+M\|_q = \|M\|_q = \|0+M\|_q$

$= \inf\{\|0+m\| : m \in M\}$

$= 0$



$\Leftarrow$ Conversly let $\|x+M\|_q = 0$

$\Rightarrow 0 = \inf\{\|x+m\| : m \in M\}$

$\Rightarrow \exists$ a seq. $\{m_k\} \subset M$

Such that $\lim_{k\to\infty} \|x+m_k\| = 0$

$\Rightarrow m_k \to -x$ as $k \to \infty$

$\Rightarrow -m_k \to x$ as $k \to \infty$

Now each m_k (and hence $-m_k$) $\in M$ and M is closed, it follows that

$\lim_{k\to\infty}(-m_k) \in M$

i.e $x \in M$ $\therefore x+M = M$

* * *

(iii) for $x, y \in X$

$\|(x+M)+(y+M)\|_q = \|(x+y)+M\|_q$

$= \inf\{\|x+y+m\| : m \in M\}$

$= \inf\{\|x+m_1+y+m_2\| : m = m_1+m_2 \in M\}$

$\leq \inf\{\|x+m_1\| + \|y+m_2\| : m_1, m_2 \in M\}$

$= \|x+M\|_q + \|y+M\|_q$

Next let $\alpha \neq 0 \in K$ and $x \in X$, then

$\|\alpha(x+M)\|_q = \|\alpha x+M\|_q$
$= \inf\{\|\alpha x+m\| : m\in M\}$
$= \inf\{\|\alpha x+\frac{\alpha}{\alpha}m\| : m\in M\}$
$= \inf\{\|\alpha x+\alpha m_1\| : m_1\in M\}$
$= \inf\{|\alpha|\|x+m_1\| : m_1\in M\}$
$= |\alpha|\|x+M\|_q$

further this result is trivially true if $\alpha=0$.

$\therefore (X/M, +, \cdot, \|\cdot\|_q)$ is a n.l.s.

Next let X be a B.S. To prove X/M is a B.S.

Let $\{x_n+M\}_{n=1}^{\infty}$ be a cauchy sequence in X/M.

$\therefore$ We can find a subsequence $\{x_{n_k}\}$ from $\{x_n\}$ s.t

$\|(x_{n_2}+M)-(x_{n_1}+M)\|_q < 1/2$ — (*)

$\|(x_{n_3}+M)-(x_{n_2}+M)\|_q < 1/2^2$ — (**)

— — — —

$\|(x_{n_{k+1}}+M)-(x_{n_k}+M)\|_q < 1/2^k$

Now choose any vector $y_1 \in x_{n_1}+M$

$\therefore y_1 = x_{n_1}+m_1$ where $m_1\in M$ (clearly $y_1 \in M$)

choose y_2 : $\|y_2-y_1\| < 1/2$

next select y_3 : $\|y_3-y_2\| < 1/2^2$

$\vdots$

$\|y_{n+1}-y_n\| < 1/2^n$

Let $y_2 = x_{n_2}+m_2$ where m_2 is selected so that

$\|y_2-y_1\| = \|(x_{n_2}+m_2)-(x_{n_1}+m_1)\| < 1/2$ (by (*))

Next, similarly select $y_3 = x_{n_3}+m_3$ so that $\|y_3-y_2\| < \frac{1}{2^2}$ by (**)

Proceeding in this way, we get a sequence $\{y_k\}$ in X such that —

$x_{n_k}+M = y_k+M \quad (x_{n_k}+m_k+M) \in x_{n_k}+M$ (***)

and $\|y_{k+1}-y_k\| < \frac{1}{2^k}$, $k=1,2,\ldots$

$\because$ Any two coset are either identical or both of them contain ~~k=1,2,~~ $\ldots x_{n_k}+m$ when $m\in M$, $k\in I$

Claim: $\{y_k\}_{k=1}^{\infty}$ is a cauchy seq. in X.

Let $k>r$, then —

$\|y_k-y_r\| = \|y_k-y_{k-1}+(y_{k-1}-y_{k-2})+\cdots+(y_{k-j}-y_r)\|$ where $k-j=r+1$

$\le \|y_k-y_{k-1}\|+\|y_{k-1}-y_{k-2}\|+\cdots+\|y_{k-j}-y_r\|$

$< \frac{1}{2^r}+\frac{1}{2^{r+1}}+\cdots+\frac{1}{2^r}$

$< \frac{1}{2^{r-1}}+\frac{1}{2^{r-2}}+\cdots+\frac{1}{2^r}$

$\to 0$ as $r\to\infty$

$\therefore \{y_k\}_{k=1}^{\infty}$ is a cauchy sequence in X which is complete.

$\therefore \exists$ some $y\in X$: $\lim_{k\to\infty}\|y_k-y\|=0$

$\therefore$ Now consider —

$\|(x_{n_k}+M)-(y+M)\|_q = \|(y_k+M)-(y+M)\|_q$
$= \|(y_k-y)+M\|_q$
$\le \|y_k-y\| \to 0$

$\therefore$ The subsequence $\{x_{n_k}+M\}$ converges in X/M and hence the original seq. $\{x_n+M\}$ converges in X/M

$\therefore X/M$ is a B.S

Note: • Now from here we write $\|x+M\|$ instead of $\|x+M\|_q$.
• $\|\cdot\|_q$ on X/M is called Quotient norm.
Since we shall always be using the norm only, we shall write $\|\ \|$ instead of $\|\cdot\|_q$.

* * *

Tue 16/8/16

CHAPTER-3, B.ded lin. operator (16-8-16 Tuesday)

Def^n: Let V & W be linear spaces over the same scalar field $\mathbb{K}$. A f^n $T: V \to W$ is s.t.b a linear map (linear transformation or linear operator) if —
$$T(x+y) = T(x) + T(y) \quad \forall x, y \in V$$
$$T(\alpha x) = \alpha T(x) \quad \forall x \in V \ \& \ \alpha \in F.$$

NB we write $T(x)$ by Tx.
(field may be $\mathbb{R}$ or $\mathbb{C}$).
Eg. if $F = \mathbb{C}$ $\alpha = 2+3i$ then
$$T((2+3i)x) = (2+3i)T(x).$$

Def^n: Let X & Y be n.l.s over the same field $\mathbb{K}$. An operator $T: X \to Y$ (linear or not) is s.t.b continuous at a point $x_0 \in X$, if for a ~~subspace~~ sequence $\{x_n\} \subseteq X$ s.t $x_n \to x_0$ in X implies $T(x_n) \to T(x_0)$ in Y.
(As for $\varepsilon > 0$, $\exists\ \delta(\varepsilon)$ s.t $\|x - x_0\| < \delta \Rightarrow \|T(x) - T(x_0)\| < \varepsilon$)

Alternately: $T: X \to Y$ is s.t.b continuous at $x_0 \in X$ if given $\varepsilon > 0$, $\exists\ \delta > 0$; $\forall x \in X$ with
$$\|x - x_0\|_X < \delta \Rightarrow \|Tx - Tx_0\|_Y < \varepsilon$$

Def^n: $T: X \to Y$ is s.t.b continuous on X iff T is continuous at every $x_0 \in X$.

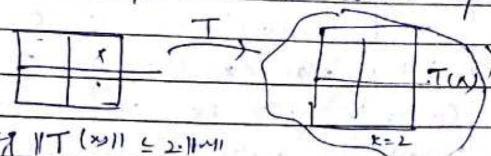
Theorem: 3.3-1/101 Let X & Y be n.l.s over K. Let $T: X \to Y$ be a linear operator. Then T is continuous on X iff T is continuous at a point in X.

$T: X \to Y$

Proof $\Rightarrow$ obvious.
$\Leftarrow$ Let T be continuous at $x_0 \in X$.

Let $x \in X$ be arbitrary.
Let $\{x_n\} \subseteq X$ be s.t $x_n \to x$.
Then $\{x_n - x + x_0\}$ is a sequence in X converging to x_0.
Thus $x_n - x + x_0 \to x_0$ as $n \to \infty$
$\therefore T(x_n - x + x_0) \to T(x_0)$ ($\because$ T is cts at x_0)
i.e $T(x_n) - T(x) + \cancel{T(x_0)} \to \cancel{T(x_0)}$
$\Rightarrow T(x_n) - T(x) \to 0$
or $T(x_n) \to T(x)$.
$\therefore$ T is cts at x.
But $x \in X$ is arbitrary.
$\therefore$ T is continuous on X.

Def^n: 3.2/101 Let X & Y be n.l.s over field K. Let $T: X \to Y$ be a linear operator. Then T is s.t.b a BOUNDED LINEAR operator on^t iff $\exists$ $K > 0$ s.t
$$\|T(x)\|_Y \le K\|x\|_X \quad \forall x \in X.$$
If T is NOT b.dd linear operators. Then we say that T is unb.dd linear operator.



Eg $\|T(x)\| \le 2\|x\|$

Eg $\|T(x)\| \le 8.75\|x\|$
$\|T(x_0)\| > 50\|x_0\|$
~~$\|T(x_1)\|$~~ $\|T(x_n)\| > 1000\|x_n\|$

Thus if T is unb.dd linear operator, then given any $n \in \mathbb{N}$, $\exists$ some $x_n \in X$ s.t $\|T(x_n)\| > n\|x_n\|$

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N.B.: A Bounded linear operator is NOT the same as b.dd real valued f^n Because for $f:\mathbb{R}\to\mathbb{R}$ to be b.dd We mean $\exists$ a constant k: $|f(x)| \le k \;\forall x \in \mathbb{R}$.

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* * *

Theorem: (3.1.3/102) Let X & Y be n.l.s over the field $\mathbb{K}$ and $T: X \to Y$ be a linear operator, then T is continuous iff T is Bounded.

proof: $\Leftarrow$ (converse part)

Let T be bounded.

Claim: T is continuous at 0.

Let $\{x_n\}_{n=1}^{\infty}$ be any seq. in X s.t $x_n \to 0$ as $n\to\infty$.

Now T is b.dd (by hypothesis) and hence $\exists$ some constant k s.t

$\| T(x_n) \| \le k\|x_n\|$. $\forall\; n = 1,2,\ldots$
$\to 0$ as $n\to\infty$ by (*)

(and hence $\|x_n\| \to 0$ as $n\to\infty$ ∴ $\|\cdot\|$ is cts f^n)

$\therefore T(x_n) \longrightarrow \underset{(T(0))}{0}$

$\therefore T$ is continuous at 0.

$\therefore$ By earlier theorem T is cts on X.

$\Rightarrow$ Conversely let T be cts on X.

Claim: T is b.dd linear operator.

Suppose T is NOT b.dd linear operator. Then $\forall n \in \mathbb{N}$ $\exists$ some $x_n \ne 0$

$\| T(x_n)\|_Y > n\|x_n\|_X$

$$\therefore \left\| T\left(\frac{x_n}{n\|x_n\|_X}\right)\right\|_Y = \left\| \frac{1}{n\|x_n\|_Y} T(x_n)\right\| = \frac{1}{n\|x_n\|}\|T(x_n)\| > 1$$

Setting $y_n = x_n/(n\|x_n\|_X)$. observe that $y_n \to 0$

$\ni \|Ty_n\| > 1 \;\forall n \in \mathbb{N} \Rightarrow Ty_n \not\to 0$ as $n\to\infty$

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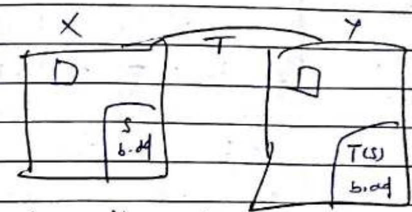
$\nexists$ T cont at origin. be cts $\frac{1}{n\|x\|}\cdot n\|x_n\| = 1$ — (1)

Theorem: (3.1.4/103) Let X & Y be n.l.s over a field $\mathbb{K}$ and $T: X \to Y$ be a linear operator. Then T is b.dd linear operator iff T maps b.dd set in X into b.dd set in Y.



Remark: Thus from this and earlier theorems, we have —

Theorem: (3.1.5/103) Let X & Y be n.l.s over $\mathbb{K}$. Let $T: X \to Y$ be a linear operator. Then the following statements are equivalent —

(i) T is cts at the origin.
(ii) T is cts on X.
(iii) T is b.dd
(iv) T maps b.dds sets in X into b.dd sets in Y.

to be proved

* * *

Q.1 Let X & Y be n.l.s over the field $\mathbb{K}$. Let $T: X \to Y$ be b.dd lin. op. then P.T.

$M_1 = M_2 = M_3 = M_4$.

Where

$M_1 = \sup\{\|T(x)\|_Y : x \in X, \|x\|_X \le 1\}$

$M_2 = \sup\{\|T(x)\|_Y : x \in X, \|x\|_Y = 1\}$

$M_3 = \sup\left\{\frac{\|T(x)\|_Y}{\|x\|_X} : x \in X, x \ne 0\right\}$

$M_4 = \inf\{K : K > 0 \text{ and } \|Tx\|_Y \le K\|x\|_X \;\forall x \in X\}$.

(Hint: $M_2 \le M_1 \le M_4 \le M_3 = M_2$).

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Q.2 (2.1.4 / 103) Let X & Y be N.L.S over the field K & $T : X \to Y$ be a lin. op. Then P.T. T is a b.dd linear operator $\Leftrightarrow$ T maps bounded sets in X into b.dd sets in Y.

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Pf: Let T be a b.dd operator.

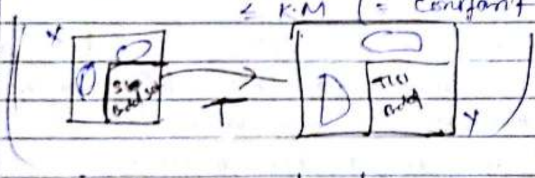
$\Rightarrow \exists$ a constant k : $\|T(x)\|_Y \leq k\|x\|_X \ \forall x \in X$.

Let S be any b.dd set in X.

$\Rightarrow \exists$ a constant say $M \ni \|x\|_X \leq M \ \forall x \in S$.

Now for any $x \in S$,

$$\|T(x)\|_Y \leq k\|x\|_X \leq k.M \ (= \text{constant})$$



$\therefore T(S)$ is a b.dd set.

$\Leftarrow$ Conversely

Let T map b.dd set in X into B.dd set into Y.

clearly $S(0,1) = \{x : \|x\| \leq 1\}$ is a b.dd closed set in X.

$\therefore T(S)$ is b.dd in Y.

$\Rightarrow \exists$ a constant K : $\|T(s)\| \leq K$ where $s \in S$.

Now consider any $x \in X$ $(x \neq 0)$.

Then $\frac{x}{\|x\|} \in S$.

$$\left\| T\left(\frac{x}{\|x\|}\right) \right\| \leq K$$

$$\Rightarrow \frac{1}{\|x\|} \|T(x)\| \leq K$$

$$\Rightarrow \|T(x)\| \leq K.\|x\|$$

And if $x = 0$, then this result is trivially true.

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$\therefore \|T(x)\| \leq K\|x\| \ \forall x \in X$.

Def: Let X & Y be n.l.s over a field K. Let T be b.dd linear operator. Then —

$$\|T\| = \sup\{\|T(x)\| : x \in X, \|x\| \leq 1\}$$

is called NORM (or BOUND) of T

Theorem: [3.1.7/104] Let X, Y be n.l.s over a field K and let $T : X \to Y$ be a bounded linear operator. Then $\|T\|$ can be expressed as any one of the following:—

(3.1.6) (a) $\sup\{\|T(x)\|_Y : x \in X, \|x\|_X \leq 1\} = M_1$

(b) $\sup\{\|T(x)\|_Y : x \in X, \|x\|_X = 1\} = M_2$

(c) $\sup\left\{\frac{\|T(x)\|}{\|x\|} : x \in X, x \neq 0\right\} = M_3$

(d) $\inf\{k : k > 0 \text{ and } \|T(x)\|_Y \leq k\|x\|_X \ \forall x \in X\} = M_4$

Pf: — We claim $M_1 = M_2 = M_3 = M_4$. To do this, we shall prove $M_2 \leq M_1 \leq M_4 \leq M_3 \leq M_1$.

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Clearly

$$\sup\{\|T(x)\| : x \in X, \|x\|_X = 1\} \leq \sup\{\|T(x)\| : x \in X, \|x\| \leq 1\}$$

$$\therefore M_2 \leq M_1 \quad \text{—(1)}$$

Next

$$M_3 = \sup\left\{\frac{\|T(x)\|}{\|x\|} : x \in X, x \neq 0\right\}$$

$$= \sup\left\{\left\|\frac{T(x)}{\|x\|}\right\| : x \in X, x \neq 0\right\}$$

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$= \sup \left\{ \left\| T\left(\frac{x}{\|x\|}\right) \right\| : x \in X,\ x \neq 0 \right\}$

$= \sup \{ \|T(y)\| ;\ y \in X\ \&\ \|y\| = 1 \} = M_2$

from (i). $\frac{\|T(x)\|}{\|x\|} \leq M_3 \quad \forall x \in X, x \neq 0$

$\Rightarrow \|T(x)\| \leq M_3\|x\| \ \forall\ x \in X, x \neq 0$

$\therefore$ By definition of $\inf^m$, $\underline{M_4 \leq M_3}$

But by def^n M_4 is the smallest value of K.

$\therefore \|Tx\| \leq K\|x\|$

$\therefore \boxed{M_4 \leq M_3}$

Next from def^n of M_4 (as a special case)

$\|T(x)\| \leq M_4\|x\| \quad \forall x \in X.$

In particular, for any x; $\|x\| \leq 1$, $x \neq 0$.

