IIT Bombay
PhD Maths Screening Test

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Department of Mathematics, IIT Bombay

Screening Test for PhD Admissions (2 Dec, 2015)

Time allowed: 2 hours and 30 minutes **Maximum Marks: 40**

Note: Q.1-12 carry 3 marks each. Q.13 carries 4 marks.

Q.1 Let A be a 5×5 matrix s.t. $A^2 = 0$. Compute the least upper bound for its rank.

Q.2 Gram-Schmidt process is applied to the *ordered* basis $\{\mathbf{i}+\mathbf{j}+\mathbf{k}, \mathbf{i}+\mathbf{j}, \mathbf{i}\}$ in $\mathbb{R}^3$. Find the resulting orthonormal basis.

Q.3 Let $A = [a_{ij}]$ be a square matrix of order n whose entries are given as follows. For $1 \leq i, j \leq n$ we have

$$a_{ij} = \begin{cases} ij & \text{if } i \neq j, \\ 1 + ij & \text{if } i = j. \end{cases}$$

Evaluate the determinant of A.

Q.4 Arrange the following matrices with their ranks in a non-decreasing order.

$$P = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \quad Q = \begin{bmatrix} 2 & 2^2 & 2^3 \\ 2^4 & 2^5 & 2^6 \\ 2^7 & 2^8 & 2^9 \end{bmatrix}, \quad R = \begin{bmatrix} 1^2 & 2^2 & 3^2 \\ 4^2 & 5^2 & 6^2 \\ 7^2 & 8^2 & 9^2 \end{bmatrix}.$$

Q.5 If $a_1 \geq 0, \;\; a_2 \geq 0$ and $a_{n+2} = \sqrt{a_n a_{n+1}}$, find the limit of the sequence $\{a_n\}$.

Q.6 Let $\{P_n\}$ be a sequence of polynomials such that for $n = 0, 1, 2, ...$

$$P_0 = 0 \text{ and } P_{n+1}(x) = P_n(x) + \frac{x^2 - P_n^2(x)}{2}.$$

Assuming the fact that $\{P_n\}$ is convergent point-wise, find the limit function $\lim_{n\to\infty} P_n(x)$.

Q.7 Find the values of x, $(x \in \mathbb{R})$ for which the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{x^2 + n}{n^2}$$

is convergent.

Q.8 Find the range of values of α for which $\displaystyle\int_0^\infty \frac{\sin x}{x^\alpha}$ is convergent (i.e. is finite).

Q.9 A point X is picked uniformly at random from the perimeter of a unit circle. Find the expected value of $|X|$.

Q.10 Suppose the distribution of Y, conditional on $X = x$, is Normal(x, x^2) and the marginal distribution of X is Uniform$(0, 1)$. Find the $Cov(X, Y)$.

Q.11 Let X be an observation from the probability density function

$$f(x|\theta) = \left(\frac{\theta}{2}\right)^{|x|} (1-\theta)^{1-|x|}, \quad x = -1, 0, 1; \quad 0 \leq \theta \leq 1.$$

Find the maximum likelihood estimator (mle) of θ and its expectation.

Q.12 Let $U_1, U_2, \ldots, U_n$ be i.i.d. Uniform$(0, \theta)$, $\theta > 0$, random variables. Find the uniformly minimum variance unbiased estimator (UMVUE) of $\cos\theta$.

Q.13 Let $A = \begin{bmatrix} 1 & r & r \\ r & 1 & r \\ r & r & 1 \end{bmatrix}$. Find the range of values of r so that A is positive definite.

Department of Mathematics, IIT Bombay

Screening Test for PhD Admissions (9 May, 2016)

Time allowed: 2 hours and 30 minutes **Maximum Marks: 40**

Name: Choice: | Math | Stat |

- *Write your name in the blank space at the top of this question-paper, and also tick 'Math' or 'Stat' to indicate your first choice of one of the* 2 *PhD programs. Do the same on the attached answer-sheet.*
- *All questions carry 2 marks.*
- *The answer to each question is a number, function, set, inequality, random variable etc. Record only your answers on the answer-sheet as indicated. You are being given a separate work-sheet for doing rough work (this will not be collected).*

Q.1 Consider the vector space $\mathbb{R}^{2016}$ over the field $\mathbb{R}$ of real numbers. What is the smallest positive integer k for which the following statement is true: given any k vectors $v_1, \ldots, v_k \in \mathbb{R}^{2016}$, there exist real numbers $a_1, \ldots, a_k$, not all zero, such that $a_1v_1 + \cdots + a_kv_k = 0$ and $a_1 + \cdots + a_k = 0$.

Q.2 Find all complex triples (a, b, c) such that the following matrix is diagonalizable

$$\begin{bmatrix} 1 & a & b \\ 0 & 2 & c \\ 0 & 0 & 1 \end{bmatrix}.$$

Q.3 For what values of k does the linear system

$$x - 3z = -3$$
$$2x + ky - z = -2$$
$$x + 2y + kz = 1$$

in unknowns x, y, z have no solution?

Q.4 Let V and W be subspaces of the vector space $\mathbb{R}^9$ over the field $\mathbb{R}$ of real numbers with $\dim V = 5$ and $\dim W = 6$. Then what is the smallest possible dimension of $V \cap W$?

Q.5 Consider the inner product

$$\langle (a_1, a_2), (b_1, b_2) \rangle = 2a_1b_1 - a_1b_2 - a_2b_1 + 5a_2b_2$$

on $\mathbb{R}^2$. Write down a vector which is orthogonal to $(1, 0)$ and has norm 1.

Q.6 Find all values $\alpha \in \mathbb{R}$ for which the matrix

$$\begin{bmatrix} \alpha & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 4 \end{bmatrix}$$

is positive definite.

Q.7 Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ be a real matrix and c_1, c_2, c_3 be the scalars

$$c_1 = \det\left(\begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix}\right), \; c_2 = \det\left(\begin{bmatrix} a_{13} & a_{11} \\ a_{23} & a_{21} \end{bmatrix}\right), \; c_3 = \det\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right).$$

Find all triples (c_1, c_2, c_3) so that rank $A = 2$.

Q.8 Let A be the complex 3×3 matrix

$$A = \begin{bmatrix} 2 & 0 & 0 \\ a & 2 & 0 \\ b & c & -1 \end{bmatrix}.$$

Find all triples (a, b, c) for which the characteristic and minimal polynomials of A are different.

Q.9 Given $\epsilon > 0$, what is the largest δ which fits the definition of continuity of the function

$$f(x) = \begin{cases} \frac{x+3}{2} & \text{if } x \leq 1 \\ \frac{7-x}{3} & \text{if } 1 \leq x \end{cases}$$

at $x = 1$, that is, the largest $\delta > 0$ for which the implication $|x - 1| < \delta \implies |f(x) - f(1)| < \epsilon$ holds?

