Kalika
Central University of Rajasthan
M.Sc Tech Maths 2015-18
Discrete mathematics
klkaprsd@gmail.com

www.awgp.org
ॐ भूर्भुवः स्वः तत्सवितुर्वरेण्यं भर्गो देवस्य धीमहि धियो यो नः प्रचोदयात्
नाम Kalika
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DS - $\frac{K}{80}$ | $\frac{M}{83}$

$\frac{10}{20}$, $\frac{18.5}{20}$

K.S. Rao

MTM-104 Discrete Mathematics Structure

1. Logic
2. Graph theory
3. Lattices
4. Boolean Algebra.

Syllabus

(1) Discrete mathematical structure
-Kolman, Busby & Ross — Pearson

I. Formal Logic - statements, Symbolic Representation and Tautologies, Quantifiers, Predicates and validity, Propositional logic, Lattices; Partially ordered sets and lattices, Hasse Diagrams, Lattices as algebraic systems, sub-lattices, Direct product and Homomorphism, Complete lattices, modular lattices, Distributed lattices, the complemented lattices, Convex sub lattices, Congruence relations in lattices.

II. Graphs, complete graphs, Regular graphs, Bipartite Graphs, Vertex degree, Subgraphs, paths and cycles, the matrix Representation of Graphs, fusion, trees & Connectivity, bridge - spanning trees, connector problems, Shortest path problems, cut vertices & Connectivity

MTMA-104 | Discrete Mathematics Structure

1. Logic
2. Graph theory
3. Lattices
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Syllabus

(4) Discrete Mathematics Structure
- Kenneth Rosen & Kenneth Rosen

I. Formal Logic - statements, Symbolic Representation and Tautologies, Quantifiers, Predicates and Validity, Propositional Logic, Lattices: Partially ordered sets and Lattices, Hasse Diagrams, Lattices as algebraic systems, Sub-Lattices, Direct Product and Homomorphism, Complete Lattices, Modular Lattices, Distributed Lattices, The Complemented Lattices, Bounded Lattices, Congruence relations in Lattices.

II. Graphs, Complete graphs, Regular graphs, Bipartite Graphs, Vertex degree, Subgraphs, Paths and Cycles, The matrix Representation of Graphs, Fusion, Trees & Connectivity, bridge, Spanning trees, Connector problems, Shortest path problem, Cut vertices & Connectivity

Handshaking Lemma: In any graph, the sum of degree of all the vertices is equal to twice the no. of edges.

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→ Simple Graph: A graph G without loop and parallel edges.

→ In a graph, no. of odd degree vertices are always in even no.

$$(2e = \sum_{u \in V_1} d(v) + \sum_{v \in V_2} d(v))$$

→ If G is a simple graph with n-vertices then degree of every vertices will be at most $(n-1)$.

→ The maximum no. of edges in a simple graph with n vertices is $\frac{n(n-1)}{2}$.

Sol. $2e = \sum_{i=1}^{n} d(v_i) = d(v_1) + d(v_2) + d(v_3)$

$\leq (n-1) + (n-1) + (n-1)$

$\leq n(n-1)$

$2e \leq n(n-1)$

$$\boxed{e \leq \frac{n(n-1)}{2}}$$

Pigeonhole Principle :—

If there are n item that to be distributed in m ~~holes~~ where $n > m$ then at least one hole will contain more than one elements.

Theorem: In a simple connected graph ~~at least two~~ ~~vertices~~ (there are) at least two vertices of same degree.

Proof: Let G be a simple graph with n vertices. Let v is arbitrary vertex of G.

$$d(v_i) \in \{1, 2, 3, \ldots, (n-1)\}$$

Since these n vertices have to be ~~distinct~~ distributed $(n-1)$ degrees, then by the Pigeonhole Principle at least two vertices should have same degree.

Theorem: The maximum no. of edges in a simple graph with n-vertices and 'k' components are

$$\frac{(n-k)(n-k+1)}{2}.$$

(Int 2014)

Proof: Let G be a graph with 'n' vertices and k components namely $G_1, G_2, G_3, \ldots, G_k$ and $n_1, n_2, n_3, \ldots n_k$ are number of ~~not~~ vertices.

$$n = n_1 + n_2 + \cdots + n_k$$

and $$\sum_{i=1}^{k}(n_i - 1) = \sum_{i=1}^{k} n_i - k = n - k$$

$$\Rightarrow \left[\sum_{i=1}^{k}(n_i - 1)\right]^2 = (n-k)^2$$

$$\Rightarrow \sum_{i=1}^{k}(n_i - 1)^2 + 2\sum_{i=1}^{k}\sum_{j>i}(n_i - 1)(n_j - 1) =$$

$$= n^2 - 2nk + k^2$$

$$\Rightarrow \sum_{i=1}^{k}(n_i-1)^2 \le n^2-2nk+k^2$$

$$\Rightarrow \boxed{\sum_{i=1}^{k} n_i^2 \le n^2-2nk+k^2+2n-k}$$

The maximum no. of edges in the i^{th} Component (G_i) $\Rightarrow \dfrac{n_i(n_i-1)}{2}$ edges

The maximum no. of edges in G are

$$\sum_{i=1}^{k}\frac{n_i(n_i-1)}{2} = \frac{1}{2}\left(\sum_{i=1}^{k} n_i^2 - n\right)$$

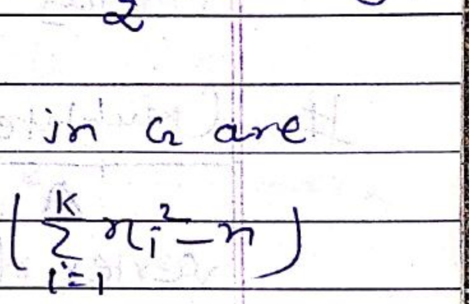
$$\le \frac{1}{2}\left(n^2-2nk+k^2+2n-k-n\right)$$

$$= \frac{(n-k)(n-k+1)}{2}$$

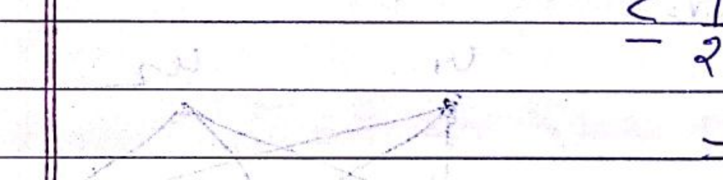
Bipartite Graph :-

$E_1 = \{v_1, v_4, v_6, v_7\}$

$E_2 = \{v_2, v_3, v_5\}$



Bipartite Graph



not Bipartite graph

* Def^n :-

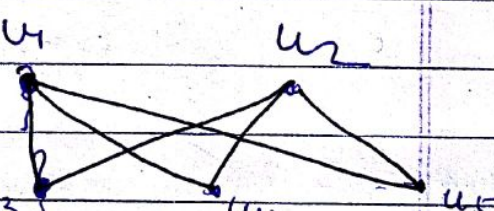
A graph $G=(V,E)$ is called a Bipartite Graph if its vertex set of 'V' can be partitioned in two sets V_1 & V_2 s.t not any two vertices of V_1 are adjacent, also not any two vertices of V_2 are adjacent and edges are possible only from vertices of V_1 to V_2 (or V_2 to V_1)

Complete Bipartite Graph

A bipartite graph s.t each vertex of V_1 is adjacent to all the vertices of V_2.

$V_1 = \{$

$V_2 = \{$



Theorem: The number of edges in complete bipartite graph with n vertices is at most $n^2/4$.

Proof:- Let G be a complete bipartite graph with n-vertices.

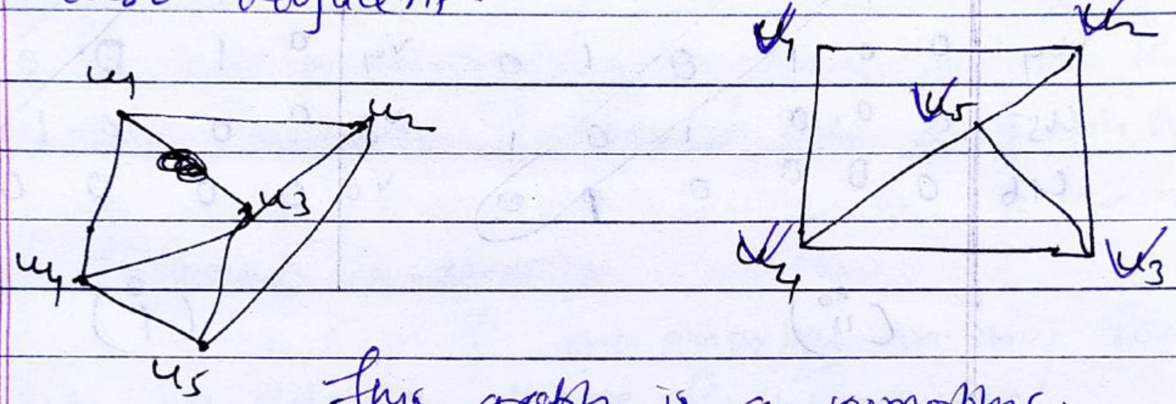
The maximum no. of edges to be G is a complete bipartite graph are

$f(x) = x(n-x)$.

Two graphs are s.t.b ISOMORPHIC if there is one-to-one correspondence b/w their vertices & their edges, s.t incidences are preserved.

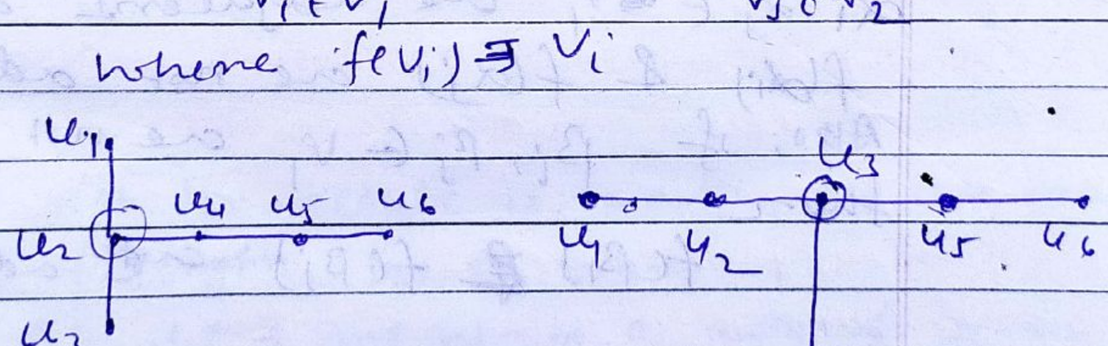
Isomorphic Graph

Let $G_1 = (V_1, E_1)$ & $G_2 = (V_2, E_2)$ are two graphs. Then G_1 and G_2 are said to be isomorphic if $\exists$ a bijection $f(v): V_1 \to V_2$ s.t if $\alpha_i, \alpha_j \in V_1$ are adjacent then $f(\alpha_i)$ & $f(\alpha_j)$ are also adjacent.



This graph is a isomorphic

(i) no. of (V_1) = No. of (V_1)
(ii) no of (E_1) = No. of E_2
(iii) No. of $d(v_i)$ = no. of (v_j)
$v_i \in V_1$ $\quad$ $v_j \in V_2$
where $f(v_i) \ni v_i$



This graph is not isomorphic.

+) **Adjacency matrix:**— A matrix $[a_{ij}]_{n \times m}$ of a graph is a matrix s.t

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \,\&\, v_j \text{ are adjacent} \\ 0 & \text{if } v_i \,\&\, v_j \text{ are not adjacent} \end{cases}$$

	u_1	u_2	u_3	u_4	u_5	u_6
u_1	0	1	0	0	0	0
u_2	1	0	1	0	0	0
u_3	0	1	0	1	0	0
u_4	0	0	1	0	1	0
u_5	0	0	0	1	0	1
u_6	0	0	0	0	1	0

(ii)

	v_1	v_2	v_3	v_4	v_5	v_6
v_1	0	1	0	0	0	0
v_2	1	0	1	0	0	0
v_3	0	1	0	0	0	0
v_4	0	1	0	0	0	0
v_5	0	0	0	1	0	1
v_6	0	0	0	0	1	0

(i)

Complement of Graph

Let $G_1 = (V_1, E_1)$ is a graph, then a graph $G_2 = (V_2, E_2)$ is said to be complement of a graph G_1 if $\alpha_i, \alpha_j \in V_1$ are adjacent then $f(\alpha_i)$ & $f(\alpha_j)$ are not adjacent. Also, if $\beta_i, \beta_j \in V_1$ are not adjacent then,

$f(\beta_i)$ & $f(\beta_j)$ are adjacent.

Walk, Path & trail.

A walk in a graph $G = (V, E)$ is a seq. of pairs (V_{TT}, E_{TT}) where

$V_{TT} = v_1\ v_2, v_3 \ldots v_n$

$E_{TT} = e\ e_2\ e_3 \ldots e_n$

walk – $v_1 e_1 v_2 e_2 v_3 e_3$

trail – A walk in which no edges repeated

path – A walk in which no vertices repeated

s.t v_i vertex next to v_j then v_i and v_j should be adjacent.

Path is a walk s.t edges are not repeating.

Some properties of adjacency matrix:-

The adjacency matrix of a digraph is not a usually symmetrical about the main diagonal. Also if the digraph has no loops, then each entry on the main diagonal is zero '0'.

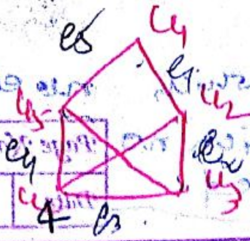
The sum of the entries in any row is the outside degree of the vertex corresponding to that row, and the sum of the nos in any column is the in-degree of the vertex corresponding to that column.

Theorem: (1) If $u \neq w$, then every walk connecting u and w contains a subsequence which is a path connecting u & w.

Proof:- Let the result is true for, for the walk of length $1, 2, 3, \ldots, n-1$.

let $u \neq w$ and W is a walk of length n, connecting u & w, if the vertices during the walk are unique, then it is a path.

let $u \neq w$ & W is a walk of length n connecting u and w if any vertices during the walk is repeating then $\exists\ i < j$ s.t $u_i = u_j$ then the walk - -

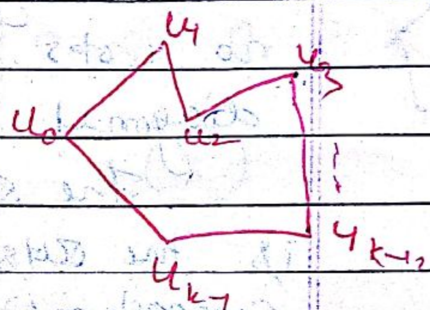


$u_1 e_1 u_2 e_2 u_3 e_3 u_5 e_6 u_4 e_4 u_4 e_5$

$u_0 u_1 \ldots u_i u_{i+1} \ldots u_n$ connects u and w and its length is less than and equal to $n-1$.

Theorem: A graph in which every vertex has deg. at least 2 contain a cycle.

Pf:— Let p be path of max. length in G and let its vertices are $v_0, v_1, v_2, \ldots v_k$. If v_k is terminal vertex and $u_0 = v_k$ Then degree of v_k is '2' and it contains a cycle. If v_k is not a terminal vertex then $\exists$ a $w \neq v_{k-1}$ (if graph is simple)



Then $p \cup \{v_k, w\}$ is a graph of length greater than p, which is contradiction of our hypothesis. Here $v_k = v_0$ i.e graph contains a cycle.

* **Eularian Graph:** In a graph, a circuit that contain every edges of G is called Eular Circuit & graph G is called Eularian graph.

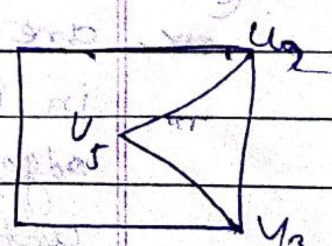
A graph G is called Eularian if it contains a closed path in G.

Theorem: Every closed trail containg a given ~~trail~~ vertex has a cycle containg that vertex.
(1)

Proof: Let the result is true for the trail of length 1, 2 ... (n-1) closed.

Let T be a closed trail of length n containg a specific vertex v. If closed trail containing all the vertices. Therefore the Result is proved.

If closed trail T is not containg all the vertices. Then there will be two cycles appeared in the trail T within atleast of a common vertex, where $i < j$ s.t. $v_i = v_j$



So that, there will be two cycles as

$$v_0, v_1, \dots, v_i, v_{i+1}, v_{i+2} \dots v_n = v_0$$

and $v_i v_{i+1} \dots v_j$

Both the closed trail in G of length $\leq n-1$ and at least one must contain the specified vertex. Hence by I.H these closed trail will contain a cycle, so that result is proved.

(mt 2014)

Theorem: ② A non-empty connected graph G is Eulerian $\Leftrightarrow$ all of its vertices are of even degree.

Proof: Let G is an Eulerian Graph.

$\Rightarrow$ Then it must contain a closed Eulerian ~~path~~ graph i.e initial and final vertices are same and it contains all the edges exactly once. If we ~~able~~ travel along a path P, then each time we visit a vertex v, we use two edges one in and one out.

(p-110)

Therefore $d(v)$ = even. Since v is arbitrary, Hence each vertices have even degree.

$\Leftarrow$ Again let degree of each vertices are even. We can construct a path in G starting from the edge e. We add edges in the path P one by one until it reaches to the initial vertex.

If P is not closed at any step

i.e P start at u but ends at $v \neq u$, then Many odd no. of edges appeared at v. which is not possible as per our hypothesis. Hence we can extend P by adding another edges incident with v.

Continue this process until P returns to u if P contains all the edges of G. then the result is proved.

Suppose that P does not contain all the edges of G. Then we may delete all the edges of G to get a subgraph T of G, since degree of each vertices of G are even, then degree of each vertices of T are even.

T may not be connected, but at least one vertex of T is common with G. say v.

If we start a path in T starting from e. If $G = P \cup T$ has an Eulerian (~~Eulerian path/trail~~)

A path in a graph G consists a pair (V_π, E_π) of sequence: a vertex sequence V_π: $v_1, v_2, \ldots v_k$ and edge seq. E_π: $e_1, e_2, \ldots, e_{k-1}$ for which

(i) Each successive pair v_i, v_{i+1} of vertices is adjacent in G and e_i has v_i and v_{i+1} as end points for $i=1, \ldots k-1$.

(ii) No edge occurs more than once in the edge seq.

Eular Path

A path in a graph G is Euler path if it includes every edge exactly once.

Eular Circuit

A graph having Eular Circuit is known as Eularian.

Hamiltionian Path

A path is a Hamiltionian path that contains/consists each vertex exactly once.

Theorem: A connected graph G has an Eulerian path iff it has at most two vertices of odd degree.

1st 2014

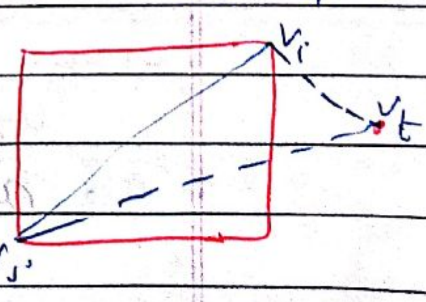
Proof:- Suppose G has an Eulerian path. In the middle of path, degree of each vertex should be even.

and there is only one edge associated with ~~even~~ each end vertex of the path. Then end vertex must with be odd degree. Hence the graph has exactly two vertices of odd degrees.

$\Leftarrow$ Suppose that G is a connected graph with at most two vertices of odd degree. If G has no odd vertices then G is Eulerian. and hence it has an Euler path. If not then let v_i and v_j be only two vertices with odd degree then there are two cases.