$\|T(x)\| \leq M_4\|x\| \leq M_4.$)

Also. $\|T(x)\| \leq M_4$ is trivially true if $x = 0$.

$\therefore \sup \|T(x)\| \leq M_4$.

i.e $\boxed{M_1 \leq M_4}$

So from all above conclusions, we have

$M_2 \leq M_1 \leq M_4 \leq M_3 \leq M_2$

Corollary: [3.1.8|105] let X, Y be n.l.s over a field K. Let $T: X \to Y$ be a bdd linear transfor.. Then $\|Tx\| \leq \|T\|\|x\| \quad \forall x \in X.$

Proof: By prev theorem,

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$\|T(y)\| \leq \|T\| \quad \forall y, \|y\| = 1$

for only y, $\|y\| = 1$

Let $x \in X$ be arbitrary $(x \neq 0)$.

then $\left\| T\left(\frac{x}{\|x\|}\right) \right\| \leq \|T\| \qquad y = \frac{x}{\|x\|}$

$\therefore \left\| \frac{1}{\|x\|} T(x) \right\| \leq \|T\|$

$\therefore \frac{1}{\|x\|} \|Tx\| \leq \|T\|$

$\therefore \|T(x)\| \leq \|T\|\|x\| \quad$ — (1)

(1) is trivially true if $x = 0$.

Thus $\boxed{\|T(x)\| \leq \|T\|\|x\|.}$

Def^n: (107) Let $T: X \to Y$ be a linear transformation where X & Y are V.S. Let

$N(T) = \{x \in X : T(x) = 0\}$

then $N(T)$ is called **NULL SPACE** of the linear operator T.

N.B (Note that Null space is indeed a v.s)

Theorem: 3.1.10 (107-8) Let $T: X \to Y$ be a bdd linear transformation from n.l.s X into n.l.s Y. Then the NULL space $N(T)$ is closed.

Sol^n: We shall prove that

$\overline{N(T)} \subset N(T)$

We know that $N(T) \subseteq \overline{N(T)}$.

$\therefore N(T) = \overline{N(T)}$

$\therefore \overline{N(T)}$ will be closed.

So let $x \in \overline{N(T)}$

$\Rightarrow \exists$ a seq. $\{x_n\} \subseteq N(T)$ s.t $x_n \to x$

Now T is cts (being b.dd)

$\frac{1}{n}$...

$\therefore T(x_n) \to T(x)$

But $T(x_n) = 0 \quad \forall\, n = 1,2,3 \cdots$

$\therefore T(x) = 0 \quad \therefore x \in N(T)$

$\therefore \overline{N(T)} \subseteq N(T)$

$N(T)$ is closed

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Note: The Null space is closed. (Prev. thm)
But the whole range need not be closed.

Ex.— (3.1.11) Let $T: l^\infty \to l^\infty$ be defined by
$$T(\{\alpha_j\}) = \left\{\frac{\alpha_j}{j}\right\}$$

(verify) → Then clearly T is linear.
$$T(\{\alpha_i\}) + T(\{\beta_i\}) = \left\{\frac{\alpha_j}{j}\right\} + \left\{\frac{\beta_j}{j}\right\}$$
$$= \left\{\frac{\alpha_j + \beta_j}{j}\right\}$$
$$= T(\{\alpha_j + \beta_j\})$$

and $T(\{A\alpha_j\}) = \left\{\frac{A\alpha_j}{j}\right\}$
$$= A\left\{\frac{\alpha_j}{j}\right\} = A\,T(\{\alpha_j\})$$

Also T is b.dd linear operator.

$\therefore \|T(x)\| = \|\cancel{T(\{\alpha_j\})}\| \quad \|T(\{\alpha_j\})\|$
$$= \left\|\left\{\frac{\alpha_j}{j}\right\}\right\| \le \|\{\alpha_j\}\|$$
$$= \|x\|$$

$\Rightarrow \|T(x)\| \le 1 \cdot \|x\| \quad (k = 1)$

$\therefore T$ is bounded.

Hint: In the range side, consider
$\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\}$
$\{\frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5} \cdots\}$
$\{\frac{1}{3}, \frac{1}{9}, \frac{1}{5}, \frac{1}{6}, \cdots\}$
$\{\frac{1}{4}, \frac{1}{2}, \frac{1}{6}, \frac{1}{7} \cdots\}$

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$\{0, \frac{1}{2}, 0, 0 \cdots\}$

on domain side $\{1, 1, 1, \cdots\}$
$\{\frac{1}{2}, 1, \frac{3}{4}, \cdots\}$
$\{\frac{1}{3}, 1, \frac{3}{5}, \cdots\}$
$\{\frac{1}{4}, 1, \frac{3}{6}, \cdots\}$

Note: $\nexists$ any α:
$T(\alpha) = \{0, \frac{1}{2}, 0, 0, \cdots\}$? so T is NOT closed

$(0,1,0,0 \cdots) \longrightarrow$

Def^n: A linear transformation $T: X \to Y$ is s.t.b invertible if $\exists$ a linear transf. $S: Y \to X$ s.t SOT and TOS are identity operators. We denote this S by T^{-1}.

Theorem: (3.1.12) Let X & Y be n.l.s. over a field K and $T: X \xrightarrow{\text{onto}} Y$ be a linear transf. Then T^{-1} exists and is bounded operator iff $\exists$ a constant k_1:
$$\|T(x)\| \ge k_1 \|x\| \quad \forall\, x \in X \quad \text{—— } (*)$$

Pf: Let $T^{-1}: Y \to X$ be a b.dd linear operator
$$\|T^{-1}(y)\| \le k\|y\| \quad \forall\, y \in Y$$

$y = T(x)$

In particular $\forall\, x \in X$.
$$\|T^{-1}(Tx)\| \le k\|T(x)\|$$
$$\Rightarrow \|x\| \le k\|T(x)\|$$
$$\Rightarrow \frac{1}{k}\|x\| \le \|T(x)\| \quad \text{Denote } \frac{1}{k} = k_1$$
$$\Rightarrow \|T(x)\| \ge k_1\|x\| \quad \text{—— } (*)$$

$\Leftarrow$ Conversely, let condition hold, then
T is 1-1 : Let $T(x) = T(y)$
$\Rightarrow T(x-y) = 0$ by linearity prop.
from $(*)$, with x replaced by $x-y$.

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$\|T(x-y)\| \geq k_1 \|x-y\|$

$\therefore \|0\| \geq k_1\|x-y\|$

$k_1\|x-y\| = 0$

$\Rightarrow x = y.$

$\therefore$ T is one-one.

* * *

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let $x_1 = \{1, 0, 0 \ldots 0, \ldots 0\} \in l^\infty$

$x_2 = \{1, 2, 0 \ldots 0\} \in l^\infty$

$\vdots$

$x_n = \{1, 2, 3 \ldots n, 0, \ldots 0\} \in l^\infty$

Then $y_1 = T(x_1) = \{1, 0, \ldots, 0\} \in R(T)$

$y_2 = T(x_2) = \{1, 1, \ldots 0\} \in R(T)$

$\ldots$

$y_n = T(x_n) = \{1, 1, 1, \ldots 1, \ldots 0\} \in R(T)$

Eg. 3.1.11 (108) Let l^∞ be the n.l.s of all bdd seq. in $\mathbb{K}$. & $T: l^\infty \to l^\infty$ be defined by

$$T\left(\{\alpha_i\}_{i=1}^\infty\right) = \left\{\frac{\alpha_i}{i}\right\}_{i=1}^\infty$$

then clearly T is linear & B.dd

now, $\lim_{n\to\infty} y_n = \{1, 1, 1, \ldots\} = y$ say

Note that y which is limit of seq. $\{y_n\}$ $\notin R(T)$

$\because$ the only possibility is $\{1, 2, 3 \ldots\}$ s.t.

$T(\{1, 2, 3 \ldots\}) = \{1, 1, 1 \ldots\} \notin l^\infty$

$x_1 \to y_1$

$x_2 \to y_2$

$\ldots$

$x_n \to y_n$ i.e $\{1, 2, 3 \ldots\} \to \{1, 1, 1 \ldots\}$

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Continue of Th 3.1.12

onto $T: X \to Y$ L.T. T^{-1} exists & bdd L.T. iff $\exists$ K_1 s.t

$\|T(x)\| \geq K_1 \|x\| \quad \forall x \in X$

$\Rightarrow$ done

$\Leftarrow$ Let the condition be true. Then T is 1-1 (done).

By hypothesis T is onto, $\therefore$ T^{-1} will exist.

clearly T^{-1} is linear.

$T^{-1}y = x$

$T^{-1}(y_1+y_2) = T^{-1}[T(x_1) + T(x_2)]$ for some $x_1, x_2 \in X$

$= T^{-1}(T(x_1+x_2))$

$= x_1 + x_2$

$= T^{-1}(y_1) + T^{-1}(y_2)$

Similarly $T^{-1}(\alpha y) = \alpha T^{-1}(y)$

$\therefore$ T^{-1} is linear map.

Claim: T^{-1} is bounded

$\|Tx\| \geq K_1\|x\|$

By hypothesis -

$\|T(T^{-1}(y))\| \geq K_1 \|T^{-1}(y)\| \quad \forall y \in Y$

$\therefore \|T^{-1}(y)\| \leq \frac{1}{K_1}\|y\|$

$\therefore$ T^{-1} is bounded, LT.

Theorem: (3.1.14 | 110) Let X & Y be B.S over the same field K and $T: X \to Y$ be a b.dd LT. If $\exists$ a constant $K > 0$:

$\|T(x)\| \geq K\|x\| \quad \forall x \in X$

then $R(T)$ is closed.

Proof: Let $\{y_n\} \in R(T)$. Let $\{y_n\}$ converge to y.

Let $\{y_n\} \in R(T) \to y$

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(To prove: $y \in R(T)$. To find $x \in X : T(x) = y$)

$\Rightarrow \exists\, x_n \in X \;;\; T(x_n) = y_n$

Claim: $\{x_n\}$ is a cauchy sequence in X.

Consider $\|y_n - y_m\| = \|T(x_n) - T(x_m)\|$

$= \|T(x_n - x_m)\|$

$\geq k\|x_n - x_m\| \quad \forall\, n, m$

But $\|y_n - y_m\| \to 0$ as $n, m \to \infty$

$\|x_n - x_m\| \leq \frac{1}{k}\|y_n - y_m\| \leq \epsilon/k$

$\because \{y_n\}$ is a convergent and hence cauchy sequence.

$\therefore \|x_n - x_m\| \to 0$ as $n, m \to \infty$

$\therefore \{x_n\}$ is a cauchy sequence in X.

and hence $\{x_n\}$ converging to say x.

($\because X$ is B.S)

Thus $x_n \to x$

$\therefore T(x_n) \to T(x)$

But $T(x_n) \xrightarrow{\;y_n\;} y$

$\therefore T(x) = y \quad \therefore y \in R(T)$.

$\Rightarrow R(T)$ is closed.

Theorem 3.2.1/111 :- let X & Y be n.l.s over field K. Then the set $B(X,Y)$ of all b.dd L.T from X into Y is a linear subspace of $L(X,Y)$ (the space of all linear transformation)

Proof:- clearly $B(X,Y) \subseteq L(X,Y)$

Also $B(X,Y) \neq \phi \quad (\because \theta \in B(X,Y))$

let $S, T \in B(X,Y)$

$\Rightarrow \exists$ constants k_1 and k_2 :

$\|S(x)\| \leq k_1\|x\| \quad \forall\, x \in X$.

$\|T(x)\| \leq k_2\|x\| \quad \forall\, x \in X$.

$\therefore \|(S+T)(x)\| = \|S(x) + T(x)\| \quad \forall x \in X$

3.2. SPACES of BOUNDED LINEAR OPERATOR

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$\leq \|S(x)\| + \|T(x)\| \quad \forall x \in X$

$\leq k_1\|x\| + k_2\|x\| \quad \forall x \in X$

$\leq 2k\|x\|$ where $k = \max\{k_1, k_2\}$

(so $k_1 \leq k$ & $k_2 \leq k$)

$\therefore S+T \in B(X,Y)$

$S \in B(X,Y)$

$\Rightarrow \alpha S \in B(X,Y)$

$\therefore B(X,Y)$ is a subspace of $L(X,Y)$.

↳ n.l.s

* * *

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Note :

Thus we observe that $B(X,Y)$ is a linear space. ~~Now~~ We shall now define a suitable norm on it, so that $B(X,Y)$ becomes a n.l.s.

Theorem 3.2.2/112: Let X & Y be n.l.s over the field K and $B(X,Y)$ be the linear space of all b.dd LO $T: X \to Y$.

Define $\|\cdot\| : B(X,Y) \to \mathbb{R}$ by

$\|T\| = \sup\{\|Tx\|_y : x \in X,\ \|x\|_x \leq 1\}$

Then $(B(X,Y), \|\cdot\|)$ is a n.l.s. Further if Y is a Banach space then so $B(X,Y)$.

Proof: (i) clearly $\|T\| \geq 0 \quad \forall\, T \in B(X,Y)$

(ii) $\|T\| = 0$ iff $\sup_{\|x\| \leq 1}\|T(x)\| = 0$

$\Rightarrow \|T(x)\| = 0 \quad \forall\, x : \|x\| \leq 1$

$\Rightarrow \|T(x)\| = 0 \quad \forall\, x : \|x\| = 1$

Now for any $v \in X$, $v \neq 0$

$\left\|T\left(\frac{v}{\|v\|}\right)\right\| = 0$

$\Rightarrow \frac{1}{\|v\|}\|T(v)\| = 0 \Rightarrow \|T(v)\| = 0$

$\Rightarrow T(v) = 0 \quad \forall v \in X, v \neq 0$

$T \equiv 0$ (zero operator)

Clearly
$T(0) = 0$
($\because T$ is linear operator)

Conversely if $T = 0$
$\Rightarrow T(x) = 0 \quad \forall x$
$\Rightarrow \sup |Tx| = 0 \quad \forall n$

In particular
$\sup_{\|x\| \le 1} |T(x)| = 0 \quad \Rightarrow \|T\| = 0$

(iii) For any $\lambda \in \mathbb{K}$

$\|\lambda T\| = \sup\{\|(\lambda T)x\| : \|x\| \le 1\}$
$= \sup\{\|\lambda T(x)\| : \|x\| \le 1\}$
$= \sup\{|\lambda| \|T(x)\| : \|x\| \le 1\}$
$= |\lambda| \sup\{\|Tx\| : \|x\| \le 1\}$
$= |\lambda| \cdot \|T\|$

(iv). $\forall S, T \in B(X, Y)$
$\|S+T\| = \sup_{x}\{\|(S+T)(x)\| : \|x\| \le 1\}$
$= \sup\{\|S(x) + T(x)\| : \|x\| \le 1\}$
$\le \sup\{\|S(x)\| + \|T(x)\| : \|x\| \le 1\}$
$\because Y$ is n.l.s
$\le \sup\{\|S(x)\| : \|x\| \le 1\} + \sup\{\|T(x)\| : \|x\| \le 1\}$
$= \|S\| + \|T\|$

$\therefore B(X, Y)$ is a n.l.s

Sol. Now to prove, that $B(X, Y)$ is B.S, we follow three steps —
(I) $T_n \in B(X, Y)$ converges to some T
(ii) T is b.dd. l.o
(iii) $T \in B(X, Y)$

Claim: $B(X, Y)$ is a B.S

given Y is a B.S (Banach Space)
let $\{T_n\}$ be a Cauchy seq. in $B(X, Y)$.
then
$\|T_n(x) - T_m(x)\| = \|(T_n - T_m)(x)\|_Y$
$\le \|T_n - T_m\| \|x\|_X$
$\longrightarrow 0 \quad$ as $n, m \to \infty$

$\therefore \{T_n(x)\}$ is a cauchy seq. in Y. which is by hypothesis is complete.
$\therefore \{T_n(x)\}$ must converge to some value (f^n) depending on x say Tx.
[Note: T at this stage is some f^n].

We claim: (i). $T \in B(X, Y)$
(ii) $T_n \to T$, so that $B(X, Y)$ will be complete.

Evening

T is linear

(i) for any $x, y \in X$, $\lambda \in \mathbb{K}$
$T(x+y) = \lim_{n \to \infty} T_n(x+y)$
$= \lim_{n \to \infty} \{T_n(x) + T_n(y)\}$
$= \lim_{n \to \infty} T_n(x) + \lim_{n \to \infty} T_n(y)$

$T(x+y) = T(x) + T(y)$
$\|^{ly}$ $T(\alpha x) = \alpha T(x)$
it verifies that T is linear.
$\Rightarrow T \in B(X, Y)$

Claim: T is b.dd
for each n, T_n is b.dd linear operator
$\therefore \|T_n(x)\| \le \|T_n\| \|x\|$
Also, $\{T_n\}$ is a cauchy seq in $B(X, Y)$ & is b.dd.
$\therefore \exists$ a constant k, $\because \|T_n\| \le k \quad \forall n \in \mathbb{N}$
$\therefore \|T_n(x)\| \le \|T_n\| \|x\|$

2pr | mon

$\|T_n(x)\| \le k\|x\|$

$$\|T(x)\| = \|T(x) - T_n(x) + T_n(x)\|$$
$$\le \|T(x) - T_n(x)\| + k\|x\|$$
$$\longrightarrow k\|x\| \quad \text{as } n \to \infty$$

$\therefore T$ is bdd.