Q.10 A point ω is said to be a *fixed point* of a function f if $f(\omega) = \omega$. Given that the function

$$f(x) = \frac{x^3 + 1}{3}$$

has three fixed points α, β, γ in $(-2, -1)$, $(0, 1)$ and $(1, 2)$ respectively, let us define a sequence of real numbers $\{x_n\}$ as

$$x_1 = \gamma - 0.01 \; , \; x_{n+1} = f(x_n) \quad n = 1, 2, 3, \ldots .$$

Given that the sequence converges, find

$$\lim_{n\to\infty} x_n.$$

Q.11 Suppose f is a real valued continuously differentiable function on $[0,1]$ with $f(0) = f(1) = 0$ and

$$\int_0^1 f^2(x)\, dx = 1.$$

Find the value of $\int_0^1 x f(x) f'(x)\, dx$.

Q.12 Let

$$y_n = \frac{n^2}{n^3+n+1} + \frac{n^2}{n^3+n+2} + \cdots + \frac{n^2}{n^3+2n}.$$

Find $\lim_{n\to\infty} y_n =$?

Q.13 Find all values of $x > 0$ for which the series

$$1 + \frac{x}{1!} + \frac{2^2x^2}{2!} + \frac{3^3x^3}{3!} + \cdots + \frac{n^nx^n}{n!} + \cdots$$

converges.

Q.14 Find all values of (p, q) for which the integral

$$\int_0^1 x^p \, \log^q(1/x)\, dx$$

converges.

Q.15 Suppose X, Y, Z are i.i.d. Uniform$[0,1]$ random variables. What is the probability $\mathbb{P}(XY < Z^2)$? Write down your answer as a fraction.

Q.16 Suppose $X_1, X_2, \ldots, X_n$ are i.i.d. random variables for which $\mathbb{E}(X_1^{-1}) < \infty$, where $\mathbb{E}$ denotes expectation. Let $S_i := X_1 + X_2 + \cdots + X_i$. For $m < n$ calculate $\mathbb{E}(S_m/S_n)$.

Q.17 Suppose the distribution of Y, conditional on $X = x$, is Poisson(x) and random variable X is exponentially distributed with rate parameter 1. Find the correlation between X and Y.

Q.18 Let $U_1, U_2, \ldots, U_n$ be i.i.d. Uniform$(0, \theta)$, $\theta > 0$ random variables. Find the uniformly minimum variance unbiased estimator (UMVUE) of θ^2.

Q.19 Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables with one of two possible probability density functions $f(x|\theta)$. If $\theta = 0$, then $f(x|\theta) = I_{(0,1)}(x)$ while if $\theta = 1$, then $f(x|\theta) = \frac{1}{2\sqrt{x}} I_{(0,1)}(x)$. Find the maximum likelihood estimator $\hat{\theta}$ of θ.

Q.20 Suppose that the random variables $Y_1, \ldots, Y_n$ satisfy

$$Y_i = \beta x_i + \epsilon_i, \qquad i = 1, \ldots, n,$$

where $x_1, \ldots, x_n$ are fixed known constants, and $\epsilon_1, \ldots, \epsilon_n$ are i.i.d. $N(0, \sigma^2)$, σ^2 known. What is the distribution of MLE of β?

Department of Mathematics, IIT Bombay

Screening Test for PhD Admissions (Dec 1, 2016)

Time allowed: 2 hours and 30 minutes **Maximum Marks: 40**

Name: Choice: | Math | Stat |

- *Write your name in the blank space at the top of this question-paper, and also tick 'Math' or 'Stat' to indicate your first choice of one of the* 2 *PhD programs. Do the same on the attached answer-sheet.*
- *All questions carry 2 marks. There will be no partial credit. Simplify all your answers. In particular, the answer should not be in the form of a sum or product.*
- *The answer to each question is a number or a set or a yes/no statement. Record only your answers on the answer-sheet as indicated. You are being given a separate work-sheet for doing rough work (this will not be collected).*

1. Let

$$A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}.$$

(i) What is the minimal polynomial of A?

(ii) Is A diagonalizable over $\mathbb{R}$?

2. For each real number α, we define the bilinear form $F_\alpha : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ by

$$F_\alpha\big((x_1, x_2, x_3), (y_1, y_2, y_3)\big) = 2x_1x_2 + (\alpha+5)x_1y_2 + x_1y_3 + (\alpha+5)x_2y_1$$

$$-(2\alpha+4)x_2y_2 + 2x_2y_3 + x_3y_1 + 2x_3y_2 + 2x_3y_3.$$

Find the set of $\alpha \in \mathbb{R}$ such that F_α is positive definite.

3. What is the number of non-conjugate 6×6 complex matrices having the characteristic polynomial $(x-5)^6 = 0$?

4. Let S be a subspace of the vector space of all 11×11 real matrices such that (i) every matrix in S is symmetric and (ii) S is closed under matrix multiplication. What is the maximum possible dimension of S?

5. Let A be a 55×55 diagonal matrix with characteristic polynomial

$$(x - c_1)(x - c_2)^2(x - c_3)^3 \ldots (x - c_{10})^{10},$$

where $c_1, \ldots, c_{10}$ are all distinct. Let V be the vector space of all 55×55 matrices B such that $AB = BA$. What is the dimension of V?

6. Let A be the complex square matrix of size 2016 whose diagonal entries are all -2016 and off-diagonal entries are all 1. What are the eigenvalues of A and their geometric multiplicities?

7. Let V be a subspace of $\mathbb{R}^{13}$ of dimension 6, and W be a subspace of $\mathbb{R}^{31}$ of dimension 29. What is the dimension of the space of all linear maps from $\mathbb{R}^{13}$ to $\mathbb{R}^{31}$ whose kernel contains V and whose image is contained in W?

8. Let V (resp. W) be the real vector space of all polynomials in two commuting (resp. noncommuting) variables with real coefficients and of degree strictly less than 100. What are the dimensions of V and W?

9. Find the number of connected components of the set

$$\left\{x \in \mathbb{R} : x^3\left(x^2 + 5x - \frac{65}{3}\right) > 70x^2 - 300x - 297\right\}$$

under the usual topology on $\mathbb{R}$.

10. Let $P_n(x)$ be the Taylor polynomial at $x = 0$ for the exponential function e^x. Compute the least n such that $|e - P_n(1)| < 10^{-5}$.

