Real Analysis CSIR-NET Solution

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Year-wise Solution

-2019 December
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Que (3M) What is the sum of the following series?

$\left(\frac{1}{2\cdot 3}+\frac{1}{2^2\cdot 3}\right)+\left(\frac{1}{2^2\cdot 3^2}+\frac{1}{2^3\cdot 3^2}\right)+\cdots+\left(\frac{1}{2^n 3^n}+\frac{1}{2^{n+1}\cdot 3^n}\right)+\cdots$

(1). $\frac{3}{8}$ (2). $\frac{3}{10}$ (3). $\frac{3}{14}$ (4). $\frac{3}{16}$

(Ans: 2)

Sol^n Write it in summation form, then Solve using geometric series.

$$l = \sum_{n=1}^{\infty}\left(\frac{1}{2^n 3^n}+\frac{1}{2^{n+1}\cdot 3^n}\right)$$

$$\Rightarrow l = \sum_{n=1}^{\infty}\left(\frac{1}{2^n\cdot 3^n}+\frac{1}{2}\cdot\frac{1}{2^n\cdot 3^n}\right) = \sum_{n=1}^{\infty}\left(1+\tfrac{1}{2}\right)\left(\frac{1}{6^n}\right)$$

$$= \frac{3}{2}\sum_{n=1}^{\infty}\frac{1}{6^n} = \frac{3}{2}\cdot\left[\frac{1/6}{1-1/6}\right] = \frac{3}{2}\times\frac{1}{5} = \frac{3}{10}$$

$= \frac{3}{10}$ $\Rightarrow$ op(2). TRUE

$\left(\text{Recall } \sum_{n=1}^{\infty} a^n = a + a^2 + \cdots = \frac{a}{1-a}\right)$

— P. Kalika Notes

Que. Let n be a fixed natural no. Then the series –

$$\sum_{m \geq n} \frac{(-1)^m}{m} \quad \text{is —}$$

(CSIR-NET DEC-2019 (4.75 M))

(1). Absolutely Convergent
(2). Divergent
(3). Absolutely convergent if $n > 100$
(4). Convergent.

(Ans: 4)

Sol^n. We have given a Alternate series –

$$\sum_{m \geq n} \frac{(-1)^m}{m} \qquad —(*)$$

(Recall: Lebnitz Test, ~~a_n~~, If $\langle a_n \rangle$ is $\downarrow$ s.t $\lim_{n \to \infty} a_n = 0 \Rightarrow \sum (-1)^n a_n$ is cgt)

Here, $a_m = 1/m$ & $\sum \frac{(-1)^m}{m} = \sum_{m \geq n} (-1)^m . a_m$

So, by Lebnitz Test,

given series $(*)$ is cgt. $\Rightarrow$ op(4). TRUE

So, op(2). FALSE

For Absolutely

$\because \sum_{m \geq n} \left| \frac{(-1)^m}{m} \right| = \sum_{m \geq n} 1/m \Rightarrow$ divergent

(whatever you take n, it will be still dgt.)

$\Rightarrow$ op(1) & op(3). FALSE

P. Kalika | CSIR-NET- June 2019 | K. Munesh

Real Analysis

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① Let $f:\mathbb{R}\to\mathbb{R}$ be a cts. & one-one f^n. Then w.o.t.f is true ?

C/2

(1). f is onto

(2). f is either strictly $\downarrow$ or strictly $\uparrow$

(3). $\exists$ $x\in\mathbb{R}$ s.t $f(x)=1$

(4). f is unbounded.

Ans: 2

Sol^n We solve this problem by discarding op.s with Counter examples.

e.g. let $f(x)=e^x$, then $f(x)$ is cts + one-one

But $f(x)\neq 0$ $\forall x\in\mathbb{R}$ $\Rightarrow$ f is not onto $\Rightarrow$ op(1) FALSE

Next op(3). Let $f(x)=1+e^x$ which is cts + one-one

But $f(x)\neq 1$ $\forall x\in\mathbb{R}$ $\Rightarrow$ op(3). false.

(Hint: op(1) & op(3) are Related)

Next op(4). Let $f(x)=\frac{1}{1+e^x}$ then f is cts & one-one

As $0<e^x+1<\infty$ $\Rightarrow$ $0<\frac{1}{1+e^x}<1$ thus bounded

So op(4). FALSE as $f(x)$ is bdd.