$\|T_n - T_m\| < \varepsilon \ \forall n,m \ge n_0$

now fix m, $m = n_0$

$\therefore \|T_n - T_{n_0}\| < \varepsilon \ \forall n > n_0$

$\therefore \|T_m\| = \|T_m - T_{n_0} + T_{n_0}\|$
$\le \|T_m - T_{n_0}\| + \|T_{n_0}\|$
$\le \underbrace{\varepsilon + \|T_{n_0}\|}_{k}$

thus T is a bdd linear operator.

(ii) Since $\{T_n\}$ is a Cauchy seq. in $B(X,Y)$

$\therefore$ for $\varepsilon > 0$, $\exists N \in \mathbb{N}$:

$$\|T_n - T_m\| < \varepsilon \quad \forall n,m \ge N$$

$\therefore \|T_n(x) - T_m(x)\| \le \|T_n - T_m\| \|x\|$

$\hookrightarrow \|(T_n - T_m)(x)\|$

$$< \varepsilon\|x\| \quad \forall n,m \ge N$$

~~for~~ making $m \to \infty$;

$\|T_n(x) - T(x)\| < \varepsilon\|x\|$

$\therefore \sup_{\|x\| \le 1} \|T_n(x) - T(x)\| \le \sup_{\|x\| \le 1} \frac{\varepsilon}{2}\|x\| \le \frac{\varepsilon}{2}$

$\|T_n - T\| \le \frac{\varepsilon}{2} < \varepsilon \quad (\text{as } \|x\| \le 1)$

$\therefore T_n \to T$

$\therefore B(X,Y)$ is a Banach space

Theorem: 3.2.3 | 114

Let X be a n.l.s over a field $\mathbb{K}$. If $S, T \in \mathcal{B}(X,X)$, then $S \circ T \in B(X,X)$ and $\to \mathcal{B}(X)$

$$\|ST\| \le \|S\| \|T\|$$

Proof: clearly ST is LT.

As $\quad \alpha, \beta \in \mathbb{K}$, $x,y \in X$

$$(ST)(\alpha x + \beta y) = S(T(\alpha x + \beta y))$$
$$= S(T(\alpha x) + T(\beta y))$$
$$= S(\alpha T(x) + \beta T(y))$$
$$= S(\alpha T(x)) + S(\beta T(y))$$
$$= \alpha S(T(x)) + \beta(S(T(y)))$$
$$= \alpha(ST)(x) + \beta(ST)(y)$$

$\therefore ST$ is linear

Now,

We shall show that, ST is continuous at 0 (and so by earlier theorem ST is cts everywhere

$\therefore ST$ is bdd linear operator)

$\because T \in \mathcal{B}(X,X)$

$\therefore T$ is bounded

$\therefore T$ is continuous

Let $x_n \to 0$

$\Rightarrow T(x_n) \to T(0)$

$\Rightarrow S(T(x_n)) \to S(T(0)) \quad (\because S \in \mathcal{B}(X,X)$. $\therefore S$ is cts$)$

$\therefore ST$ is continuous at 0. $\quad (S(0) = 0)$

$\therefore ST$ is cts.

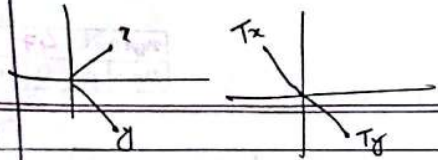
$\therefore ST$ is bounded $\Rightarrow ST \in B(X,Y)$

Next,

$$\|(ST)(x)\| = \|S(T(x))\|$$
$$\le \|S\| \|T(x)\|$$
$$\le \|S\| \|T\| \|x\|$$

$\therefore \sup_{\|x\| \le 1} \|ST(x)\| \le \sup_{\|x\| \le 1} \{\|S\| \|T\| \|x\|\}$

$$\|ST\| \le \|S\| \|T\|$$



Def^n: Let X and Y be n.l.s over a field $\mathbb{K}$. A mapping (not necessarily linear) $T: X \to Y$ is called isometry if
$$\|T(x)\| = \|x\| \quad \forall x \in X.$$

Theorem: Let X & Y be n.l.s over a field $\mathbb{K}$. Let
3.3.2/117 $T: X \to Y$ be LT. If T is an isometry, then T is a bounded and $\|T\| = 1$.
$\|T(x)\| = 1$

pf: $\because$ ~~sup~~ T is isometry
$$\Rightarrow \|T(x)\| \le \|x\| \quad \forall x \in X$$
$$\le k\|x\| \quad (\text{i.e } 2\|x\|)$$
for $k \ge 1$

(∃ at least one k s.t. $\|Tx\| \le k\|x\|$)

$\therefore T$ is bounded and further
$$\sup_{\|x\|=1} \|T(x)\| = \sup_{\|x\|=1} \|x\|$$
$$\Rightarrow \sup_{\|x\|=1} \|T(x)\| = 1 \quad (\because \|x\| = 1)$$
i.e $\|T\| = 1$

Proof

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* * *

Eg. 3.3.3/117 Example of a linear transformation which is NOT an isometry (i.e converse 3.3.2)

Let $T: (C[0,1], \|\cdot\|_\infty) \to \mathbb{K}$ is defined by (consider as V.S)
$$T(x) = x(0) \quad \forall x \in C[0,1]$$

Clearly T is linear map.

claim: T is bdd.

$\forall x \in X$,
$$|T(x)| = |x(0)| \le \sup\{|x(t)| : t \in [0,1]\} = \|x\|_\infty$$
$$\Rightarrow \sup_{\|x\|_\infty \le 1} |T(x)| \le \sup_{\|x\|_\infty \le 1} \|x\|_\infty \le 1$$
$\sup_{\|x\|_\infty \le 1}$??

$\therefore$ i.e $\|T\| \le 1 \Rightarrow T$ is bounded

Next let $y: [0,1] \to \mathbb{K}$ by $y(t) = 1-t$. Clearly, y is cts on $[0,1]$ and hence $y \in C[0,1]$. Also —
$$\|y\|_\infty = \sup\{|y(t)| : t \in [0,1]\}$$
$$= \sup\{|1-t| : t \in [0,1]\}$$
$$= 1$$
And $|T(y)| = |y(0)| = 1$
$$\therefore 1 = |T(y)| \le \|T\| \|y\| \le \|T\| \cdot 1 \le 1$$
$\therefore \|T\| = 1$

Claim: T is NOT an isometry.

Consider the identity f^n $p(t) = t$, $\forall t \in [0,1]$. Then clearly, $p \in C[0,1]$
$$T(x) = x(0) \quad \forall x \in C[0,1]$$
Also $\|p\|_\infty = \sup\{|p(t)| : t \in [0,1]\}$
$$= \sup\{|t| : t \in [0,1]\}$$
$$= 1$$
While $|T(p)| = |p(0)| = |0| = 0$
$$\therefore |T(p)| \ne \|p\|$$
i.e. Converse is NOT True.

Def^n: 3.3.4/118 Let X & Y be n.l.s & $T: X \to Y$ be linear one-one & onto and also an isometry. Then T is called an isometric isomorphism from X to Y. And we say that X is isometrically

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isomorphic to Y

Def^n: 3.3.5|118 Let X & Y be n.l.s & $T: X \to Y$ be linear, 1-1 & onto, And let T & T^{-1} be both continuous. Then T is called an topological isomorphism from X to Y. And we say that X is topologically isomorphic to Y.

Theorem 3.3.6|118 Let X & Y be n.l.s over a field $\mathbb{K}$ & let $T: X \xrightarrow{onto} Y$ be a linear transformation. Then T is topological isomorphism iff $\exists$ constants k_1 & k_2 s.t

$$k_1 \|x\|_X \le \|T(x)\|_Y \le k_2 \|x\|_X$$

Proof: Let $T: X \xrightarrow{onto} Y$ be a topological Isomorphism.

$\Rightarrow T^{-1}$ exists and is continuous linear. Trans.
(and hence T^{-1} bdd bcz T is also cts).

$\therefore \exists$ Constant k_2 $(=\|T\|)$: $\|T(x)\|_Y \le k_2 \|x\|_X \;\; \forall x \in X$.
($\because T$ is bdd)

(By thm 3.1.12 p-77) $\exists$ some k_1: $k_1\|x\| \le \|T(x)\|_Y \;\; \forall x \in X$
(proved earlier)

$\therefore k_1 \|x\|_X \le \|T(x)\|_Y \le k_2 \|x\|_X \;\; \forall x \in X$.

$\Leftarrow$ Conversely, let $\underbrace{k_1\|x\|_X \le \|T(x)\|_Y}_{(b)} \underbrace{\le k_2\|x\|_X}_{(a)}$

from (a): we have
T is continuous
and from (b), T^{-1} exists & is continuous $\Rightarrow$ bdd due to (By 3.1.12)
so from this both conditions —
$\Rightarrow T$ is top. iso.

Def^n: 3.3.7|119 Two norms $\|\cdot\|$ & $\|\cdot\|'$ on the same

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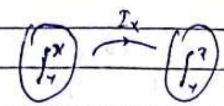
linear space are s.t.b equivalent norms iff the identity mapping

$$I_X : (X, \|\cdot\|) \to (X, \|\cdot\|') \text{ is a top. iso.}$$



Theorem 3.3.8|119 Two norms $\|\cdot\|$ & $\|\cdot\|'$ on the same linear space X are equivalent norms iff $\exists$ positive constant k_1 and k_2 s.t

(pf-74) $$k_1\|x\| \le \|x\|' \le k_2\|x\| \quad \forall x \in X$$

Pf: self from prev theorem, we have
$$k_1\|x\| \le \|T(x)\| \le k_2\|x\|$$
Now replace T by I, we have
$$\boxed{k_1\|x\| \le \|I(x)\|' \le k_2\|x\|}$$

* * *

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Eg. 3.3.9|120 The norms $\|\cdot\|_1, \|\cdot\|_2, \|\cdot\|_\infty$ are equivalent Norms on $\mathbb{K}^n$.
$\left(\|x\|_\infty = \max_{i=1,2,\dots n} |x_i|\right)$

Proof: Let $x = (x_1, x_2, \dots x_n) \in \mathbb{K}^n$

then $\|x\|_\infty \le \sum_{i=1}^{n} |x_i|$ — (a)
$\le n \cdot \left(\max_{i=1,2,\dots n} |x_i|\right)$
$= n \cdot \|x\|_\infty$

$\therefore \|x\|_\infty \le \|x\|_1 \le n\|x\|_\infty$

$\therefore \|\cdot\|_\infty$ and $\|\cdot\|_1$ are equivalent

Next, we know that
$$|x_1|^2 + |x_2|^2 \le (|x_1| + |x_2|)^2$$

Evening

$\Rightarrow (|x_1|^2+|x_2|^2)^{1/2} \le |x_1|+|x_2|$

Continuing —

$$\left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2} \le \sum_{i=1}^{n} |x_i|$$

Clearly

$\|x\|_2 \le \sum_{i=1}^{n} |x_i|$ (see below NB)

$\le \sqrt{n}\underbrace{\left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}}_{=\|x'\|_2}$ (what ?? if there n ≠ last)

?? $\therefore \|x\|_2 \le \|x\|_1 \le \sqrt{n}\,\|x\|_2$

$\Rightarrow \|x\|_2$ is equivalent to $\|\cdot\|_1$

$\therefore$ All three norms are equivalent.

NB

$\sum |\alpha_i \beta_i| \le \left(\sum|\alpha_i|^2\right)^{1/2}\left(\sum|\beta_i|^2\right)^{1/2}$ (Schwarz ineq)

put $\alpha_1=\alpha_2=\dots=\alpha_n=1$

$\Rightarrow \sum|\beta_i| \le (1^2+1^2+\dots+1^2)^{1/2}\left(\sum|\beta_i|^2\right)^{1/2}$

$\le \sqrt{n}\left(\sum|\beta_i|^2\right)^{1/2}$

Theorem: 3.3.11 / 121 Let $\|\cdot\|$ and $\|\cdot\|'$ be equivalent norm on a same linear space X. Then —

(a) $(X,\|\cdot\|)$ is a B.S iff $(X,\|\cdot\|')$ is B.S

(b) A set is bdd in $(X,\|\cdot\|)$ iff it is bdd in $(X,\|\cdot\|')$

Hint: $(X,\|\cdot\|) \xrightarrow{\text{complete}} \text{B.S}$ $\quad (X,\|\cdot\|') \to$ B.S

Proof: (a) Let $(X,\|\cdot\|)$ be a B.S.

Let $\{x_n\}$ be a ~~convergent series~~ Cauchy sequence in $(X,\|\cdot\|')$

$\Rightarrow$ given $\varepsilon>0$, $\exists$ some integer N:

$\|x_n-x_m\|' < \varepsilon \quad \forall\, n,m \ge N$

But $\|\cdot\|$ & $\|\cdot\|'$ are equivalent.

$\therefore \exists$ a constant k_1:

$k_1\|x_n-x_m\| \le \|x_n-x_m\|' < \varepsilon \quad \forall\, n,m \ge N$

$k_1\|x\| \le \|x\|' \le k_2\|x\|$

$\therefore \{x_n\}$ is a ~~convergent series~~ Cauchy sequence in $(X,\|\cdot\|)$ which is complete by hypothesis

$\therefore x_n \to x$ in $(X,\|\cdot\|)$

$\Rightarrow \|x_n - x\| < \varepsilon \quad \forall\, n \ge n_1$

Now again $\|\cdot\|$ & $\|\cdot\|'$ are equivalent

$\therefore \exists\, k_2$:

$\|x_n - x\|' \le k_2\|x_n-x\|$

$\le k_2\varepsilon \quad \forall\, n \ge n_1$

$\Rightarrow \{x_n\}$ converges to x in $(X,\|\cdot\|')$

$\therefore (X,\|\cdot\|')$ is a B.S.

Conversely $\Rightarrow$ The converse follows similarly by interchanging the roles of $\|\cdot\|$ and $\|\cdot\|'$

(b) Without loss of generality, let $A \subseteq X$, be bounded in $(X,\|\cdot\|)$.

$\Rightarrow \exists$ some $M>0$: $\|x\| \le M \quad \forall\, x\in A$

$\therefore \|x\|' \le k_2\|x\| \le k_2 M$ (By thm 3.3.8)

$\therefore A$ in $(X,\|\cdot\|')$ is bounded (the converse follows similarly by applying other inequality of thm 3.3.8)

* * *

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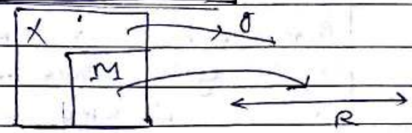
Application of Hahn-Banach Theorem (4.3.?)

Theorem: Hahn-Banach Theorem

4.3.3 | 183 Let X be a n.l.s over $\mathbb{K}$ & M be a subspace of X. Then for every b.dd linear functional f of M, $\exists$ a b.dd linear functional g on X s.t

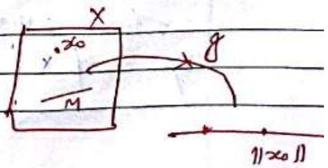
$$\boxed{g|_M = f \quad \& \quad \|g\| = \|f\|}$$



Application Theorem: 4.3.4 | 186 Let X be a N.L.S over a field $\mathbb{K}$. Let $0 \ne x_0 \in X$. Then $\exists$ a b.dd linear functional g on X such that —

(i) $g(x_0) = \|x_0\|$

(ii) $\|g\| = 1$



Proof: – Consider $M = \{\alpha x_0 \mid \alpha \in \mathbb{K}\}$

Then clearly M is a subspace of X.

Define:

$$f : M \to K \quad \text{by}$$

$$f(\alpha x_0) = \alpha \|x_0\|$$

Then clearly f is a linear functional on M.

Also, $f(x_0) = f(1.x_0) = 1\|x_0\| = \|x_0\|$

further $\forall \underbrace{\alpha x_0 \in M}_{\text{say } y}$

$$\text{then } |f(y)| = |f(\alpha x_0)| = |\alpha \|x_0\||$$
$$= |\alpha| \|x_0\|$$
$$= \|\alpha x_0\|$$
$$= \|y\| \quad \text{—— (1)}$$

$\therefore f$ is b.dd linear functional.

Also, $\|f\| = \sup_{\|y\|=1} |f(y)|$
$= \sup_{\|y\|=1} \|y\| = 1$

Now, by Hahn Banach Theorem $\exists$ a bdd linear functional g on X s.t

$g|_M = f$ and $\|g\| = \|f\| = 1$

$\therefore g(x_0) = f(x_0) \quad \because x_0 \in M$
$= \|x_0\|$

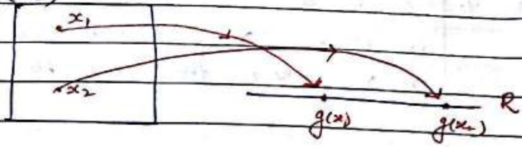
Defn: Let X be a N.L.S over a field $\mathbb{K}$. Then the Banach Space $\mathcal{B}(X,K)$ of all bdd linear functional's on X is called Dual space (or conjugate space or adjoint space) of X and is denoted by X^*.

Note: $\mathcal{B}(X,\mathbb{K})$ is always complete irrespective wether X is complete or NOT.