11. Find the set of values of the real number a for which $\sum_{n=1}^{\infty}\left(\frac{1}{n} - \sin\frac{1}{n}\right)^a$ converges.

12. Let $p(x)$ be a polynomial of degree 7 with real coefficients such that $p(\pi) = \sqrt{3}$ and

$$\int_{-\pi}^{\pi} x^k p(x)dx = 0 \quad \text{for } 0 \leq k \leq 6.$$

What are the values of $p(0)$ and $p(-\pi)$?

13. Let $f : [0, 1] \to \mathbb{R}$ be defined by

$$f(x) = \frac{57^{(x^2+1)} + 3}{e^{x^2} + 1113337x^2 + 1113339x^{3/2} + 1113341x + 1}.$$

Find the value of $\lim_{n\to\infty}\left(\int_0^1 f(x)^n\, dx\right)^{\frac{1}{n}}$.

14. Consider $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^{1/3}$. Let $g(x) = \sum_{n=0}^{\infty} a_n(x - 3/2)^n$, where $a_n = \frac{f^{(n)}(3/2)}{n!}$ for $n \geq 0$. What is the largest open set contained in $\{x \mid f(x) = g(x)\}$?

15. An urn contains 11 balls numbered $1, 2, \ldots, 11$. We remove 4 balls at random without replacement and add their numbers. Compute the mean of the total.

16. Let X, Y and Z be independent, identically distributed random variables, each having the Bernoulli distribution with parameter p, $0 < p < 1$. Put $T = X + Y + Z$ and $S = XYZ$.

    Find $P(T = 2|S = 0)$.

17. Let $\{X_n; n \geq 1\}$ be a sequence of identically and independently distributed random variables with uniform distribution on $(0, 1)$. Suppose $Y_n = (X_1X_2 \ldots X_n)^{1/n}$, (i.e., Y_n is the geometric mean of $X_1, X_2, \ldots, X_n$). Find the number c such that Y_n converges to c with probability 1.

18. Let $\{X_n; n \geq 1\}$ be a sequence of identically and independently distributed random variables having Poisson distribution with mean 1. Let $\bar{X}_n = \frac{X_1+\ldots+X_n}{n}$. Find the limit of $P(\bar{X}_n \geq 1)$ as n goes to ∞.

19. Suppose that the joint probability function of two random variables X and Y is

$$f(x, y) = \frac{xy^{x-1}}{3}, \quad x = 1, 2, 3 \text{ and } 0 < y < 1.$$

    Find the variance of X.

20. Suppose that the random variables X and Y are independent and identically distributed and that the moment generating function (mgf) of each is

$$\psi(t) = e^{t^2+3t}, \text{ for } -\infty < t < \infty.$$

    Find the mgf of $Z = 2X - 3Y + 4$ at $t = 1$.

Department of Mathematics, IIT Bombay

Screening Test for PhD Admissions (May 9, 2017)

Time allowed: 2 hours and 30 minutes **Maximum Marks: 40**

Name: Choice: | Math | Stat |

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- *Use of calculators is not allowed. Please keep aside your notes and mobile phones, whose use during the examination is prohibited. Any candidate found to be adopting any unfair means will be disqualified.*

1. Let d_1, d_2, d_3, d_4, d_5 denote the dimensions of the subspaces of all real 29×29 matrices which are diagonal, upper triangular, trace zero, symmetric, and skew symmetric, respectively. Write down the 5-tuple $(d_1, d_2, d_3, d_4, d_5)$.

2. Find all real α for which the following quadratic form

$$Q(x_1, x_2, x_3) := x_1^2 + 2x_2^2 + 2x_3^2 + 2x_1x_3 + 2\alpha x_2 x_3$$

is positive definite.

3. If $I \neq T \in M_4(\mathbb{C})$ has $(X-1)^4$ as its characteristic polynomial then what is the largest possible dimension of the centraliser of T in $M_4(\mathbb{C})$ (= the subspace of all matrices that commute with T)?

4. Let V denote the (complex) vector space of complex polynomials of degree at most 9 and consider the linear operator $T : V \to V$ defined by

$$T(a_0 + a_1X + \cdots + a_9X^9) =$$
$$a_0 + (a_2X + a_1X^2) + (a_4X^3 + a_5X^4 + a_3X^5) + (a_7X^6 + a_8X^7 + a_9X^8 + a_6X^9).$$

   (a) What is the trace of T^4?

   (b) What is the trace of T^2?

5. What is the signature of the symmetric bilinear form defined by the following matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & -5 & 0 & 0 \\ 0 & -5 & 0 & 0 & 0 \end{pmatrix}?$$

6. Let A be the 3×3 matrix

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

Determine all real numbers a for which the limit $\lim_{n\to\infty} a^n A^n$ exists and is non-zero. [For a sequence of 3×3 matrices $\{B_n\}$ and a 3×3 matrix B, $\lim_{n\to\infty} B_n = B$ means that, for all vectors $x \in \mathbb{R}^3$, we have $\lim_{n\to\infty} B_n x = Bx$ in $\mathbb{R}^3$.]

7. Let V denote the vector space consisting of all polynomials over $\mathbb{C}$ of degree at most 2017. Consider the linear operator $T : V \to V$ given by $T(f) = f'$, that is, T maps a polynomial f to its derivative f'. Write down all eigenvalues of T along with their algebraic and geometric multiplicities.

8. Let $A = (a_{ij})$ be the square matrix of size 2018 defined by

$$a_{ij} = \begin{cases} 2 & \text{if } i+1=j, \\ 1/3 & \text{if } i=j+1, \\ 0 & \text{otherwise.} \end{cases}$$

Let B be the leading principal minor of A of order 1009 (=the submatrix of A formed by the first 1009 rows and columns).

(a) What is the determinant of A?

(b) What is the rank of B?

9. For $\alpha \in (0,1)$, let the sequence $\{x_n\}$ be such that $x_0 = 0$, $x_1 = 1$ and $x_{n+1} = \alpha x_n + (1-\alpha)x_{n-1}$, $n \geq 1$. Find $\lim_{n\mapsto\infty} x_n$.

10. For what values of p is the series

$$\sum_{k=1}^{\infty} (-1)^k \, k^p \, \log k$$

(a) absolutely convergent?

(b) conditionally convergent?

11. Determine the radius of convergence of the following two power series.

(a) $\sum_{n=1}^{\infty} \frac{x^{6n+2}}{\left(1+\frac{1}{n}\right)^{n^2}}$.

