Real Analysis

(Handwritten Classroom Study Material)

(Sequence & Series)

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Real Analysis

Sequence :- A mapping $a_n : N \longrightarrow R$ is called Sequence, denoted by $\langle a_n \rangle$ or $\{a_n\}$ or (a_n)

limit of a Sequence :- Let $\langle a_n \rangle$ be a real Sequence, then a real number 'l' is said to be limit of $\langle a_n \rangle$ for a given $\varepsilon_0 > 0$, $\exists$ a +ve integer $N(\varepsilon_0)$ such that

$$|a_n - l| < \varepsilon_0 \quad \forall \quad n \geq N(\varepsilon_0)$$

$$\text{or } \forall \quad n \geq N(\varepsilon) \Rightarrow |a_n - l| < \varepsilon$$

$$\lim_{n \to \infty} a_n = l$$

Q Find limit of

(i) $\langle \frac{n}{n+1} : n \in N \rangle \longrightarrow 1$

(ii) $\left(1 + \frac{a}{n}\right)^n = e^{n \log\left(1 + \frac{a}{n}\right)}$

$$= e^{n\left(\frac{a}{n} - \left(\frac{a}{n}\right)^2 \cdot \frac{1}{2} + \left(\frac{a}{n}\right)^3 \cdot \frac{1}{3} + \cdots\right)}$$

$$= e^{a\left(1 - \frac{a}{n \cdot 2} + \cdots\right)} = e^a$$

(iii) $\langle \frac{1}{2^n} : n \in N \rangle \longrightarrow 0$

Convergent Sequence (cgt) :- Let $\langle a_n \rangle$ be a Sequence of real number then a_n is said to be Convergent at a point 'a' if

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for a given $\varepsilon_0 > 0$ $\exists$ a +ve integer $N(\varepsilon_0)$ such that

$$|a_n - a| < \varepsilon_0 \quad \forall \quad n \geq N(\varepsilon_0)$$

$$\Rightarrow \lim_{n \to \infty} a_n = a.$$

Oscillatory Sequence :- let $\langle a_n \rangle$ be a seqⁿ. of real number is said to be oscillatory If it is neither cgt nor divergent.

Eg- Convergent

① $\left\langle \frac{n}{n+1} : n \in N \right\rangle \longrightarrow 1$ as $n \to \infty$

② $\left\langle \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n} \right\rangle$

③ ~~$\left\langle \frac{1}{n} : n \in N \right\rangle \longrightarrow 0$ as $n \to \infty$~~

④ $\left\langle \frac{n^n}{\lfloor n} ; n \in N \right\rangle$

⑤ $\langle n^{1/n} : n \in N \rangle$

Eg :- Divergent

① $\langle n! \ n \in N \rangle$

② $\langle 2^n : n \in N \rangle$

③ $\langle (\lfloor n)^{1/n} : n \in N \rangle$

④ $\left\langle 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} : n \in N \right\rangle$

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Eg:- Oscillatory

① $\langle (-1)^n : \rangle_{n \in N}$

② $\langle 1, 2, 1, 2, \ldots \rangle$

③ $\langle \sin n : n \in N \rangle$

Sandwitch Theorem:- If $a_n < b_n < c_n$
then $\lim_{n \to \infty} a_n \le \lim_{n \to \infty} b_n \le \lim_{n \to \infty} c_n$

If $\lim_{n \to \infty} a_n = l = \lim_{n \to \infty} c_n \Rightarrow \lim_{n \to \infty} b_n = l$

Q If $\lim_{n \to \infty} \left(\frac{1}{n} + \frac{1}{n+1} + \ldots \frac{1}{2n} \right) = l$

Sol:- Clearly $\frac{1}{2n} + \frac{1}{2n} + \ldots$ upto n times

$\le a_n \le \left(\frac{1}{n} + \frac{1}{n} + \ldots \text{ upto } n \text{ times} \right)$

$\frac{\cancel{n}}{2\cancel{n}} \le a_n \le \frac{\cancel{n}\,1}{\cancel{n}}$

$\frac{1}{2} \le a_n \le 1$

$\Rightarrow \frac{1}{2} \le \lim_{n \to \infty} a_n \le 1$

$\Rightarrow \frac{1}{2} \le l \le 1$

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Theorem :- Prove that a convergent sequence has unique limit.