Thus when we say X^* is Conjugate space of X, we mean

$$X^* = \mathcal{B}(X, K)$$

Corollary 4.3.5 | 186 ⇒ Let X be a N.L.S over K. and let $x_1 \neq x_2$. Then $\exists$ a $g \in X^*$ s.t $g(x_1) \neq g(x_2)$



Proof: Let $y = x_1 - x_2 \neq 0$ (as $x_1 \neq x_2$)

$\therefore$ By prev. thm $\exists$ $g \in X^*$ s.t

$g(x_1) - g(x_2) = g(x_1 - x_2) = g(y) = g(y) = \|y\| = \|x_1 - x_2\| \neq 0$

$\therefore g(x_1) \neq g(x_2)$

Corollary 4.3.6 | 186: Let $X \neq \{0\}$ be a n.l.s then $\exists$ a non-trivial bdd linear functional on X

Proof: $X \neq \{0\}$

$\Rightarrow \exists$ $x_0 \neq 0 \in X$.

$\therefore$ By Prev. theorem, $\exists$ a b.dd linear functional g.

Note: $\therefore g$ is Non-trivial

$\because$ by prev. theorem, $\|g\| = 1$.

Corollary 4.3.7 | 186: Let X be a n.l.s. If $f(x) = 0 \ \forall f \in X^*$ then $x = 0$.

Proof: Suppose $x \neq 0$

$\therefore$ By prev. corollary, $\exists$ $g \in X^*$ s.t

$g(x) \neq g(0) = 0$

$\Rightarrow g(x) \neq 0$

Here we get contradiction (bcz $g \in X^*$)

$\therefore x = 0$

Corollary 4.3.7 may be restate as —

Cor. 4.3.7 : If all bdd linear functional on a n.l.s X vanish on a given vector in X, then the vector must be zero.

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Theorem 4.3.8) 187: Let X be a n.l.s over a field K. Let $x \in X$, then —

$$\|x\| = \sup\left\{ \frac{|f(x)|}{\|f\|} : f \in X^{*}, f \neq 0 \right\}$$

$B(X, K)$

Proof: If $x = 0$, the result is trivially true. (Both side zero)

So let $x \neq 0$.

Then $\exists\ g \in X^{*} : g(x) = \|x\|$ and $\|g\| = 1$ (First appⁿ of H.B. Theorem)

$$\Rightarrow \sup\left\{ \frac{|f(x)|}{\|f\|} : f \in X^{*}, f \neq 0 \right\} \geq \frac{|g(x)|}{\|g\|} = \|x\| \quad \text{——— (1)}$$

next —

$|f(x)| \leq \|f\|\,\|x\| \quad \forall f \in X^{*}$

$\Rightarrow \frac{|f(x)|}{\|f\|} \leq \|x\| \quad \forall f \in X^{*}$

$$\Rightarrow \sup\left\{ \frac{|f(x)|}{\|f\|} : f \in X^{*}, f \neq 0 \right\} \leq \|x\| \quad \text{— (2)}$$

from (1) & (2) inequality

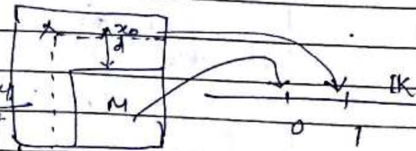
$$\Rightarrow \sup\left\{ \frac{|f(x)|}{\|f\|} : f \in X^{*}, f \neq 0 \right\} = \|x\|$$

Theorem: 4.3.10) 187 Let X be a n.l.s over a field IK. Let M be a subspace of X. Let $x_0 \in X$ be s.t $d(x_0, M) = d > 0$. Then $\exists\ g \in X^{*}$ such that —

(i) $g(x_0) = 1$

(ii) $g(M) = 0$

(iii) $\|g\| = \frac{1}{d}$ $\|f\|\|x\| \leq \frac{\|x\|}{d}$



Proof: Let $M_0 = M + [x_0]$ be the space spanned by M and x_0.

$\because m_0 \in M_0 \Rightarrow m_0 = m + \alpha x_0$, where α = scalar,

$m \in M$, $\alpha = 0$

$\Rightarrow m_0 = m + 0.x_0$

$\Rightarrow m \in M_0 \qquad \Rightarrow M \subseteq M_0$

$\therefore$ If $x \in M_0$, then we can write

$x = m + \alpha x_0$, where $m \in M$, $\alpha \in IK$

Now consider the functional $f : M_0 \to IK$ defined by

$$f(m + \alpha x_0) = \alpha$$

then clearly $f(x_0) = f(0 + 1.x_0) = 1$

$f(m) = f(m + 0.x_0) = 0 \Leftrightarrow f(m) = 0$, $m \in M$.

Also, clearly f is linear functional on M_0

Exercise: Required to Prove

$$f(pa + qb) = pf(a) + qf(b)$$

where $a = m + \alpha x_0$

$b = m + \beta x_0$

Also if $\alpha \neq 0$, then $\|x\| = \|m + \alpha x_0\|$

$\in M_0$

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$\|x - (-\frac{m}{\alpha})\| \geq d$

$\|x\| = \|m + \alpha x_0\| = |\alpha| \, \|\frac{m}{\alpha} + x_0\|$ $\quad (d = d(x_0, M))$

$\geq |\alpha| d$

$= |f(x)| d$

Also if $\alpha = 0$, then

$\|x\| = \|m\| \geq 0 = d|f(x)|$

$\Rightarrow f$ is bounded on M_0 and

$\sup_{\|x\|=1} |f(x)| \leq \sup_{\|x\|=1} \frac{\|x\|}{d}$

$\Rightarrow \|f\| \leq \frac{1}{d}$ —(iii) Proved

Claim: $\|f\| \geq \frac{1}{d}$

consider a seq. $\{m_k\} \subseteq M$: $\|x_0 - m_k\| \to d$ as $k \to \infty$

Then $f(x_0 - m_k) \leq |f(x_0 - m_k)| \leq \|f\| \|x_0 - m_k\|$

$f(-m_k + 1 \cdot x_0) = 1$ $\quad \longrightarrow \|f\| d$

& $\because f(x_0 + \alpha x_0) = \alpha$

$\therefore 1/d \leq \|f\|$

Thus we have established that $\exists$

$f \in M_0^*$: $f(M) = 0$

$f(x_0) = 1$

and $\|f\| = 1/d$

$\therefore$ By H.B thm. $\exists g \in X^*$:

$g|_{M_0} = f$ and $\|g\| = \|f\| = 1/d$

But $g|_{M_0} = f \Rightarrow \begin{cases} g(M) = f(M) = 0 \\ g(x_0) = f(x_0) = 1 \end{cases}$

* * *

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4.3.18|190 **Corollary:** let X be a n.l.s over K and let M be a closed subspace of X and $x_0 \in X - M$. Then $\exists g \in X^*$:

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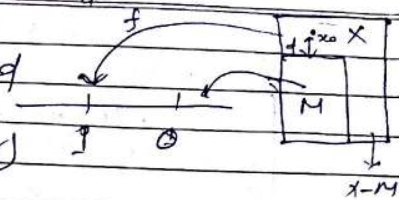
$g(x_0) \neq 0$, $\quad \|f\| = 1/d$

$g(M) = 0$

Hint: Since M is closed & $x_0 \notin M$

$\therefore d(x_0, M) > 0$ (= d say)



Def^n: A topological space X is s.t.b separable iff $\exists$ a countable dense subset of X (i.e iff $\exists$ a countable subset of E of X : $\overline{E} = X$)

Eg: $\mathbb{R}$ is separable $\because \overline{\mathbb{Q}} = \mathbb{R}$

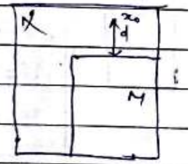
$\mathbb{Q}$ is countable.

4.3.17|193 **Theorem:** let X be a n.l.s and X^* be its dual space. Then X^* is separable $\Rightarrow X$ is separable.

proof: X^* is separable.

$\therefore \exists$ a countable dense subset of X^*.

i.e $\exists S = \{f_n \in X^*, n \in \mathbb{N}\}$ s.t $\overline{S} = X^*$



Now $\|f_n\| \overset{def^n}{=} \sup_{\|x\|=1} \{|f_n(x)|\}$

$\|f_1\| = \sup_{\|x\|=1} \{f_1(x)\}$

$\leq |f_1(x)| + \epsilon$

$\geq |f_1(x)| - \epsilon$

$\leq |f_1(x)| + |f_1(x)|$

$\leq 2|f_1(x)|$

We choose an x_n : $\|x_n\| = 1$ and $\|f_n\| \leq 2 \cdot |f_n(x_n)|$

Let $M = \overline{[x_n]}$

Claim $M = X$

$(0,1)$ is Not countable

suppose it is countable

Let $x = .3279$

$y = .10123$

$z = .3925$

A. uncountable
B. is countable
A - B is uncountable,
Suppose non-countable $\Rightarrow B \cup (A-B)$ is countable
A is countable

Let $x = 1/2 = .5 = .4999$
$\therefore 10x = 4.9999$
$100x = 49.9999$

$\times$

$* \; * \; *$

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Claim: $M = [x]$
$\Rightarrow [x_n] = X \Rightarrow X$ has countable dense subset of X
$\therefore X$ is separable

Clearly $M \subseteq X$
Claim ~~$X \subseteq M$~~ $X \subseteq M$
~~Suppose $X \not\subseteq M$.~~

Suppose $M \neq X$
$\Rightarrow \exists x_0 \in X - M$
$\therefore$ By earlier corollary, $\exists 0 \neq g \in X^*$
s.t. $g(x_0) = 1 \neq 0$ & $g(M) = 0$
$\Rightarrow g(x_n) = 0 \quad \forall n = 1, 2, 3 \ldots$ $(\because g(x_n) = 0)$

$\therefore \frac{1}{2}\|f_n\| \leq |f_n(x_n)| \leq |(f_n - g)(x_n)|$
$\leq \|f_n - g\| \, \|x_n\|$
$= \|f_n - g\|$

$\therefore \|g\| = \|g - f_n + f_n\| \leq \|g - f_n\| + \|f_n\|$
$\leq \|g - f_n\| + 2\|g - f_n\|$
$= 3\|g - f_n\|$
$\to 0$ as $n \to \infty$

$\therefore \Rightarrow g = 0$
which is contradiction
$\therefore M = X$ and hence X is separable.

(Next $X^* \Rightarrow X$ is separable but converse NOT true)

Note:- The converse of the above thm is NOT true.

Eg. l^1 is separable but $(l^1)^*$ is NOT.
4.3.19 | 192 $\because (l^1)^*$ is isometrically isomorphic to l^∞ and l^∞ is NOT separable.
$\therefore (l^1)^*$ is NOT separable.

$X^* = B(X, \mathbb{K})$ $B(X^*, \mathbb{K}) = X^{**}$
$f : X \to \mathbb{K}$ $\phi : X^* = \mathbb{K}$
$g : X \to \mathbb{K}$ $\psi : X^* = \mathbb{K}$
$h : X \to \mathbb{K}$ $\eta : X^* = \mathbb{K}$
$f(x) = 5$ $\phi(f) =$ real no.
$\Rightarrow \phi(f) = f(x) = 5$

[4.4]: Embedding & Reflexivity of N.S. 195

Recall: Given a n.l.s $X \neq \{0\}$, then the dual space X^* is a n.l.s with norm $\|\cdot\| : X^* \to \mathbb{R}$ defined by

$$\|f\| = \sup\{|f(x)| : x \in X, \|x\| = 1\}$$

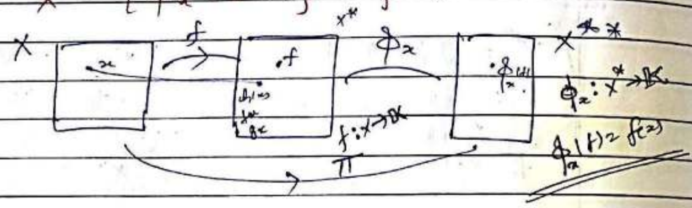
Also recall, X^* is complete irrespective whether X is complete or NOT.
Since X^* is n.l.s we can speak of $(X^*)^*$, where the norm $\|\cdot\|$ in this case is defined by

$$\|\phi\| = \sup\{|\phi(f)| : f \in X^* \;\&\; \|f\| = 1\}$$

We denote $(X^*)^*$ by X^{**} and call it the second conjugate space or bidual space of X

Note: X^{**} is Banach space.

Theorem 4.4.1 | 195: Let $X \neq \{0\}$ be a n.l.s over a field $\mathbb{K}$. Given $x \in X$, define $\phi_x(f) = f(x)$ $\forall f \in X^*$. Then ϕ_x is a bdd linear functional on X^*. (i.e $\phi_x \in X^{**}$). Further the mapping $x \to \phi_x$ is an isometric isomorphism of X onto the subspace $\hat{X} = \{\phi_x : x \in X\}$ of X^{**}.



Evening — $*$ $*$ $*$ —

ϕ_x is linear.

$\phi_x(\alpha f + \beta g) = (\alpha f + \beta g)(x)$
$= \alpha f(x) + \beta g(x)$
$= \alpha \phi_x(f) + \beta \phi_x(g)$

$\therefore \phi_x$ is linear.

ϕ_x is bounded

$|\phi_x(f)| = |f(x)| \leq \|f\| \|x\|$ $\forall f \in X^*$

$\therefore \phi_x$ is bdd linear functional

$\therefore \phi_x \in X^{**}$

Also, ϕ_x is unique.

$\because$ suppose for some $x \in X$,

$\exists$ functionals ϕ_x & $\psi_x \in X^{**}$

why unique ??

$\phi_x(f) = f(x)$ and $\psi_x(f) = f(x)$ $\forall f \in X^*$

$\Rightarrow \phi_x(f) = \psi_x(f)$ $\forall f \in X^*$

$\Rightarrow \phi_x = \psi_x$

Thus given any $x \in X$, $\exists$ unique f^n $\phi_x \in X^{**}$

$\Rightarrow \exists$ a mapping $\pi : X \to X^{**}$ defined by

$\pi(x) = \phi_x$

claim: π is an isometric isomorphism from X onto a subspace of X^{**}

π is linear

$\pi(\alpha x + \beta y)(f) = \phi_{(\alpha x + \beta y)}(f)$ $\forall f \in X^*$

$= f(\alpha x + \beta y)$
$= \alpha f(x) + \beta f(y)$
$= \alpha \phi_x(f) + \beta \phi_y(f)$
$= (\alpha \phi_x + \beta \phi_y)(f)$
$= (\alpha \pi(x) + \beta \pi(y))(f)$

$\therefore \pi$ is linear as $\forall f \in X^*$

$\Rightarrow \pi(\alpha x + \beta y) = \alpha \pi(x) + \beta \pi(y)$

π preserves norms: (i.e π is isometry)

For each $x \in X$

$\|\pi(x)\| = \|\phi_x\| = \sup\left\{\frac{|\phi_x(f)|}{\|f\|} : f \in X^*, f \neq 0\right\}$

$= \sup\left\{\frac{|f(x)|}{\|f\|} : f \in X^*, x \neq 0\right\}$

$= \|x\|$ (proved earlier)

$\Rightarrow \pi$ is an isometry.

π is 1-1

let $x \neq y$ where $x, y \in X$ then

$\|x - y\| \neq 0$

$\Rightarrow \|\pi(x - y)\| \neq 0$ $\because \pi$ is isometry (proved Above)

$\Rightarrow \|\pi(x) - \pi(y)\| \neq 0$

$\Rightarrow \pi(x) \neq \pi(y)$

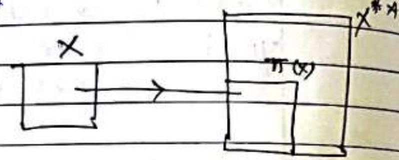
$\therefore \pi$ is one-one.

$\therefore \pi$ is an isometric isomorphism from X onto $\pi(X) \subseteq X^{**}$

Thus X can be embedded into a subspace $\pi(X)$ of X^{**}

or in "loose" language, we say that —

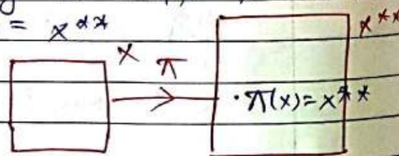
$X \subseteq X^{**}$



Def^n: (4.4.2 | 197) Let X be a n.l.s over a field $\mathbb{K}$. Then the isometric isomorphism $\pi : X \to X^{**}$ defined by $\pi(x) = \phi_x \quad \forall x \in X$ is called natural embedding of X into its second conjugate space X^{**}. And the function $\phi_x \in X^{**}$ is called induced functional.

Def^n: (4.4.3 | 197) A n.l.s X is s.t.b reflexive if the natural embedding π maps X onto X^{**}

i.e if $\pi(X) = X^{**}$



Note: We know that X^{**} is complete.

$\therefore$ If X is reflexive, then $\pi(X) = X^{**}$ and π is isometric isomorphism.

$\therefore \pi(X)$ is complete.

But $\pi(X) = X \quad (\because$ Reflexive$)$

$\Rightarrow X$ is complete.

p. 212 [4.6] - Weak convergence

Def^n: (4.6.1 | 212) A sequence $\{x_n\}$ in a n.l.s X is s.t.b weakly convergent if there exist an elt $x \in X$:

$$\lim_{n\to\infty} f(x_n) = f(x) \quad \forall f \in X^*$$

The vector x is called weak limit of the seq. $\{x_n\}$. We say that x_n converges to x weakly

We write $x_n \xrightarrow{w} x$

(i.e $f(x_n) \to f(x)$
$g(x_n) \to g(x)$)

Note: Since $f \in X^*$, $f(x_n)$ is scalar & $f(x)$ is also a scalar and hence we are speaking of scalar convergence.