$\Rightarrow$ op(2). is Correct.

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② Let $g_n(x)=\frac{nx}{1+n^2x^2}$, $x\in[0,\infty)$. w.o.t.f is true as $n\to\infty$?

(1). $g_n\to 0$ pointwise but NOT uniformly?

(3M)

① Consider the f^n $\tan x$ on the set

$$S = \{x \in \mathbb{R} : x \geq 0, x \neq k\pi + \pi/2 \text{ for any } k \in \mathbb{N} \cup \{0\}\}$$

We say that it has a fixed point in S if $\exists\, x \in S$ s.t $\tan x = x$. Then —

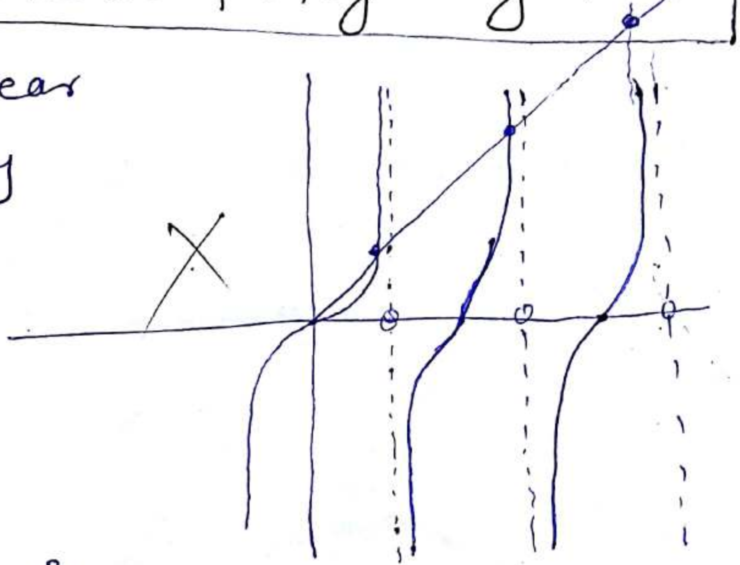
1. There is a unique fixed pt.
2. There is no fixed pt.
3. There are infinitely many fixed pt.s
4. There are more than one but finitely many fixed pt.

Ans: 3

Sol^n Geometrically, it is clear that, $\exists$s infinitely many (or countably many) pt.s $x_0 \in S$ s.t

$\tan x_0 = x_0$

Thus only op (3). true



NB: Fixed Pt., $\exists\, x_0 \in$ Domain s.t $f(x_0) = x_0$ then x_0 is s.t.b fixed point.

geometrically, look at the points of intersection of $y_1 = x$ & $y_2 = f(x)$, where $y_1 = y_2$, those pt.s are fixed point.

(3M)

(2). Define $f(x) = \frac{1}{\sqrt{x}}$ for $x > 0$. Then f is uniformly cts

1. on $(0, \infty)$
2. on $[r, \infty)$ for any $r > 0$
3. on $(0, r]$ for any $r > 0$
4. Only on intervals of the form $[a, b]$ for $0 < a < b < \infty$

Ans: 2

Sol^n $\because f'(x) = (x^{-1/2})' = -\frac{1}{2} x^{-3/2} = \frac{-1}{2x\sqrt{x}}$

& $|f'(x)| < K$ for finite K when $x \to \infty$ & $x \neq 0$

thus $f(x)$ is uniformly cts on $[r, \infty)$, $r > 0$

[We leave $x=0$ case, bcz f^n is not defined at $x=0$ i.e $\lim_{x\to 0} f(x) = \infty$, i.e not exists]

Thus op(2). True.

II-method on $I = (a, b]$ or $[a, b]$ or $[a, \infty)$

$\because \lim_{x\to 0} f(x) = \infty \Rightarrow f(x)$ is not U.cts on $(0, \infty)$

So, take any $a > 0$,

Now $\lim_{x\to\infty} f(x) = 0$ & $\lim_{x\to a} f(x) = \frac{1}{\sqrt{a}}$ = finite

so $f(x)$ is uniformly cts on $[a, \infty)$, $a > 0$.