I. Suppose v_i & v_j are not adjacent.
If we join an edges e_{ij}, which is incident with v_i & v_j, then this new graph is Eulerian, hence has a closed Euler path. Now deleting e_{ij} from the closed Euler path, we get a Euler path

II. Suppose v_i & v_j are adjacent. In this case ~~the vertex~~ we adjoin the vertex set of G with a new vertex v_t and the edge set of G with two new edges $e_{it} = (v_i, v_t)$ and $e_{jt} = (v_j, v_t)$



Ref
H.W

Fleury's Algorithm

Let $G = (V, E)$ be a graph ~~whaving~~ having degree of each verticy even.
To find ~~path~~ Eularian circuit, we follows:

Step-I, Choose any arbitrary vertex v_0 from the vertex set V as a starting vertex.

Step-II, Select an edge $e = (v_0, v)$. If there are many edges then select an edge which is not a bridge. Extend the path $v_0 v$ and delete the edge e from the edge set.

$E_1 = E - \{e\}$

* * *

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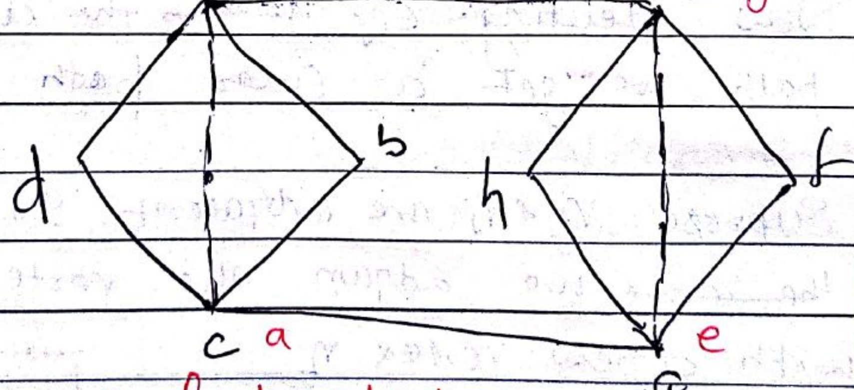
Start with odd deg vertex.

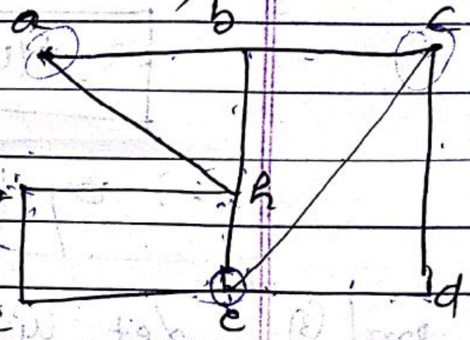


agefghecbodca

aegfehgcbacda

Hamiltonian Path :-

Theorem: (1) If G is a connected simple graph with $n \geq 3$ vertices and $d(u) + d(v) \geq n$ for each pair of non adjacent vertices u & v in G. Then G is Hamiltonian



$d(v) \geq 6$

$n(v) = 8$
$n(e) = 11$

$n = 8$

Not Hamil

$d(u) \geq 4$

Theorem: (2) Let G be a connected simple graph with $n \geq 3$ vertices. If for every vertex u of G, $d(u) \geq n/2$, then G is Hamiltonian

Theorem: (3) Let the no. of edges in a connected simple graph G are m and no. of vertices be n. Then G have Hamiltonian circuit if

$$m \geq \frac{1}{2}(n^2 - 3n + 6)$$

Eg: In above graph $n(e) = 11$, $n = 8$

$$11 \geq \frac{1}{2}(64 - 24 + 6)$$

$$\geq \frac{1}{2} \times 46 = 23$$

$$\ngeq 23.$$ (so not Hamiltonian)

→ graph is also Hamiltonian path.

Proof (2) Let (u_i, u_j) is arbitrary pair of non-adjacent vertices in G then

$$\deg(u_i) = d(u_i) \geq \frac{n}{2}, \quad d(u_j) \geq n/2$$

$$\boxed{\therefore d(u_i) + d(u_j) \geq \frac{n}{2} + \frac{n}{2} = n}$$

$\therefore$ By theorem (1), G is Hamiltonian

Proof (3) Let u_i and u_j are any non-adjacent vertices in G. If we remove u_i and u_j from G, we get a subgraph H with $n-2$ vertices and $m - d(u_i) - d(v_j)$ edges.

A graph, min $(n-2)$ vertices has atmost ${}^{n-2}C_2$ edges.

$$\therefore m - d(u_i) - d(v_j) \leq \frac{(n-2)(n-3)}{2}$$

$$\Rightarrow d(u_i) + d(u_j) \geq m - \frac{1}{2}(n^2 - 5n + 6)$$

$$\geq \frac{1}{2}(n^2 - 3n + 6) - \frac{1}{2}(n^2 - 5n + 6)$$

$$\geq n$$

$\Rightarrow$ G is hamiltonian

Proof (1)

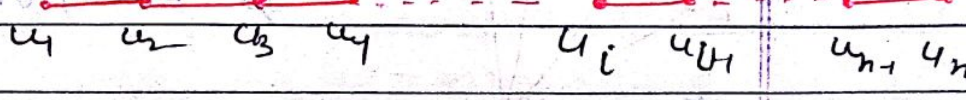
We prove it by contradiction.

Suppose G satisfies the given condition but not Hamiltonian.

but it may become Hamiltonian by just adding one edge.

Now,

Let u & w are non-adjacent vertices in G. Since graph is connected, then their is a path connecting u_1 and u_n of the graph.



(i) make u_1 and u_{i+1} adjacent.

(ii) Not making u_1 & u_n adjacent.

* * *

Mon 10/08/15

Theorem: If G is a simple graph (1) with $(n \geq 3)$ vertices and if —— thm (1)

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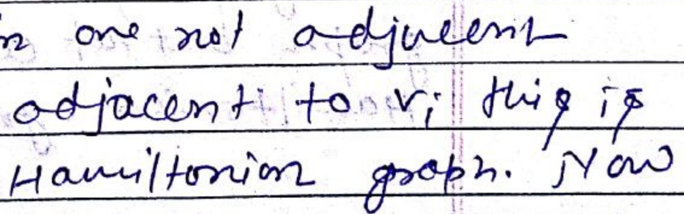
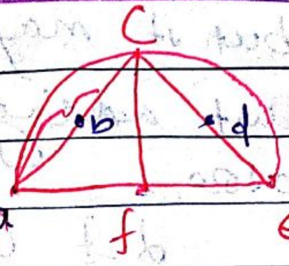


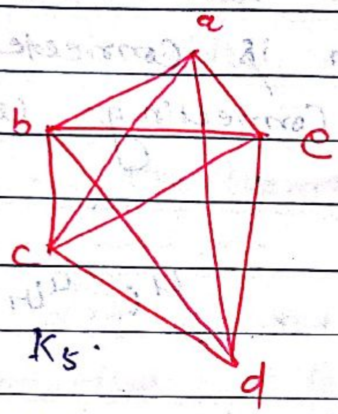
Now let v_1 & v_n are not adjacent then let v_1 is adjacent to v_i this is not going a Hamiltonian graph. Now make v_n adjacent to v_{i-1} as it was taken that by adding just one edge graph becomes Hamiltonian. This ~~becomes~~ gives Hamiltonian graph.

* Graph:

1. Hamiltonian but not Eularian.



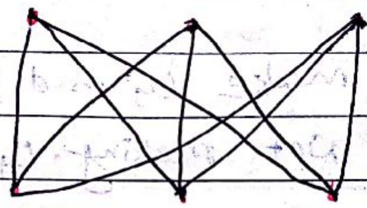
2.



K_5.

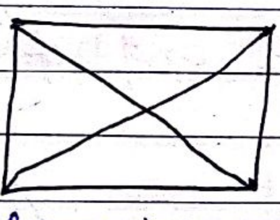
Every complete graph with vertices $n \geq 3$ is Hamiltonian.

3. This graph is Hamiltonian.



Every complete Bipartite graph is Hamiltonian.

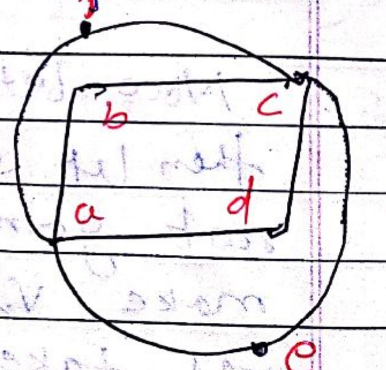
4.



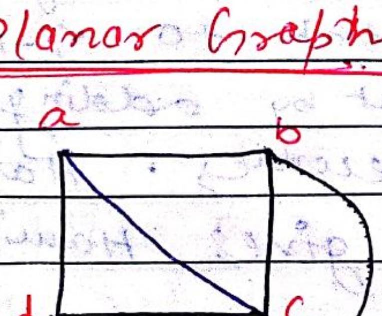
Complete graph is not Hamiltonian graph

$d_{max}(v) = max.\{d(v) \mid v \text{ is vertices of } G\}$

5. This graph is Eularian graph but not a Hamiltonian graph



Planar Graph



K_4 is planar graph.
K_5 is not planar graph.

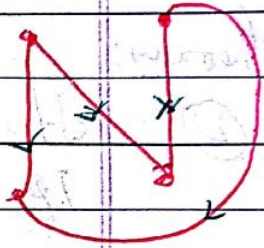
planner graph.

A ~~graph~~ complete graph with $n \geq 5$ is ~~not~~ not a planar

~~$K_{2,2}$~~ (i.e A graph is called planar graph if its edges not cross other edges).

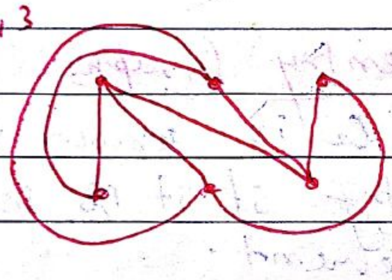
$K_{2,2}$

Planar graph



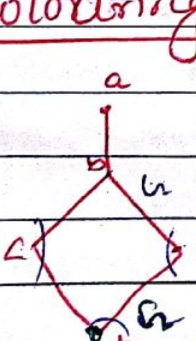
$K_{2,3}$

$K_{3,3}$



A complete Bipartite graph is planar if $m \leq 3$ then $n \leq 2$, where m and n are no. of vertices in the two partition of the

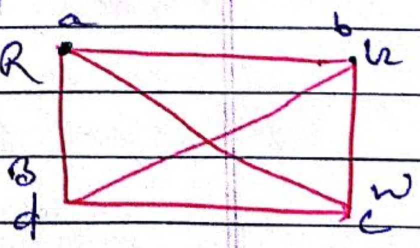
Colouring of Graph :



chromatic number is the min. number of colours needed to colour a graph.

$K(G) = 2$

K_4

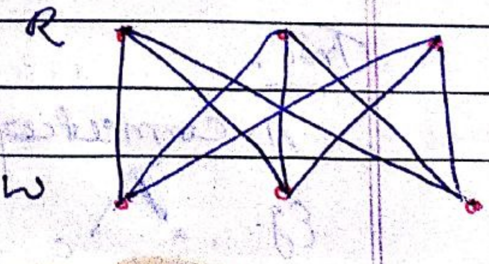


$K_4(G) = 4$

$K(K_n) = n$

$K(K_{m,n}) = 2$

$K_{3,3}$



Theorem:

(1) A graph is s.t.b connected iff its vertex set V can be partitioned into two subsets V_1 and V_2 s.t their exists an edge in G whose one vertex in subsets V_1 and the other is in subset V_2.

Theorem:

(2) If G is not connected then $\bar{G}$ is connected.

(3) Self-complementry Graph.

A graph G is known to be self complementry if it is isomorphic with its complement.

Then

(3) The self complementry graph G has 4n or 4n+1 vertices.

* * *

Tree:

A connected acyclic graph.

Eg. (f=1)

H.W

Theorem: If G is not-connected, then $\bar{G}$ is connected.

(1)

Proof: Let G is not connected and u, v are any vertices of G.

1. u and v are adjacent.
2. u and v are not-adjacent.

(2) ⇒ Hence they must be adjacent in $\bar{G}$. We are getting a path joining u and v.

(1) ⇒ u and v may be in different components and u and v may be in same components.

If u & v are in different components. Hence there is not any path connecting u & v in G. Then there must be path joining u and v in $\bar{G}$.

Again, if u & v in same component, let w be any vertex in second component, then there must be a path $u-w-v$ in $\bar{G}$.

Hence $\bar{G}$ is connected.

H.W

Theorem: A self complementary graph has $4n$ or $4n+1$ vertices.

(2)

Proof: Let G be a graph with m-vertices.

Then

$$E(G) + E(\bar{G}) = \frac{m(m-1)}{2}$$

$(\because E(G) = E(\bar{G}))$

$$2E(G) = \frac{m(m-1)}{2}$$

$$\Rightarrow E(G) = \frac{m(m-1)}{4} = \text{integer}$$

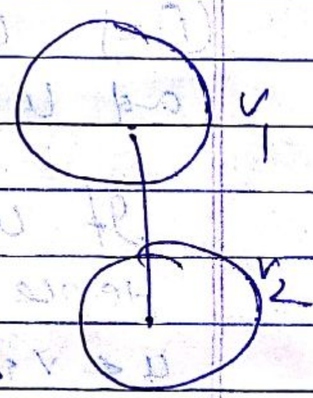
$E(G) = \frac{m(m-1)}{4}$ = integer

$\Rightarrow$ m or $m-1$ is multiple of 4.

$\Rightarrow$ $m = 4n$ or $m-1 = 4n$

$\Rightarrow$ $m = 4n$ or $m = 4n+1$

Theorem (3): A graph G is s.t.b connected iff its vertex set V can be partitioned into two components with vertices ~~with non-empty subsets~~ V_1 & V_2 s.t $\exists$ an edge in G whose one end in V_1 and other in V_2.



Proof: Let G is not connected. then it must have at least two components. Let G has two components (say) V_1 and V_2. Then
$V = V_1 \cup V_2$

Start $\Rightarrow$ Let $V = V_1 \cup V_2$, If there is an edge connecting V_1 and V_2, then G must be connected.

$\Leftarrow$ Let G be a connected graph and u & v are any two vertices of G. then there is a path connecting u and v. let it is $\underbrace{u, u_1, u_2, \dots u_i}_{V_1}, u_{i+1}, \underbrace{\dots v}_{V_2}$

it breaks

Tree: A connected acyclic graph

n vertices, $e = n-1$ edges

Euler's formula

Theorem (4) If G be a planar graph with n-vertices e-edges and f-faces there, then

$n-e+f=2$

(Hint: mathematical Induction $f=1,2,\ldots$)

* * *

or then $e-n+2$ faces (Regions).

i.e $f=e-n+2$;

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proof: If there is only one one region. in the graph (i.e $f=1$): If graph is tree, then it must has $(n-1)$ edges Then the given condition satisfied.

$$1=n-1+2-n \Rightarrow 1=1 \text{ check}$$

Let the graph has $(f>1)$ faces and satisfying the given condition. Suppose that graph has $(f+1)$ faces. If we remove one edge from graph then graph is still planar and connected. ~~then given condition satisfies~~

~~$f = e-n+2$~~

having f faces, hence satisfies the given condition. Again adding that edge, we get $(f+1)$ faces and $(e+1)$ edges and the given condition

$$f+1 = (e+1)-n+2$$

$$\Rightarrow \boxed{f = e-n+2}$$ Ans.

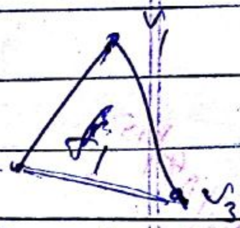
Euler's formula ($f = e - n + 2$)

Theorem (5): Let G be a connected planar graph with n-vertices and e-edges where $n \geq 3$. Then (i) $2e \geq 3f$

(ii) $e \leq 3n - 6$

Proof: First let $n = 3$, $e = 2$, $f = 1$

~~$e = 3$~~



If G = tree: $n - e + f = 3 - 2 + 1$
$= 2$

$n = 3,\ e = 3,\ f = 2$

If $G \neq$ tree $\Rightarrow n - e + f = 3 - 3 + 2 = 2$

(i) $2 \times 2 \geq 3 \times 1 \Rightarrow 4 \geq 3$

(ii) $2 \leq 3 \cdot 3 - 6 \Rightarrow 2 \leq 3$

(i) $3 \times 2 \geq 3 \times 2 \Rightarrow 6 \geq 6$

(ii) $3 \leq 3 \times 3 - 6 \Rightarrow 3 \leq 3$

Again let $n \geq 4$. If G is tree, then $e = n - 1$

~~$2(n-1) \geq 3 \Rightarrow 2n \geq 5$~~

$e = n - 1 \geq 4 - 1 = 3 > 3/2 \Rightarrow \boxed{2e \geq 3f}$

Also,
$(3n - 6) - e = (3n - 6) - (n - 1)$
$= 2n - 5 \geq 2 \cdot 4 - 5$
$= 3 > 0$

$\Rightarrow \boxed{3n - 6 \geq e}$

* * *

Euler's formula

$f = e - n + 2$ $\quad (\because 2e \geq 3f)$

$e = f + n - 2 < \frac{2e}{3} + n - 2$

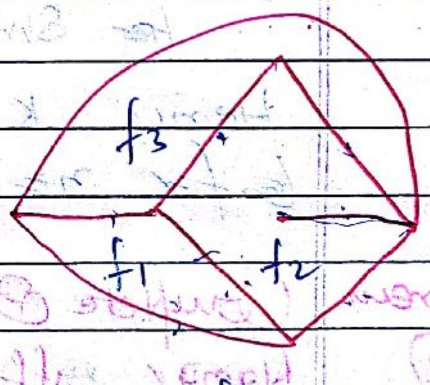
$= \frac{2e + 3n - 6}{3}$

$\Rightarrow 3e \leq 2e + 3n - 6$

$\Rightarrow$ ~~3e~~ $\underline{e \leq 3n - 6}$

Again let $n \geq 4$ and G is not a tree
$\therefore$ Therefore G have faces more than or equal to 2.

$d(f_1) = 3$
$d(f_2) = 6$
$d(f_3) = 4$
$d(f_4) = 3$



Suppose $b = \sum_{\varphi} d(\phi)$,

where ϕ denotes the faces of G, φ denotes set of faces of G. we have

~~$b \leq 3f$~~ $\quad b \geq 3f$

(In above example, $\varphi = \{f_1, f_2, f_3, f_4\}$
$b = \sum_{i=1}^{4} d(f_i) = 3 + 6 + 4 + 3 = 16$
& $16 \geq 3 \cdot 4$)

Also, $\boxed{b \leq 2e} \Rightarrow \boxed{3f \leq b \leq 2e}$

Theorem ① A complete graph K_n is not planar iff $n \geq 5$.

$\frac{5(5-1)}{2} = 10$

Let $n=5$, then K_5 has 10 edges. If K_5 is planar, then $e \leq 3n-6$. Hence
$10 \nleq 3 \cdot 5 - 6$, which is contradiction.
Hence K_5 is not a planar graph.

$\frac{6 \times 5}{2} =$

Let $n=6$, then K_6 has 15 edges. If K_6 is planner, then $e \leq 3n-6$. Hence
$15 \nleq 3 \cdot 6 - 6$, which is contradiction.

~~He~~ Similarly for $n=7$.

Hence K_n is not a planar graph for $n \geq 5$.

Theorem ② Complete Bipartite graph $K_{m,n}$ is not a planar iff $n \geq 3$, $m \geq 3$.

The Result can be proved if we show it for $K_{3,3}$. Here $n=6$ and $e=9$

$e \leq 3n-6 \Rightarrow 9 \leq 3 \cdot 6 - 6 \Rightarrow 9 \leq 12$

In $K_{3,3}$, at least 4 edges need to make a face.