Theorem: (4.6.2 | 213) Let $\{x_n\}$ be a weakly convergent seq. in a n.l.s X. Then —

(i). The weak limit of $\{x_n\}$ is unique

(ii). $\{\|x_n\|\}$ is a b.dd seq. in $\mathbb{R}$.

(iii). Every subseq. of $\{x_n\}$ converge weakly to the limit of $\{x_n\}$.

Pf: (i) Suppose $x_n \xrightarrow{w} x$ and y.

$\Rightarrow \lim_{n\to\infty} f(x_n) = \begin{cases} f(x) & \forall f \in X^* \\ f(y) & \forall f \in X^* \end{cases}$

$\Rightarrow f(x) - f(y) = 0 \quad \forall f \in X^*$

$\Rightarrow f(x-y) = 0 \quad \forall f \in X^*$

$\Rightarrow x - y = 0 \quad (\because f$ is linear$)$

$\Rightarrow x = y$.

(ii) Since $x_n \xrightarrow{w} x$

we have $\lim_{n\to\infty} f(x_n) = f(x)$ $\forall f \in X^*$

Thus $\forall f \in X^*$, $\{f(x_n)\}_{n=1}^{\infty}$ is a cgt. seq. in $\mathbb{R}$ and hence it is bdd.

$\Rightarrow |f(x_n)| \le K_f$ $\forall n = 1, 2, \ldots$
for some constant K.

Now let $x \to \phi_{x}$ be a natural embedding of X into X^{**}.

Thus for each n
$\|x_n\| = \|\phi_n\|$

Also, $|\phi_{x_n}(f)| = |f(x_n)| \le K_f$ $\forall n \in \mathbb{N}$

Thus $\{|\phi_{x_n}(f)|\}$ is a bdd seq. for each $f \in X^*$
But X^* is a Banach space.
$\therefore$ By Principle of uniform convergence (To be proved later)
it follows that $\{\|\phi_n\|\}$ is bdd
$\Rightarrow \|x_n\|$ is bdd

(iii) Every subseq. of $\{x_n\}$ converges weakly to the limit of $\{x_n\}$.

Next, Since $x_n \xrightarrow{w} x$
$\Rightarrow f(x_n) \to f(x)$ $\forall f \in X^*$
$\therefore \{f(x_n)\}$ converges to $f(x)$ in $\mathbb{R}$.
But In $\mathbb{R}$, every subsequence converges to the same limit.
i.e $f(x_{n_k}) \to f(x)$
$\Rightarrow x_{n_k} \to x$ weakly.

Def^n: 4.6.4 | 214 A seq. $\{x_n\}$ in a n.l.s X is s.t. converge strongly (or convergent in Norm) if $\exists$ a vector $x \in X$:

$$\lim_{n\to\infty} \|x_n - x\| = 0$$

and x is called strong limit of the convergent sequence $\{x_n\}$.

Notation $\lim_{n\to\infty} x_n = x$

or $x_n \to x$ as $n \to \infty$

Theorem: 4.6.5 | 214 let $\{x_n\}$ be a seq. in n.l.s X. Then $x_n \to x$ in $X \Rightarrow x_n \xrightarrow{w} x$ in X.

Pf:- let $x_n \to x$
then $\forall f \in X^*$
$|f(x_n) - f(x)| = |f(x_n - x)|$
$\le \|f\| \underbrace{\|x_n - x\|}_{\to 0}$
$\longrightarrow 0$

$\therefore \lim_{n\to\infty} f(x_n) = f(x)$ $\forall f \in X^*$
$\therefore x_n \xrightarrow{w} x$

Note: The converges of the above theorem is NOT true.

Eg. 4.6.6 | 215 let $X = \ell^p$, $1 < p < \infty$
Then clearly$^{H.W}$ $(\ell^p)^* = \ell^q$
Now consider the basis
$e_1 = \{1, 0, 0, \ldots\}$
$e_2 = \{0, 1, 0, \ldots\}$
$\vdots$
$e_n = \{0, 0, \ldots 1 \ldots 0, 0\}$
$\vdots$
then $e_i \xrightarrow{w} 0$
and $e_i \not\to 0$

Theorem: In a finite dim. n.l.s X, weak
4.6.7 | 2/6 convergence implies strong convergence.

Pf:- Let ~~{e,~~ dim $X = m$

Let $\{e_1, e_2, \ldots e_m\}$ be a basis for X

Let $x_n \to x$ (T.P $x_n \to x$ in norm)

Since, $x_n \in X$,

$$x_n = \lambda_1^{(n)} e_1 + \lambda_2^{(n)} e_2 + \cdots + \lambda_m^{(n)} e_m$$

$$x = \lambda_1 e_1 + \lambda_2 e_2 + \cdots \lambda_m e_m$$

Consider the functionals $\{f_1, f_2, \ldots f_m\}$ in X^* defined by

$$f_i(e_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Then

$$f_i(x_n) = f_i\left(\sum_{j=1}^{m} \lambda_j^{(n)} e_j\right)$$

$$= \sum_{j=1}^{m} \lambda_j^{(n)} f_i(e_j)$$

$$= \lambda_i^{(n)}$$

Similarly $f_i(x) = \lambda_i \qquad 1 \le i \le m$

Now by def^n of weak convergence

$g(x_n) \to g(x) \quad \forall\, g \in X^*$ in particular for f_i

$\therefore f_i(x_n) \to f_i(x)$

$\Rightarrow \lambda_i^{(n)} \to \lambda_i \qquad 1 \le i \le m$

$$\therefore \| x_n - x \| = \left\| \sum_{i=1}^{m} (\lambda_i^{(n)} - \lambda_i) e_i \right\|$$

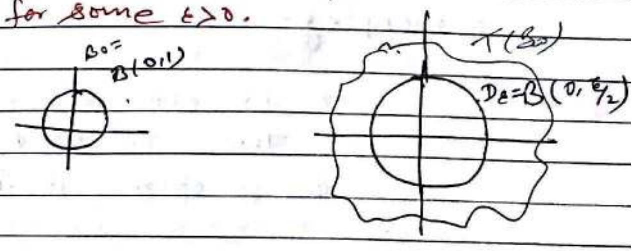
$$\le \sum_{i=1}^{m} |(\lambda_i^{(n)} - \lambda_i)| \, \|e_i\|$$

[3.5] Open Mapping theorem And its Consequences (131)

$\therefore x_n \to x$ in norm.

Lemma: Let X & Y be Banach spaces over a field
3.5.4 (134) $\mathbb{K}$. Let $T : X \xrightarrow{\text{onto}} Y$ be a bdd linear operator. Let $B_0 = B(0,1)$ and $D_\varepsilon = B(0, \varepsilon/2)$ be the balls in X & Y respectively. Then $T(B_0) \supseteq D_\varepsilon$ for some $\varepsilon > 0$.

Sol^n



self:

Pf: (H.W)

Theorem (Open Mapping theorem)
3.5.2 Let X & Y be Banach spaces over a field $\mathbb{K}$. Let $T : X \xrightarrow{onto} Y$ be a b.d.d linear operator. Then T is an open mapping.

Pf: — Let $G \subseteq X$ be open set. ~~If $G \neq \phi$~~

→ If $G = \phi$, Then $T(G) = \phi$
Since ϕ is open, it follows $T(G)$ is open.

→ If $G \neq \phi$
(T.P $T(G)$ is open)
Let $y \in T(G) \Rightarrow y = T(x_0)$ for some $x_0 \in G$
But G is open
$\therefore \exists$ a ball $B(x_0, \alpha)$: $x_0 \in B(x_0,\alpha) \subseteq G$
$\Rightarrow x_0 + \alpha B(0,1) \subseteq G$
$\Rightarrow \alpha B(0,1) \subseteq G - x_0$
$\Rightarrow B(0,1) \subseteq \frac{1}{\alpha}(G - x_0)$
$\Rightarrow T(B(0,1)) \subseteq T\left[\frac{1}{\alpha}(G-x_0)\right]$ ($\because T$ is linear)

In general it is not possible that if $A \subseteq B \Rightarrow f(A) \subseteq f(A)$
for eg: let $f(x) = \frac{1}{x}$
then $f(A) \subset$

~~then f(B)~~ Here $A \subseteq B$
But $f(B) \subseteq f(A)$

$= \frac{1}{\alpha}\{T(G - x_0)\}$

$= \frac{1}{\alpha}[T(G) - T(x_0)]$ $\because T$ is linear

Now by prev. lemma, we obtain
$D_\varepsilon \subseteq T(B(0,1)) \subseteq \frac{1}{\alpha}(T(G) - T(x_0))$ $D_\varepsilon = B(0, \varepsilon/2)$
$\therefore D_\varepsilon + \frac{1}{\alpha}T(x_0) \subseteq \frac{1}{\alpha}T(G)$
$\Rightarrow T(G) \supseteq \alpha\left(D_\varepsilon + \frac{1}{\alpha}T(x_0)\right)$
$= \alpha D_\varepsilon + T(x_0)$
$= B(Tx_0, \alpha\varepsilon)$
$\Rightarrow T(x_0)$ is an interior pt. of $T(G)$.
$y =$
But $y \in T(G)$ is arbitrary
$\therefore T(G)$ is open

(NB: Here B.S is not used, But is required in prev. lemma, thats reason we use in Def^n of Banach space)

→ * * * —

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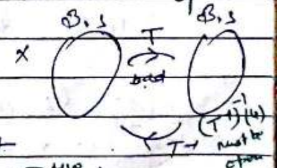
Banach Inverse Mapping Theorem

Corollary: 3.5.6
Let X & Y be B.S & $T : X \to Y$ be a bdd linear operator. If T is bijective then T^{-1} is also a b.dd linear operator.
Note that



Pf: Since T is 1-1 & onto,
$\Rightarrow T^{-1}$ will exist.
It is enough to show that T^{-1} is continuous (linear operator)
i.e $T^{-1}(\alpha x + \beta y) = \alpha T^{-1}(x) + \beta T^{-1}(y)$ $Tx = y$, $x = T^{-1}y$
as $\because T$ is linear
$\Rightarrow T(a_1x_1 + a_2x_2) = a_1T(x_1) + a_2T(x_2)$

$\to a_1x_1 + a_2x_2 = T[a_1T^{-1}(x_1) + a_2T^{-1}(x_2)]$ 1pr | FR1
$= T^{-1}[a_1y_1 + a_2y_2]$

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Now operating T^{-1} on both side
$T^{-1}(T[a_1x_1+a_2x_2]) = T^{-1}[a_1T[x_1] + a_2T[x_2]]$

let G be any open set in X.
Then $T(G)$ is open in Y (by open mapping theorem)
$\Rightarrow (T^{-1})^{-1}(G)$ is open in Y.
$\Rightarrow T^{-1}$ is continuous
$\therefore T^{-1}$ is b.dd linear operator.

Corollary: 3.5.7 / 137
let $(X, \|\cdot\|_1) = X_1$ & $(X, \|\cdot\|_2) = X_2$ be two B.S.
Suppose $\exists$ a constant $k>0$:
$\|x\|_1 \le k\|x\|_2 \quad \forall x \in X$. —— (a)
then $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent norms.

pf: let I be the identity operator
i.e $I : X_2 \to X_1$ defined by $I(x) = x \quad \forall x \in X$
Then clearly I is b.dd linear operator. (H.W)
which is bijective.
$\therefore I^{-1} : X_1 \to X_2$ is also a b.dd lin. operator
so $\exists k_1$ s.t. $\to \|I^{-1}(x)\|_2 \le k_1\|x\|_1, \quad \forall x \in X_1 (= X)$
$\therefore \|x\|_2 \le k_1\|x\|_1$ —— (b)
from (a) & (b) —
$$\frac{1}{k}\|x\|_1 \le \|x\|_2 \le k_1\|x\|_1$$
$\therefore \|\cdot\|_1$ & $\|\cdot\|_2$ are equivalent norms.

Def^n: let X and Y be n.l.s over a field $\mathbb{K}$.
Let $T : X \to Y$ be a linear operator. Then the set
$G(T) = \{(x, Tx) : x \in X\}$ is called Graph of T.
$\subseteq X \times Y = X \times Y$

Note ① clearly $G(T)$ is subspace of $X \times Y$.
Note: ② Let X & Y be a n.l.s. Then clearly $X \times Y$ is a linear space. and is indeed a n.l.s with the norm defined by
(i) $\|(x,y)\| = \|x\|_x + \|y\|_y$

1pr | FR1
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(ii) $\|(x,y)\| = \max\{\|x\|_x, \|y\|_y\}$
(iii) $\|(x,y)\| = \|x\|_x^p + \|y\|_y^p$. $1 \le p < \infty$
HW(1)

Def^n: ~~let~~ Note that above three are different norms on the same linear space $X \times Y$. However it can be prove ~~that~~ those are equivalent norms.

Def^n: Let X and Y be n.l.s and let the graph of T (i.e $G(T)$) be closed in the n.l.s $X \times Y$. Then we say that T is closed lin. operator

Theorem: 3.6.2 / 139 Let X & Y be n.l.s over $\mathbb{K}$. Then the lin. operator $T : X \to Y$ is closed iff for every seq. $\{x_n\} \subseteq X$ with $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} T(x_n) = y$.
We have $T(x) = y$. $\forall x \in X$

let $\{x_n\}$ be a seq. in X with
Proof: ~~Let~~ $x_n \to x$ and $Tx_n \to y$
$\Rightarrow T(x_n) \to y$
T.P $T(x) = y$
Then
$\|(x_n, T(x_n)) - (x,y)\| = \|(x_n - x, T(x_n) - y)\|$
$= \max\{\|x_n - x\|_x, \|T(x_n) - y\|\}$
$\to 0$ as $n \to \infty$

Pf: (Start) Let T be closed.
$\Rightarrow$ graph of T is closed.
$\Rightarrow G(T)$ is a closed subspace of $X \times Y$
and $X \times Y$ is a n.l.s with the norm defined by
$\|(x,y)\| = \max\{\|x\|_x, \|y\|_y\}$

$\therefore \lim_{n\to\infty} \underbrace{(x_n, T(x_n))}_{\in G(T)} = (x,y)$

$\therefore (x,y) \in \overline{G(T)} = G(T)$ ($\because G(T)$ is closed)

$\therefore (x,y) \in G(T)$

$\Rightarrow y = T(x)$ by def^n of $G(T)$.

* * *

self. $\Leftarrow$ let (x,y) be a limit pt. of $G(T)$. Then $\exists$ a seq. $\{(x_n, Tx_n)\}$ in $G(T)$ such that —

$$\lim_{n\to\infty} (x_n, Tx_n) = (x,y)$$

$$\Rightarrow \lim_{n\to\infty} \|(x_n - x, Tx_n - y)\| = 0$$

$$\Rightarrow \text{Max}\{\|x_n - x\|_X, \|Tx_n - y\|_Y\} \to 0 \text{ as } n\to\infty$$

$$\Rightarrow \lim_{x_n\to\infty} x_n = x \text{ and } \lim_{n\to\infty} Tx_n = y$$

$\therefore$ By hypothesis, it follows that $x \in X$ & $Tx = y$. Consequently, the limit pt. $(x,y) \in G(T)$.

$\Rightarrow G(T)$ is closed

$\Rightarrow T$ is closed.

* * *

Theorem (3.3.8) Two norms $\|\cdot\|$ & $\|\cdot\|'$ on the same linear space X are equivalent iff $\exists$ positive constants k_1 & k_2 s.t.

$$k_1\|x\| \le \|x\|' \le k_2\|x\| \quad \forall x \in X$$

Pf: By the def^n of equivalent Norms (3.3.7 | 50-51)

$\|\cdot\|$ & $\|\cdot\|'$ are equivalent Norms on X

$\Leftrightarrow$ the id. mapping I_X is a topological isomorphism of $(X, \|\cdot\|)$ onto $(X, \|\cdot\|')$

$\Leftrightarrow \exists$ constants k_1 & $k_2 > 0$ s.t.

$$k_1\|x\| \le \|I(x)\|' \le k_2\|x\| \quad \forall x \in X$$

$$\Leftrightarrow k_1\|x\| \le \|x\|' \le k_2\|x\| \quad \forall x \in X$$

This completes the proof.

* * *

(four fundamental thm of n.l.s —
Hahn-Banach thm —
Open-mapping thm —
Closed graph thm —
Uniform b.dd thm —

Closed Graph Theorem

Let X & Y be banach spaces over a field $\mathbb{K}$. Let $T: X \to Y$ be a linear operator. Then T is ~~called~~ closed iff T is bounded.

Pf:— Let T be bounded.

(TP: T is closed i.e. $G(T)$ is closed)

Let $\{x_n\} \subseteq X$ be such that $x_n \to x$ & $T(x_n) \to y$ (A)

[we shall prove $y = T(x)$, then by prev. thm, T will be closed]

Now $x_n \to x$ and $x_n \in X$ $\therefore x \in X$ ($\because X$ is B.S)

Also T is b.dd & hence cts.

$\therefore T(x_n) \to T(x)$ — (B).

from (A) & (B), we have $T(x) = y$

$\therefore T$ is closed.

$\Leftarrow$ Conversely let T be closed.

$\Rightarrow$ The graph of T is closed

i.e. $G(T)$ is closed.

Also, clearly $G(T)$ is a B.S [$\because X$ & Y are B.S $\therefore X\times Y$ is B.S]

(And we know that closed subsp. of B.S is B.S)

Now consider a mapping $P: G(T) \to X$ defined by $P(\underbrace{x, T(x)}_{\in G(T)}) = x$

Clearly P is bijective & P^{-1} will exist.