(b) $\sum_{n=0}^{\infty} a_n(x-2017)^n$ with $a_n = \begin{cases} 1/2 & \text{if } n \text{ is even,} \\ 1/3 & \text{if } n \text{ is odd.} \end{cases}$

12. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given below:

$$f(t) = \begin{cases} |t/2| & t < -2, \\ |t+3/2| + 1/2 & -2 \le t < -1, \\ |t^3| & -1 \le t < 1, \\ |t-3/2|^2 + 3/4 & 1 \le t < 2, \\ |t/2| & t \ge 2. \end{cases}$$

What is the number of connected components of the set $\{t \in \mathbb{R} : f \text{ is differentiable at } t\}$?

13. Determine the set of all points where the Taylor series of the function

$$f(x) = \sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n}$$

around the point $x = e$ converges to $f(x)$.

14. Consider the sequence of real-valued functions $\{f_n\}$ defined by

$$f_n(x) = \frac{1}{1+nx^2}.$$

Assuming the fact that $\{f_n\}$ converges uniformly to a function f find out all real numbers x for which

$$f'(x) = \lim_{n\to\infty} f_n'(x).$$

15. A bowl contains 5 strands of spaghetti. We select two ends at random and join them together. We repeatedly do this until there are no ends left. What is the expected number of loops in the bowl? Write the answer as a fraction.

16. The zero truncated random variable X_T has probability mass function,

$$P(X_T = x) = \frac{P(X = x)}{P(X > 0)}, \quad x = 1, 2, \ldots$$

What is the mean of X_T when $X \sim \text{Poisson}(\lambda)$.

17. An electronic device has lifetime denoted by T. The device has value $V = 5$ if it fails before time $t = 3$; otherwise, it has value $V = 2T$. The probability density function of T is $f(t) = \frac{1}{1.5}e^{-t/(1.5)}, t > 0$. Determine $P(V \leq v)$ for

(a) $0 \leq v < 6$.

(b) $v \geq 8$.

18. If the random variable X has probability density function

$$f(x) = \begin{cases} \frac{x-1}{2} & \text{if } 1 < x < 3 \\ 0 & \text{otherwise,} \end{cases}$$

find a monotone function $u(x)$ such that the random variable $Y = u(X)$ has a uniform distribution.

19. Let X have density $f(x) = 3x^2$, $0 \leq x \leq 1$, and let $Y_i = e^{X_i^2}$. What is the constant m to which $(Y_1 Y_2 \ldots Y_n)^{1/n}$ converges with probability 1 as $n \to \infty$?

20. Let $\{Y_n \ : \ n \geq 1\}$ be a sequence of independent standard normal random variables and let

$$X_n = \frac{Y_{2n}}{Y_{2n-1}}, \quad n = 1, 2, \ldots$$

If $\overline{X_n} = \frac{X_1 + X_2 + \cdots + X_n}{n}$, find

$$\lim_{n \to \infty} P\{\sqrt{n}\,(\overline{X_n} - 1) \leq 0\}.$$

Department of Mathematics, IIT Bombay

Screening Test for PhD Admissions (December 1, 2017)

Time allowed: 2 hours and 30 minutes **Maximum Marks: 40**

Name: Choice: | Math | Stat |

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- *Use of calculators is not allowed. Please keep aside your notes and mobile phones, whose use during the examination is prohibited. Any candidate found to be adopting any unfair means will be disqualified.*

1. A coin is selected at random from a collection of 10 coins numbered 1 to 10 where the ith coin has probability $\frac{1}{2^i}$ for getting a head. The selected coin was tossed and resulted in a head. Find the probability that the selected coin was the coin numbered 1.

2. Let (X, Y) be uniform $(0,\ 1) \times (0, 1)$ random vector and $Z = \min\{X, Y\}$. Find $M(1)$, where $M(t)$ is the moment generating function of Z.

3. A woman leaves for work between 8 am and 8.30 am and takes between 40 to 50 minutes to get there. Let the random variable X denote her time of departure, and the random variable Y the travel time. Assuming that these variables are independent and uniformly distributed, find the probability that the woman arrives at work before 9 am.

4. Let $X_1, \ldots, X_n$ be iid uniform$(0, \theta)$, where $\theta > 0$. Suppose $T_n = \max\{X_1, \ldots, X_n\}$. Then $f(x)$, the density of the limiting distribution of $n(1 - \frac{T_n}{\theta})$ as n goes to ∞, is

5. Let U, V, W be independent random variables with mean and variance both equal to 1. Let $X = U + V$ and $Y = V + W$. Then the covariance between X and Y is

6. Let $X_1, X_2, \ldots$ be a sequence of independent Poisson random variables with mean 1. For $n \geq 1$ set

$$Y_n = \begin{cases} 1 & \text{if } X_n = 0 \text{ or } 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then the constant to which $\frac{\sum_{i=1}^{n} Y_i}{n}$ converges with probability 1 is

7. (a) For what real values of x does the series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{5^{n-1}}(x+2)^n$ converge?

(b) What is the radius of convergence of the Taylor series of $\dfrac{2\tan x}{1+4x^2}$ around $x = 0$?

8. For any positive real numbers α and β, define

$$f(x) = \begin{cases} x^{\alpha} \sin\left(\frac{1}{x^{\beta}}\right) & , \text{ if } x \in (0,1], \\ 0 & , \text{ if } x = 0. \end{cases}$$

   (a) For a given $\beta > 0$, find all values of α such that $f'(0)$ exists.

   (b) For a given $\beta > 0$, find all values of α such that f is of bounded variation on $[0,1]$.

9. Let $\{a_n\}$ be a strictly increasing sequence of positive real numbers (i.e. $a_n < a_{n+1}$, for all natural number n) with $\lim_{n\to\infty} a_n = (\sqrt{2})^e$ and let $s_n = \sum_{k=1}^{n} a_k$. If $\{x_n\}$ is a strictly decreasing sequence of real numbers with $\lim_{n\to\infty} x_n = e^{\sqrt{2}}$, then find the value of

$$\lim_{n\to\infty} \frac{1}{s_n} \sum_{i=1}^{n} a_i x_i \, .$$

10. Let $\{f_n\}$ be a sequence of polynomials with real coefficients defined by $f_0 = 0$ and for $n = 0, 1, 2, \ldots,$

$$f_{n+1}(x) = f_n(x) + \frac{x^2 - f_n^2(x)}{2}.$$

Find $\lim_{n\to\infty} f_n$ on $[-1,1]$, where the limit is taken in the supremum norm of f_n over the interval $[-1,1]$.

11. (a) Find the number of connected components of the set

$$\left\{x \in \mathbb{R} : x^5 + 60x \geq 15x^3 + 10x^2 + 20\right\}.$$

   (b) How many of the connected components in part (a) above are compact?