$4f \leq 2e \Rightarrow f \leq \frac{e}{2}$

Euler's formula, $f = e-n+2$

$\Rightarrow e = f+n-2 \leq \frac{e}{2} + 6 - 2$

$\Rightarrow \frac{e}{2} \leq 4 \Rightarrow \frac{9}{2} \leq 4$ This is a contradiction

Similarly for $K_{3,4}$

2-Q

Polyhedral Graph:

Polyhedral Graph is a ~~graph~~ connected planar graph, in which each face has degree greater than or equal to 3. And each vertex has degree greater than or equal to 3.

Eg: There is no polyhedral graph with exactly 30 edges and 11 faces.

here $e = 30, f = 11$

$\therefore$ By Euler's formula

$$f = e - n + 2 = 30 - n + 2$$

$$11 = 32 - n$$

$$\Rightarrow n = -11 + 32$$

$$= 21$$

$3n < 3f < 2e$ $e \le 3n - 6$

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Here

$n = 21, \; f = 11, \; e = 30$ $(f = e - n + 2)$

$2 \cdot 30 \ge 3f$

$60 \ge$

$\Rightarrow 2e \ge 3f$

$2 \times 30 \ge 3 \cdot 11$

$60 \ge 33$

$n \le 3 \cdot 21 - 6$

$\le 63 - 6$

≤ 57

$e = \frac{21 \times 10}{2}$

$= 210$

$3n \le 2e$

$63 \le 60$

contradiction

$d(f) \ge 3$

$d(v) \ge 3$

$3n \le 2e$

$63 \le 60$

$\therefore \nexists$ any such polyhedral graph.

(2) ST, a connected planar graph with less than 30 edges has a vertex of deg ≤ 4.

(3) If G is a polyhedral graph, then there is a region of deg ≤ 5

(3)

Suppose that every vertex has deg ≥ 6. Then the sum of the deg. is greater than or equal to $6n$.

$2e \ge 6n$

$e \ge 3n$

or $e \ge 3n$

Since $\boxed{e \le 3n - 6}$

$3f \ge 2e$

$d(f) \ge 3$

$3n \le 3n - 6$

which is a contradiction

$f = e - n + 2$

(1) G is polyhedral graph.

$\because e = 30, \; f = 11$

$\Rightarrow$ G is planar and connected

$\Rightarrow f = e - n + 2$

$\Rightarrow n = 21$

$\because d(v_i) \geq 3$

$\therefore 3n \leq 2e$

$3 \cdot 21 \leq 2 \cdot 30 \Rightarrow 63 \nleq 60$

$\Rightarrow$ get a contradiction

(2) Let G be a ~~graph wit~~ connected planar graph, with edges ≤ 30.
we will prove it by contradiction.

here $e \leq 30$,

If graph has all the vertices with degree 5 or more. Then max. no. of

$5n \leq 2e \Rightarrow \boxed{n \leq \frac{2e}{5}}$

$e \leq 30, \; 5n$
$20 = 5n$
$e = 5n/2$
$e \leq 3n - 6$
$\frac{5n}{2} \leq 3n - 6$

$f = e - n + 2$.

for a planar graph

$3f \leq 2e$

$\Rightarrow \boxed{f \leq \frac{2e}{3}}$

$\Rightarrow e = f + n - 2 \leq \frac{2e}{3} + \frac{2e}{5} - 2$

$= \frac{16e - 30}{15}$

$\Rightarrow$ $15e \leq 16e - 30$

$\Rightarrow$ $30 \leq e$

But we have $30 \geq e$

So we get a contradiction.

Hence
vertex has deg ≤ 4,

(3) Let all the regions has deg ≥ 6.

$6f \leq 2e \Rightarrow f \leq \frac{e}{3}$

Also we have

$3n \leq 2e \Rightarrow n \leq \frac{2e}{3}$

$\therefore$ $e = f + n - 2$

$\leq \frac{e}{3} + \frac{2e}{3} - 2 = e - 2$

$\Rightarrow$ ~~$3e \leq e - 6$~~ $0 \leq -2$

which is a contradiction

So we have at least one vertex of degree ≥ 5.

* * *

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Note:-

→ Colouring of graph is possible viz simple graph

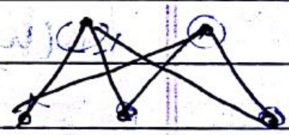
→

The minimum no. of colours need to colour a graph

Chromatic Number, $K(G)$

Theorem: For any graph G, Chromatic Number $\leq 1 + d_{max}(G)$, where $d_{max}(G) = \max\{d(v_i) : v_i$ are vertices of $G\}$

for eg. Consider $K_{2,3}$



$d_{max}(G) = 3$

$K(K_{2,3}) \leq 1 + d_{max}(G)$

$1+3 = 4$

$(2 < 4)$

Proof:

We will prove this result by the mathematical induction. Let G has only one vertex, then $d_{max}(G) = 0$, $K(G) = 1$

$n = 1,$ | $1 \leq 1 + 0$ ✓ satisfies

$e = 0$

$d(v) = 0$

~~and~~ Hence $1 \leq 1+0$ result is valid for $n = 1$.

Let it valid to graph G_1 with $(n-1)$ vertices. Suppose also that G is a graph with n vertices, having a vertex v, such that $G_1 = G - \{v\}$. By hypothesis $K(G_1) \leq d_{max}(G_1)$ th

$\boxed{K(G) \leq 1 + d_{max}(G)}$

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Also, $d_{max}(G_1) \leq d_{max}(G)$

Case-I, $K(G_1) = K(G)$,

~~Then we can assign a colour to v from one colour of G_1.~~

$d_{max}(G_1) = d_{max}(G)$,

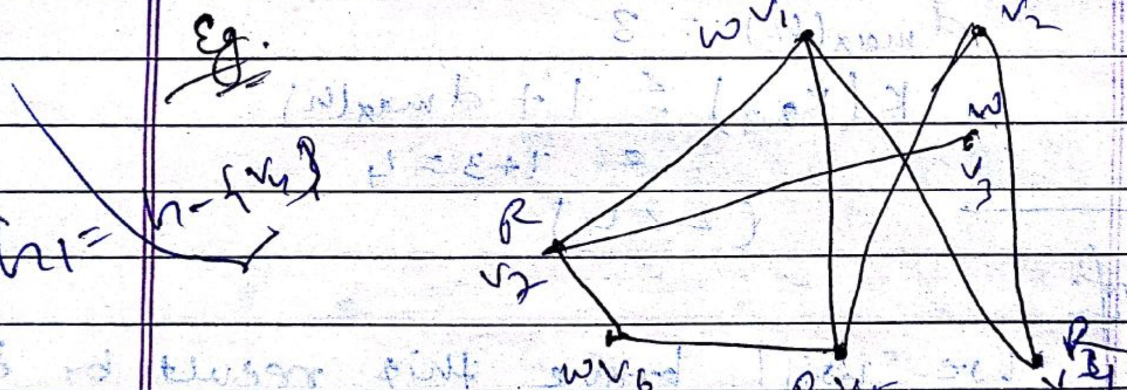
Since there are $d_{max}(G_1)+1 = d_{max}(G)+1$ colours, then there is at least one colour not used by neighbours of v.

$$K(G) = d_{max}(G) + 1$$

$$\left(K(G) = K(G_1) \leq d_{max}(G_1) + 1 = d_{max}(G) + 1\right)$$

Eg.

$G_1 = G - \{v_4\}$



Case-II, If $d_{max}(G_1) < d_{max}(G)$. then we use a new colour to colour v, hence we are using $K(G_1)+1$ colours for colouring of G. Hence

$$K(G) = K(G_1) + 1 \leq [d_{max}(G_1) + 1] + 1$$
$$\leq [d_{max}(G) + 1] + 1$$
$$\Rightarrow K(G) < d_{max}(G) + 2 \leq d_{max}(G) + 1$$

Theorem: there is only one path between every
(1) pair of vertices in a tree.

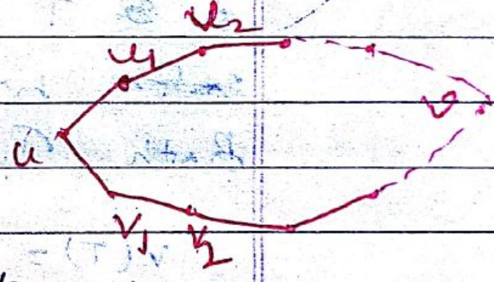
Proof: – Let there are two paths b/w a pair of vertices (u, v) of G. These pairs are

$$P_1 = (u, u_1, u_2, u_3, \ldots u_n, v)$$
$$P_2 = (u, v_1, v_2, v_3, \ldots v)$$

Case-I, If $(u_1, u_2, \ldots u_n) \cap (v_1, v_2, \ldots, v_n) = \phi$
It means the P_1 followed by P_2, hence is a cyclic

$$(u, u_1, u_2, \ldots u_n, v, v_n, v_{n-1}, \ldots v_1, u)$$

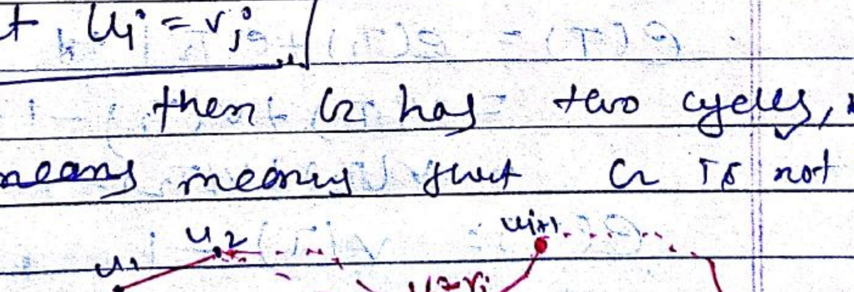
Hence G is not a tree



Case-II,
If $(u_1, u_2, \ldots u_n) \cap (v_1, v_2, \ldots v_n) \neq \phi$

$\boxed{\text{let } u_i = v_j}$

then G has two cycles,
~~It means~~ means that G is not a tree



Hence to be a tree, there should ~~only~~ only one path b/w every pair of vertices of G.

Theorem (2)

A tree of n vertices has $(n-1)$ edges.

Pf: we prove it by M. Induction — Let T has $n=1$ vertex, then T has 0 edges. Then the statement is true for $n=1$ vertex.

Suppose the statement is true for $n=K$ vertices, means that T has $(K-1)$ edges. ~~Again take~~

Again take T with $(K+1)$ vertices. If we delete one edges from T, then it gives two components both are tree (wt T_1 & T_2). Both T_1 and T_2 has K or less vertices.

$$V(T) = V(T_1) + V(T_2) \quad \text{---} \; (1)$$

$$e(T_1) = V(T_1) - 1 \quad \& \quad e(T_2) = V(T_2) - 1$$

$$e(T) = e(T_1) + e(T_2) + 1$$

$$= V(T_1) + V(T_2) - 1 - \cancel{1} + \cancel{1}$$

$$= V(T_1) + V(T_2) - 1$$

$$e(T) = V(T) - 1 \qquad (\text{By } (1))$$

sh. sew (8.14)

$$\Rightarrow \boxed{e(T) = K}$$

Theorem (3): Let T be a graph with n-vertices, then following statements are equivalent

(i) T is a tree.

(ii) Every two vertices of T are joined by a unique path.

(iii) T is a connected graph with $(n-1)$ ~~vertices~~ edges

(iv) T has $(n-1)$ edges and has ~~any~~ not any cycle.

* * *

Pf:- (i) $\Rightarrow$ (ii) obviously
(Pf in thm-2)

(ii) $\Rightarrow$ (iii)

Every two vertices of T are joined by a unique path, means T has not a cycle and T is connected. Hence T is a tree. Hence it has $(n-1)$ edges.

(iii) $\Rightarrow$ (iv)

T is a connected graph with $(n-1)$ ~~cycles~~ edges
we need to PT, T has not any cycle.

On contrary,

Let T has ~~a~~ cycles. Let T has cycles. Let e be an edge of a cycle of T. Then $T_1 = T - \{e\}$ is also connected. Again let T_1 has cycles and suppose e_1 be an edge of a cycle of T_1, hence $T_2 = T - \{e_1, e_2\}$ remain connected. and T_2 may have cycles. we may continue this process unless we get a connected acyclic graph. The deleted ~~graph~~ edges are finitely, let they are (k). The remaining graph has n-vertices (connected acyclic) and $n-1$ edges.

$\therefore n-1+k = n-1 \Rightarrow k=0$

$(iv) \Rightarrow (i)$

Let T has (n-1) edges has not any cyles.

* * *

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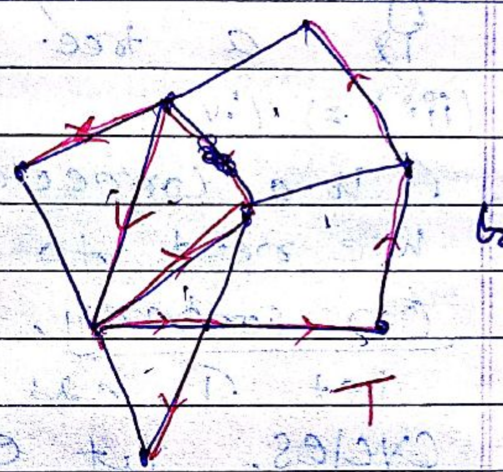
Spanning tree :-

The spanning tree of a connected graph G is a tree containing same vertices of G but having only (n-1) edges of G.

Eg



$\rightarrow = 8$

$n(V) = 9$

① Theorem: A simple graph G has a spanning tree if and only if G is connected.

Pf: let G ~~is~~ has a spanning tree T then T is acyclic, having all tree vertices of G and (n-1) edges of G. Hence there is an unique path between any pair of vertices of G. Therefore that path is also a path in G - connecting that vertices. Hence G is connected.

Now, suppose that G is a connected graph. Here we have two cases.

Case-I, If G is acyclic, then it has a spanning tree.

Case-II, If G is a cyclic graph (i.e. not a acyclic graph). Then we can remove an edge e_1 from G, s.t $G_1 = G - \{e_1\}$ is still connected.

If G_1 is acyclic, then it is the spanning tree. If G_1 is cyclic, then we can remove an edge e_2, from G_1 s.t $G_2 = G_1 - \{e_2\}$ is still connected.

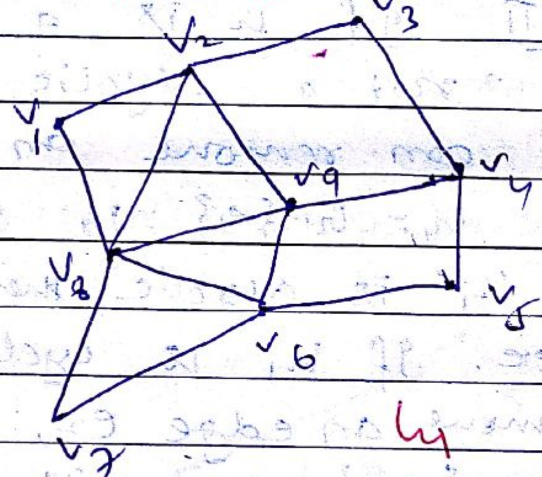
Weighted graph

In any graph, if we assign numerical value to edges, then it is known as weighted graph.

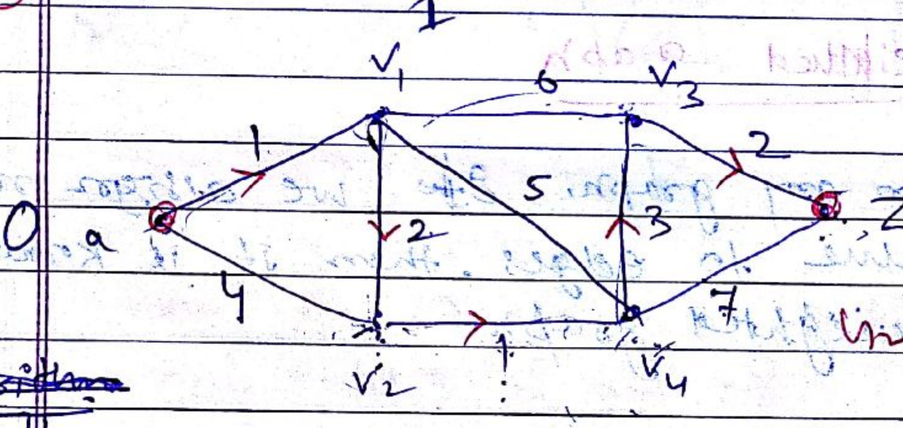
minimal spanning tree

In the weighted graph, the spanning tree with minimum weight is known as minimal spanning tree.

Eg. ①



Eg. ②



~~Algorithm~~

* Algorithm

Let $G = (V, E)$ be a connected weighted graph, let v_i and v_j are initial and final vertex between which the shortest path has to find. Then follows

Step-1 Assign a permanent level zero to the starting vertex v_0 and temporary level ∞ to the remaining vertices

Step-2 Each vertex v_j that is not got a permanent level, will given a new temporary level of

$L(v_j) = \min\{$ old level of v_j, level of $v_i + w(v_i, v_j)\}$

$L(v_1) = \min\{\infty, 0+1\} = 1$ (perm)

$L(V_2) = \min\{\infty, 0+4\} = 4$ (still temp)

Step-3.

Select an smallest value among all the temporary levels, this becomes new permanent level.

Step-4. Repeat the step 2 and step-3

$L(V_3) = \min\{\infty, 1+6\} = 7$ (temp)
$L(V_2) = \min\{4, 1+2\} = 3$ (perm)
$L(V_4) = \min\{\infty, 1+5\} = 6$ (temp)

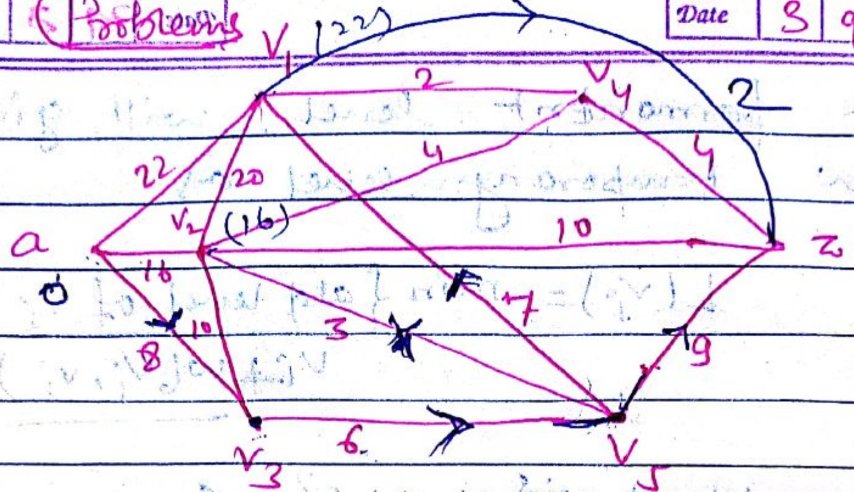
$L(V_4) = \min\{6, 3+1\} = 4$ (perm)
$L(V_3) = \min\{7, 4+3\} = 7$ (temp)
$L(z) = \min\{\infty, 4+7\} = 11$ (temp)

* * *

$a - v_3 - v_5 - z$

$a - v_3 - v_5 - v_1 - z$

(1) Problems



find the shortest path between a-z

Sol^n

$L(v_1) = \min\{\infty, 0+22\} = 22$

$L(v_2) = \min\{\infty, 0+16\} = 16$

$L(v_3) = \min\{\infty, 0+8\} = 8$ (perm)

Here $L(v_3)$ belongs permanent

Again Repeat same process.