Also P is bounded.

$$\because \|P(x, T(x))\| = \|x\|' \le \max\{\|x\|, \|T(x)\|\} = \|(x, T(x))\|$$

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Thus P is bdd linear transformation from $G(T)$ onto X.

$\therefore$ By inverse mapping theorem

$P^{-1} : X \to G(T)$ is going to be continuous linear transformation where

$$P^{-1}(x) = (x, T(x))$$

$$\therefore \|Tx\|_Y \le \max\{\|x\|, \|Tx\|\}$$
$$= \|(x, Tx)\|$$
$$= \|P^{-1}(x)\|$$
$$\le \underbrace{\|P^{-1}\|}_{K} \|x\|$$

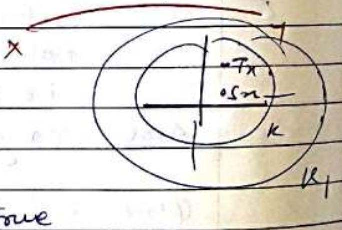
$\Rightarrow T$ is bdd

Def^n: 3.7.1 / 143

Let X and Y be n.l.s and $\mathcal{F} = \{T : X \to Y$ s.t T is bdd lin. operator$\}$ Then $\mathcal{F}$ is s.t.b

(i) Pointwise bdd if $\forall x \in X$, the set $\{T(x) / T \in \mathcal{F}\}$ is bdd in Y. (i.e $\|T(x)\| \le K\|x\|$; $\|T(y)\| \le K_y$; $\|S(x)\| \le K\|x\|$; $\|S(y)\| \le K\|y\|$ – etc)

(ii). Uniform bdd if $\mathcal{F}$ is bdd set in the normed linear space $B(X, Y)$
(i.e $\exists k : \|T\| \le K \;\; \forall T \in \mathcal{F}$) T, S, U



NB (1) If $\mathcal{F}$ is a uniformly bdd set, then $\mathcal{F}$ is pointwise bdd but the converse need not be true

Pf:–

Let $\mathcal{F}$ is be unif bdd
$\Rightarrow \exists k : \|T\| \le k \quad \forall T \in \mathcal{F}$

Let $x \in X$ be given then $\forall \; T \in \mathcal{F}$

$$\|T(x)\| \le \|T\| \|x\|$$
$$\le k \|x\|$$

$\Rightarrow \mathcal{F}$ is pt-wise bdd.

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Converse in next page

For converse some preliminary –

Let S be the space of the sequences $x = \{\xi_1, \xi_2, \dots \xi_n, 0, 0 \dots\}$ in $\mathbb{K}$ where $\xi_n \ne 0$ for only finitely many term

Let c_0 be the ~~the~~ space of all sequences $x = \{\xi_n\}$ converging to 0 with $\|x\|_\infty$

$$\|x\|_\infty = \sup_{1 \le i \le \infty} \{|\xi_i|\}$$

The c_0 is a Banach space.

Then clearly $S \subseteq c_0 \subseteq \ell^\infty$
[$\because \{1, 1/2, \dots\} \in c_0 - S$]

Also,
$\bar{S} = c_0$ in $(\ell^\infty, \|\cdot\|_\infty)$

$\therefore S$ is NOT closed in ℓ^∞.
[$\because$ if S is closed, then $S = \bar{S} = c_0$ # to $S \subsetneq c_0$

$\therefore S$ is an incomplete n.l.s with the norm $\|\cdot\|_\infty$ of ℓ^∞]

Consider $\underset{\text{incomplete}}{(S, \|\cdot\|_\infty)}$ 4 the space $x \in X$

$(K, |\,|)$ wins

* * *

AFTER - II - Internal

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~~INNER PRODUCT SPACE~~

2.3.9 | 97

c_0 is the space of all seq. $x = \{\xi_n\}$. $\{\xi_n\}$ converge, & $\|x\|_\infty = \sup\{\xi_i\}$, $1 \le i < \infty$

$S = \{x = (\xi_1, \xi_2, \xi_3, \ldots, \xi_n, 0, \ldots)\}$ then

clearly $S \subseteq c_0$ & $S \ne c_0$. Note that c_0 is closure of S in $(l^\infty, \|\cdot\|_\infty)$. Thus S is NOT closed in l^∞ and hence S is an incomplete n.l.s equipped with the norm induced by $\|\cdot\|_\infty$ in l^∞

→ S is complete n.l.s (with $\|\cdot\|_\infty$) of l^∞.

consider $(S, \|\cdot\|_\infty)$ and space $(\mathbb{K}, \|\cdot\|)$

let (family) $\mathcal{F} = \{T_n : S \to K \mid T_n(x) = n\lambda_n\}$

Where $x = \{\lambda_n\}_{n=1}^{\infty}$

$T_1(x) = \lambda_1$, $T_2(x) = 2\lambda_2$, $T_3(x) = 3\lambda_3 \ldots$

Then clearly $T_n \in B(S, K)$.

Claim: $\mathcal{F}$ is pointwise bounded

Let $x_0 \in S$, then $\exists\ n_0 \in N : \lambda_n = 0\ \forall\ n \ge n_0$.

$\therefore$ $$\|T_n(x_0)\| = \begin{cases} n\lambda_n & \text{if } n < n_0 \\ 0 & \text{if } n \ge n_0 \end{cases}$$

$$\le n_0 \|x\|_\infty$$

$\therefore$ The family of $\mathcal{F}$ is pt-wise cgt. On the other hand, for any $n \in N$,

$$\|T_n\| = \sup\{|T_n(x)|, x \in S, \|x\|_\infty \le 1\}$$
$$= \sup\{n\lambda_n \mid x = \{\lambda_n\} \in S, \|x\| \le 1\}$$
$$\to \infty \text{ as } n \to \infty$$

$\therefore \mathcal{F}$ is not uniformly bounded

3.7-3 | 14

Uniform Boundedness Principle (Banach stein

Let X be a B.S & Y be n.l.s over field $\mathbb{K}$. If a set $\mathcal{F}$ of b.dd lin. op. from X to Y is pt-wise b.dd then it is uniformly b.dd.

Proof: For each $n \in N$, define a family —

$$F_n = \{x \in X : \|Tx\|_Y \le n\ \forall\ T \in \mathcal{F}\}$$

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claim: F_n is closed

For this $F_n = \overline{F_n} = F_n \cup d(F_n) \subseteq F_n$

It is enough, if we show $d(F_n) \subseteq F_n$

$\Rightarrow F_n \cup d(F_n) \subseteq F_n \cup F_n$

$\Rightarrow \overline{F_n} = F_n \Rightarrow F_n$ is closed.

Let $\{x_k\}$ be a seq. in F_n s.t $x_k \to x$ then

$$\|Tx_k\|_Y \le n \quad \forall\ T \in \mathcal{F}$$

Also, T is cts, & $x_k \to x$, so we have $(as\ k \to \infty)$

$\therefore T(x_k) \to T(x)$

$\therefore \|Tx\| = \lim \|T(x)\| \le n \quad \forall\ T \in \mathcal{F}$

$\therefore x \in F_n$

$\therefore F_n$ is closed.

Now, by hypothesis $\mathcal{F}$ is pt-wise b.dd.

$\therefore \forall\ x \in X$, the set $\{T(x) : T \in \mathcal{F}\}$ is b.dd set in n.l.s Y.

$\therefore$ each $x \in X$ belong to some F_n.

$\therefore X = \bigcup_{n=1}^{\infty} F_n$

But X is complete.

$\therefore$ By Baire Category theorem (Next page), $\exists$ some $n_0 \in N$, s.t $\overline{F_{n_0}}$ has a n.e interior, But F_{n_0} is closed, $\Rightarrow F_{n_0} = \overline{F_{n_0}}$

$\therefore F_{n_0}$ has n-e interior.

$\therefore \exists\ B(x_0, r_0) : B(x_0, r_0) \subseteq F_{n_0}$

$\therefore$ By definition. $\|T(B(x_0, r_0))\| \le n_0$

$\therefore \|T(B(0,1))\| \le n_0$

$$\therefore \|T(B(0,1))\| = \left\|T\left(\frac{B(x_0, r_0) - x_0}{r_0}\right)\right\|$$
$$= \frac{1}{r_0}\|T(B(x_0, r_0) - x_0)\|$$
$$\le \frac{1}{r_0}\|T(B(x_0, r_0))\| + \|T(x_0)\|$$
$$\le \frac{1}{r_0}\{n_0 + n_0\} \quad \forall\ T \in \mathcal{F}$$

$$\therefore \|T(x)\|_Y \le \frac{2n_0}{r_0} \quad \forall\ x \in B(0,1), \forall\ T \in \mathcal{F}$$

~~$\mathcal{F}$ sup~~

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$$\Rightarrow \sup_{\|x\|\le 1} \|Tx\| \le \frac{2\eta_0}{r_0} \quad \forall T \in \mathcal{F}$$

$= \|T\|$

$$\Rightarrow \|T\| \le \frac{2\eta_0}{r_0} \quad \forall T \in \mathcal{F}$$

Hence $\mathcal{F}$ is bdd subset of $B(X,Y)$.

~~Theorem: Cauchy-Schwartz Inequality~~

$\Rightarrow$ ~~$\mathcal{F}$~~ ~~$\mathcal{F}$~~ is uniformly bounded.

~~1.3.27~~ 1.7.32 | 53 **Theorem (Baire Category theorem)**

If a complete metric space is the union of a seq. of ~~thip~~ its subset, then the closure of at least one of the sets in the seq. must have a non-empty interior.

$X = \bigcup F_i$

~ (Baire Category theorem)

Every complete metric space is of second category.

~ In a complete metric space, the intersection of a countable no. of dense open sets is itself dense.

$\because \langle x, \alpha y\rangle = \bar{\alpha}\langle x, y\rangle$

$\langle x+y, z\rangle = \langle x,z\rangle + \langle y,z\rangle$

$\langle \alpha x, z\rangle = \alpha\langle x,z\rangle$

INNER PRODUCT SPACE

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Theorem 5.1.7 | 237 **(Cauchy-Schwartz Inequality)**

If x & y are any two vectors in an IPS, then $|\langle x,y\rangle| \le \|x\|\|y\|$.

Further the equality occurs iff x & y are linearly dependent.

Pf: Case-I. Let $y = 0$

then R.H.S $= 0$

And $\langle x,y\rangle = \langle x, 0\rangle = \langle x, 0y\rangle = \bar{0}\langle x,y\rangle$

$= 0\langle x,y\rangle \quad (\because \bar{0} = 0)$

$= 0 = $ L.H.S

$\therefore$ Equality holds.

Next let $y \neq 0$, then for any scalar λ,

$$0 \le \|x - \lambda y\|^2 = \langle x - \lambda y, x - \lambda y\rangle$$

$$= \langle x, x-\lambda y\rangle + \langle -\lambda y, x - \lambda y\rangle$$

$$= \langle x,x\rangle - \bar{\lambda}\langle x,y\rangle - \lambda\left(\langle y, x - \lambda y\rangle\right)$$

$$= \langle x,x\rangle - \bar{\lambda}\langle x,y\rangle - \lambda\langle y,x\rangle + \lambda\bar{\lambda}\langle y,y\rangle$$

$$= \|x\|^2 - \bar{\lambda}\langle x,y\rangle - \lambda\{\langle y,x\rangle - \bar{\lambda}\langle y,y\rangle\}$$

Since λ is arbitrary, so select λ s.t

$$\bar{\lambda} = \frac{\langle y,x\rangle}{\langle y,y\rangle} \quad \left(\text{i.e choose } \lambda = \frac{\langle x,y\rangle}{\langle y,y\rangle}\right)$$

$\therefore$ We have

$$0 \le \|x\|^2 - \frac{\langle y,x\rangle}{\|y\|^2}\langle x,y\rangle - \lambda\{0\}$$

$$\therefore 0 \le \|x\|^2 - \frac{\langle x,y\rangle}{\|y\|^2}\cdot\overline{\langle x,y\rangle}$$

$$\therefore 0 \le \|x\|^2 - \frac{|\langle x,y\rangle|^2}{\|y\|^2} \quad (\because z\bar{z} = |z|^2)$$

$$\therefore 0 \le \|x\|^2\|y\|^2 - |\langle x,y\rangle|^2$$

$$\Rightarrow |\langle x,y\rangle| \le \|x\|\|y\|$$

further if $y=0$ then clearly $\{x,y\}$ is L.D. ∵ any set of vectors having zero is always a dependent set, & in the second case equality will occur only when $0=\|x-\lambda y\|^2$ (See (1))

$\Rightarrow x=\lambda y$

∴ $\{x,y\}$ is L. dependent set.

Cor: (5.1.8|238) If $X\neq\{0\}$ is an IPS, then

$$\|x\| = \sup\{|\langle x,y\rangle| : \|y\|=1\}$$

Pf: if $x=0$, then clearly $\|x\|=0$

& R.H.S $= \sup\{\langle 0,y\rangle \mid \|y\|=1\}$

$= \sup\{0 : \|y\|=1\}$ $\left(∵ \langle 0,y\rangle = \langle 0.x,y\rangle = 0\langle x,y\rangle = 0\right)$

∴ Trivially true.

Next let $x\neq 0$, then

$$\|x\| = \langle x, \frac{x}{\|x\|}\rangle \quad \left(= \frac{1}{\|x\|}\langle x,x\rangle = \frac{\|x\|}{\|x\|}=1\right)$$

$$\left(\text{or } \langle \|x\|, \frac{x}{\|x\|}\rangle \text{ or } \langle x, \frac{\|x\|}{x}\rangle\right)$$

(∵ $x \le |x|$)

~~Step~~ $\|x\| \le \sup\{\langle x,y\rangle : \|y\|\le 1\}$

$\le \sup\{\|x\|\|y\| : \|y\|=1\}$

$= \sup\{\|x\|\} = \|x\|$

∵ $\|x\| \le \sup\{\langle x,y\rangle : \|y\|=1\} \le \|x\|$

$\Rightarrow \|x\| = \sup\{\langle x,y\rangle : \|y\|=1\}$ Proved

~~We now~~

* We now show that $\|x+y\| \le \|x\|+\|y\|$

we have

$\|x+y\|^2 = \langle x+y, x+y\rangle$

$= \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle$

$= \|x\|^2 + \langle x,y\rangle + \overline{\langle x,y\rangle} + \|y\|^2$ (∵ $z+\bar z = 2Re(z)$)

$= \|x\|^2 + 2\cdot Re(\langle x,y\rangle) + \|y\|^2$

(∵ $Re(z) \le |z|$)

$\|x+y\|^2 \le \|x\|^2 + 2|\langle x,y\rangle| + \|y\|^2$

$\le \|x\|^2 + 2\|x\|\|y\| + \|y\|^2$ (By Schwarz inequality)

$\le (\|x\|+\|y\|)^2$

$\Rightarrow \|x+y\| \le \|x\|+\|y\|$

Remark: Thus we see that, an IPS can be made into a n.l.s by defining $\|x\| = \sqrt{\langle x,x\rangle}$. Such a n.l.s is called n.l.s induced by IPS.

Thus, in this space, we can speak of convergence, limit, continuity etc.

(Now Next Friday)

* * *

(M.L. due to going home)

07/10/16

Inner Product Space

Def^n: (5.1.1|234) let X be a linear space over a field $\mathbb{K}$. A f^n $\langle\cdot\rangle : X\times X \to \mathbb{K}$ satisfying $\langle\cdot\rangle(x,y) = \langle x,y\rangle$ is called an INNER PRODUCT SPACE if it satisfies the following conditions

(i) $\langle x,x\rangle \ge 0 \quad \forall x\in X$ (positive definite property)

(ii) $\langle x,x\rangle = 0$ iff $x=0$

(iii) $\langle x,y\rangle = \overline{\langle y,x\rangle} \quad \forall x,y\in X$

(iv) $\langle \alpha x,y\rangle = \alpha\langle x,y\rangle \quad \forall x,y\in X$, α is scalar

(v) $\langle x+y, z\rangle = \langle x,z\rangle + \langle y,z\rangle \quad \forall x,y,z\in X$

The scalar $\langle x,y\rangle$ is called IP of x & y.

The pair $(X, \langle\cdot\rangle)$ is called IPS (or Pre-Hilbert space)

Note: (5.1.2)/235

(i) If X is a real space, then $\langle x,y\rangle = \langle y,x\rangle$

(ii) $\langle \alpha x+\beta y, z\rangle = \langle \alpha x, z\rangle + \langle \beta y, z\rangle$

$= \alpha\langle x,z\rangle + \beta\langle y,z\rangle$

(iii). $\langle x, \alpha y\rangle = \overline{\langle \alpha y, x\rangle}$
$= \overline{\alpha \langle y, x\rangle} = \bar{\alpha}\, \overline{\langle y,x\rangle}$
$= \bar{\alpha}\langle x, y\rangle$

(iv). $\langle z, \alpha x+\beta y\rangle = \bar{\alpha}\langle z,x\rangle + \bar{\beta}\langle z,y\rangle$

as $\langle z, \alpha x+\beta y\rangle = \overline{\langle \alpha x+\beta y, z\rangle} = \overline{\langle \alpha x, z\rangle} + \overline{\langle \beta y, z\rangle}$
$= \bar{\alpha}\overline{\langle x,z\rangle} + \bar{\beta}\overline{\langle y,z\rangle}$
$= \bar{\alpha}\langle z,x\rangle + \bar{\beta}\langle z,y\rangle$

Example :
(5.1.3) For $x = (x_1, x_2, \dots x_n)$, $y = (y_1, y_2, \dots y_n)$ $\in \mathbb{R}^n$

define
$$\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i$$
then clearly $(\mathbb{R}^n, \langle\cdot\rangle)$ is an inner product space.