12. Let $P_n(x)$ be the Taylor polynomial for the exponential function, e^x, at $x = 0$. Compute the least n such that $|e - P_n(1)| < 10^{-4}$.

13. Consider $\mathbb{R}^3$ (column vectors) with the standard inner product. Let L be the one dimensional subspace of $\mathbb{R}^3$ spanned by the column vector $(2,1,2)^t$. Let A be the 3×3 matrix such that the linear transformation of $\mathbb{R}^3$ given by $x \mapsto Ax$ is orthogonal projection onto the line L. Then the sum of the entries of A equals

14. For $\alpha \in \mathbb{R}$, let $q(x_1, x_2) = x_1^2 + 2\alpha x_1 x_2 + \frac{1}{2}x_2^2$, for $(x_1, x_2) \in \mathbb{R}^2$.

    (a) Take $\alpha = 1/4$ and let B be the symmetric matrix of q with respect to the basis $\{(1,0),(1,1)\}$ of $\mathbb{R}^2$. Then the entry in row 1, column 2 of B equals

    (b) Find all values of α for which the signature of q is 1.

15. Let $M_n(\mathbb{R})$ denote the set of all $n \times n$ real matrices. Let

$$\mathcal{A} := \left\{ A \in M_8(\mathbb{R}) \;\middle|\; \begin{array}{l} \text{the characteristic polynomial of } A \text{ is } (x-1)^2(x-2)^6 \\ \text{and the minimal polynomial of } A \text{ is } (x-1)(x-2)^2. \end{array} \right\}$$

and

$$G := \left\{ k \in \mathbb{N} \;\middle|\; \begin{array}{l} \text{there exists a matrix } A \in \mathcal{A} \text{ for which} \\ k \text{ is the geometric multiplicity of the eigenvalue 2} \end{array} \right\}.$$

Then the set G equals

16. Consider the vector space $\mathbb{R}^3$ with coordinates (x_1, x_2, x_3) equipped with the inner product

$$\langle (a_1,a_2,a_3),(b_1,b_2,b_3)\rangle = 2(a_1b_1 + a_2b_2 + a_3b_3) - (a_1b_2 + a_2b_1 + a_2b_3 + a_3b_2).$$

Write down all vectors in $\mathbb{R}^3$ which are orthogonal to the plane $x_1 - 2x_2 + 2x_3 = 0$ and have norm 1.

17. Let A be a 10×10 matrix defined by $A = (a_{ij})$ where $a_{ij} = 1 - (-1)^{i-j}$. If $P(x)$ is the minimal polynomial of A then

(a) what is the degree of $P(x)$?

(b) what is the coefficient of x in $P(x)$?

18. Let A be a diagonal matrix whose characteristic polynomial is

$$P(x) = (x-15)(x-14)^2(x-13)^3 \cdots (x-2)^{14}(x-1)^{15}.$$

Let V be the set of all 120×120 matrices commuting with A. Then the dimension of V is

19. For each real number α, let B_α be the bilinear form

$$B_\alpha\big((x_1, y_1, z_1), (x_2, y_2, z_2)\big) =$$

$$x_1x_2 + 2y_1y_2 + 11z_1z_2 + (\alpha+1)(x_1y_2 + y_1x_2) + 3(x_1z_2 + z_1x_2) + (\alpha+2)(y_1z_2 + z_1y_2).$$

Find the set $\{\alpha \in \mathbb{R} : B_\alpha \text{ is positive definite}\}$.

20. Let A and B be $(2017)^2 \times (2017)^2$ complex matrices with A being invertible. Find the maximum number of complex numbers α for which the matrix $\alpha A + B$ is not invertible.

Department of Mathematics, IIT Bombay
Screening Test for PhD Admissions (May 10, 2018)
Time allowed: 2 hours and 30 minutes

Departmental Reg No. : **Maximum Marks: 40**

Name : **Choice:** | Math | Stat |

Instructions :

- Write your name and registration number in the blank space at the top of this question-paper, and also tick 'Math' or 'Stat' to indicate your first choice of one of the 2 PhD programs.
- All questions carry 2 marks. If there are 2 parts to a question, then each part carries 1 mark.
- There will be no partial credit. Simplify all your answers. In particular, the answer should not be in the form of a sum or product.
- The answer to each question is a number, function or a set.
- Only the question paper will be graded. Write only the final answers on the question paper at the space provided below the questions.
- You are being given a separate work-sheet for solving the problems. This work sheet will not be graded.
- If your marks in the written test is ≥ 16, then you will surely be called for the interview. Final selection will be based on written test marks and interview marks.

Probability - 6 Questions

1. Suppose we have a random sample of size n from the probability density function $f(y) = 1/2$, $0 < y < 2$. Find the moment generating function of $\sum_{i=1}^{n} Y_i$ at $t = 1/2$.

   ---

2. Let $\{X_n : n \geq 1\}$ be a sequence of independent random variables with mean 2 and variance 1. Take $Y_n = (X_{2n-1} + X_{2n})^2$, $n = 1, 2, \ldots$ and let $Z_n = \sum_{i=1}^{n} Y_i/n$. The constant c to which Z_n converges with probability 1 as $n \to \infty$ is

   ---

3. Let X_1, X_2 and X_3 be independent random variables with common mean μ and common variance σ^2. Define $U = X_1 + X_2 + X_3$ and $V = X_1 + X_2 - 2X_3$. The correlation coefficient between U and V is

   ---

4. Suppose X_1 and X_2 are independent Poisson random variables with mean θ. Given $X_1 + X_2 = 2$, the conditional expectation of X_1 is

   ----

5. A student has to sit for an examination consisting of 3 questions selected randomly from a list of 100 questions. To pass, he needs to answer all three questions correctly. What is the probability that the student will pass the examination if he knows the correct answers to 90 questions on the list?

   ----

6. The pdf of Y, where $Y = 1 - X^2$ if $X \le 0$ and $Y = 1 - X$ if $X > 0$ for $f_X(x) = \frac{3}{8}(x+1)^2$, $-1 < x < 1$ is

   ----

Linear Algebra - 8 Questions

7. Consider the vector space $\mathbb{R}^3$ with coordinates (x_1, x_2, x_3) equipped with the inner product

$$\langle (a_1, a_2, a_3), (b_1, b_2, b_3) \rangle = 2(a_1b_1 + a_2b_2 + a_3b_3) - (a_1b_2 + a_2b_1 + a_2b_3 + a_3b_2).$$

   Find all vectors in $\mathbb{R}^3$ of norm 1 which are orthogonal to the plane $x_1 - 2x_2 + 2x_3 = 0$.