$L(v_2) = \{\min\{16, 10+8\} = 16$

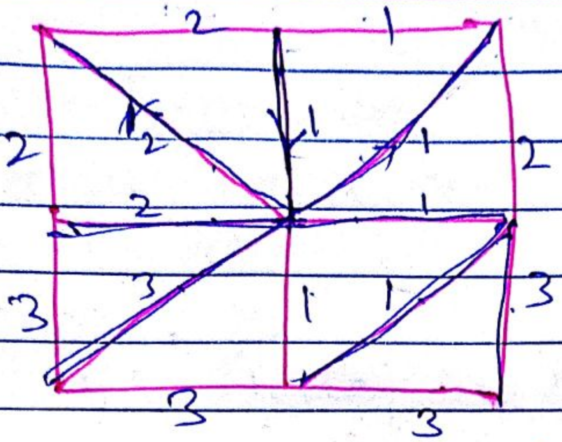
$L(v_5) = \min\{\infty, 8+6\} = 14$

Here $L(v_5)$ is belong perm

Dijkstra algorithm

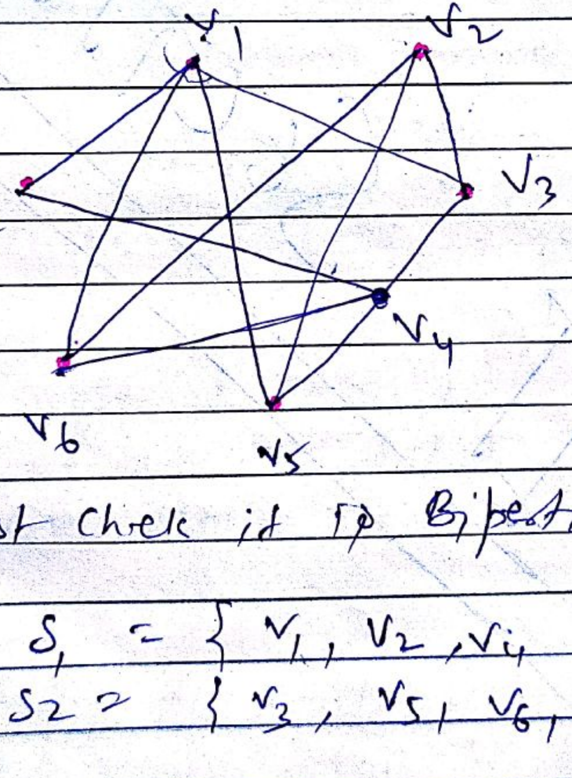
$V \rightarrow$	a	v_1	v_2	v_3	v_4	v_5	z
$L(v)$	$\boxed{0}$	∞	∞	∞	∞	∞	∞
	0	22	16	$\boxed{8}$	∞	∞	∞
		22	16		∞	$\boxed{14}$	∞
		21	$\boxed{16}$		∞		23
		21			$\boxed{20}$		23
		$\boxed{21}$					23
							$\boxed{23}$

(2) Find the minimal spanning tree



$n(V) = 9$ so we may select at most 8 edges to make spanning tree

(3) ~~find shortest path b/w a to z.~~

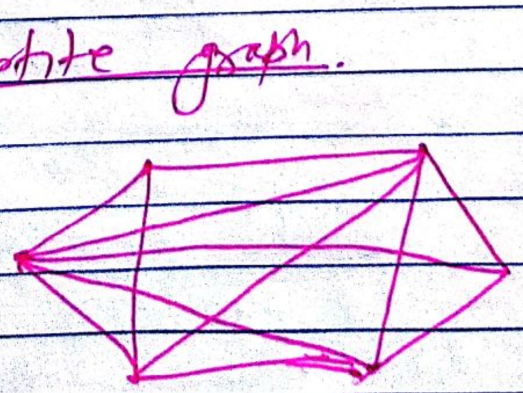


~~not~~ yes Bipartite

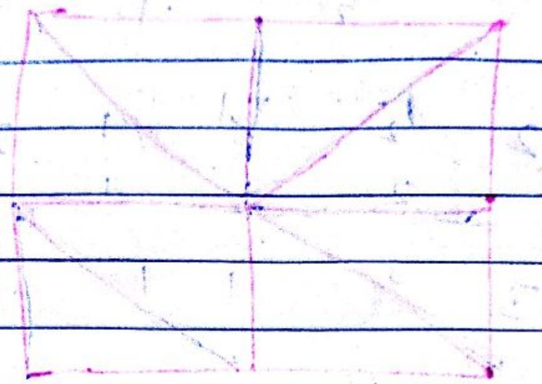
first check it is Bipartite graph or not

$S_1 = \{ v_1, v_2, v_4 \}$
$S_2 = \{ v_3, v_5, v_6, v_7 \}$

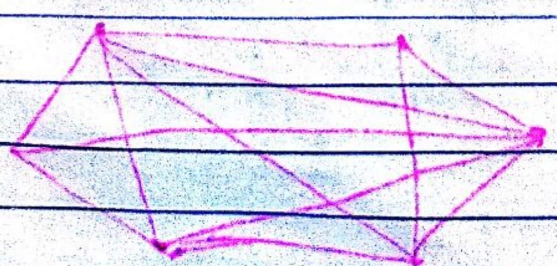
(4) Check Bipartite graph.



Not Bipartite graph



II SECTION



Statement (Proposition)

sentences which are either true or false (but not both)

① ~ Negation

p: Today is monday

~p: Today is not monday.

Truth Table

p	~p
T	F
F	T

② → ∧ (and) ~~conjunction~~ conjunction.

p: Today is monday.

q: We are setting in class.

$\boxed{p \wedge q}$

→ p: Today is monday

q: we are not setting our class.

p	q	p∧q
~~T~~	~~F~~	F
T	T	T
F	F	F
F	T	F

③ → ∨ (or) disjunction :—

p	q	p∨q	~(p∨q)
T	T	T	F
F	F	F	T
T	F	T	F
F	T	T	F

④ → ⇒ (implies)

p⇒q	T	F	T	T

2 2

Pf :- $x, y \in \mathbb{R}$,

(a)

p	then $x+y = y+x$	T
q	$3+2=5$	T

$\{x, y \in \; x+y = y+x, \; 3+2=5\} = T$

(b) p | $x, y \in R, \; y \neq 0$
then

p	$x+y = x$	F
q	$3+2=5$	T

$\{x, y \in R, y \neq 0; \; \underbrace{x+y = x}_{F} \; \underbrace{3+2=5}_{T}\} = T$

(5) ($\Leftrightarrow$) (iff) (equivalence)

$p \Leftrightarrow q \qquad (p \Rightarrow q) \wedge (q \Rightarrow p)$

p	q	$p \Leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

(6) $(p \Rightarrow q) \Leftrightarrow (\sim q \Rightarrow \sim p) \Rightarrow$ Tautology

eg. p. Today is monday
q. We are sitting in our class.

$p \Rightarrow q$ | If today is monday then we are sitting in our class.

$\sim q \Rightarrow \sim p$ | If we are not sitting in our class then today is not monday.

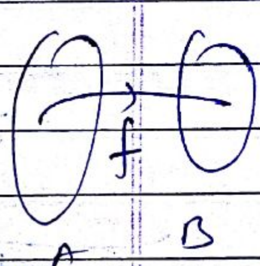
p, q are in Delhi
q, q are in India

p	q	$p \Rightarrow q$	$\sim p$	$\sim q$	$\sim q \Rightarrow \sim p$	$(p \Rightarrow q) \Leftrightarrow (\sim q \Rightarrow \sim p)$
T	T	T	F	F	T	T
T	F	F	F	T	F	T
F	T	T	F	F	T	T
F	F	T	F	T	T	T

* $p \Rightarrow q \Leftrightarrow \sim q \Rightarrow \sim p$

Eg. A f^n f is one-one

$y = f(x) ; A \to B$



$\underbrace{x_i = x_j}_{p} \Rightarrow \underbrace{f(x_i) = f(x_j)}_{q}$

$p \Rightarrow q$

$f(x_i) \neq f(x_j) \Rightarrow x_i \neq x_j$

$(\sim q \Rightarrow \sim p)$

Theorem: If n is any integer and n^2 is odd
(1) then n is odd.

Pf. If $\underbrace{n^2 \text{ is odd}}_{p}$ then $\underbrace{n \text{ is odd}}_{q}$

$p \Rightarrow q$

I method

or n is not odd $\Rightarrow$ n^2 is not odd

n is odd means, n is even

as we know that product of two even no. is always even

so if n is even $\Rightarrow$ n^2 is even

i.e if n is not odd $\Rightarrow$ n^2 is not odd

II method

proof

p = if n^2 is odd.

then we can write n^2 as

$n^2 = 2k+1$ when $k \in Z^+$

~~$\Rightarrow n = \sqrt{2k+1}$~~

~~$n^2 = (\sqrt{2k}-1)(\sqrt{2k}+1)$~~

$n^2 = 4k^2 + 4k + 1$

$= (2k+1)^2$

$\Rightarrow n = (2k+1)$

Again 3rd method

$\sim q$ = n is even

$\sim p$: n^2 is even

Now, we will prove $\sim q \Rightarrow \sim p$

let $n = 2k \mid k \in N$

$\Rightarrow n^2 = (2k)^2 = 4k^2 = 2(2k^2)$

$n^2 = 2p \mid p = 2k^2$

the $\sim q \Rightarrow \sim p \Rightarrow p \Rightarrow q$.

Idempotent Law

(7) $p \vee p \Leftrightarrow p$ $\quad p \wedge p \Leftrightarrow p$ Idempotent law

(i)

p	$p \vee p$	$p \wedge p$
T	T	T
F	F	F

Association Low

6(ii) $(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$ — (a)
$(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$ — (b)

p	q	r	p∨q	(p∨q)∨r	q∨r	p∨(q∨r)
T	T	T	T	T	T	T
T	T	F	T	T	T	T
T	F	T	T	T	T	T
F	T	T	T	T	T	T
F	F	T	F	T	T	T
F	T	F	T	T	T	T
T	F	F	T	T	F	T
F	F	F	F	F	F	F

both are same

Dual statement

A statement is dual statement obtained from to statement by replacing '∨' by '∧', '∧' by '∨', 'T' by 'F' and 'F' by 'T'.

for proof (b). Replace '∧' by '∨' and '∨' by '∧' in (a), then we are done.

Distributive Low

(iii) $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$
$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$
Distribution Low.

→ for this proof, replace ∧ by ∨ & ∨ by ∧ in first statement.

Identity Law

(6)(iv). $p \wedge T \Leftrightarrow p$, $p \vee F \Leftrightarrow p$ | Identity Law

p	T	$p \wedge T$
T	T	T
F	T	F

p	F	$p \vee F$
T	F	T
F	F	F

Domination Law

(6)(v) $p \vee T \Leftrightarrow T$, $p \wedge F \Leftrightarrow F$ | Domination Law

p	T	$p \vee T$
T	T	T
F	T	T

Negation Law

(vi) $p \vee (\sim p) \Leftrightarrow T$, $p \wedge (\sim p) \Leftrightarrow F$

(Negation Law)

P	$\sim p$	T
T	F	

Demorgon's Law

(vii) $\sim (p \wedge q) \Leftrightarrow \sim p \vee \sim q$ | Demorgon's Law

$\sim (p \vee q) \Leftrightarrow \sim p \wedge \sim q$

p	q	$p \land q$	$\sim(p \land q)$	$\sim p$	$\sim q$	$\sim p \lor \sim q$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

equivalent

Absorption Law

(viii) $p \lor (p \land q) \Leftrightarrow p$, $p \land (p \lor q) \Leftrightarrow p$, Absorption Law

Q.1 Show that

$\sim p \land (\sim q \land r) \lor (q \land r) \lor (p \land r) \Leftrightarrow r$

Sol^n L.H.S

$\sim p \land (\sim q \land r) \lor [(q \lor p) \land r]$ (Distributive Law)

$\Rightarrow [(\sim p \land \sim q) \land r] \lor [(q \lor p) \land r]$ (Associative Law)

$= [\{\sim(p \lor q)\} \land r] \lor [(q \lor p) \land r]$ (Demorgan's Law)

$= [\sim(p \lor q) \lor (p \lor q)] \land r$ (Distributive Law)

$= T \land r$ (Negation Law)

$= r$ (Identity Law)

(No need to prove from both sides bcz it is not 'iff' condition)

* * *

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(1) $p \vee p = p$

(2) $p \vee (q \vee r) = (p \vee q) \vee r$

(3) $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$

(4) $p \wedge T \Leftrightarrow p$; $p \vee F \Leftrightarrow p$

(5) $p \vee T \Leftrightarrow T$, $p \wedge F \Leftrightarrow F$

(6) $p \vee (\sim p) \Leftrightarrow T$, $p \wedge (\sim p) \Leftrightarrow F$

(7) $\sim (p \wedge q) = \sim p \vee \sim q$

(8) $p \vee (p \wedge q) \Leftrightarrow p$

(9) $p \wedge (p \vee q) \Leftrightarrow p$

(10) $p \Rightarrow q \Leftrightarrow \sim p \vee q$

(1) $\underbrace{(p \vee q)}_{B} \wedge \underbrace{[\sim(\sim p \wedge (\sim q \vee \sim r))]}_{A} \vee \underbrace{[(\sim p \wedge \sim q) \vee (\sim p \wedge \sim r)]}_{C}$ is tautology.

(A) $\Leftrightarrow \sim (\sim p \wedge \sim (q \wedge r))] \Leftrightarrow p \vee (q \wedge r)$

$\Leftrightarrow (p \vee q) \wedge (p \vee r)$

(B) $\underbrace{(p \vee q) \wedge (p \vee q)}$ $\wedge (p \vee r)$

$\Leftrightarrow (p \vee q) \wedge (p \vee r)$ (By (1))

(C) $\Leftrightarrow \sim (p \vee q) \vee \sim (p \vee r)$

$\Rightarrow$ $\Leftrightarrow (p \vee q) \wedge \{(p \vee r) \vee [\sim (p \vee q) \vee \sim (p \vee r)]$

$\Leftrightarrow (p \vee q) \wedge \{\underbrace{(p \vee r) \vee [\sim (p \vee r)]}_{T \text{ by (6)}} \vee [\sim (p \vee q)$

$\Leftrightarrow$ ~~$(p \vee q) \wedge \{(p \vee q) \vee T$~~

$(p \vee q) \wedge \{\underbrace{T \vee (p \wedge q)\}}_{\text{By (5)}}$

$\Leftrightarrow (p \vee q) \wedge T$ (~~$p \wedge q$~~) $= (p \wedge q)$

II- method

starting form (C).

$(C) \Leftrightarrow \sim(p \vee q) \vee \sim(p \vee r)$

$\Leftrightarrow \sim[(p \vee q) \wedge (p \vee r)]$

$\Leftrightarrow \sim[p \vee (q \wedge r)]$

$\Rightarrow \Leftrightarrow [(p \vee q) \wedge (p \vee r)] \vee \sim [p \vee (q \wedge r)]$

$\Leftrightarrow [p \vee (q \wedge r)] \vee \sim [p \vee (q \wedge r)]$

$\Leftrightarrow A \vee \sim A \Leftrightarrow T$

(2) S.T. $(p \wedge q) \Rightarrow (p \vee q)$ is tautology

Eg. p : 0111 010110
q : 1100011 0101

$p \wedge q$ = 0100010000

$(p \wedge q = p \times q)$

$(\rightarrow)$

Eg. $p \Rightarrow q \Leftrightarrow \sim p \vee q$

p	q	$p \Rightarrow q$	$\sim p$	$\sim p \vee q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

$\Leftrightarrow \sim(p \wedge q) \vee (p \vee q)$

$\Leftrightarrow (\sim p \vee \sim q) \vee (p \vee q)$

$\Leftrightarrow \sim p \vee (p \vee (\sim q \vee q))$

$\Leftrightarrow \sim p \vee (p \vee T)$ $\Leftrightarrow$ T

$\Leftrightarrow \sim p \vee T \Leftrightarrow T$

(3) Show that —

$[(p \vee q) \wedge r] \Rightarrow (p \vee r)$ is tautology

pf:- $[(p \vee q) \wedge r]$

$\Leftrightarrow (p \wedge r) \vee (q \wedge r)$

$\sim [(p \vee q) \wedge r] \vee (p \vee r)$

$\Leftrightarrow \sim [(p \wedge r) \vee (q \wedge r)] \vee (p \vee r)$

$\Leftrightarrow [\sim (p \wedge r) \wedge \sim (q \wedge r)] \vee (p \vee r)$

$\Leftrightarrow [(\sim p \vee \sim r) \wedge (\sim q \vee \sim r)] \vee (p \vee r)$

$\Leftrightarrow$

$\sim [(p \vee q) \wedge r] \vee (p \vee r)$

$\Leftrightarrow [\sim (p \vee q) \vee \sim r] \vee (p \vee r)$

$\Leftrightarrow [\sim r \vee [\sim (p \vee q) \vee (p \vee r)]]$

$\Leftrightarrow \sim r \vee T \Leftrightarrow T$

(4) Show that $\sim[p \Leftrightarrow (q \Rightarrow (r \vee p))]$ is tautology or not.

Pf $\sim[p \Leftrightarrow (q \Rightarrow (r \vee p)]$

$\Leftrightarrow \sim[p \Leftrightarrow (\sim q \vee (r \vee p)]$

$\Leftrightarrow \sim[p \Leftrightarrow [\_\_\_\_\_\_]]$

$(\because p \Leftrightarrow q \equiv (p \Rightarrow q) \wedge (q \Rightarrow p))$

$\Leftrightarrow \sim[p \Rightarrow (\sim q \vee (r \vee p))] \wedge [(\sim q \vee (r \vee p)) \Rightarrow p]$

$\Leftrightarrow \sim[\sim p \vee (\sim q \vee (r \vee p))] \wedge [\sim[\sim q \vee (r \vee p)] \vee p]$

$\Leftrightarrow p \wedge \sim(\sim q \vee (r \vee p)) \wedge [q \vee \sim(r \vee p) \vee p]$

$\Leftrightarrow p \wedge (q \wedge \sim(r \vee p)) \wedge (q \vee (\sim r \wedge \sim p) \vee p]$

$\Leftrightarrow p \wedge (q \wedge \sim r \wedge \underset{F}{\sim p}) \wedge [q \vee \sim r \wedge \sim p \vee \underset{T}{p}]$

$\Leftrightarrow (q \wedge \sim r \wedge F) \wedge [q \vee \sim r \wedge T]$

$\Leftrightarrow$

II. method

$\Leftrightarrow \sim[(p \Rightarrow \{q \Rightarrow (r \vee p)] \wedge \{(q \Rightarrow (r \vee p)) \Rightarrow p\}]$

$\Leftrightarrow \sim[(p \Rightarrow \{\sim q \vee (r \vee p)\}] \wedge (\{\sim q (r \vee p)\} \Rightarrow p)\}$

$\Leftrightarrow \sim[\{\sim p \vee \{q \vee (r \vee p)\}\}] \wedge \{\{q \wedge (\sim r \wedge \sim p)\} \vee p$

$\Leftrightarrow \sim(T \wedge A)$ $= \sim q \vee (r \vee p) \wedge \sim p$

$= \sim q \vee [(r \wedge \sim p) \vee (p \wedge \sim p)]$

$= \sim q \vee (r \wedge \sim p)$

* * *

(1) XOR : Exclusive or $\oplus$, $P \oplus Q$

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

$P \oplus Q$: is true only when any one of p or q are true else it is false.