Example.
(5.1.4)/h.w.
For $x = (x_1, x_2)$, $y = (y_1, y_2)$, $\in \mathbb{R}^2$
define $\langle x, y\rangle = x_1y_1 - x_2y_1 - y_2x_1 + 3x_2y_2$
then clearly $(\mathbb{R}^2, \langle\cdot\rangle)$ is a real IPS.
This IP $\langle\cdot\rangle$ is of course, NOT the std. IP on $\mathbb{R}^2$.

NB Thus, we observe that on the same linear space we can have more than one IP.

Example (5.1.5)/236
Consider the linear space $\mathbb{C}^n (= l^2(n))$. For vectors
$x = (x_1, x_2, \dots x_n)$
$y = (y_1, y_2, \dots y_n)$
define $\langle x, y\rangle = \sum_{i=1}^{n} x_i \bar{y}_i$
then $(\mathbb{C}^n, \langle\cdot\rangle)$ is a IPS (Complex IPS).

Example 5.1.6(3).
Let $C[a,b]$ be the set of all cts. real (or complex) f^n on $[a,b]$. for $x, y \in C[a,b]$, define —
$$\langle x, y\rangle = \int_a^b x(t)\cdot y(t)\,dt$$
then clearly $(C[a,b], \langle\cdot\rangle)$ is an IPS.

Note: Every IPS can be made into a n.l.s. let $(X, \langle\cdot\rangle)$ be IPS, define
~~$\langle x, y\rangle = \int_a^b x(t)\cdot y(t)\,dt$~~
~~Then clearly~~
~~$(C[a,b], \langle\cdot\rangle)$ is an IPS.~~
$\|\cdot\|$ on X by
$$\|x\|^2 = \langle x, x\rangle$$
A metric on X is given by
$$d(x,y) = \|x-y\| = (\langle x-y, x-y\rangle)^{1/2}$$
then clearly $(X, \|\cdot\|)$ is a n.l.s.

Pf: —
(i) $\|x\| \geq 0$ $\quad (\because \langle x,x\rangle \geq 0)$
(ii) $\|x\| = 0$ iff $\|x\|^2 = 0 \iff \langle x,x\rangle = 0 \iff x = 0$
(iii) $\|\alpha x\|^2 = \langle \alpha x, \alpha x\rangle = \alpha\bar{\alpha}\langle x,x\rangle = |\alpha|^2\|x\|^2$
$\Rightarrow \|\alpha x\| = |\alpha|\|x\|$
(iv) $\|x+y\| \leq \|x\| + \|y\|$ (By schwarz Ineq.)
Hence,
$(X, \|\cdot\|)$ is n.l.s.

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Defn: An IPS $(X, \langle\cdot\rangle)$ over a field $\mathbb{K}$ is s.t.b a **Hilbert space** if $(X, \langle\cdot\rangle)$ is complete w.r.t $\|\cdot\|$ given by $\|x\| = \sqrt{\langle x,x\rangle}$.
The Hilbert space X is called real / complex Hilbert space according as $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$.

Note: Every Hilbert space is a Banach space but the converse is NOT true.

Note: If ~~a~~ $(X, \langle\cdot\rangle)$ is an IPS which induces a normed linear space ~~which is~~ induced by the induced NORM., must satisfy the parallelogram equality viz
$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

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Reason for Note:

We have $\|x+y\|^2 = \langle x+y, x+y\rangle$
$= \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle$
$= \|x\|^2 + \langle x,y\rangle + \langle y,x\rangle + \|y\|^2$ —— (i)

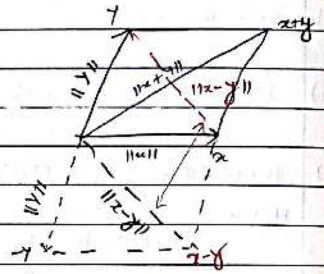
Replace y by $-y$ in the above ——
$\|x-y\|^2 = \|x\|^2 + \langle x,-y\rangle + \langle -y,x\rangle + \|y\|^2$ —— (ii)

Now adding (i) & (ii), we get ——

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

Note: In a parallelogram, the sum of squares of the diagonals is equal to the twice the sum of squares of its sides.

Thus,
$\|x+y\|^2 + \|x-y\|^2$
$= 2(\|x\|^2 + \|y\|^2)$
and hence the name "parallelogram equality".



Note:— We know that ℓ^p, $1 \le p < \infty$ is normed linear space which is equipped with the norm
$$\|x\|_p = \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p}$$
where $x = \{x_i\}_{i=1}^{\infty}$

* We claim that this space (for $p \neq 2$) is not an IPS. (and hence not a Hilbert space)

For suppose ℓ^p is an IPS $(p \neq 2)$
$\Rightarrow$ ℓ^p satisfies parallelogram identity.
Let $x = \{-1, -1, 0, 0 \ldots\}$
$y = \{-1, 1, 0, 0, \ldots\}$, then clearly
$x, y \in \ell^p$ $\therefore$ By the induced norm, must satisfy the parallelogram equality
$(\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2))$.

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Then $\|x\|_p = \{|-1|^p + |-1|^p + |0|^p + |0|^p + \cdots\}^{1/p} = 2^{1/p}$
$\|y\|_p = \{|-1|^p + |1|^p + |0|^p + |0|^p + \cdots\}^{1/p} = 2^{1/p}$

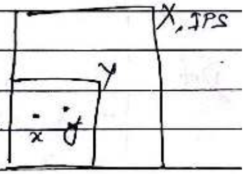
$\therefore \|x+y\| = \{|-2|^p + |0|^p + |0|^p + \cdots\}^{1/p} = 2$
$\|x-y\| = \{|0|^p + |2|^p + |0|^p + \cdots\}^{1/p} = 2$

$\therefore$ By parallelogram equality ——
L.H.S $= \|x+y\|^2 + \|x-y\|^2 = 4+4 = 8$
R.H.S $= 2\cdot(2^{2/p} + 2^{2/p}) = 4\cdot 2^{2/p}$
clearly L.H.S $\neq$ R.H.S
$\therefore$ ℓ^p, $(p \neq 2)$ is NOT an IPS.

(ℓ^p is IPS,
ℓ^p, $p \neq 2$, IP , ℓ^p, $p \neq 2$ is not IPS)

5.1.12 (24^0)
Defn: (VVI) A non-empty subset Y of an IPS X is s.t.b a subspace of X, if —
(i) Y is linear subspace of X.
(ii) Y is equipped with IP $\langle\cdot\rangle_Y$ induced by the IP $\langle\cdot\rangle$ of X, i.e
$$\langle x,y\rangle_Y = \langle x,y\rangle, \quad \forall x, y \in Y.$$

A subspace of an IPS is also an IPS



Note: A subspace of a Hilbert space need not be a Hilbert space. However a closed subspace of a Hilbert space is a Hilbert space. In fact the following is true.

(VVI) Defn
~~Theorem~~:
5.1.1.10 (24^0)
Let H be a Hilbert space and Y be a subspace of H, then
(i) Y is complete iff Y is closed ~~ele~~ in H.
(ii) $\overline{Y}$ is a closed subspace of H and hence $\overline{Y}$ is always a Hilbert-space.

Theorem: Let X be an IPS. If $x_n \to x$, $y_n \to y$ in X, then $\langle x_n, y_n\rangle \to \langle x, y\rangle$

Proof: We have

$$|\langle x_n, y_n\rangle - \langle x, y\rangle| = |\langle x_n, y_n\rangle - \langle x_n, y\rangle + \langle x_n, y\rangle - \langle x, y\rangle|$$

$$= |\langle x_n, y_n - y\rangle + \langle x_n - x, y\rangle|$$

$$\le |\langle x_n, y_n - y\rangle| + |\langle x_n - x, y\rangle|$$

$$\le \|x_n\| \underbrace{\|y_n - y\|}_{\to 0} + \underbrace{\|x_n - x\|}_{\to 0} \|y\| \quad \text{as } n \to \infty$$

(By Schwartz Ineq.)

≤ 0

$\therefore \lim_{n\to\infty} \langle x_n, y_n\rangle = \langle x, y\rangle$

Corollary: If $x = \sum_{i=1}^{\infty} x_i$ is an element of IPS X, then

$$\langle x, y\rangle = \sum_{i=1}^{\infty} \langle x_i, y\rangle \quad \forall y \in X$$

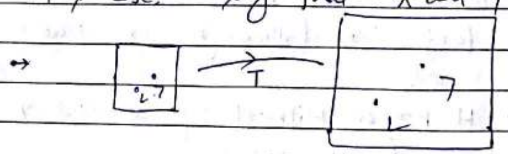
Hint: let $S_n = x_1 + \cdots + x_n$

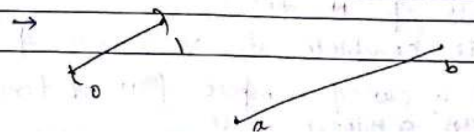
$$\langle x, y\rangle = \langle \lim S_n, y\rangle$$
$$= \lim \langle S_n, y\rangle$$
$$= \lim \left[\sum_{i=1}^{n} \langle x_i, y\rangle\right]$$

Def^n: Let X, Y be IPS over a field $\mathbb{K}$. An operator $T: X \to Y$ is s.t.b an isomorphism of X onto Y if T is bijective and

$$\langle Tx, Ty\rangle = \langle x, y\rangle \quad \forall x, y \in X$$

In this case, we say that X and Y are isomorphic





Note: An inner product isomorphism is an isometry also.

$\because$ If T is an isomorphism

$$\|T(x)\|^2 = \langle Tx, Tx\rangle = \langle x, x\rangle \quad (\because T \text{ is iso})$$
$$= \|x\|^2$$

$\therefore \|T(x)\| = \|x\| \quad \therefore$ isometry.

Note: An IPS can be made into a n.l.s by defining $\|x\| = \sqrt{\langle x, x\rangle}$

Once this has been done, We can define

(*)
$$\langle x, y\rangle = \begin{cases} \frac{1}{4}\left[(\|x+y\|^2 - \|x-y\|^2) + i(\|x+iy\|^2 - \|x-iy\|^2)\right] & \text{if } \mathbb{K} = \mathbb{C} \\ \frac{1}{4}\left[\|x+y\|^2 - \|x-y\|^2\right] & \text{if } \mathbb{K} = \mathbb{R} \end{cases}$$

Then it is easy to see that $\langle \cdot \rangle$ is indeed an IP. (* is known as polarization identity)

(Hint: Start from R.H.S

$$\|x+y\|^2 = \langle x+y, x+y\rangle$$
$$= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle$$
$$\|x-y\|^2 = \langle x, x\rangle - \langle x, y\rangle - \langle y, x\rangle + \langle y, y\rangle$$

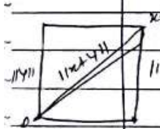
(

Def^n: Let X be an IPS. A vector $x \in X$ is s.t.b orthogonal to a vector $y \in X$ if $\langle x,y\rangle = 0$.
In this case we say that x and y are orthogonal vectors.
Notation: $x \perp y$. (x is perpendicular to y)

Note: (i) $x \perp A$ iff $x \perp a$ $\forall a \in A$.
(ii) $A \perp B$ iff $a \perp b$ $\forall a \in A, b \in B$
(iii) $x \perp y$ iff $y \perp x$
(iv) $x \perp 0$ $\forall x \in X$
(v) 0 is only vector orthogonal to itself.

Theorem: (Pythagorean Theorem)
In an IPS if $x \perp y$, then $\|x+y\|^2 = \|x\|^2 + \|y\|^2$

Pf:- $\|x+y\|^2 = \langle x+y, x+y\rangle$
$= \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle$
$= \|x\|^2 + 0 + 0 + \|y\|^2$



Def^n: For any subset A of IPS X.
Define $A^{\perp} = \{x \in X : x \perp A\}$
Notation: $(A^{\perp})^{\perp} = A^{\perp\perp}$

Note: (i) $\{0\}^{\perp} = X$, $X^{\perp} = \{0\}$
(ii) If $A \subseteq X$, $A \neq \phi$, then $A^{\perp}$ is a closed subspace of X. Further,
(iii) $A \cap A^{\perp} = \phi$ or $\{0\}$ ← possible when $A = \phi$
(iv) $A \subseteq A^{\perp\perp}$ for all subsets A of X.
(v) If $A \subseteq B$, then $B^{\perp} \subseteq A^{\perp}$
(vi) If A is n-e subset of X then $A^{\perp} = A^{\perp\perp\perp}$

pf:(v) given: $c \in B^{\perp}$
$\langle c,b\rangle = 0$ $\forall b \in B$
$\langle c,b\rangle = 0$ $\forall b \in A \Rightarrow c \perp A$
$\Rightarrow c \in A^{\perp}$

$\Rightarrow B^{\perp} \subseteq A^{\perp}$ Hence proved

(vi) from (iv), $A \subseteq A^{\perp\perp}$
in particular $(A^{\perp}) \subseteq (A^{\perp})^{\perp\perp}$
and by (v) $A^{\perp\perp\perp} \subseteq A^{\perp}$
from both, we have $A^{\perp} = A^{\perp\perp\perp}$

***

IPs | TVED

02/11/16

Def^n: Let X be an IPS. If M and N are subspaces of X with $M \perp N$, then the vector sum $M+N = \{x+y \mid x \in M, y \in N\}$ and is called Orthogonal sum of M & N.

Theorem: Let H be a Hilbert space and M & N be orthogonal closed subspaces of H, then $M+N$ is also closed subspace of H.

Pf:- Clearly $M+N$ is a subspace of H w.r.t. the IPS of X.
claim: $M+N$ is closed (T.P. $\overline{M+N} = M+N$)
(As we know that $A \subseteq \bar{A} \Rightarrow M+N \subseteq \overline{M+N}$
Now we have to ST $\overline{M+N} \subseteq M+N$) $\Rightarrow =$
let $z \in \overline{M+N}$
$\Rightarrow \exists$ a sequence $\{z_n\} \in M+N$ such that $z_n \to z$ as $n \to \infty$
$\therefore$ Each z_n can be uniquely written in form
$z_n = x_n + y_n$

Uniqueness.
$z_n = \begin{cases} x_n + y_n \\ t_n + w_n \end{cases}$
$\Rightarrow x_n + y_n = t_n + w_n \Rightarrow \underbrace{x_n - t_n}_{\in M} = \underbrace{w_n - y_n}_{\in N}$
$x_n - t_n \in M \cap N = \{0\}$
$\Rightarrow x_n = t_n$
lly $y_n = w_n$

Also $M \perp N$ and hence by ~~poly~~ Pythagorean thm,

$$\|z_n - z_m\|^2 = \|(x_n - x_m) + (y_n - y_m)\|^2$$
$$= \|x_n - x_m\|^2 + \|y_n - y_m\|^2$$

($\because M \perp N$ By hypothesis) — (A)

Now $\{z_n\}$ being convergent, it is a Cauchy seq. and hence from (A), $\{x_n\}$, $\{y_n\}$ are Cauchy seq. in M and N respectively.

But M and N are closed and hence these Cauchy seq. must converge, say to x and y respectively.

Thus

$$z = \lim_{n\to\infty} z_n = \lim_{n\to\infty} (x_n + y_n)$$
$$= \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n$$
$$= x + y \in M+N$$

$\therefore \overline{M+N} \subseteq M+N$

and hence $\overline{M+N} = M+N$

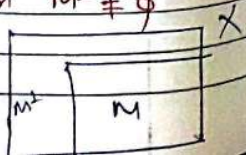
$\Rightarrow M+N$ is closed.

Def^n: Let X be a linear space and M & N be subspace of X. Then X is s.t.b direct sum of M and N written as $X = M \oplus N$, if every element can be uniquely expressed as $x = m+n$ with $m \in M$ & $n \in N$.

Equivalently $X = M \oplus N$ iff $X = M+N$ and $M \cap N = \{0\}$

Theorem: Let X be IPS and $M \neq \phi$ be a complete proper subspace of X. Then $M^\perp \neq \phi$.

Pf: Assume



Theorem: Projection Theorem

Statement: Let M be a closed subspace of a Hilbert space H. Then $H = M \oplus M^\perp$.

Pf:

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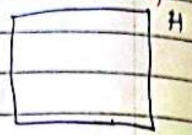
Projection Theorem

Let M be a closed subspace of a Hilbert space H. Then $H = M \oplus M^\perp$



Proof: We know that $M^\perp$ is a closed subspace of H. Thus M and $M^\perp$ are closed subspaces of H

$\therefore M+M^\perp$ is closed subspace of H by earlier theorem

$\therefore M+M^\perp$ is complete

($\because$ ~~Complete~~ closed subspace of a complete metric space is complete)

We now prove $H = M \oplus M^\perp$

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Clearly $M + M^\perp \subseteq H$.