(3M)

(3). Consider the map $f: \mathbb{Q} \to \mathbb{R}$ defined by (i) $f(0) = 0$ (ii) $f(r) = \frac{p}{10^q}$ where $r = \frac{p}{q}$, $p \in \mathbb{Z}$, $q \in \mathbb{N}$ and $\gcd(p,q) = 1$. Then the map f is— see on P-32

1. 1-1 & onto
2. not 1-1 but onto
3. onto but not 1-1
4. ✓ neither 1-1 nor onto.

sol^n $\because f(0) = 0$ & $f(\frac{p}{q}) = \frac{p}{10^q}$, $p \in \mathbb{Z}$, $q \in \mathbb{N}$

~~let~~ $\because r_1 = \frac{p}{q} = \frac{1}{10}$ & $r_2 = \frac{10}{11}$

Since $r_1 \neq r_2$ but $f(r_1) \neq f(r_2)$. so NOT 1-1

$\because r \in \mathbb{R}$ & $r = \frac{p}{q} \neq \frac{p}{10^q}$ $\forall p, q$, ~~thus~~ f is NOT ONTO.

~~thus f^n is not well-defined~~

Hence, f is neither 1-1 nor onto

$\Rightarrow$ op(4). True.

From Here: 2018-JUNE Solution

P.KALIKA

JUNE- 2018 || Real

C (1)

[Part-B]

3M (28) ① Let $f(x,y) = \log(\cos^2(e^{x^2})) + \sin(x+y)$ then $\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} f(x,y)\right)$ is ——

1. $\frac{\cos(e^{x^2}) - 1}{1+\sin^2(e^{x^2})} - \cos(x+y)$
2. 0
3. $-\sin(x+y)$ ✓
4. $\cos(x+y)$

Sol^n

$$f(x,y) = \underbrace{\log(\cos^2(e^{x^2}))}_{\text{free from } y} + \sin(x+y)$$

so $\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}(\log(\cos^2(e^{x^2})))\right) = 0$

$$\Rightarrow \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = 0 + \frac{\partial}{\partial y}\left(\frac{\partial(\sin(x+y))}{dx}\right)$$

$$= \frac{\partial}{\partial y}(\cos(x+y))$$

$$= -\sin(x+y) \text{ — } ③ \checkmark$$

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as $\frac{\partial}{\partial y}\left(\frac{\partial(\cos^2(e^{x^2}))}{dx}\right) = \frac{\partial}{\partial y}\left(2\cdot\cos(e^{x^2})\cdot -\sin(e^{x^2})\cdot 2x e^{x^2}\right)$

$$= \frac{\partial}{\partial y}\left(-\sin(2e^{x^2})\cdot 2xe^{x^2}\right)$$

$$= 0$$

3M

(2)

(28) If $\{x_n\}$ is cgt seq. in $\mathbb{R}$ and $\{y_n\}$ is a b.dd seq. in $\mathbb{R}$, then we can conclude that —

1. $\{x_n+y_n\}$ is cgt
2. $\{x_n+y_n\}$ is b.dd ✓
3. $\{x_n+y_n\}$ has no cgt subseq. ✗
4. $\{x_n+y_n\}$ has no b.dd subseq. ✗

Sol^n let $\{x_n\} = 2$, $\{y_n\} = 3$ then $\forall n \in \mathbb{N}$
$\{x_n+y_n\}$ is cgt.

Also, we know that a cgt seq. has all cgt subseq. , so (3) ✗

Since it is cgt. ⇒ also bdd subseq.

⇒ (4) discarded

Now (1) & (2)

$\because$ $\{x_n\}$ is cgt ⇒ $\{x_n\}$ is b.dd

Also, given that $\{y_n\}$ is b.dd

⇒ $\{x_n+y_n\}$ is b.dd — (2) ✓

But it is not necessary that it is cgt.

eg. let $\{x_n\} = 1$, $\{y_n\} = (-1)^n$
$= \{1, -1, 1, -1, \ldots\}$

then $\{x_n+y_n\} = \{2, 0, 2, 0 \ldots\}$

which is bdd but NOT cgt

(NOT cgt. bcz limit pt. is NOT unique)

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