(2) NAND : NOT And : $p \uparrow q \equiv \sim(p \wedge q)$

$p \uparrow q$ is false only when both true. (p & q are true) else it is true

p	q	$p \wedge q$	$p \vee q$	$p \downarrow q$ $\sim(p \vee q)$	$p \uparrow q$ $\sim(p \wedge q)$
T	T	T	T	F	F
T	F	F	T	F	T
F	T	F	T	F	T
F	F	F	F	T	T

$\uparrow$ = NOR
$\downarrow$ = NAND

(3) NOR : NOT or | $p \downarrow q = \sim(p \vee q)$

(4) Statement ;

NAND and NOR operators are not - Associative

T.S. $(p \uparrow q) \uparrow r = p \uparrow (q \uparrow r)$, (not)

L.H.S. $(p \uparrow q) \uparrow r \Leftrightarrow \sim(p \wedge q) \uparrow r$

$\Leftrightarrow \sim[\sim(p \wedge q) \wedge r]$

$\Leftrightarrow [(p \wedge q) \vee \sim r]$

$\Leftrightarrow (p \vee \sim r) \wedge (q \vee \sim r)$

R.H.S $\Leftrightarrow \sim [p \wedge (q \uparrow r)] \Leftrightarrow \sim [p \wedge \sim (q \wedge r)]$
$\Leftrightarrow \sim p \vee (q \wedge r) \Leftrightarrow (\sim p \vee q) \wedge (\sim p \vee r)$

Since L.H.S $\neq$ R.H.S $\Rightarrow$ Not Asso

(5) NAND & NOR operations are ~~not~~ commutative
$(p \uparrow q) = (q \uparrow p)$ (yes)

(6) Statement
Sonia is watching T.V. If sonia is watching T.V, then she is not studying. If she is not studying, then her father will not buy her a scooty. Therefore sonia's father will not buy a scooty.

Sol^n p: Sonia is watching T.V
q: sonia is studying.
r: Sonia's father buy a scooty for her.

Logically
$[p,\ p \Rightarrow \sim q,\ \sim q \Rightarrow \sim r] \Rightarrow \sim r$
$\Rightarrow [\underbrace{p \wedge (p \Rightarrow \sim q)}_{R} \wedge (\sim q \Rightarrow \sim r)] \Rightarrow \sim r$

$(\because p \Rightarrow q = \sim p \vee q)$

$\therefore p \wedge [(p \Rightarrow \sim q)] = p \wedge [\sim p \vee \sim q]$
$= [p \wedge \sim p] \vee [p \wedge \sim q]$
$= F \vee (p \wedge \sim q)$
$= F \vee \sim q$ $\sim q$
$=$

$p \wedge [\sim p \vee \sim q] \Rightarrow \sim q = [(p \wedge \sim p) \vee (p \wedge \sim q)] \Rightarrow \sim q$
$= [F \vee (p \wedge \sim q)] \Rightarrow \sim q$
$= (p \wedge \sim q) \Rightarrow \sim q$

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$= \sim(p \wedge \sim q) \vee \sim q$

$= (\sim p \vee q) \vee \sim q$

$= \sim p \vee (q \vee \sim q) = \sim p \vee T = T$

$p \wedge (p \to \sim q) \wedge (\sim q \to \sim r)$

$\Rightarrow \sim q \wedge (\sim q \to \sim r) \Rightarrow \sim r$

(7) Statement

If there was a ball game, then travelling was difficult. If they arrived on time, then travelling was not difficult. They arrived on time. Therefore, there was no ball game.

Soln p: There was a ball game.

q: Travelling was difficult.

r: They arrived on time.

$[p \to q ; r \to \sim q , r] \to \sim p$

$[(p \to q) \wedge (r \to \sim q) \wedge r] \to \sim p$

$\Rightarrow (p \to q) \wedge (\sim q)$

$\Rightarrow (\sim q \to \sim p) \wedge (\sim q)$

$\Rightarrow \sim p$

* * *

Q. ① Show that the following statements consist a valid argument.

"If A work hard, then either B or C will enjoy. If B enjoy then A will not work hard. If D enjoy, then C will not. Therefore if A works hard D will not enjoy.

Sol^n p: A works hard
q: B enjoy.
r: C enjoy
s: D enjoy.

$\to$ $p \Rightarrow q \vee r$, $q \to \sim p$
$s \to \sim r$,

Tip $\{\underbrace{p \to q \vee r \cdot q \to \sim p \cdot s \to \sim r}\} \Leftrightarrow \{\underbrace{p \to \sim s}\}$

$\because$ $p \to q \Leftrightarrow \sim p \vee q$

L.H.S: $(p \to q \vee r) \cdot (q \to \sim p) \cdot (s \to \sim r)$
~~$\{[\sim p \vee (q \vee r)] \cdot [\sim q \vee \sim p] \cdot [\sim s \vee \sim r]\}$~~

I. $p \to q \vee r \Leftrightarrow [\sim q \to r]$

II. $q \to \sim p \Leftrightarrow p \to \sim q$

$\therefore$ from I. & II.,
we have, $p \to r$. — (iii)

III. $s \to \sim r \Leftrightarrow r \to \sim s$
$\therefore$ from (iii) & (iv)
$p \to \sim s$. = R.H.S Proved

Normal form

1. $\vee \rightarrow$ sum $(+)$
   $\wedge \rightarrow$ Product $(\times)$

I. Disjunctive Normal form

If a formula is equivalent to given formula consisting sum of elementary products. (DNF)

II. Conjunctive Normal form

If a formula is equivalent to given formula consisting product of elementary sum. (CNF)

* $\sim(p \vee q) \Leftrightarrow (p \wedge q)$

$\Rightarrow [\sim(p \vee q) \Rightarrow (p \wedge q)] \wedge [(p \wedge q) \Rightarrow \sim(p \vee q)]$

$\Rightarrow [(p \vee q) \vee (p \wedge q)] \wedge [\sim(p \wedge q) \vee (\sim p \wedge \sim q)]$

$\Rightarrow [(p \vee q \vee p) \wedge (p \vee q \vee q)] \wedge [(\sim p \vee \sim q) \vee (\sim p \wedge \sim q)]$

$\underbrace{(p \vee q) \wedge (p \vee q)} \wedge (\sim p \vee \sim q) \wedge (\sim p \vee \sim q \vee \sim q)$

$p \vee p = p$
$p \wedge p = p$

$(p \vee q) \wedge (\sim p \vee \sim q)$

$\underbrace{(p \vee q)}_{\text{sum}} \underbrace{\wedge}_{\text{product}} \underbrace{(\sim p \vee \sim q)}_{\text{sum}} \rightarrow$ CNF

$\sim \Rightarrow [(p \vee q) \wedge \sim p] \vee [(p \vee q) \wedge \sim q]$

$\Rightarrow [\underbrace{(p \wedge \sim p)} \vee \underbrace{(q \wedge \sim p)}] \vee [\underbrace{(p \wedge \sim q)} \vee \underbrace{(q \wedge \sim q)}]$

product

Sum of elementary product.

$\Rightarrow \underbrace{(q \wedge \sim p)}_{\text{product}} \underset{\text{sum}}{\vee} \underbrace{(p \wedge \sim q)}_{\text{product}} = \text{DNF}$

* (1) $p \wedge (p \to q)$ ∴ write down its CNF & DNF

$\Rightarrow p \wedge (\sim p \vee q) \Leftrightarrow \cancel{(p \wedge p) \vee (p \wedge q)}$

$\Leftrightarrow (p \wedge \sim p) \vee (p \wedge q)$

$\longrightarrow$ DNF

$\Rightarrow p \wedge q$ ~~CNF~~

$\Rightarrow (p \vee p) \wedge (q \vee q) = \text{CNF}$

(2) ~~this is~~ the $[p \wedge \sim(q \vee r)] \vee [((p \wedge q) \vee \sim r) \wedge p]$

writes down its CNF / DNF

$[p \wedge \sim(q \vee r)] \vee [((p \wedge q) \vee \sim r) \wedge p]$

$\rightarrow \underbrace{[p \wedge (\sim q \wedge \sim r)]}_{\text{product}} \vee \underbrace{[\sim(\sim(p \wedge q) \wedge r) \wedge p]}_{\text{product}}$

$\xrightarrow{\text{sum}}$ DNF

$\rightarrow$ ~~$\sim[\sim(p \wedge \sim(q \vee r))] \wedge [\sim((p \wedge q) \vee \sim r) \wedge p]$~~

$(p \wedge \sim q \wedge \sim r) \vee [[\sim r \vee (p \wedge q)]] \wedge p]$

$\vee [(\sim r \vee p) \wedge (\sim r \vee q)] \wedge p]$

$\Rightarrow \vee [(\sim r \vee p) \wedge (\sim r \vee q)] \wedge p]$

$\vee [(\sim r \vee p) \wedge p \wedge (\sim r \vee q)]$

$\vee [\sim(r \wedge \sim p) \wedge p \wedge \sim(r \wedge \sim q)]$

$\vee \sim [(r \wedge \sim p) \vee \sim p \vee (r \wedge \sim q)]$

Rule of ~~Interfere~~ Inferences

Rule: P: A premise may be introduced at any pt. in the derivation.

Rule T: A formula S may be introduced in a derivation of S is ~~tautology~~ tautologically implied by one or more of the preceeding formullas in the derivation.

Eg. (1) Demonstrate that r is valid inferences from the permises $\underbrace{p \to q,\ q \to r\ \&\ p}_{\text{permises}}$

{1}	(1) p	rule P.
{2}	(2) $p \to q$	rule. P
{1,2}	(3) q	Rule T. [$\because p \wedge (p \to q) \Rightarrow q$]
{4}	(4) $q \to r$	Rule P
{1,2,4}	(5) r	Rule T

Eg. (2) ST, $(r \vee s)$ follows logically from premises $c \vee d$, $c \vee d \to \sim h$, $\sim h \to (a \wedge \sim b)$ and $(a \wedge \sim b) \to (r \vee s)$.

{1}	(1) $c \vee d$	Rule P
{2}	(2) $(c \vee d) \to \sim h$	Rule P
{1,2}	(3) $\sim h$	Rule T
{4}	(4) $\sim h \to (a \wedge \sim b)$	Rule P

$\{1,2,4\}$ (5) $a \wedge \sim b$ Rule T
$\{6\}$ (6) $(a \wedge \sim b) \to (r \vee s)$ Rule P
$\{1,2,4,6\}$ (7) $r \vee s$ Rule T

(3) A graph with n vertices has $(n-1)$ edges
Let $n=1$
Let $\boxed{n=k}$ premice
$n=k+1$
p: A graph with k vertices
q: ———— $k-1$ edges. $\Big] p \Rightarrow q$

$p, q, p \Rightarrow q$

s: A graph with $k+1$ vertices
t: A ———— k edges $\Big] s \Rightarrow t$

$s, t, s \Rightarrow t$
$p, q \} \Rightarrow p \Rightarrow q$
$\{p, q, p \Rightarrow q, s, t\} \Rightarrow s \Rightarrow t$

self:
$\{1\}$ (1) p rule P
$\{2\}$ (2) q rule P
$\{3\}$ (3) $p \Rightarrow q$ rule P
$\{1,2,3\}$ (4) q rule T
(5) s rule P
(6) t rule ~~P~~

??

(4) If n is an add no., then n^2 is add

p: n is an add no.] $p \to q$
q: n^2 is an add no.

(i) p Rule P
(ii) q Rule P
?? (iii) $p \to q$ Rule T

(checking : If n is add, n^2 is add
$n = 2k+1$, k is an integer
$n^2 = (2k+1)^2$ $k \in Z$
$=$ add)

(5) ST $r \to s$ follows logically from premises $\underbrace{p \to (q \to s)}_{A}$, $\sim r \vee p$ and q

~~(1) q Rule P~~

(A $\Rightarrow$) $r \to s$
↳ include this r in our premises

$p \to (q \to s)$, $\sim r \vee p$, q & r

{1} (1) r Rule P
{2} (2) $\sim r \vee p$ Rule P
{1,2} (3) $r \to p$ Rule T
{1,3} (4) p Rule T
{5} (5) $p \to (q \to s)$ Rule P
{1,3,5} (6) $q \to s$ Rule T
(7) q ← ??
(8) s

$p \vee q$
$\sim p \to q$

Lattices

Relation:

$A \times B = \{(x,y) \mid x \in A \ \& \ y \in B\}$

Relation between A & B is any subset of $A \times B$.

I. Reflexive: A Relation R in a set S is reflexive if for each $x \in S$ and xRx and xRx.

Eg: Z^+, a/a : a is divided by b.
— reflexive ✓

II. Symmetric:
A relation R in a set S is symmetric if for $x, y \in S$, xRy and yRx.
Eg.
(i) Z^+, a/b : a is divided by b
— Not symmetry. ✓
(ii) $\leq$: $x \leq y$, $y \leq x$, Not ✓
(iii) Blood Relation — Not symmetric ✓

III. Transitive
A relation R in a set S is transitive if for $x, y, z \in S$, xRy and yRz then xRz.
Eg: (i) Blood Relation: Not transitive

IV. Anti-symmetric
A Relation R in a set S is anti-symmetric for $x, y \in S$, xRy and yRx then $x = y$.

Eg:-

(i) $x \le y$, $y \le x \Rightarrow x = y$

(ii) $\frac{x}{y}$, $\frac{y}{x} \Rightarrow x = y$

Partial Order Relation

A Relation R in a set S is partial Order Relation (POR) ($\le$) if it is ~~sym~~ reflexive, anti-symmetric and transitive.

Eg.

(1) S, all subsets of $S = P(S)$ — Power set

$P(S) \subseteq$ POR

$S = \{1,2,3\}$

$P(S) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, S\}$

$x, y \in P(S)$, (for anti-symmetric)

$x \subseteq y$, $y \subseteq x$ then $x = y$.

for eg. $y \subseteq \{1,2\}$, $\{1,2\} \subseteq y$

$\Rightarrow \{1,2\} = \{1,2\}$

The set S alongwith POR defined in S is known as partial order set (Poset)

Eg

→ $(Z^+, /)$ → Poset.

→ $(P(S), \subseteq)$

→ $(R, \le)$ or $(R, \ge)$

In general, POR is denoted by '$\le$'.

$(S, \le)$

$(S, \le)$ Poset, if $x, y \in S$ then y is known to be cover of x. if $x \le y$. and $\nexists$ (There not exists) any $z \in S$ s.t $x \le z$, $z \le y$

Eg.

(1) $(S = \{1, 2, 4, 5, 3, 6, 7, 8\}, /)$

2 is cover of 1

4 is cover of 2

4 is cover of 1 $\longrightarrow \times$ (B/c $\exists$ 2 $1/2$, $2/4$)

(2) $(S = \{1, 2, 3, 4, 5, 6\}, /)$

$\{(1,2), (1,3), \cancel{(1,4)}, (1,5), \cancel{(1,6)}, (2,4), (2,6), (3,6),$
$\cancel{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6)} \}$

$= \{(1,2), (1,3), (1,5), (2,4), (2,6), (3,6)\}$

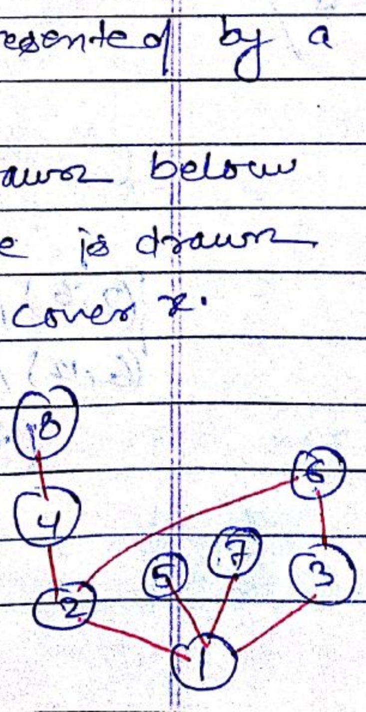
Hasse Diagram

If $(P, \le)$ is a poset, then a diagram known as Hasse Diagram is drawn by —

(i) (i) Each element is represented by a small circle of dot.

(ii) The circle for $x \in P$ is drawn below $y \in P$ if $x \le y$ and a line is drawn between x and y if y cover x.

Eg. $(S = \{1, 2, 3, 4, 5, 6, 7, 8\}, /)$:

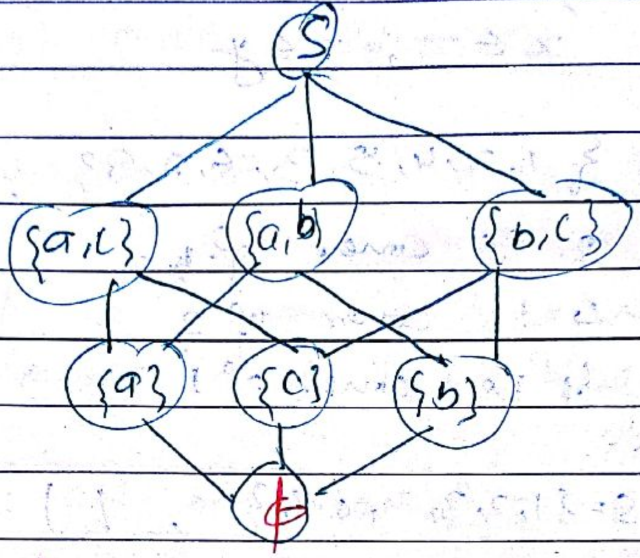


* * *

(1) Create Hasse Diagram

$S = \{a, b, c\}$, $(P(S), \subseteq)$

$P(S) = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}, S\}$



$\phi \subseteq \{a\}$

$\{a\} \subseteq S$

(2) Create Hasse diagram

$X = \{2, 3, 6, 12, 24, 36\}$

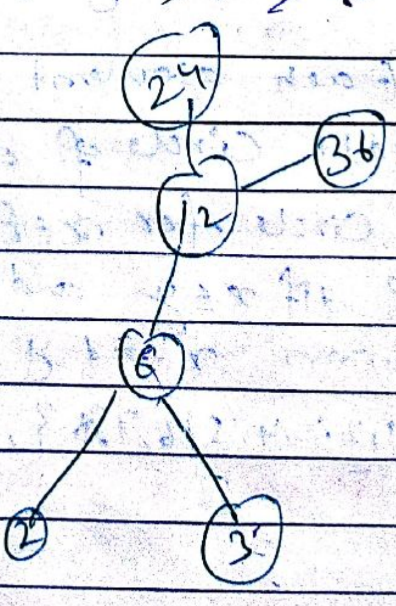
$p = (X, /)$, x/y = x is divides by y

(a,b) (b,c) (a,c)

$P = \{(2,6), \cancel{(2,12)}, \cancel{(2,24)}, (2,36), (3,6), \cancel{(3,12)}, \cancel{(3,24)}, \cancel{(3,36)}, \cancel{(6,12)}, (6,24), (6,36), (12,24), (12,36), 12\ldots, (36,36)\}$

Diagram

$\{(2,6), (3,6), (6,12), (12,24), (12,36)\}$



Maximal/minimal Elements

* let $(P, \leq)$ is a poset. An element $x \in P$ is known as **maximal element** of P. If for each $y \in P$, $y \leq x$.

$\{2, 3, 6, 12, 36\} = 36$

* Let $(P, \leq)$ is a poset. An element $x \in P$ is known as **minimal element** of P. If for each $y \in P$, $x \leq y$.

$\{2, 6, 12, 24, 36\} = 2$

LOWER / UPPER BOUNDS

* let $(P, \leq)$ is a poset. An element $x \in P$ is known as <u>lower Bound</u> of P if is not any $y \in P$ s.t $y \leq x$.

* let $(P, \leq)$ is a poset. An element $x \in P$ is known as <u>upper Bound</u> of P. If $\nexists$ any $y \in P$ s.t $x \leq y$.