Suppose $M+M^\perp \subsetneq H$

As $M \neq \phi$ & $M \subsetneq H$ $\Rightarrow M^\perp \neq \phi$

Now by prev. theorem, $(M+M^\perp)^\perp \neq \phi$

$\Rightarrow \exists$ some $z_0 \in H$, i.e. $z_0 \perp M+M^\perp$

$\Rightarrow \langle z_0, y+z\rangle = 0 \quad \forall y \in M, z \in M^\perp$

$\Rightarrow \langle z_0, y\rangle + \langle z_0, z\rangle = 0 \quad \forall y \in M, \forall z \in M^\perp$

In particular

taking $y=0$, $z=0$ separately

$\langle z_0, 0\rangle + \langle z_0, z\rangle = 0 \quad \forall z \in M^\perp$

& $\langle z_0, y\rangle + \langle z_0, 0\rangle = 0$

$\Rightarrow \langle z_0, z\rangle = 0$ & $\langle z_0, y\rangle = 0 \quad \forall z \in M^\perp, \forall y \in M$

$\Rightarrow z_0 \in (M^\perp)^\perp$ & $z_0 \in M^\perp$ $\} \Rightarrow z_0 = 0$ which is contradiction.

$\therefore M + M^\perp = H$

Also since $M + M^\perp = \{0\}$

it follows that $M \oplus M^\perp = H$

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Theorem:* A subspace M of a Hilbert space H is closed in H iff $M = M^{\perp\perp}$

Proof: Let $M = M^{\perp\perp}$, then $(M^\perp)^\perp$ is closed.

$\Rightarrow M$ is closed

$\Leftarrow$ Conversely, let M be closed subspace of H. Now we know that $M \subseteq M^{\perp\perp}$ (see p-90, Th.) (*)

Hence it only remains to show that $M^{\perp\perp} \subseteq M$.

Let $x \in M^{\perp\perp}$

Obviously, $x \in H = M \oplus M^\perp$, by projection thm

$\Rightarrow \exists\, y \in M$ and $z \in M^\perp$ s.t

$x = y + z \Rightarrow x - y = z$

Now $x \in M^{\perp\perp}$, also $y \in M \subseteq M^{\perp\perp}$ (See *)

$\Rightarrow x - y \in M^{\perp\perp}$

Also, $z \in M^\perp \Rightarrow x - y \in M^\perp$

Thus $x - y \in M^\perp \cap (M^\perp)^\perp = \{0\}$

$\Rightarrow x = y$, But $y \in M$;

$\therefore x \in M$

$\Rightarrow \therefore M^{\perp\perp} \subseteq M$ —— (**)

$\therefore$ Using (*) and (**), Equality holds

i.e. $M = M^{\perp\perp}$

Theorem: Let A be a non-empty subset of a Hilbert space H. Then span of A is dense in H iff $A^\perp = \{0\}$.

[$\overline{[A]} = H \Leftrightarrow A^\perp = \{0\}$]

* * *

Proof: let $[A]$ denote the span of A. Let $\overline{[A]} = H$

To prove: $A^\perp = \{0\}$

obviously $\{0\} \subseteq A^\perp$
Hence it only remains to prove $A^\perp \subseteq \{0\}$
Let $x \in A^\perp$, Now $x \in H$ always $= \overline{[A]}$

$\Rightarrow \exists$ a sequence $\{x_n\} \in [A]$ s.t $\lim_{n\to\infty} x_n = x$
Now
$x \in A^\perp$ and $A^\perp \perp [A]$
$\therefore \langle x, x_n \rangle = 0 \quad \forall\ n = 1,2,\ldots$

$0 = \lim_{n\to\infty} \langle x, x_n \rangle = \langle x, \lim_{n\to\infty} x_n \rangle$ (∵ ⟨,⟩ is cts fn)
$= \langle x, x \rangle$
$\Rightarrow \boxed{x = 0}$
$\Rightarrow \boxed{A^\perp = \{0\}}$

$\Leftarrow$ Conversely let $A^\perp = \{0\}$ ——— (α)
Let $x \in [A]^\perp \Rightarrow x \perp [A]$.
$\Rightarrow$ In particular, $x \perp A$
$\Rightarrow x \in A^\perp = \{0\}$
$\Rightarrow x = 0$
$\therefore [A]^\perp = 0$ (from (α))
Now $\overline{[A]}$ is a closed subspace of H.
$\therefore$ By earlier theorem —
$H = \overline{[A]} \oplus \overline{[A]}^\perp$
$= \overline{[A]}$ (bcz $[A]^\perp = 0$
$\therefore \overline{[A]}^\perp = 0$)

Hence proved

Def^n: (5.5.1/262) Let X be IPS and A be a subset of X. Then A is s.t.b
(a) Orthogonal set in X if $x \perp y$ whenever $x, y \in A$, with $x \neq y$
(b) Ortho-Normal set in X if A is orthogonal in X and $\|x\| = 1 \ \forall\ x \in A$.

(5.5.4 | 264)
Theorem: An orthonormal set in an IPS X is LI.
pf:
Let $A = \{e_1, e_2, \ldots e_n\}$ be a finite Orthonormal set.
Let $\sum_{i=1}^{n} \alpha_i e_i = 0$, where α_i's are scalars.
Then $0 = \langle \sum_{i=1}^{n} \alpha_i e_i, e_j \rangle = \langle \alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n, e_j \rangle$
$= \langle \alpha_1 e_1, e_j \rangle + \langle \alpha_2 e_2, e_j \rangle + \cdots + \langle \alpha_j e_j, e_j \rangle + \cdots + \langle \alpha_n e_n, e_j \rangle$
$= 0 + 0 + \cdots + \alpha_j \cdot 1 + \cdots + 0 \quad \forall\ j = 1, 2, \ldots n$
$= \alpha_j$
Thus $\alpha_j = 0$ $(j = 1 \ldots n)$. Hence A is LI

NB For the infinite case use the limit concept with the fact that ⟨,⟩ is continuous

NB Recall that an orb. set (A) (is finite or infinite) is said to be LI if every non-empty finite subset of A is LI. Hence it follows that our assertion is valid for the case when A is infinite.

Theorem: (5.5.6/264) Let $\{e_1, e_2, \ldots e_n\}$ be a finite O.N. set in an IPS X, Then for any x in X, we have —
(i) $\sum_{i=1}^{n} |\langle x, e_i \rangle|^2 \leq \|x\|^2$ (Bessel's Inequality)
(ii) $x - \sum_{i=1}^{n} \langle x, e_i \rangle e_i \perp e_j \quad \forall\ j = 1, 2, \ldots n$

Proof: — (i) We have —
$0 \leq \| x - \sum_{i=1}^{n} \langle x, e_i \rangle e_i \|^2$

$0 \le \langle x - \sum_{i=1}^{n} \langle x, e_i\rangle e_i, x - \sum_{j=1}^{n} \langle x, e_j\rangle e_j \rangle$

$(\because \|x\|^2 = \langle x, x\rangle)$

for sake of simplicity, we shall work for $n=2$

$= \langle x - \langle x, e_1\rangle e_1 - \langle x, e_2\rangle e_2, x - \langle x, e_1\rangle e_1 - \langle x, e_2\rangle e_2 \rangle$

$= \langle x, x\rangle - \sum_{i=1}^{n} \langle x, e_i\rangle \overline{\langle x, e_i\rangle} - \sum_{j=1}^{n} \langle x, e_j\rangle \overline{\langle x, e_j\rangle}$

$+ \sum_{i=1}^{n} \sum_{j=1}^{n} \langle x, e_i\rangle \langle x, e_j\rangle \langle e_i, e_j\rangle$

$0 \le \|x\|^2 - \sum_{i=1}^{n} |\langle x, e_i\rangle|^2$

$\Rightarrow \sum_{i=1}^{n} |\langle x, e_i\rangle|^2 \le \|x\|^2$

$= \|x\|^2 - \langle e_1, x\rangle \langle x, e_1\rangle - \langle e_2, x\rangle \langle x, e_2\rangle$

$0 \le \|x\|^2 - |\langle x, e_1\rangle|^2 - |\langle x, e_2\rangle|^2$

$\therefore |\langle x, e_1\rangle|^2 + |\langle x, e_2\rangle|^2 \le \|x\|^2$

(b) To prove $\langle x - \sum_{i=1}^{n} \langle x, e_i\rangle e_i, e_j \rangle = 0$ — (1)

Observe that $\langle x - \sum_{i=1}^{n} \langle x, e_i\rangle e_i, e_j\rangle$

$= \langle x, e_j\rangle - \sum_{i=1}^{n} \langle x, e_i\rangle \langle e_i, e_j\rangle$

$(0 + 0 + \cdots + \langle x, e_j\rangle \langle e_j, e_j\rangle + 0 + \cdots + 0)$

$= \langle x, e_j\rangle - \langle x, e_j\rangle = 0$

$\Rightarrow$ (1) will holds here

$\Rightarrow x - \sum_{i=1}^{n} \langle x, e_i\rangle e_i \perp e_j \quad (j = 1, 2, \cdots n)$

Theorem: [5.5.7 | 268] let $\{e_i\}_{i \in I}$ be an ON set in an IPS X. let $x \in X$, then the set $S = \{e_i ; \langle x, e_i\rangle \ne 0, i \in I\}$ is either empty or ~~not~~ countable.

Pf:- let $S_n = \{e_i : |\langle x, e_i\rangle|^2 > \frac{\|x\|^2}{n}, i \in I\}$

then clearly $S = \bigcup_{n=1}^{\infty} S_n$

$\frac{\|x\|^2}{n} \quad \|x\|^2$

We claim, each S_n contains at most $n-1$ terms. For suppose S_n contain m-terms where $m \ge n$. Then—

$$\left.\begin{array}{l} |\langle x, e_1\rangle|^2 > \frac{\|x\|^2}{n} \\ |\langle x, e_2\rangle|^2 > \frac{\|x\|^2}{n} \\ \vdots \\ |\langle x, e_m\rangle|^2 > \frac{\|x\|^2}{n} \end{array}\right\} \Rightarrow \sum_{i=1}^{m} |\langle x, e_i\rangle|^2 > \frac{m}{n} \|x\|^2 \ge \|x\|^2$$

get contradiction to Bessel Ineq

$\therefore$ Each S_n is finite.

$\therefore S$ being countable union of finite sets is finite.

$\therefore S$ is at most countable. ***

Note: if $\{e_1, e_2, \cdots e_n\}$ is an ON set and $x \in \text{span}\{e_1, e_2, \cdots e_n\}$

$\Rightarrow x = \sum_{i=1}^{n} \alpha_i e_i$

$\therefore \langle x, e_j\rangle = \langle \sum_{i=1}^{n} \alpha_i e_i, e_j \rangle$

$= \langle \alpha_1 e_1, e_j\rangle + \langle \alpha_2 e_2, e_j\rangle + \cdots$

$+ \langle \alpha_j e_j, e_j\rangle + \cdots + \langle \alpha_n e_n, e_j\rangle$

$= \alpha_j \langle e_j, e_j\rangle = \alpha_j$

Substituting $x = \sum_{i=1}^{n} \langle x, e_i \rangle e_i$

CHAPTER-6 * * * 08/11/16

Def^n: Let H be Hilbert space. The space $B(H,K)$ of all bdd linear functionals on H over a scalar field K is called dual space of H and is denoted by H^*.

Theorem: Let X be an IPS. Then for each $z \in X$, $\exists$ a unique bdd linear functionals f_z on X s.t. $\|f_z\| = \|z\|$

Pf:- define $f_z : X \to K$ by $f_z(x) = \langle x, z \rangle$ —(*)

Clearly, f_z is a functional

→ f_z is linear

$$f_z(\alpha x + \beta y) = \langle \alpha x + \beta y, z \rangle$$
$$= \alpha \langle x, z \rangle + \beta \langle y, z \rangle$$
$$= \alpha f_z(x) + \beta f_z(y)$$

→ f_z is bdd and $\|f_z\| = \|z\|$

We have

$$\|f_z(x)\| = |\langle x, z \rangle|$$
$$\leq \|x\| \|z\| \quad \text{by Schwarz Ineq.}$$

$\therefore f_z$ is bounded.

def^n of Norm → $\therefore \sup_{\|x\|=1} |f_z(x)| \leq \sup_{\|x\|=1} \|x\| \|z\|$

$$\Rightarrow \boxed{\|f_z\| \leq \|z\|} \quad \text{— (A)}$$

Next, for $z \neq 0$

$$\|f_z\| = \sup_{\|x\|=1} |f_z(x)|$$
$$= \sup_{\|x\|=1} |\langle x, z \rangle|$$
$$\geq \left| \left\langle \frac{z}{\|z\|}, z \right\rangle \right|$$

$$\geq \left| \frac{1}{\|z\|} \langle z, z \rangle \right|$$
$$= \frac{1}{\|z\|} \cdot \|z\|^2 = \|z\| \quad \text{— (B)}$$

$\therefore \|f_z\| \geq \|z\|$ — (B)

Also trivially true for $z = 0$

from (A) & (B), $\Rightarrow$ Equality

(verify) Clearly, f_z is unique.

Suppose f_z defined by (*) is not unique

for $z \neq 0$ $f_z(x) = \begin{cases} \langle x, z \rangle \\ \langle y, z \rangle \end{cases}$

$\Rightarrow \langle x, z \rangle = \langle y, z \rangle$

$\Rightarrow \langle x, z \rangle - \langle y, z \rangle = 0$

$\Rightarrow \langle x - y, z \rangle = 0 \Rightarrow$ either $x - y = 0$ or $z = 0$

$\because z \neq 0 \Rightarrow x - y = 0$

$x = y$

in particular

$z = x - y$ we

$\langle x - y, x - y \rangle = 0$

$\Rightarrow \|x - y\|^2 = 0$

$\Rightarrow x - y = 0 \Rightarrow x = y$

Theorem 6.1.1 /287 (Riesz-Frechet-Representation theorem)

Let H be a Hilbert space & H^* be its dual space. If $f \in H^*$ is an arbitrary ~~then $\exists$~~ (but fixed functional), then $\exists$ a unique vector $z \in H$ s.t.

$$f(x) = \langle x, z \rangle \quad \forall x \in H$$

Where z depends on f and has the norm

$$\|z\| = \|f\|$$

Pf:- Case-I. $f \equiv 0$, take $z = 0$

{Then thm is obviously satisfied}.

Then $f(x) = 0$

$\langle x, z\rangle = \langle x, 0\rangle = 0$

$(\because f(x) = \langle x, z\rangle)$

then thm is satisfied

Case-II, Let $f \neq 0$

We know that the null space $N = N(f)$ is closed subspace of H.

Also, since $f \neq 0$, it follows that $N(f) \neq H$.

$\therefore$ By projection theorem

$H = N \oplus N^{\perp}$ (Here $N = N(f)$)

$\therefore N^{\perp} \neq \phi$

$\therefore \exists\ z_0 \neq 0$; $z_0 \in N^{\perp} \Rightarrow z_0 \perp N$ — $(*)$

Let $S = \{v = z_0 f(x) - x f(z_0) : x \in H\}$

Claim.

$S \subseteq N(f)$

$\therefore f(v) = f[\underbrace{z_0}_{\text{scalar}} f(x) - x \underbrace{f(z_0)}_{\text{scalar}}]$ $(\because f$ is linear$)$

$= f(x) f(z_0) - f(x) f(z_0)$

$= 0$ $\forall x \in H$

$\therefore S \subseteq N$

$\therefore z_0 \perp S$ (using $(*)$)

$\therefore \langle \underbrace{z_0 f(x) - x f(z_0)}_{\in S}, \underbrace{z_0}_{\in S}\rangle = 0$

$\Rightarrow f(x) \|z_0\|^2 = f(z_0) \langle x, z_0\rangle$

$\therefore f(x) = \dfrac{f(z_0)}{\|z_0\|^2} \langle x, z_0\rangle$

$= \left\langle x, \dfrac{\overline{f(z_0)}}{\|z_0\|^2} z_0\right\rangle$

$= \langle x, z\rangle$ where $z = \dfrac{\overline{f(z_0)}}{\|z_0\|^2} z_0$

Uniqueness

Let $z_1, z_2 \in H$ s.t.

$f(x) = \langle x, z_1\rangle$ & $f(x) = \langle x, z_2\rangle$ $\forall x \in H$

$\Rightarrow \langle x, z_1\rangle = \langle x, z_2\rangle$ $\forall x \in H$

$\Rightarrow \langle x, z_1 - z_2\rangle = 0$ $\forall x \in H$

In particular —

$\langle z_1 - z_2, z_1 - z_2\rangle = 0$

$\Rightarrow \|z_1 - z_2\|^2 = 0 \Rightarrow \|z_1 - z_2\| = 0$

$\Rightarrow z_1 = z_2$.

Claim $\|f\| = \|z\|$

We have $f(x) = \langle x, z\rangle$ $\forall x \in H$

In particular,

$f(z) = \langle z, z\rangle = \|z\|^2$

$\therefore \|z\|^2 = |f(z)| \leq \|f\| \|z\|$

$\therefore \|z\| \leq \|f\|$ — $(*\alpha)$

Next using Schwarz's inequality —

$|f(x)| = |\langle x, z\rangle| \leq \|x\| \|z\|$

$\therefore \sup_{\|x\|=1} |f(x)| \leq \sup_{\|x\|=1} \|x\| \|z\|$

$\Rightarrow \|f(x)\| \leq \|z\|$. — (α_2)

$\therefore$ Equality from $(*\alpha)$ & (α_2)

Theorem: 6.2.3/296 Let H_1 & H_2 be Hilbert space, $T : H_1 \to H_2$ be a bdd linear operator. Then $\exists$ a unique bdd linear operator T^*, $T^* : H_2 \to H_1$ s.t.

$\langle Tx, y\rangle_2 = \langle x, T^* y\rangle_1$ $\forall x \in H_1, y \in H_2$

Furthermore $\|T^*\| \leq \|T\|$