   ----

8. Let $A = \begin{bmatrix} 0 & 4 & 1 & -2 \\ -1 & 4 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ -1 & 3 & 0 & 0 \end{bmatrix}$.

   The minimal polynomial of A is

   ----

9. For each integer $j \ge 1$, let V_j denote the real vector space of all polynomials in two variables of degree strictly less than j.

   (a) The dimension of V_{100} is

   (b) The dimension of the space of all linear maps from V_5 to V_{11} whose kernel contains V_3 and whose image is contained in V_7, is

   ----

10. Let A be the complex square matrix of size 2018 whose diagonal entries are all -2018 and off-diagonal entries are all 1.

    (a) The eigenvalues of A are

    (b) What are the geometric multiplicities of the eigenvalues of A?

    ----

11. Consider the vector space $V = \left\{ a_0 + a_1 x + a_2 x^2 + \cdots + a_{11} x^{11} : a_i \in \mathbb{R} \right\}$.

    Define a linear operator A on V by $A(x^i) = x^{i+4}$ where $i+4$ is taken modulo 12.

    Find (a) the minimal polynomial of A and (b) the characteristic polynomial of A.

    (a) ---------------------- (b) ----------------------

12. Let A be a diagonal matrix whose characteristic polynomial is

$$P(x) = (x-16)^8(x-15)^8(x-14)^7(x-13)^7(x-12)^6(x-11)^6 \cdots (x-2)(x-1).$$

    Let V be the set of all 72×72 matrices commuting with A. Let W be the subset of V consisting of all diagonal matrices.

    (a) The dimension of W is ----------------------------

    (b) The dimension of V is ----------------------------

13. Let $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 10 & 4 & 4 \\ 1 & 4 & 27 & 7 \\ 1 & 4 & 7 & 52 \end{pmatrix}$.

    It is given that $A = GG^T$, where G is a 4×4 lower triangular matrix with positive diagonal entries, and G^T denotes the transpose of G.

    Find (a) the smallest and (b) the largest eigenvalue of G.

    (a) -------------------- (b) ------------------------

14. Consider the matrix $A_\alpha = \begin{pmatrix} 4 & 2 & 2 \\ 2 & \alpha^2+4\alpha+5 & \alpha+3 \\ 2 & \alpha+3 & -4\alpha^2+4\alpha+14 \end{pmatrix}$.

    Find the set of all real numbers α such that A_α is positive definite.

Real Analysis - 6 Questions

15. (a) All real numbers p for which the series $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ converges are

– –

(b) All real values of x for which the series $\sum_{n=2}^{\infty} \frac{x^n}{n(\log n)^2}$ converges absolutely are

– –

16. Let $f_1 : [-1, 1] \to \mathbb{R}, \quad f_1(0) = 0$ be a continuously differentiable function and $\lambda > 1$. Consider the sequence of functions defined inductively by

$$f_k(x) := \lambda f_{k-1}(x/\lambda), \qquad k \geq 2, \quad x \in [-1, 1].$$

The pointwise limit of the sequence of functions (f_n) is

– –

17. Find the exact number of real roots of the following polynomials:

(a) $x^3 + 50x + 35$. (b) $x^3 + 50x - 105$.

(a) – – – – – – – – – – – – – – – – – – (b) – – – – – – – – – – – – – – – – – –

18. Let $f_n : [0, 1] \to \mathbb{R}$ be defined by $f_n(x) = \dfrac{n + x^3 \cos x}{ne^x + x^5 \sin x}$, $n \geq 1$. Find $\lim_{n\to\infty} \int_0^1 f_n(x)\, dx$.

– –

19. Evaluate: $\displaystyle\int_0^1 x^5 \left\{ \log\left(\frac{1}{x}\right) \right\}^3 dx$.

– –

20. You can use the Fourier sine series of 2π-periodic odd function $f(x) = \frac{1}{8}\pi x(\pi - x)$ for $x \in [0, \pi]$.

The sum of the series $1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \ldots$ is

– –

Department of Mathematics, IIT Bombay
Screening Test for PhD Admissions (December 4, 2018)
Time allowed: 2 hours and 30 minutes

Departmental Reg No. : **Maximum Marks : 40**

Name : **Choice :** | Math | Stat |

Instructions :

- Write your name and registration number in the blank space, and also tick 'Math' or 'Stat' to indicate your first choice of one of the 2 PhD programs.
- All questions carry 2 marks. If there are 2 parts to a question, then each part carries 1 mark.
- There will be no partial credit. Simplify all your answers and if your answer is a fraction, then do not convert it into a decimal number.
- The answer to each question is a number, function, matrix or a set.
- Only the question paper will be graded. Write only the final answers on the question paper at the space provided below the questions.
- You are being given a separate work-sheet for solving the problems. This work sheet will not be graded.
- If your marks in the written test is ≥ 16, then you will surely be called for the interview. Final selection will be based on written test marks and interview marks.
- For a function f, f' denotes its derivative and for a matrix A, A^T denotes its transpose.

– –

Probability - 6 Questions

1. Suppose the distribution of a random variable Y, conditional on $X = x$ is Normal (x, x^2) and the marginal distribution of X is Uniform $(0, 1)$. Then Var(Y) is

   Ans. –

2. Let X be a random variable with probability density function $f_X(x) = \begin{cases} e^{-x}, & \text{if } x > 0 \\ 0, & \text{otherwise} \end{cases}$, and let $A = \{X \geq 1\}$. Then $E\big(X \big| A\big)$ is

   Ans. –

3. Let $X_1, X_2, \cdots$ be independent and identically distributed random variables such that X_1 is a Poisson (1) random variable. Let $Y_n = X_1^2 + X_2^2 + \cdots + X_n^2$, $n \geq 1$. Then $\lim_{n\to\infty} P\{Y_n > 2n\}$ is

Ans. --

4. Let $X_1, X_2, \cdots$ be independent and identically distributed uniform (0, 1) random variables and N denote the smallest integer $n \geq 1$ such that $X_1 + X_2 + \cdots + X_n > \frac{1}{2}$. Then $P(N > 3)$ is

Ans. --

5. Consider two coins, an unbiased coin (i.e. probability of head $\frac{1}{2}$) and a biased coin with probability of head $\frac{1}{3}$. A coin is selected at random and the toss resulted in a head. The probability that the selected coin is unbiased is

Ans. --

6. Let $X_1, X_2 \cdots, X_{100}$ be positive identically distributed random variables and let $S_n = X_1 + X_2 + \cdots + X_n$ for $n = 1, 2, \cdots, 100$. Then $E\Big[\frac{1}{S_{100}} \sum_{k=1}^{100} 2S_k\Big]$ is

Ans. --

Linear Algebra - 8 Questions

7. (a) Let V be a real vector space and $v_1, \ldots, v_{16} \in V$. Assume that $\sum_{i=1}^{8} a_{2i-1}v_{2i-1} = 0$ has infinitely many solutions and $\sum_{i=1}^{8} a_{2i}v_{2i} = 0$ has a unique solution. Find the maximum possible dimension of $W := \text{Span}\{v_1, \ldots, v_{16}\}$.