LEAST UPPER / GREATEST LOWER BOUND,

* let $(P, \leq)$ is a poset. An element $x \in P$ is known as l.u.b of P if x is an upper bound of P and $x \leq y$ for each upper bound of P.

GLB (glb)

* let $(P, \leq)$ is a poset. An element $x \in P$ is known as glb of P if x is lower bound of P and $y \leq x$ for each lower bounds of P.

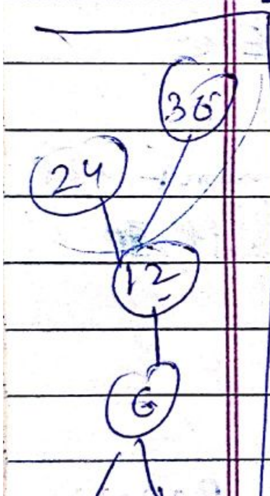
upper bound of 6 = {6, 12, 24, 36}
　　　　　　　　 12 = {12, 24, 36}
lub{6, 12} = 12

glb {6, 12} = 6
b/c lower bound of 6 = {2, 3, 6}
　　　　　　　　 12 = {2, 3, 6, 12}

Lattice

A poset $(P, \leq)$ is known as lattice if each pair of P has least upper bound and greatest lower bound.

Eg-

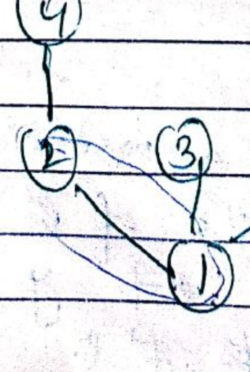


glb{2, 3} = not exists
lub{24, 36} = not exists

lb{6} = {2, 3, 6}, lb{3} = {3} | = {3}

Eg
(1) $S = \{1, 2, 3, 4, 5\}$
$P = (S, /)$
a/b = a divided by b.

$P(S) = \{(1,2), (1,3), (1,~~4~~), (1,5), (2,4)\}$



(2) $T = \{1, 2, 4, 8, 16\}$

* * *

Mon 12/10/15

Theorems:

I. ~~Idempotent~~ laws: Let $(L, \leq)$ be a lattice and $a \in L$, then $a \in L$, then $a \vee a = a$ and $a \wedge a = a$.

$a \vee b = lub\{a, b\}$
$a \wedge b = glb\{a, b\}$

Pf:- $a \vee a = \cancel{lub}\{a, a\} = a$

$a \wedge a = glb\{a, a\} = a$

II. Commutative law: Let $(L, \leq)$ be a Lattice and $a, b \in L$, then $a \vee b = b \vee a$ and $a \wedge b = b \wedge a$.

Pf:- $\rightarrow$ $a \vee b = lub\{a, b\}$

$= lub\{b, a\} = b \vee a$

and $a \wedge b = glb\{a, b\}$

$= glb\{b, a\} = b \wedge a$

III. Associative law: let $(L, \leq)$ be a lattice and $a, b, c \in L$, then

$a \vee (b \vee c) = (a \vee b) \vee c$

and $a \wedge (b \wedge c) = (a \wedge b) \wedge c$.

Pf :- let $x = a \vee (b \vee c)$ and $y = (a \vee b) \vee c$

$x = \text{lub}\{a, b \vee c\}$

$\Rightarrow a \le x \,,\; b \vee c \le x$ —— (1)

Also, $b \vee c = \text{lub}\{b, c\}$

$\therefore b \le b \vee c$ and $c \le b \vee c$ —— (2)

from (i) & (ii) ——

$a \le x$ and $b \le x$ and $c \le x$

$\Rightarrow (a \le x \;\&\; b \le x)$ and $c \le x$

$\Rightarrow (a \vee b) \le x$ and $c \le x$

$\Rightarrow (a \vee b) \vee c \le x$

$\Rightarrow \boxed{y \le x}$ —— A

Now Converse part i.e (to show $x \le y$)

$y = \text{lub}\{a \vee b, c\}$

$\Rightarrow y \ge a \vee b,\; y \ge c$ —— (iii)

Also, $a \vee b = \text{lub}\{a, b\}$

$\therefore a \le a \vee b$ & $b \le a \vee b$ (iv)

$\therefore$ from (iii) & (iv) ——

$a \le y$ and $y \ge c$ and $b \le y$

$\Rightarrow a \le y$ and $(c \le y$ and $b \le y)$

$\Rightarrow a \le y$ and $(c \vee b \le y)$

$\Rightarrow a \le y$ & $c \vee b \le y$

$\Rightarrow a \vee (b \vee c) \le y$

$\Rightarrow x \le y$ —— B

$\therefore$ By A & B

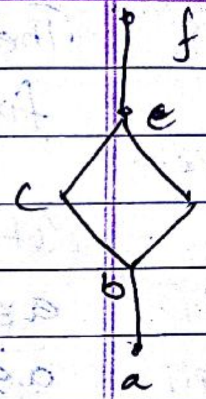
$\underline{x = y}$

i.e $a \vee (b \vee c) = (a \vee b) \vee c$.

$glb(e,f) = e$

As. $LB\ f = \{f, e, d, c, b, a\}$

$LB\ e = \{e, d, c, b, a\}$



Here

$glb(e,f) = e$

Again

$UB\ f = \{f\}$

$UB\ e = \{e, f\}$

$\therefore lub\{e,f\} = f$

IV. Absorption law:

Let $(L, \leq)$ be a lattice and $a, b \in L$

then $a \vee (a \wedge b) = a$ and $a \wedge (a \vee b) = a$

pf:

Here we have to show that –

$a \vee (a \wedge b) \leq a$ and $a \leq a \vee (a \wedge b)$

Now,

$a \vee (a \wedge b) = lub\{a, a \wedge b\}$

$\Rightarrow a \leq a \vee (a \wedge b)$ ——— (*)

And $a \wedge b = glb\{a, b\}$

$\Rightarrow a \wedge b \leq a$ & $a \wedge b \leq b$

Also $a \leq a$

$\Rightarrow a \wedge b \leq a$ & $a \leq a$

$\Rightarrow a \geq lub\{a, a \wedge b\}$

$a \vee (a \wedge b) \leq a$ ——— (**)

$\therefore$ by (*) & (**) —

$a = a \vee (a \wedge b)$

And $\therefore$ by Duality (i.e. Replace $\wedge$ by $\vee$ & $\wedge$ by $\vee$)

$a = a \wedge (a \vee b)$

The second statement is just dual of first i.e a valid statement.

Theorem

Let $(L, \leq)$ be a lattice and $a, b \in L$, then

(i) $a \leq b$ iff $a \vee b = b$

(ii) $a \leq b$ iff $a \wedge b = a$.

(iii) $a \vee b = b$ iff $a \wedge b = a$

(i) → By (i) & (ii) we have →

$a \wedge b = a$ and $a \leq b$

$\Rightarrow a \wedge b \leq b$

(ii) → $a \vee b = b$ and $a \leq b$

$\Rightarrow a \leq a \vee b$

→ $a \wedge b = a$ and $a \leq b$

$\Rightarrow a \wedge b \leq b$, $a \wedge b = a$

→ $a \wedge b \geq a$ and $a \leq a \wedge b$

(iii) $a \vee b = b \Rightarrow a \leq b$ (by (i))

$\Rightarrow a \wedge b = a$ (by (ii))

Tue 12/10/15

* * *

(i) $a \leq b$ and also $b \leq b$

so $a \vee b \leq b$ — (a)

And $a \vee b = \text{lub}\{a, b\}$

$a \leq a \vee b$ & $b \leq a \vee b$ — (b)

By (a) & (b)

$a \vee b = b$

(ii) $a \leq b$ and also $a \leq a$

So $a \wedge b \geq a$ — (a)

Also $a \wedge b = \text{glb}\{a, b\}$

$\Rightarrow a \wedge b \leq a$ & $a \wedge b \leq b$ (b) by ⓐ & ⓑ

$$a \wedge b = a$$

for converse part

(i) let $a \vee b = b = \text{lub}\{a, b\}$

$$\Rightarrow a \leq b$$

(ii) let $a \wedge b = a = \text{glb}\{a, b\}$

$$\Rightarrow a \leq b$$

Distributive Inequalities

$(L, \leq)$ be a lattice, If $a, b, c \in L$, then following inequalities holds.

ⓐ $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$

ⓑ $a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$

TH

1 pm 15|10|15 No class — due to Seminar

Tpm ERP 16|10|15

Theorem:

let $(L, \leq)$ is a lattice $a, b, c \in L$, then

(i) $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$

(ii) $a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$

Pf:- ① $a \vee b = \text{lub}\{a, b\}$

$$\Rightarrow a \leq a \vee b \text{ and } b \leq a \vee b$$

$$b \wedge c = \text{glb}\{b, c\}$$

$$\Rightarrow b \wedge c \leq b \text{ and } b \wedge c \leq c$$

$$\Rightarrow b \wedge c \leq a \vee b$$

As $(x \leq y,\ y \leq z \Rightarrow x \leq z)$

$\Rightarrow a \vee (b \wedge c) \leq a \vee b$ — (1)

So as $a \leq a \vee b$
& $b \wedge c \leq a \vee b$

As
$\alpha \leq x$
$\beta \leq x$
$\Rightarrow \alpha \vee \beta \leq x$

$a \vee c = lub\{a, c\}$

$c \leq a \vee c$
$b \wedge c \leq c$

$\Rightarrow a \leq a \vee c$ and $c \leq a \vee c$

$b \wedge c \leq a \vee c$

$\Rightarrow a \vee (b \wedge c) \leq a \vee c$ — (2)

∴ from (1) & (2), we get

$a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$

(ii) It is dual of (i)
i.e we get it by just replacing $\wedge$ by $\vee$ & $\vee$ by $\wedge$.

Theorem: Modular Inequality:
If $(L, \leq)$ is a lattice, $a, b, c \in L$.
Then $a \leq c \Leftrightarrow a \vee (b \wedge c) \leq (a \vee b) \wedge c$

Pf: → Let $a \leq c \Leftrightarrow \underline{a \vee c = c}$ — (1)

Now, $a \vee (b \wedge c) \leq (a \vee b) \wedge \underline{(a \vee c)}$
$= (a \vee b) \wedge c$ — (2)

If $a \vee (b \wedge c) \leq (a \vee b) \wedge c$,
$a \vee (b \wedge c) = lub\{a, b \wedge c\}$

$\Rightarrow a \leq a \vee (b \wedge c) \leq (a \vee b) \wedge c$
$= glb\{a \vee b, c\}$
$\Rightarrow a \leq c$

Sublattice

If $(L, \vee, \wedge)$ is a lattice and $S \subseteq L$, then $(S, \vee, \wedge)$ is sublattice of $(L, \vee, \wedge)$ if it is closed under $\vee$ and $\wedge$. ~~If~~
If

$a, b \in S \Rightarrow a \vee b \in S$ and $a \wedge b \in S$.

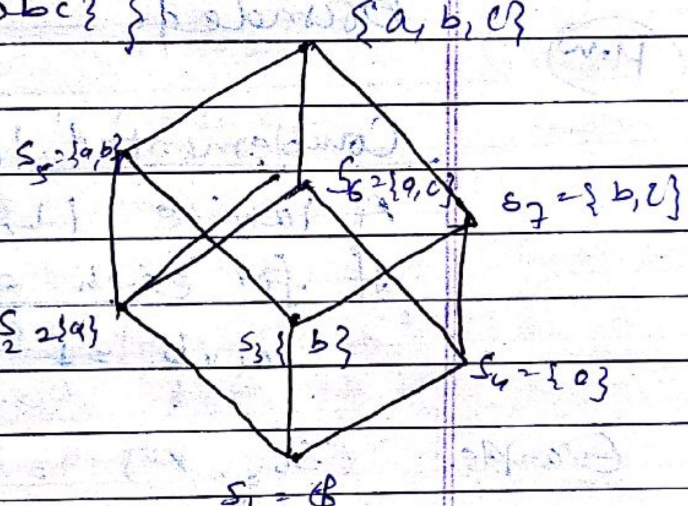
Example

(1) $S = \{a, b, c\}$

$P(S) = \{\phi, \{a\}, \{b\}, \{c\}, \{ab\}, \{bc\}, \{ac\}, \{abc\}\}$

$D_1 = \{S_1, S_2, S_6, S_4\}$

$D_2 = \{S_1, S_5, S_8, S_7\}$



$S_1 = S_1, S_2, S_4, S_6$

$S_6 = S_6$

$\rightarrow$ lub $\{S_1, S_6\} = S_6$

glb $\{S_5, S_7\} = S_3 \notin D_2$

Complete lattice :

Let lattice $(L, \vee, \wedge)$. If each of its n-e subsets of L has lub or glb.

Bounded lattice: An elt. 1 is known of upper bound of lattice $(L, \vee, \wedge)$ if

$x \leq 1 \quad \forall x \in L$

An element 0 is known as lower bound of a lattice L if
$0 \leq x \ \forall \ x \in L$.

And, if $(L, \vee, \wedge)$ has 1 and 0, then it is bounded.

Theorem: Every finite lattice ~~has join~~ bounded.

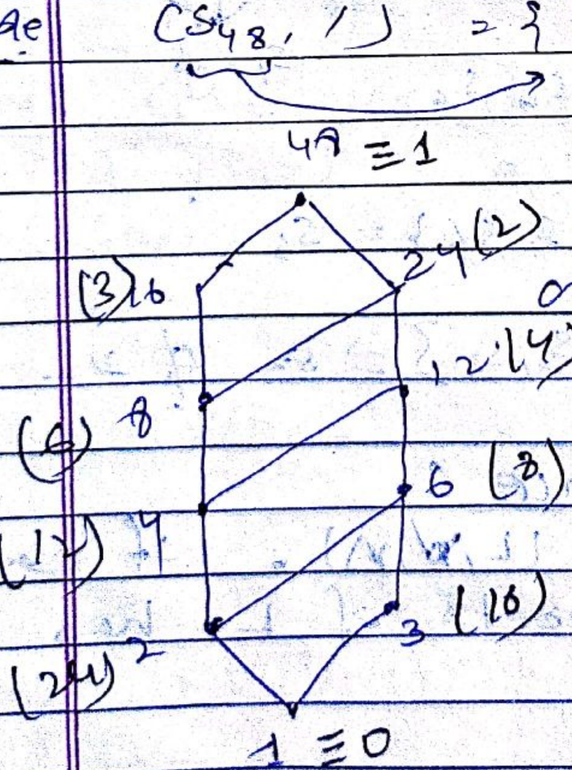
(H.W)

Complemented lattice

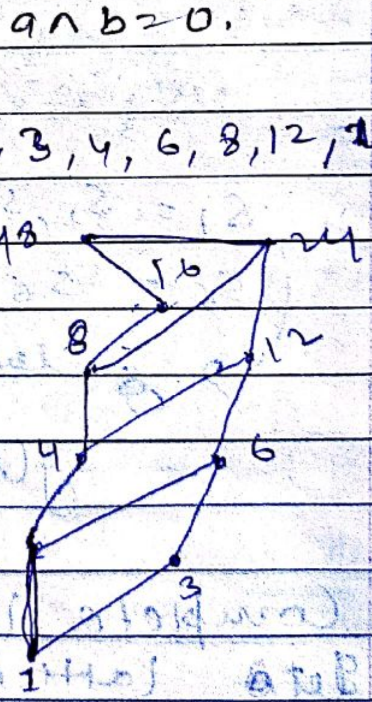
A lattice $(L, \vee, \wedge)$ is a complemented if for each $a \in L$, $\exists$ $b \in L$, s.t
$a \vee b = 1$ and $a \wedge b = 0$.

Example $(S_{48}, /) = \{1, 2, 3, 4, 6, 8, 12, 16, 24, 48,$

or



or



$8, 12, 16, 24, 48 \}$

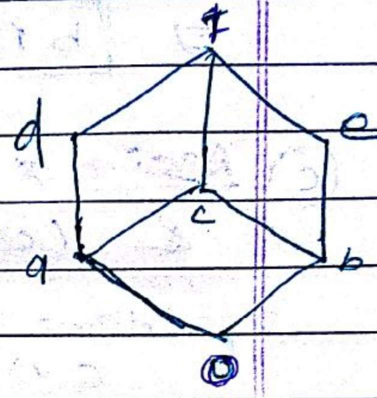
$16 \vee a = 48$
$16 \wedge a = 1$ $\Rightarrow a = 3$

$24 \vee a = 48$
$24 \wedge a = 48$ } $\therefore a = 2$

* * *

18/ mon
19/10/15

Eg- 9.2.4



(1) $a \vee b = 1$
$a \wedge b = 0$

(2) $a \wedge e = 0$
$a \vee e = 1$

(3) $c \vee a = c$, $c \wedge a = a$ ✗ not complement

(4) $c \vee d = 1$, $c \wedge d = a$ ✗ Not complement

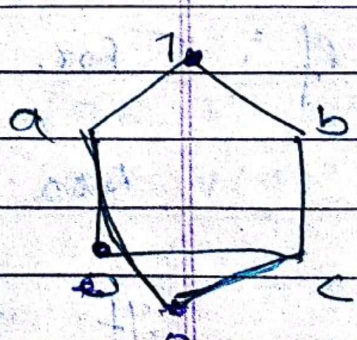
Here a is not complent of c
so this graph is not complemented graph. (i.e. if any elt. has no complent paint then it the ~~graph~~ lattice is not complemented ~~graph~~ lattice)

DISTRIBUTIVE LATTICE :-

A lattice $(L, \vee, \wedge)$ is s.t.b Distributive lattice if for $a, b, c \in L$, then

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$
$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

—x—



\# (a) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
$\because a \wedge b = 0$ & $a \wedge c \neq 0$
$\Rightarrow (a \wedge b) \vee (a \wedge) = 0 \vee 0$
$= 0$

(b) ~~A4~~ ~~b~~ $b \wedge 1 (c \vee a) = b \wedge 1 = b$

& $(b \wedge c) \vee (b \wedge a) = c \vee 0 = c$

$\Rightarrow \boxed{b \wedge (c \vee a) \neq (b \wedge c) \vee (b \wedge a)}$

(c) ~~Again~~ $c \wedge (a \vee b) = c \wedge 1 = c$

& $(c \wedge a) \vee (c \wedge b) = 0 \vee c = c$

Hence

$c \wedge (a \vee b) = (c \wedge a) \vee (c \wedge b)$

Here in case of (b), Distribution prop. not holds. So the given lattice is not a Distributive lattice.

(d) $a \wedge (1 \vee b) = a \wedge 1 = a$

$(a \wedge 1) \vee (a \wedge b) = a \vee 0 = a$

NB In general, where 1 or 0 takes part, then the lattice mostly becomes Distributive lattice

Theorem: If $(L, \wedge, \vee)$ be a distributive lattice s.t $a \vee b = a \vee c$ and $a \wedge b = a \wedge c$, then $b = c$. And this lattice ~~property~~ is also holds ~~known as~~ Cancellation law.

(9.27)

Pf: for, $a, b, c \in L \Rightarrow$

$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

Also let

$a \vee b = a \vee c$ and $a \wedge b = a \wedge c$.

If $b \wedge (a \vee c) = b \wedge (a \vee b) = b \wedge (b \vee a)$

$= b$

$b \wedge b' = 0, \; a \wedge a' = 0$

$b \vee b' = 1, \; a \vee a' = 1$

Also,

$(b \wedge a) \vee (b \wedge c) = (a \wedge c) \wedge (b \wedge c)$

$= (a \vee b) \wedge c$

$= (a \vee c) \wedge c = c$

$\Rightarrow \boxed{b = c}$

Theorem:
(Q. 23) De Morgan's law holds good for the complemented distributive lattice. If a' is complement of a, then

[a] $(a \vee b)' = a' \wedge b'$ and

[b] $(a \wedge b)' = a' \vee b'$.