Proof :- Let l and l' are two distinct limits. $|l - l'| < \delta$ where δ is fixed finite

$\because a_n \to l$ $\therefore$ ~~there exists~~ given $\varepsilon > 0$ there exists $m_1 \in N$

$\therefore |l - a_n| < \varepsilon \quad \forall \; n \geq m_1$

Also $a_n \to l'$

$\therefore |l' - a_n| < \varepsilon \quad \forall \; n \geq m_2$

Let $m = \max\{m_1, m_2\}$

$\therefore |l - a_n| < \varepsilon \quad \forall \; n \geq m$

$|l' - a_n| < \varepsilon$

$\therefore |l - l'| = |l - a_n + a_n - l'|$

$\leq |l - a_n| + |l' - a_n|$

$< \varepsilon + \varepsilon = 2\varepsilon \quad \forall \; n \geq m$

$\therefore |l - l'| < 2\varepsilon$

which is contrary that $l - l'$ is fixed finite.

Cauchy Sequence :- A Sequence $\langle a_n \rangle$ is said to be cauchy sequence if for a given $\varepsilon > 0$ $\exists$ a +ve integer $N(\varepsilon)$ such that

$|a_n - a_m| < \varepsilon \quad \forall \; n, m > N(\varepsilon)$

eg :- (1) $\langle \frac{1}{n} : n \in N \rangle$

$= \langle 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\ldots, \frac{1}{n}, \ldots\ldots \rangle$

Theorem :- Every Convergent Sequence is cauchy Sequence.

Q) Give an example of rational sequence which converges to irrational point.

Sol :- $\left(1+\frac{1}{n}\right)^n = e$

$= \left(\frac{n+1}{n}\right)^n = \cancel{\frac{(n+1)^n}{n^n} = \frac{p}{q} \neq p}$

we know $\left(1+\frac{1}{n}\right)^n \rightarrow e$ as $n \rightarrow \infty$
irrational.

Bounded Sequence :- Let $\langle a_n \rangle$ be Sequence, is said to be bounded if we consider the set S of terms of Sequence $\langle a_n \rangle$ which will be bounded i.e. there exist greatest lower bound (glb) of S say l and least upper bound (lub) of S say u such that

$l \le n \le u \quad \forall \; n \in S$

we write $|a_n| \le K \;\; \exists \;\; K \in N$

Theorem :- Prove that every Convergent Sequence is bounded but converse not necessarily true.

Proof :- let $\langle a_n \rangle$ be a Convergent Seq^n. Converges to a then for a given $\varepsilon > 0$ $\exists$ a +ve Integer $N(\varepsilon)$ such that

$|a_n - a| < \varepsilon \quad \forall \; n \geq N(\varepsilon)$

Clearly 'a' is finite number then $|a_n - a| < \varepsilon$

$\Rightarrow |a_n| - |a| < \varepsilon \Rightarrow |a_n| < |a| + \varepsilon$

$\Rightarrow |a_n| < k \quad , \quad k = |a| + \varepsilon \quad \forall \; n \geq N(\varepsilon)$

This shows that $\langle a_n \rangle$ is bounded but converse not necessarily true. we can see by an example let the seq^n.

$\langle a_n \rangle = \langle (-1)^n ; n \in N \rangle$
$= \langle -1, 1, -1 \ldots\ldots \rangle$

Clearly $|a_n| < 2$

But $\langle (-1)^n : n \in N \rangle$ is oscillatory seq^n. which is neither convergent nor divergent.

Theorem :- Prove that every Cauchy Sequence in real no. is convergent Sequence.

Proof :- let $\{a_n\}$ be a cauchy Sequence is real no. then for a given $\varepsilon>0$ $\exists$ +ve integer $N(\varepsilon_0)$ such that $|a_n - a_m| < \varepsilon_0$ $\forall$ $n, m \geq N(\varepsilon_0)$

then by order completeness of real no. there must exists a no. 'a' s.t.

$a-\varepsilon_0 < a_n < a+\varepsilon_0$

$\Rightarrow |a_n - a| < \varepsilon \quad \forall \ n \geq N(\varepsilon)$

Thus $\langle a_n \rangle$ converges to a.

Q If $\lim_{n\to\infty} \left(\frac{1}{n^2+1} + \frac{1}{n^2+2} + \cdots + \frac{1}{n^2+n} \right) = l$

(i) 0 (ii) 1 (iii) $\frac{1}{2} \leq l \leq 1$ (iv) $\frac{1}{2} \leq l \leq 1$

We have

$$\frac{1}{n^2+n} + \frac{1}{n^2+n} + \cdots \leq a_n \leq \frac{1}{n^2+1} + \frac{1}{n^2+1} + \cdots$$

$$\Rightarrow \frac{n}{n^2+n} \leq a_n \leq \frac{n}{n^2+1}$$

$$\lim_{n\to\infty} \frac{1}{n+1} \leq \lim_{n\to\infty} a_n \leq \frac{1}{n+\frac{1}{n}}$$

$$0 \leq a_n \leq 0$$

$$\Rightarrow a_n \to 0 \text{ as } n \to \infty$$

Monotonic Sequences :-

(i) Monotonic Increasing :- if $a_n \leq a_{n+1} \cdots \quad \forall \ n \in N$

eg $\langle \frac{n}{n+1} : n \in N \rangle$, $\langle \sqrt{2}, \sqrt{2+\sqrt{2}}, \cdots \rangle$

(ii) Monotonic decreasing :- $a_n \geq a_{n+1} \ \forall n \in N$

eg :- $\{\frac{1}{n} : n \in N\}$

$\langle n^2 : n \in N \rangle$ is m.i. but not cgt.