   (b) Let S be the real vector space consisting of all 10×10 real symmetric matrices $A = (a_{ij})$ such that $\sum_{j=1}^{10} a_{ij} = 0$ for all $i = 1, \ldots, 10$. Find the dimension of S.

Ans (a) ------------------ Ans (b) ------------------

8. Consider the vector space $\mathbb{R}^4$ with coordinates $(x_1, \ldots, x_4)$ and a symmetric form defined by

$$\langle (a_1, \ldots, a_4), (b_1, \ldots, b_4) \rangle = a_1b_1 + a_2b_2 + a_3b_3 - a_4b_4.$$

Find all vectors in $\mathbb{R}^4$ of norm 1 which are orthogonal, with respect to the above form, to the solution space of $x_1 - 2x_2 + 2x_3 + x_4 = 0$.

Ans --

9. Let A be a 2×2 matrix and I be the identity matrix. Assume that the null spaces of $A - 4I$ and $A - I$ respectively are spanned by $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ respectively. Find a matrix B such that $B^2 = A$.

Ans. –

10. Find (a) the minimal polynomial and (b) the characteristic polynomial of $A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$.

Ans (a) – – – – – – – – – – – – – – – – Ans (b) –

11. (a) Let $V = \{a_0 + a_1x + \ldots + a_7x^7 : a_i \in \mathbb{Q}\}$ be the vector space over $\mathbb{Q}$ consisting of polynomials in x of degree ≤ 7. Let $T : V \to V$ be the linear map defined by $T(p) = x^2p'' - 6xp' + 12p$. Find the dimension of image of T.

    (b) Let $T : \mathbb{R}^{21} \to \mathbb{R}^{21}$ be a linear transformation such that $T \circ T = 0$. Find the maximal possible rank of T.

Ans (a) – – – – – – – – – – – – – – – – – Ans (b) –

12. Consider the three matrices $P = \begin{pmatrix} 1 & 19 \\ 17 & 325 \end{pmatrix}$, $Q = \begin{pmatrix} 4 & 19 \\ 17 & 80 \end{pmatrix}$, $R = \begin{pmatrix} 18 & 19 \\ 17 & 18 \end{pmatrix}$.

    (a) Which among P, Q and R belong to the set $\{A \mid x^TAx > 0,\ \forall x \neq 0 \in \mathbb{R}^2\}$?

    (b) List all negative eigenvalues of P, Q and R.

Ans (a) – – – – – – – – – – – – – – – – – Ans (b) –

13. Let V be the real vector space of 2×2 matrices with a symmetric form defined by $\langle A, B\rangle = \text{trace}(A^TB)$. Find the orthogonal projection, with respect to the above form, of $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ to the subspace W consisting of all symmetric matrices.

Ans –

14. Let S be a largest possible set of 4×4 complex matrices such that each element of S has the set of eigenvalues as $\{1, 2\}$ and no two elements in S are similar. Find the number of elements in S.

Ans –

Real Analysis - 6 Questions

15. (a) Let $f : \mathbb{R} \to \mathbb{Q} \subset \mathbb{R}$ be a continuous function. Find $f'(1)$.

(b) Let $f(x) = \begin{cases} \arctan(ax+b), & x < 0 \\ \frac{\pi}{4} e^{\sin bx}, & x \geq 0 \end{cases}$.

Find all values of a and b such that f is differentiable.

Ans (a) – – – – – – – – – – – – – – – – – – Ans (b) – – – – – – – – – – – – – – – – – – –

16. (a) Let f be a continuous function on $[0, 1]$ such that f is differentiable on $(0, 1)$, $f(0) = 0$ and $f(1) = 1$. Find all integers $n \geq 1$ such that there exist some $x_0 \in (0, 1)$ with $f'(x_0) = nx_0^{n-1}$.

(b) Find all $r \in \mathbb{R}$ such that if f is any continuous function on $[1, 3]$ with $\int_1^3 f(x)\,dx = 1$, then there exist some $x_0 \in (1, 3)$ with $f(x_0) = r$.

Ans (a) – – – – – – – – – – – – – – – – – – Ans (b) – – – – – – – – – – – – – – – – – –

17. Let g be a bounded continuous function defined on $[0, 2\pi]$ such that $\int_0^{2\pi} g(x)\,dx = 1$. Suppose that $\int_0^{2\pi} p(x)g(x)\,dx = 0$ for all polynomials p with $p(\pi) = 0$. Evaluate $\int_0^{2\pi} e^{-2x} g(x)\,dx$.

Ans –

18. (a) Evaluate $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{k^2}{9n^2}$.

(b) For positive integers n and m, evaluate $\int_0^1 x^m (\log x)^n\,dx$.

Ans (a) – – – – – – – – – – – – – – – – – – Ans (b) – – – – – – – – – – – – – – – – – –

19. Find the sum of the series $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots$. You may use the Fourier cosine series of x^2 on $[0, \pi]$.

Ans –

20. Find the set of all $x \in \mathbb{R}$ such that the series $\frac{1}{(1-x)} + \frac{1}{2(1-x)^2} + \frac{1}{3(1-x)^3} + \ldots$ converges.

Ans –

Answers.

1. $\frac{5}{12}$
2. 2
3. $\frac{1}{2}$
4. $\frac{1}{48}$
5. $\frac{3}{5}$
6. 101
7. (a) 15, (b) 45
8. $\pm\frac{1}{2\sqrt{2}}(1,-2,2,-1)$
9. $\begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix}$.
10. (a) $(x+1)(x-3)$, (b) $(x+1)^3(x-3)$
11. (a) 6, (b) 10
12. (a) P and R (b) $42-\sqrt{1767}$
13. $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$
14. 10
15. (a) $f'(1)=0$, (b) $a=\pi/2$, $b=1$
16. (a) $\mathbb{N}$, (b) $1/2$
17. $e^{-2\pi}$
18. (a) $1/27$ (b) $\frac{(-1)^n n!}{(m+1)^{n+1}}$
19. $\frac{\pi^2}{12}$
20. $(-\infty,0)\cup[2,\infty)$

Registration number :

Name :

From the given list, choose the two subjects in which you are best prepared. Rank them by writing (1) and (2) against them.

In the interview, questions will be asked from these two subjects.