Pf:- [a] Here, we have to prove that

$\cancel{(a \vee b)' \wedge (a' \wedge b') = 1}$

$\cancel{(a \wedge b)' \wedge (a' \vee b') =}$

$(a \wedge b) \vee (a' \vee b') = (a \vee a' \vee b') \wedge (b \vee a' \vee b')$

$= 1$

and $(a \wedge b) \wedge (a' \vee b') = (a \wedge b \wedge a') \vee (a \wedge b \wedge b')$

$= (a \wedge a' \wedge b) \vee (a \wedge b \wedge b')$

$= (0 \wedge b) \vee (a \wedge 0)$

$= 0 \vee 0 = 0$

And [b]

$(a \vee b) \vee (a' \wedge b') = 1$

$(a \vee b) \wedge (a' \wedge b') = 0$

$(a \vee b \vee a') \wedge (a \vee b \vee b') = (a \vee a' \vee b) \vee (a \vee b \vee b')$

$= (1 \vee b) \vee (a \vee 1)$

$= 1 \vee 1$

$= 1$

||'ly $(a \vee b) \wedge (a' \wedge b') = (a \wedge b \wedge a')$

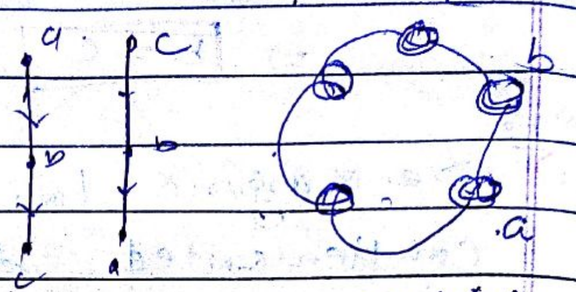
$= 0$

Def^n # Chain:

(9.10) A lattice $(L, \leq)$ has chain, if for every $a, b, c \in L$, $a \leq b \leq c$ and $a \geq b \geq c$.

In fig.

Eg.



$a \leq b \leq c$

$a \geq b \geq c$

Theorem: (9.29) Every chain is a distributive lattice

i.e. if $a, b, c \in L$ then

$\ni a \leq b \leq c$ and $a \geq b \geq c$

$a \wedge (b \vee c) = a \wedge c = a$

$(a \wedge b) \vee (a \wedge c) = a \vee a = a$

and

$a \wedge (b \vee c) = a \wedge b = b$

$(a \wedge b) \vee (a \wedge c) = b \vee c = b$

(check) $\exists$ $b \wedge (c \vee a) = (b \wedge c) \vee (b \wedge a)$

$c \wedge (a \vee b) = (c \wedge a) \vee (c \wedge b)$.

(1) Example: $(S_{16}, /)$

$S_{16} = \{1, 2, 4, 8, 16\}$

Here $2 \leq 4 \leq 8$

But no $2 \geq 4 \geq 8$

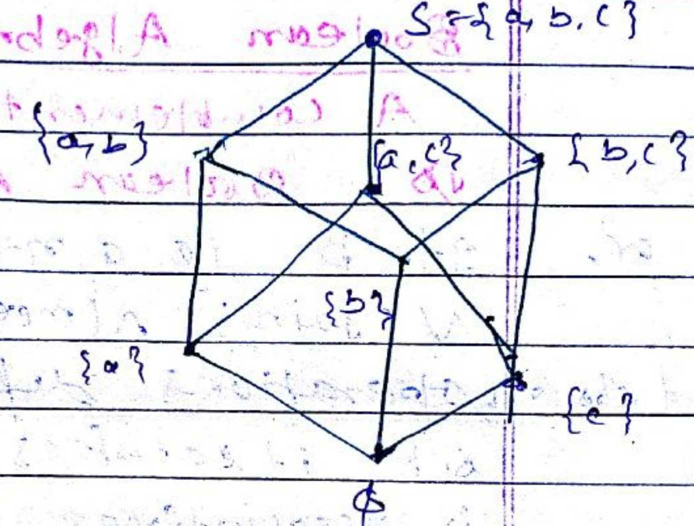
So it has not chain.

(2) $(P(S), \cap, \vee)$ distributive.

$S = \{a, b, c\}$

$P(S) = \{\phi, \{a\}, \{b\}, \{c\}, \{ab\}, \{bc\}, \{ac\}, S\}$

$\{a\} \subseteq \{a,b\} \subseteq S$



→ Next class – Boolean Algebra

(Lattice card)

Boolean Algebra:

A complemented distributive lattice is Boolean Algebra.

or, If B is a n-e set along with $\vee$ (join), $\wedge$ (meet) and $'$ (unary) and elts 0, 1, operations defined for $a, b, c \in B$ s.t. it satisfies —

(i) Associative property:

$$a \vee (b \vee c) = (a \vee b) \vee c$$
$$a \wedge (b \wedge c) = (a \wedge b) \wedge c$$

(2) Commutative property:

$$a \vee b = b \vee a \quad \text{and}$$
$$a \wedge b = b \wedge a$$

(3) Id. property:

$$a \vee 0 = a, \quad a \wedge 1 = a$$

(4) Complement prop.

$$a \vee a' = 1, \quad a \wedge a' = 0$$

(5) $$0' = 1, \quad 1' = 0$$

NB Notation of $(B, 0, 1, \vee, \wedge, ')$
or $(B, 0, 1, +, \cdot, ')$

Example Let $B = \{0, 1\}$

v or +	0	1
0	0	1
1	1	1

∧ / ·	0	1
0	0	0
1	0	1

'	0	1
	1	0

Here, Since $(B, \vee, \wedge, ', 0, 1)$ satisfy all the axioms. Hence it is Boolean Algebra.
$(B = \{0, 1\}, \wedge, \vee, +, \cdot, ', 0, 1)$

let $S = D_{16} = \{1, 2, 4, 8, 16\}$

$a \vee b = \text{lcm}\{a, b\}$

$a \wedge b = \gcd\{a, b\}$, $a' = 16/a$

$\{D_{16}, \vee, \wedge, ', 1, 16\}$

$a \vee b$

$\vee$	1	2	4	8	16
1	1	2	4	8	16
2	2	2	4	8	16
4	4	4	4	8	16
8	8	8	8	8	16
16	16	16	16	16	16

$a \wedge b$,

$\wedge$	1	2	4	8	16
1	1	1	1	1	1
2	1	2	2	2	2
4	1	2	4	4	4
8	1	2	4	8	8
16	1	2	4	8	16

a',

a	1	2	4	8	16
a'	16	8	4	2	1

$a \vee 0 = a \vee 1 = \text{lcm}\{a, 1\} = a$

$a \wedge 1 = a \wedge 16 = \gcd\{a, 16\} = a$

$a \vee a' = a \vee (16/a) = \text{lcm}\{a, 16/a\}$

$\gcd\{2, 8\} = 2 \neq 1$

~~$\gcd\{8, 2'\} = 2$~~

This is NOT a B.A.

Theorem

In a Boolean Algebra, the complement of every elt. is unique

Pf:- let b and c are two complements of a

$a \vee b = 1$, $a \vee c = 1$
$a \wedge b = 0$ $a \wedge c = 0$

$b = b \wedge 1 = b \wedge (a \vee c) = (b \wedge a) \vee (b \wedge c)$
$= 0 \vee (b \wedge c)$
$= b \wedge c$

and $c = c \wedge 1 = c \wedge (a \vee b) = (c \wedge a) \vee (c \wedge b)$
$= 0 \vee (c \wedge b)$
$= c \wedge b$

$\Rightarrow b = c$ (from both above)

Idempotent law:

$a \vee a = a$ and $a \wedge a = a$, $a \in B$.

Pf: let $a \in B$. Then

$a = a \vee 0$ (By id. law)
$= a \vee (a \wedge a')$ (By comp.)
$= (a \vee a) \wedge (a \vee a')$ (By distri.)
$= (a \vee a) \wedge 1$ (By comp.)
$= a \vee a$ (by id.)

$a = a \wedge 1$ (By id.)
$= a \wedge (a \vee a')$ (comp.)
$= (a \wedge a) \vee (a \wedge a')$
$= (a \wedge a) \vee 0$
$= a \wedge a$ (By id)

Dominance law:

$a \vee 1 = 1$, and $a \wedge 0 = 0$ $\forall a \in B$.

$a \vee 1 = a \vee (a \vee a') = (a \vee a) \vee a'$
$= a \vee a'$
$= 1$

Here $(a \wedge 0)$ is dual of $a \vee 1 = 1$, so It is obvious i.e. we get it by replacing $\vee$ by $\wedge$ and 0 by 1 & 1 by 0.

Duality

i.e. if $(B, \vee, \wedge, ', 0, 1)$ is a B.A then ~~of~~ ~~the~~ the dual is ~~ob~~obtained by
$\vee \rightleftarrows \wedge$, $0 \rightleftarrows 1$

→ $(a')' = a$, $a \vee a' = 1$

→ a is complement of a'.
$\Rightarrow (a')' = a$

Eg $0' = 1 \Rightarrow (0')' = (1)' = 0$

* * *

1 P. The Dominance Low :
$a \vee 1 = 1$ & $a \wedge 0 = 0 \; \forall a \in B$

2. Absorption Low :
$a \wedge (a \vee b) = a$ & $a \vee (a \wedge b) = a \; \forall a, b \in B$

3. Demorgon's Low : $\forall a, b \in B$
$(a \vee b)' = a' \wedge b'$ & $(a \wedge b)' = a' \vee b'$

Boolean Algebra

Boolean Expression : (BE)

$(B, \vee, \wedge, ', 0, 1)$ - BA | x_i, $1 \le i \le n$ are the Boolean variables.

Minterm :

BE of the form $\wedge_{i=1}^{n} y_i$, where y_i is x_i or its complement

Eg.

$x \wedge y$, $x \wedge y \wedge z$, $x' \wedge y' \wedge z$, $x' \wedge y \wedge z$

Maxterm :

BE of the form $\vee_{i=1}^{n} y_i$ where y_i is x_i or its complement. Eg. -

x, y, z,

$x \vee y$, $x \vee y \vee z$, $x' \vee y \vee z$

DNF

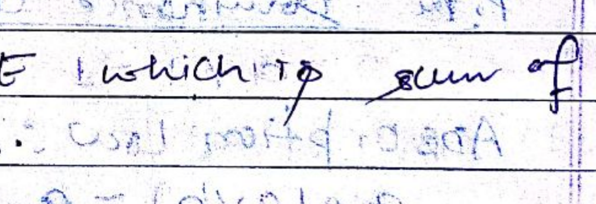
In BA, is BE which is sum of minterms.

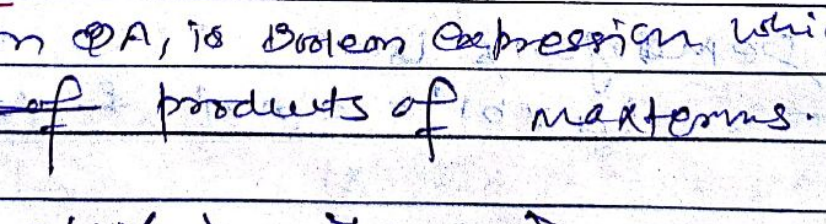
CNF :

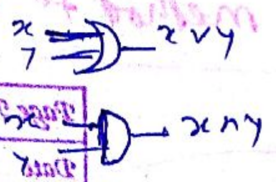
In BA, is Boolean expression which is ~~sum of~~ products of maxterms.

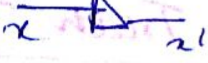
Denotation.

$x \wedge (y \vee z)$

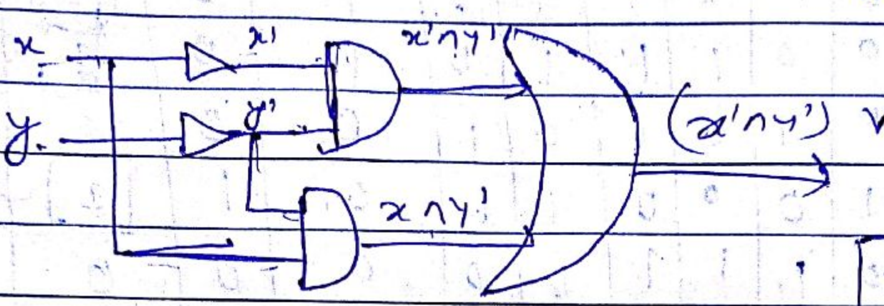








$(x' \wedge y') \vee (x \wedge y')$

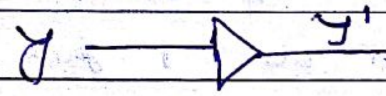


$x \vee x' = 1$
$x \wedge x' = 0$
$1 \wedge x = x$
$0 \vee x = x$

$(x' \wedge y') \vee (x \wedge y')$

$= (x' \vee x) \wedge y'$

$= 1 \wedge y' = y'$



(Both above gates are ~~equivalent~~ isomorphic)

DNF = sum of minterms

CNF = product of maxterms

To Prove Boolean Algebra

I. Truth table method

II. Algebraic method

Boolean function;

If $(B, \wedge, \vee, ', 0, 1)$ is Boolean Algebra and $B = (x_1, x_2, \ldots x_n)$

$f : B \to \{0, 1\}$

Eg. $f(x, y, z) = (x' \wedge y') \vee (x' \wedge z')$

Solving it by truth table —

I. Truth table method to write CNF/DNF

$(x' \wedge y') \vee (x \wedge z')$

x	y	z	x'	y'	z'	$x' \wedge y'$	$x \wedge z'$	
0	0	0	1	1	1	1	0	$\boxed{1}$
0	0	1	1	1	0	1	0	$\boxed{1}$
0	1	0	1	0	1	0	0	0
1	0	0	0	1	1	0	1	$\boxed{1}$
0	1	1	1	0	0	0	0	0
1	0	1	0	1	0	0	0	0
1	1	0	0	0	1	0	1	$\boxed{1}$
1	1	1	0	0	0	0	0	0

1
2
4
7

$x \wedge y$, if any one of both of x & y are zero, then 0
if both are 1 then 1.

FOR DNF

NB. Note down the row in which col. entry is 1. Corresponding to this entry, variable entry 1 is ~~replaced~~ written by corresponding variable and variable entry 0 is written by its complement.

So for first row, the min term is —
$x' \wedge y' \wedge z'$ (as $1 \wedge 1 \wedge 1 \equiv 1$)

for second row, the corresponding minterm is —
$x' \wedge y' \wedge z$

for 4th row, the corresponding min term is —
$x \wedge y' \wedge z'$

for 7th row, the corresponding minterm is —
$x \wedge y \wedge z'$

$\therefore$ DNF of $(x' \wedge y') \vee (x \wedge z')$
$= (x' \wedge y' \wedge z') \vee (x' \wedge y' \wedge z) \vee (x \wedge y' \wedge z') \vee (x \wedge y \wedge z')$.

Now checking for

L.H.S = R.H.S

L.H.S $(x' \wedge y') \vee (x \wedge z')$

$= \{(x' \wedge y') \vee x\} \wedge \{(x' \wedge y') \vee z'\}$

$= \{(x \vee x') \wedge (x \vee y')\} \wedge \{(z' \vee x') \wedge (z' \vee y')\}$

$= \{1 \wedge (x \vee y')\} \wedge \{(z' \vee x') \wedge (z' \vee y')\}$

$= (x \vee y') \wedge \{(z' \vee x') \wedge (z' \vee y')\}$

R.H.S $(x' \wedge y' \wedge z') \vee (x' \wedge y' \wedge z) \vee (x \wedge y' \wedge z') \vee (x \wedge y \wedge z')$

$= (x' \wedge y') \wedge (z' \vee z) \vee (x \wedge y' \wedge z') \vee (x \wedge y \wedge z')$

$= (x' \wedge y') \wedge (1) \vee [(x \wedge z') \wedge (y' \vee y)]$

$= (x' \wedge y') \vee (x \wedge z')$

$=$ L.H.S ✓ Verified

FOR CNF

NB Note down the rows in which col. entry is 0. Corresponding to this entry, variable entry 0 is written by corresponding ~~word~~ variable entry 1 is written by its complement.

So for 3rd Row, the maxterm is

$x \vee y' \vee z$

for 5th Row, the maxterm is —

$x \vee y' \vee z'$

for 6th Row, the maxterm is —

$x' \vee y \vee z'$

and for 9th row, the maxterm is —
$x' \vee y' \vee z'$

$\therefore$ CNF of $(x' \wedge y') \vee (x \wedge z')$
$= (x \vee y' \vee z') \wedge (x \vee y' \vee z') \wedge (x' \vee y \vee z') \wedge (x' \vee y' \vee z')$

~~II. Algebraic Method to write CNF or DNF~~

II. Algebraic Method to write CNF / DNF

DNF. $(x \wedge y') \vee (x \wedge z') = (x \wedge y' \wedge 1) \vee (x \wedge 1 \wedge z')$
$= [x \wedge y' \wedge (z \vee z')] \vee [(x \wedge z') \wedge (y \vee y')]$
$= (x \wedge y' \wedge z) \vee (x \wedge y' \wedge z') \vee (x \wedge z' \wedge y) \vee (x \wedge z' \wedge y)$

CNF

$(x' \wedge y') \vee (x \wedge z') = x \vee \cancel{(y' \vee z')}$ $(x \wedge z)$
$= [x' \vee (x \wedge z')] \wedge [y' \vee$

TPS/ MC due to manukirti

Examples :

$F(x,y,z) = (x \vee y) \wedge (x' \vee z) \wedge (y \vee z')$ → product of sum

find its CNF

Ist Method

$[(x \vee y) \vee 0] \wedge [(x' \vee z) \vee 0] \wedge [(y \vee z') \vee 0]$

$\because x \wedge x' = 0$

$[(x \vee y) \vee (z \wedge z')] \wedge [(x' \vee z) \vee (y \wedge y')] \wedge [(y \vee z') \vee (x \wedge x')]$

$= (x \vee y \vee z) \wedge (x \vee y \vee z') \wedge (x' \vee z \vee y) \wedge (x' \vee z \vee y') \wedge (y \vee z' \vee x) \wedge (y \vee z' \vee x')$

$= (x \vee y \vee z) \wedge (x \vee y \vee z') \wedge (x' \vee y \vee z) \wedge (x' \vee y' \vee z) \wedge (x' \vee y \vee z') \wedge (x \vee y \vee z')$

product of sum.

