CSIR-NET Solution Abstract Algebra

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Year-wise Solution

-2019 December
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No. of Pages: 97

(CSIR-NET Dec 2019)

Ques Let F[X] be the polynomial ring in one variable over a field F. Then W.O.T.F. statements are true?

① F[X] is a UFD.
② F[X] is a PID.
③ F[X] is ED.
(4) F[X] is a PID but is not an ED.

Ans: (1), (2), (3)

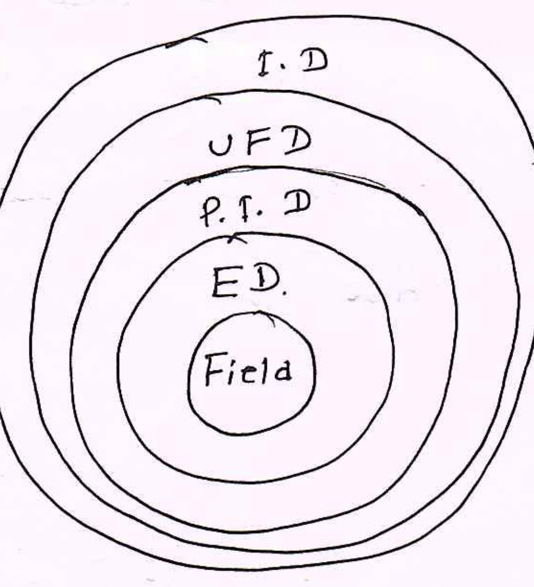
Soln Recall: ① Field ⊂ ED ⊂ PID ⊂ UFD ⊂ ID.
② If $\mathbb{F}$ is a field then $\mathbb{F}[x]$ is E.D.

So, Here $\mathbb{F}$ is a field $\Rightarrow \mathbb{F}[X]$ is E.D.
$\Rightarrow \mathbb{F}[X]$ is PID
$\Rightarrow \mathbb{F}[X]$ is UFD

∴ opt ①, (2), (3) – True
opt (4) – False.

P. Kalika Notes

Field
ED: Euclidean domain
PID: Principal Integral domain
UFD: Unique Factorization domain
I.D: Integral domain



Trick: FEPUID

(Look into Short Notes for tricks)

1

Total Marks
$3 \times 2 = 6$
$4.75 \times 4 = 19$
$25M$

Abstract Algebra

Solution CSIR-NET 2019 June

By :-
K. Munesh

Q^n.1
3 Marks

Consider the set of matrices $G = \left\{ \begin{pmatrix} s & b \\ 0 & 1 \end{pmatrix} ; b \in \mathbb{Z}, s \in \{-1, +1\} \right\}$. Then which of the following is true ?

(1) G forms a group under addition

C-36

(2) G forms an abelian group under multiplication.

(3) Every element in G is diagonalisable over $\mathbb{C}$.

✓(4) G is a finitely generated group under multiplication

Sol^n

(1) Closure not hold. Bcause, for $\begin{bmatrix} 1 & b_1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & b_2 \\ 0 & 1 \end{bmatrix} \in G$,

$$\begin{bmatrix} 1 & b_1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & b_2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & b_1 + b_2 \\ 0 & 2 \end{bmatrix} \notin G.$$

So, opt (1) is wrong.

(2) Let $A_1 = \begin{bmatrix} s_1 & b_1 \\ 0 & 1 \end{bmatrix}$ $A_2 = \begin{bmatrix} s_2 & b_2 \\ 0 & 1 \end{bmatrix} \in G$ be any two element.

Then
$$\begin{bmatrix} s_1 & b_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s_2 & b_2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} s_1 s_2 & s_1 b_2 + b_1 \\ 0 & 1 \end{bmatrix} = A_1 A_2$$

&
$$\begin{bmatrix} s_2 & b_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s_1 & b_1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} s_2 s_1 & s_2 b_1 + b_2 \\ 0 & 1 \end{bmatrix} = A_2 A_1$$

But $A_1 A_2 \neq A_2 A_1$ ($\because s_1 b_2 + b_1 \neq s_2 b_1 + b_2$).

So, G is not abelian group under multiplication.

opt (2) is wrong.

June-2019

(1). Part-B (c) 3.5

For any integer $n \geq 1$, let
$d(n)$ = no. of positive divisors of n.
$v(n)$ = no. of distinct prime divisors of n
$w(n)$ = no. of prime divisors of n counted with multiplicity
(For eg. If p is prime, then $d(p)=2$, $v(p)=v(p^2)=1$, $w(p^2)=2$).

(1). If $n \geq 1000$ & $w(n) \geq 2$ then $d(n) > \log n$.
(2). ∃ n s.t. $d(n) > 3\sqrt{n}$
✓ (3). for every n, $2^{v(n)} \leq d(n) \leq 2^{w(n)}$. ✓
(4). if $w(n) = w(m)$, then $d(n) = d(m)$.

Solⁿ (1) Let p & q be two prime s.t $n = pq \geq 1000$
then $w(n) = 2$ & $d(n) = 4$
and $\log(n) > d(n)$ for sufficiently large p & q.