- Analysis (Complex Analysis, Functional Analysis, Measure Theory)
- Algebra (Group Theory, Ring Theory, Field Theory)
- Topology (Point set tolopogy, Algebraic topology)
- ODE and PDE
- Combinatorics
- Probability
- Statistics

Department of Mathematics, IIT Bombay

Screening Test for PhD Admissions (May 10, 2019)

Time allowed: 2 hours and 30 minutes **Maximum Marks: 40**

Name: Choice: | Math | Stat |

- *Write your name in the blank space at the top of this question-paper, and also tick 'Math' or 'Stat' to indicate your first choice of one of the 2 PhD programs. Do the same on the attached answer-sheet.*
- *All questions carry 2 marks. Some of the questions have two parts of 1 mark each. There will be no partial credit.*
- *Simplify all your answers. In particular, the answer should not be in the form of a sum or product.*
- *The answer to each question is a number (or a tuple of numbers), a set, or a function. Record only your answers on the answer-sheet as indicated. You are being given a separate work-sheet for doing rough work (this will not be collected).*
- *Use of calculators is not allowed. Please keep aside your notes and mobile phones, whose use during the examination is prohibited. Any candidate found to be adopting any unfair means will be disqualified.*
- *Only the question paper will be graded. Write only the final answers on the question paper at the space provided below the questions. You are being given a separate work-sheet for solving the problems. This work sheet will not be graded.*
- *If your score at least 16 marks in the written test, then you will surely be called for the interview. Final selection will be based on written test marks and interview marks.*

Probability - 6 Questions

1. Let X and Y be independent and identically distributed random variables with common probability density function

$$f(x) = \begin{cases} e^{-x} & \text{if } x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Let $U = \min(X, Y)$. Then $E(U)$ is ____________________.

2. Let U_1, U_2, and U_3 be independent and identically distributed copies of the random variable Uniform(0,θ) for $\theta > 0$. Then $P[\max(U_1, U_2, U_3) > \theta/2]$ is equal to ____________________.

3. Let U be a Uniform(0,1) random variable. Given $U = u$, the random variable X is Poisson(u). Then, $E(X) =$____________________.

4. Assume that X_1, X_2, X_3 are discrete random variables defined on a common probability space Ω and taking values in $\{-1, 1\}$. Further, assume that $E(X_1) = E(X_2) = E(X_3) = E(X_1X_2) = E(X_2X_3) = E(X_3X_1) = 0$. Given this, what is the maximum possible value of $E(X_1X_2X_3)$?

____

5. For each $n \geq 1$, assume we have discrete random variables $X_1, \ldots, X_n$ that are independent and identically distributed; further, each X_i is uniform over the set $\{-1, 1\}$. Let $p_n = P[|\sum_{i=1}^{n} X_i| \geq (n/3)]$. Then, $\lim_{n\to\infty} p_n =$____________________.

6. Let X be a random variables with probability density function

$$f(x) = \begin{cases} e^{-x} & \text{if } x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

and let $Y = aX + b$ where $a, b > 0$. Let $M_Y(t)$ denote the moment generating function of Y. The domain of convergence of M_Y is ____________________.

Linear Algebra - 8 Questions

7. If $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 8 \\ 3 & 6 & 7 \end{pmatrix}$, the set of all $x \in \mathbb{R}^3$ satisfying $Ax = \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix}$ is given by the equation(s):

   ____

   A vector $b \in \mathbb{R}^3$ such that $Ax = b$ is inconsistent (i.e. has no solution) is:

   ____.

8. Let $\mathbb{F} = \mathbb{Z}/3\mathbb{Z}$, and V be a 4-dimensional vector space over $\mathbb{F}$.

   The number of elements in V is ________________________, and the number of 4×4 invertible matrices with entries in $\mathbb{F}$ is ________________________.

9. Let Id_5 is the 5×5 identity matrix, and A be a 5×5 real matrix with eigenvalues 0, 3, and 4, and minimal polynomial $x^2(x-3)(x-4)$.

   The eigenvalues and minimal polynomial of $B = A - 3\mathrm{Id}_5$ are

   ________________________ and

   ____ respectively.

10. A 3×3 orthogonal matrix A whose first two columns are $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ and $\begin{pmatrix} -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$ is:

$$\begin{pmatrix} \quad & \quad & \quad \\ \\ \\ \end{pmatrix}.$$

11. Consider the matrix

$$A = \begin{pmatrix} 2 & 0 & 0 \\ k & 1 & 0 \\ 5 & k-2 & 1 \end{pmatrix}.$$

If A is diagonalizable, then $k =$__________________________.

12. Consider the matrix

$$A = \begin{pmatrix} 1 & -1 \\ 1 & a \end{pmatrix}.$$

For which values of a is the matrix A diagonalizable *over* $\mathbb{R}$?

____

13. Find a 2×2 matrix A whose eigenvalues are 1 and 4, and whose eigenvectors are

$$\begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

respectively.

____

14. Let $\mathbb{F}$ be an algebraically closed field of characteristic 5. Take the 5×5 matrix over $\mathbb{F}$ all whose entries are 1. Its Jordan Canonical form is

____

Real Analysis - 6 Questions

15. Given that for certain constants A and B,

$$A\Gamma(x) = B^{x-1}\Gamma\left(\frac{x}{2}\right)\Gamma\left(\frac{x}{2}+\frac{1}{2}\right), \quad \text{for all } x > 0,$$

where the function $\Gamma(x)$ denotes the Gamma function.

Then $A =$ __________ and $B =$ __________.

16. Given $f(x) = \tan^{-1}\exp(-x - x^{-1})$ on $x > 0$, the image of the function f is

______________________________.

17. Find all real numbers c such that the equation $x^5 - 5x = c$ has three distinct real roots.

____

18. Let $\{a_n\}$ be defined as follows:

$$a_1 > 0, \quad a_{n+1} = \ln\frac{e^{a_n} - 1}{a_n} \quad \text{for} \quad n \geq 1.$$

Then the sum

$$\sum_{n=1}^{\infty} a_1 a_2 \cdots a_n$$

is ______________.

(Hint: $\{a_n\}$ is a strictly decreasing sequence of positive terms converging to 0).

19. The domain of convergence of the series

$$\sum_{n=1}^{\infty} \frac{n4^n}{3^n} x^n (1-x)^n$$

is ________________________.

20. If f is the function defined by

$$f(t) = \begin{cases} \frac{1}{t \ln 2} - \frac{1}{2^t - 1}, & t \neq 0 \\ \frac{1}{2}, & t = 0, \end{cases}$$

then the derivative $f'(0)$ of f at 0 is ________________.