II- Method (Truth table)

x	y	z	$x \vee y$	x'	$x' \vee z$	z'	$y \vee z'$	$f(x,y,z)$
0	0	0	0	1	1	1	1	$\boxed{0}$
0	0	1	0	1	1	0	0	$\boxed{0}$
0	1	0	1	1	1	1	1	1
1	0	0	1	0	0	1	1	$\boxed{0}$
0	1	1	1	1	1	0	1	1
1	0	1	1	0	1	0	0	$\boxed{0}$
1	1	0	1	0	0	1	1	$\boxed{0}$
1	1	1	1	0	1	0	1	1

values of F having 0

CNF $(x \vee y \vee z) \wedge (x \vee y \vee z') \wedge (x' \vee y \vee z) \wedge (x \vee y' \vee z) \wedge (x' \vee y' \vee z)$

Simplify the given BE

Ex. $f(x,y,z) = (x' \wedge y' \wedge z) \vee (x' \wedge y' \wedge z) \vee (x \wedge y' \wedge z) \vee (x \wedge y' \wedge z')$

$= (x' \wedge y' \wedge z) \vee [(x \wedge y') \wedge (z \vee z')]$

$= (x' \wedge y' \wedge z) \vee [(x \wedge y') \wedge 1]$

$= [x' \wedge y' \wedge z] \vee [(x \wedge y') \wedge 1]$

$= [(x' \wedge z) \vee x] \wedge y'$

$= [(x' \vee x) \wedge (z \vee x)] \wedge y'$

$= (z \vee x) \wedge y'$

$= (z \wedge y') \vee (x \wedge y')$

Example: In Boolean Algebra, ST
$(x \wedge y') \vee (x' \wedge y) = 0$, iff $x = y$

03/11/15

* * *

Sol $\Rightarrow$ let $x = y$, then we have to show only
$(x \wedge y') \vee (x' \wedge y) = 0$
Now, Consider the L.H.S part
$(x \wedge y') \vee (x' \wedge y)$
$(x \wedge x') \vee (x' \wedge x)$ $(\because x = y)$
$0 \vee 0 = 0$ $(\because x \wedge x' = 0)$
$= R.H.S$

$\Leftarrow$ let $(x \wedge y') \vee (x' \wedge y) = 0$
Now we have to prove that $x = y$
Now consider the L.H.S

$x = x \wedge 1 = x \wedge (y' \vee y)$
$= (x \wedge y') \vee (x \wedge y)$
$\cancel{= (x \wedge y') \vee (x \wedge y \wedge 1)}$

$0 \vee x = (x \wedge y') \vee (x \wedge y) \vee 0$

$x = (x \wedge y') \vee (x \wedge y) \vee [\underbrace{(x \wedge y') \vee (x' \wedge y)}_{=0}]$

$= [(x \wedge y') \vee (x \wedge y')] \vee (x \wedge y) \vee (x' \wedge y)$

$\because x \vee x = x$ $\quad = (x \wedge y') \vee (x \wedge y) \vee (x' \wedge y)$

$= (x \wedge y) \vee [\underbrace{(x \wedge y') \vee (x' \wedge y)}_{=0 \to \text{given}}]$

$= (x \wedge y) \vee 0 = x \wedge y$ ——— (i')

$|||^{ly}$ $y = (x \wedge y)$ ——— (ii')

$\therefore$ By (i') & (ii')

$x = (x \wedge y) = y$

$\Rightarrow x = y$

or Consider the following

$x \vee (x \wedge y') \vee (x' \wedge y) = x \vee 0$

$\boxed{\begin{aligned} &(x \wedge y') \vee (x' \wedge y) \vee x = x \\ &\Rightarrow (x \wedge y') \vee [(x \vee x') \wedge (x \vee y)] = x \\ &\Rightarrow (x \wedge y') \vee [1 \wedge (x \vee y)] = x \end{aligned}}$

$\underbrace{x \vee (x \wedge y')}_{} \vee (x' \wedge y) = x$ (By Absorption Law)

$x \vee (x' \wedge y) = x$

$\Rightarrow (x \vee x') \wedge (x \vee y) = x$

$\Rightarrow 1 \wedge (x \vee y) = x \vee y = x$ ——— (1)

$|||^{ly}$ for y

$(x \wedge y') \vee (x' \wedge y) \vee y = 0 \vee y = y$

$(x \wedge y') \vee y = y$ (By absorption law)

$\Rightarrow (x \vee y) \wedge (y' \vee y) = y$

$\Rightarrow (x \vee y) \wedge 1 = y$

$\Rightarrow (x \vee y) = y$ — (ii)

$\therefore$ By (i) & (ii), we have

$x = y$.

* * *

2p6 | ext
24/11/15

Predicates :

1. Ram is a man ← Predicates.
Shyam is a man

denote by M

M(R) : Ram is a man
M(S)

M(x) : x is a man.

Statement function

"P is Q" → where Q is quality of P.

Q(P) → 1 place predicates

A predicates that have m subjects are known as 'm-placed' predicates.

eg. x is sitting between y & z.

* P(x) is a representing a statement $x > 1$.

eg. x is greater than 1.

$P(x) : x > 1$

~~gf~~ universe of discourse (UD) is all Real numbers

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4 Jan

* $Q(x,y)$ denotes a symbol for $x = y + 3$
  $Q(1,2)$ — false
  $Q(3,0)$ — true

Universal Quantifier

* $(\forall x)\ P(x) \rightarrow$ for all x, $P(x)$

$P(x)$: x is a teacher ; x = man
$(\forall x)\ P(x)$ — false

eg

Here $\forall \rightarrow$ universal quantifier

Existential quantifier

$(\exists x)\ P(x)$
$\exists \rightarrow$ Existential quantifier.

eg. $x+1 > x$, UD are real
$(\exists x)\ P(x)$ — True
false if UD are complex.

→ $(\forall x)\ Q(x)$, where Q is a statement '$x < 2$' where universe of discourse consist all real no's.

→ $(\exists x)\ (Q(x)$
'$x < 2$'

→ $(\forall x)\ P(x)$
$x^2 < 10$
all positive integers non-exceeding 4.
$n =$ 1, 2, 3, 4

Ex-① verify the statement " Every living thing is a plant or animal".

~~Joseph~~ Joe's goldfish is alive and it is not a plant. All animals have hearts. Therefore, Joe's goldfish has a heart.

Sol^n

$P(x)$: x is plant.
$A(x)$: x is animal.
$H(x)$: x has heart.
f : Joe's goldfish

Now,

$(\forall x)\ P(x)$
$(\forall x)\ A(x)$

$(\forall x)\ (P(x) \vee A(x))$
$\sim P(f)$
$(\forall x)(\ A(x) \rightarrow H(x))$

$\Rightarrow H(f)$

for all x

(1) $(x)\ (P(x) \vee A(x))$
(2) $P(f) \vee A(f)$
(3) $\sim P(f) \rightarrow A(f)$ $\quad$ $(p \vee q \Rightarrow \sim p \rightarrow q)$
(4) $\sim P(f)$ $\quad$ $(p, p \rightarrow q \Rightarrow q)$
(5) $A(f)$
(6) $(x)\ (A(x) \rightarrow H(x))$
(7) $A(f) \rightarrow H(f)$
(8) $H(f)$ $\quad$ (By 5, 7)

Joe's goldfish has a heart.

(3) Eastblish the validity of argument
"All integers are rational no.s. Some integers are power of 2. Therefore, some rational numbers are power of 2"

Sol:

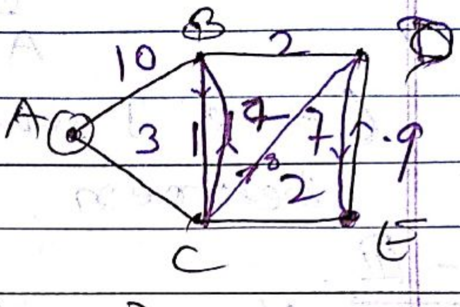
P(x) : x is an integer.
R(x) : x is a rational number
S(x) : x is power of 2

$(x)(P(x) \rightarrow R(x))$,
$(\exists x)(P(x) \wedge S(x))$, $\Rightarrow (\exists x)(R(x) \wedge S(x))$

(1) $(x) \quad (P(x) \rightarrow R(x))$
(2) $(\exists x) \quad (P(x) \wedge S(x))$
(3) $\quad (\cancel{\sim(P(x) \vee (S(x)}$ $)$

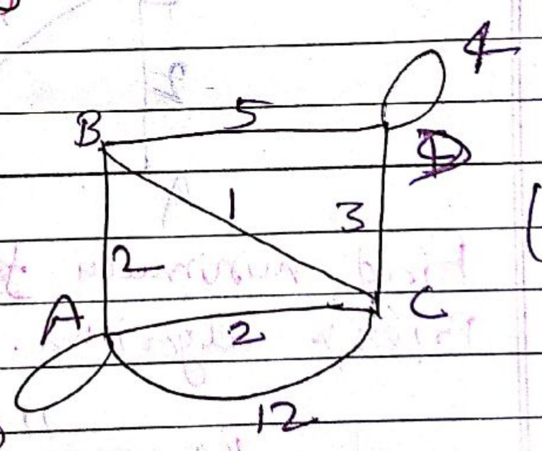
(1) Dijkstra's Algorithm

It is shortest path Algorithm.



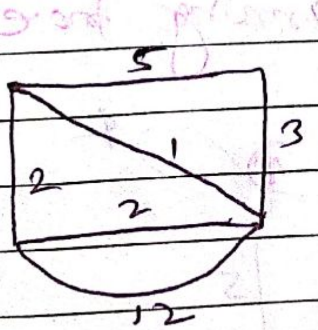
	A →	B	C	D	E
A	$\boxed{0}$	∞	∞	∞	∞
C	~~0~~	10	$\boxed{3}$	∞	∞
E		7	3	11	$\boxed{5}$
B		$\boxed{7}$	3	14	5
D		7	3	$\boxed{9}$	5

(2) Find minimum Spanning tree using 'Kruskal's algorithm'.

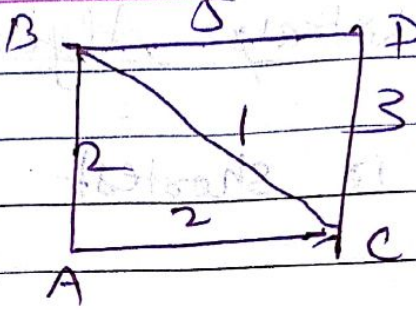


(n = 4
so n-1 = 3

Step-I Remove all the loops



Step-II Remove all the parallel edges between two vertex except the one which is one with least weight.

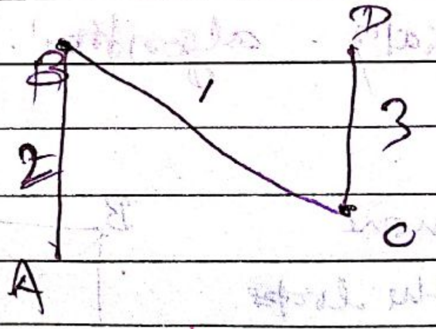


(Here I removed 12 and keep 2 between A-C)

Step - III. Now we are ready to find minimum spanning tree (MST)

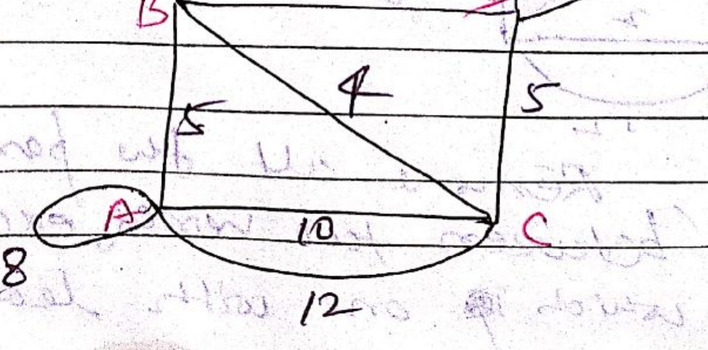
Edge	BC	AB	AC	CD	BD
Weight	1	2	2	3	5
	✓	✓	✗	✓	✗

Since the graph has 4 vertices so MST will have 3 edges.



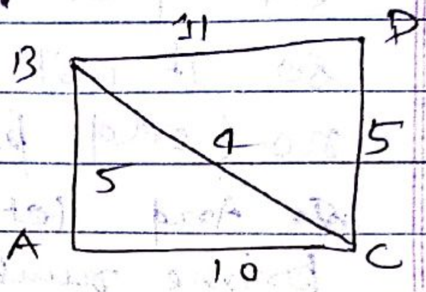
(3) Find minimum spanning tree using Prim's algorithm.

Sol^n:



Step-I, Remove all loops

Step-II Remove all parallel edges b/w two vertex except the one with least weight.



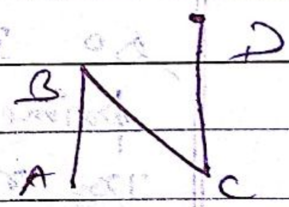
	A	B	C	D
A	0	5 ✓	10	∞
B	5 ✓	0	4 ✓	11
C	10	4 ✓	0	5 ✓
D	∞	11	✓ 5	0

(first fill the (i,j^{th}) where $i=j$ entry with 0)

Here table is completed. So MST is–

AB, BC, CD

ABCD



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(4) Prove that sum of two primes, each greater than 2, is not a prime no.

Let p be a prime greater than 2. So it will be obviously odd no. (odd prime)

And let q be another prime number.

We can write p & q as

$p = 2m+1$ where $m, n \in \mathbb{N}$

$q = 2n+1$

so $p+q = (2m+1)+(2n+1)$

$= 2(m+n)+2$

$= 2(m+n+1)$

Here whatever m & n, $2(m+n+1)$ will be always even numbers

So it implies that sum of two prime numbers greater than two is never prime no.

(5) ST

$m = 3+33+333+\ldots+33\ldots3 = (10^{n+1}-9n-10)/27$

Pf: We will prove it by ~~contradiction~~ induction

$n=1$, $3 = (10^2-9-10)/27 = 81/27$

$= 3$

Let us we assume that it is true for $n=K$ and prove for $n=K+1$

$3+33+333+\dots+333\dots3_m$

$m = k+1$

$m' = 3+33+\dots+333\dots3_{k+1}$

$m' = \left(10^{k+2} - 9(k+1) - 10\right) \times \frac{1}{27}$

or $= \frac{1}{27}\left(10^{k+2} - 9k - 19\right)$

$m' = m + 33\dots3_{k+1}$

$= m + \frac{1}{3}\left(99\dots9_{k+1}\right)$

$= m + \frac{1}{3}\left(10^{k+1} - 1\right)$

$= \frac{1}{27}\left(10^{k+1} - 9k - 10\right) + \frac{1}{3}\left(10^{k+1} - 1\right)$

$= \frac{1}{27}\left(10^{k+1} - 9k - 10\right) + \frac{9}{27}\left(10^{k+1} - 1\right)$

$= \frac{1}{27}\left(10^{k+1} - 9k - 10 + 9\left(10^{k+1} - 1\right)\right)$

$= \frac{1}{27}\left(10^{k+2} - 9k - 19\right)$

~~proved~~ for $k+1$

(6) Sqre of $(4k+1)$ is of the form same. $k \in Z$

$\therefore (4k+1)^2 = 16k^2 + 8k + 1$

~~if k = even~~ $= 8(k^2+k)+1$

let $2(k^2+k) = m$ $\forall k \in Z$

then $(4k+1)^2 = 4 \cdot m + 1$

So sq. of this type elt. is of same type.

(7) Let G be a graph with vertex set $V(G) = \{v_1, v_2, \ldots v_n\}$ and Adjacency matrix A. PT for each positive integer K, the no. of different $v_i - v_j$ walks of length ~~is~~ K in G is the $(i,j)^{th}$ entry in the matrix A^k.

Pf. We will prove it by mathematical induction on k.

Here result is true for $k=1$

Suppose the result is true for $k<K$

Any K edge walk from v_i to v_j has a K-1 edge walk from v_i to v_j followed by a one edge walk from v_h to v_j. The no. of these is

$$\sum_{v_h \in N(v_j)} A^{k-1}(i,h) = \sum_{h=1}^{n} A^{k-1}(i,h) A(h,j)$$

$$= A^k(i,j).$$

(8) A nre connected graph is Eulerian iff all of its vertices are of even degree.

(p-VI)

Proof: Let G be an Eulerian graph. Then it must contain an Eulerian (closed) ~~path~~ graph/ circuit.

Since we have a circuit, each time it visits a vertex, it does

so twice, once in and once out.
So every vertex has even degree

E let G be a connected graph s.t every vertex has even degree
T.S: G has an Eulerian circuit

first we peak a vertex 'v' as starting point.

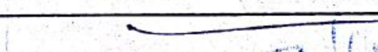
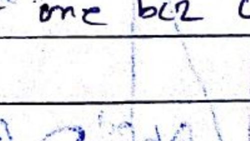
set $C = V$.

Now repeating the following:

(i) Now, Remove the edges of C from G. and any vertices which isolated. Say this graph to G' (it may not be connected).

(ii) Choose a vertex 'u' common to both C and G'. (There must one bcz G is connected)

(9) Two non-isomorphic graphs with deg. seq (2,2,2,1,1)

(i)  

(ii) (9,7,5,3,1)
— No, by Handshaking lemma
Hence $n = 9+7+5+3+1 = 25$
(Sum of deg. of graph is even)

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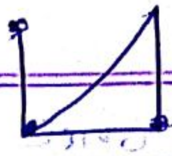
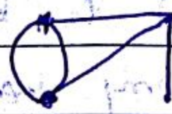
(iii) (3,2,2,1)



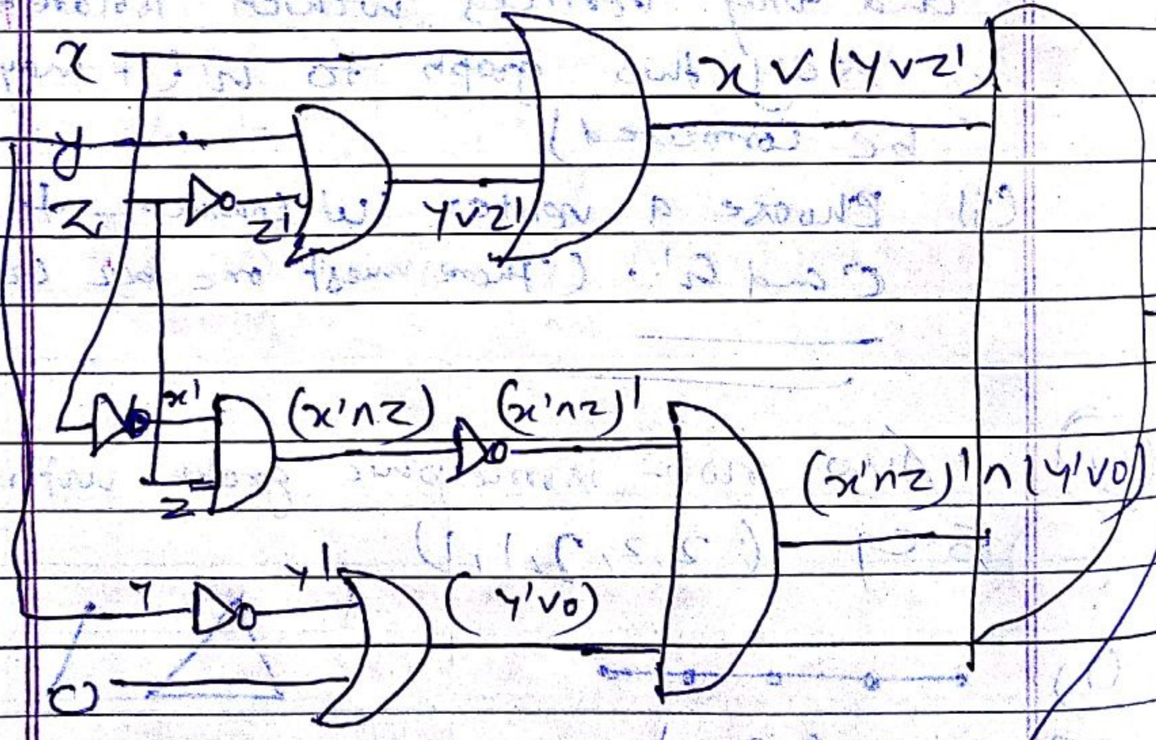
(iv) (10,3,3,2) → NO, the vertex of greatest deg. can hav can have at most $3+3+2 = 8$ edges.

(v) (3,3,3,1)



(10) Construct a logic diagram implementing the function f for

$$f(x,y,z) = (x \vee (y \vee z')) \wedge ((x' \wedge z)' \wedge (y' \vee 0))$$



$(x \vee (y \vee z')) \wedge ((x' \wedge z)' \wedge (y' \wedge 0))$