Cauchy 1st theorem on Seq^n.

Theorem :- If $\lim_{n\to\infty} a_n = l$, then show that

$$\lim_{n\to\infty} \frac{a_1+a_2+\cdots+a_n}{n} = l$$

Proof :- Given that

$$\lim_{n\to\infty} a_n = l \quad \text{—— (1)}$$

Let $b_n = a_n - l$ —— (2)

$\therefore a_n = l + b_n$

or $\frac{a_1+a_2+\cdots+a_n}{n} = \frac{l+b_1+l+b_2+\cdots+l+b_n}{n}$

or $\frac{a_1+a_2+\cdots+a_n}{n} = \frac{nl+b_1+b_2+\cdots b_n}{n}$

or $\frac{a_1+a_2+\cdots+a_n}{n} = l + \frac{b_1+b_2+\cdots b_n}{n}$ —— (3)

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Then we have to show that

$$\lim_{n\to\infty} \frac{b_1+b_2+\cdots+b_n}{n} = 0$$

Since $b_n = a_n - l$

$\Rightarrow \lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n - l = l - l$

$\Rightarrow \lim_{n\to\infty} b_n = 0$

Now $\left| \frac{b_1+b_2+\cdots+b_n}{n} - 0 \right| = \left| \frac{b_1+b_2 \cdots +b_n}{n} \right|$

$\leq \left| \frac{b_1+b_2+\cdots+b_m}{n} \right| + \left| \frac{b_{m+1}+\cdots b_n}{n} \right|$ ——— (4)

Let $K = \max\{b_1, b_2 \ldots b_m\}$

Now

$\left| \frac{b_1+b_2+\cdots+b_m}{n} \right| \leq \frac{K + K \cdots\cdots m \text{ times}}{n}$

$= \frac{mK}{n} < \frac{\varepsilon}{2}$

Now $\frac{mK}{n} < \frac{\varepsilon}{2}$

$\Rightarrow \frac{mK}{\varepsilon/2} < n$ or $n > \frac{2mK}{\varepsilon}$

Thus $\left| \frac{b_1+b_2+\cdots+b_m}{n} \right| < \frac{\varepsilon}{2} \quad \forall \; n > \frac{2mK}{\varepsilon}$

Similarly $\left| \frac{b_{m+1}+\cdots b_n}{n} \right| \leq \frac{(n-m)K_1}{n} \leq \frac{\varepsilon}{2}$

where $K_1 = \max\{b_{m+1}, b_{m+2}, \ldots b_n\}$

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let $N(\varepsilon) = \frac{2mk}{\varepsilon} + \frac{2(n-m)k}{\varepsilon}$,

Then we get

$$\left| \frac{b_1+b_2+\cdots b_n}{n} \right| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} \quad \forall\, n \geq N(\varepsilon)$$

from (4)

$$\Rightarrow \left| \frac{b_1+b_2+\cdots b_n}{n} \right| < \varepsilon \quad \forall\, n \geq N(\varepsilon)$$

$$\Rightarrow \lim_{n\to\infty} \left| \frac{b_1+b_2+\cdots b_n}{n} \right| = 0$$

Then from eqn. (3) we get

$$\lim_{n\to\infty} \frac{a_1+a_2+\cdots+a_n}{n} = l + \lim_{n\to\infty} \frac{b_1+b_2+\cdots b_n}{n}$$

$$= l + 0$$

$$= l \text{ proved}$$

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proof haid krishak book page No. 79

Cauchy 2nd theorem :- If $\lim_{n\to\infty} a_n = l$ then $\lim_{n\to\infty} (a_1 \cdot a_2 \cdots a_n)^{1/n} = l$

* If $\langle a_n \rangle$ is seqn. of +ve terms & $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = l < 1$, then $a_n \to 0$ as $n \to \infty$

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Q Evaluate

① $\lim_{n\to\infty} \left(\frac{1+2^{1/2}+3^{1/3}+\cdots+n^{1/n}}{n} \right) = ?$