[As $\log(n) = \log(p) + \log(q)$
Assume $p = 101$, $q = 103$, then $\log(n) = (\geq 2) + (\geq 2) \geq 4$]

So op(1) is NOT Correct

(2) Let $n = p_1^{r_1} p_2^{r_2} \cdots p_n^{r_n}$ then $d(n) = (r_1+1)(r_2+1)\cdots(r_n+1)$
∵ let $n = 1$, $d(n) \leq 3\sqrt{1}$ (as $1 \leq 3$ ✓)
let $n = 2^{\alpha}$, $d(2^{\alpha}) \leq 3\sqrt{2^{\alpha}}$
$\Rightarrow \alpha + 1 \leq 3 \cdot 2^{\alpha/2}$ ∀ $n \geq 1$
||'ly ∀ $p \geq 5$, $d(p^{\alpha}) = \alpha + 1 \leq \sqrt{p^{\alpha}}$
thus op(2) is NOT Correct.

(3). Let $n = p_1^{r_1} p_2^{r_2} \cdots p_m^{r_m}$
∴ $d(n) = (r_1+1)(r_2+1)\cdots(r_m+1)$, $v(n) = m$
& $w(n) = r_1 + r_2 + \cdots + r_n$
⇒ $d(n) \geq 2^m = 2^{v(n)}$
& $d(n) \leq 2^{r_1} \cdot 2^{r_2} \cdots 2^{r_m} \leq 2^{r_1+r_2+\cdots+r_m} = 2^{w(n)}$
⇒ op(3) is correct.

By P. Kalika Part-B, Unit-2

(3M)

① The no. of group homomorphisms from the A_5 to the symmetric group S_4 is —

ⓐ 1 ✓ ⓑ. 12 ⓒ. 20 ⓓ. 6

Sol^n By fundamental theorem of grp. homomorphism

If $f: A_5 \to S_4$ is group homomorphism, then

$$\frac{A_5}{Ker f} \simeq \text{Subgroup of } S_4$$

$\because$ A_5 is simple gp & Kerf is normal subgp of A_5

$\therefore$ $Ker f = \{e\}$ or A_5

Case-I, If $Ker f = \{e\}$

then $A_5 \approx$ Subgp. of S_4

$\rightarrow \leftarrow$

As $o(A_5) = 60$ ↑
$o(S_4) = 24$

Case-II If $Ker f = A_5$

then f is trivial homomorphism

$\therefore$ No. of gp. homomorphisms $= 1$

$\Rightarrow$ op(1), TRUE

② Let $p \geq 23$ be a prime no. s.t the decimal expansion (base 10) of $1/p$ is periodic with period $p-1$ (i.e $\frac{1}{p} = 0.\overline{a_1 a_2 \cdots a_{p-1}}$) with $a_i \in \{0,1,\ldots,9\}$ $\forall i$ and for any $m, 1 \leq m < p-1$, $1/p \neq 0.\overline{a_1 a_2 \cdots a_m}$). Let $(\mathbb{Z}/p\mathbb{Z})^*$ denote the multiplicative gp. of integers mod p.

Then w.o.t.f is correct?

1). The order of $10 \in \mathbb{Z}_p^*$ is proper divisor of $(p-1)$.
2). The order of $10 \in \mathbb{Z}_p^*$ is $(p-1)/2$
3). The elt. $10 \in \mathbb{Z}_p^*$ is generator of the gp. $\mathbb{Z}_p^*$.
4). The gp. $\mathbb{Z}_p^*$ is cyclic but NOT generated by elt 10.

Sol^n $\frac{1}{p} = 0.\overline{a_1 a_2 \ldots a_{p-1}}$, $a_i \in \{0,1,\ldots,9\}$

& for $1 \le m < p-1$, $\frac{1}{p} \ne 0.\overline{a_1 a_2 \ldots a_m}$

Now.

$\because$ $10^{p-1} * \frac{1}{p} = a_1 a_2 \ldots a_{p-1} . \overline{a_1 a_2 \ldots a_{p-1}}$

$= a_1 a_2 \ldots a_{p-1} + .\overline{a_1 a_2 \ldots a_{p-1}}$

$= a_1 a_2 \ldots a_{p-1} + \frac{1}{p}$

$\Rightarrow$ $10^{p-1} \equiv 1 + p * a_1 a_2 \ldots a_{p-1}$

$\Rightarrow$ $10^{p-1} \equiv 1 \pmod{p}$, $p \ge 23$ prime

$\Rightarrow$ Order $10 = p-1$ & $p-1$ is least $\longrightarrow$ op(1). FALSE

$=$ Order of $\mathbb{Z}_p^*$

$=$ $U(p)$ $= \langle 10 \rangle$

$\Rightarrow$ op(3). TRUE ✓

(3) Given integer a, b, Let $N_{a,b}$ denote the no. of positive integers $k < 100$ s.t $k \equiv a \pmod 9$ & $k \equiv b \pmod{11}$.

Then w.o.t.f is correct?

1). $N_{a,b} = 1$ $\forall$ integers a & b

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