(i) 0 (ii) 1 (iii) e (iv) $\frac{1}{e}$

by Cauchy 1st theorem $\lim_{n\to\infty} n^{1/n} = 1$

$\therefore \frac{a_1+a_2+\cdots a_n}{n} = 1$

(ii) $\lim_{n\to\infty} \frac{1}{n} \left\{ \frac{1}{2} + \left(\frac{2}{3}\right)^2 + \cdots + \left(\frac{n}{n+1}\right)^n \right\} = ?$

(i) 0 (ii) 1 (iii) e (iv) $\frac{1}{e}$

$\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty} \left(\frac{1}{1+\frac{1}{n}}\right)^n = \frac{1}{e}$

(iii) $\lim_{n\to\infty} \left(\lfloor n \right)^{1/n} = ?$ by cauchy 2nd theorem

(i) 0 (ii) 1 (iii) 1/2 (iv) ∞

Theorem :- Every m.I. bounded above Sequence Converges to its least upper bound.

Proof :- let $\langle a_n \rangle$ be a monotonic increasing seq^n. which is bounded above i.e. the upper bound of the seq^n. $\langle a_n \rangle$ exists. Then we have to show that a_n converge to its least upper bond. Suppose 'l' be the least upper bound of $\langle a_n \rangle$

$\therefore a_n \le l \quad \forall n$

then for a given $\varepsilon > 0$ there exists +ve integer m such that $l - \varepsilon < a_n \ \forall n \ge m$ —— ①

But l is lub of $\langle a_n \rangle$

$\therefore a_n < l + \varepsilon$ —— ②

from eqn. ① & ② we see that

$l - \varepsilon < a_n < l + \varepsilon \quad \forall n \ge m$

$\Rightarrow |a_n - l| < \varepsilon \quad \forall n \ge m$

This shows that $\langle a_n \rangle$ converge to l.

Theorem :- Every m.d. bounded below sequence converges to its greatest lower bound.

Theorem :- If $\langle a_n \rangle$ be a sequence of +ve terms such that $a_{n+1} = \sqrt{k + a_n}$ where $k > 0$. Then show that $\langle a_n \rangle$ cges to the +ve root of the equation $x^2 - x - k = 0$

proof :- If $\langle a_n \rangle$ be a seqn. of +ve terms then

$a_{n+1} = \sqrt{k + a_n}$ where $k > 0$ —— ①

Clearly $a_2 = \sqrt{k + a_1}$

$a_3 = \sqrt{k + a_2} = \sqrt{k + \sqrt{k + a_1}}$

This shows that $\sqrt{k + a_1} < \sqrt{k + \sqrt{k + a_1}}$

Applying Similarly

$\sqrt{k+\sqrt{k+a_1}} < \sqrt{k+\sqrt{k+\sqrt{k+a_1}}}$

$a_1 \leq a_2 \leq a_3 \cdots \leq a_{n-1} \leq a_n$

i.e. $\sqrt{k+\sqrt{k+\sqrt{k+\sqrt{k+\cdots}}}} < \sqrt{k+k+k+\cdots}$

Shows that this sequence is m.i. i.e. bounded ~~below~~ above seq^n.

Let x be the l.u.b of seq^n. $\langle a_n \rangle$

but we know that m.i. bounded ~~below~~ above seq^n. Converges to its least upper bound.

$\therefore$ $a_n \longrightarrow x$ as $n \longrightarrow \infty$

Then from (1) $x = \sqrt{x+k}$

Squaring $x^2 = x+k$

or $x^2 - x - k = 0$

proved.

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Infinite Series

let $\langle a_n ; n \in N \rangle$ be a sequence then $\sum_{n=1}^{\infty} a_n$ is called series

eg:- (i) $\sum_{n=1}^{\infty} \frac{1}{n}$ (ii) $\sum_{n=1}^{\infty} \frac{1}{n \log n}$ (iii) $\sum_{n=1}^{\infty} (\sqrt{n^4+1} - \sqrt{n^4-1})$

(iv) $\sum_{n=1}^{\infty} \frac{n}{n+1}$

Partial Sum of Series.

let $\sum_{n=1}^{\infty} a_n$ be a series then sum of 1st n terms is called partial sum of this series denoted by S_n we write

$S_n = a_1 + a_2 + \ldots + a_n$.

Nature of Convergency of Series $\sum_{n=1}^{\infty} a_n$

let $\sum_{n=1}^{\infty} a_n$ be any series & S_n denotes its partial sum then if $\langle S_n \rangle$ cges then whole series $\sum_{n=1}^{\infty} a_n$ is also cgt. and if $\langle S_n \rangle$ dges then $\sum_{n=1}^{\infty} a_n$ is also dgt.

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Sequence of function and Uniform cgf

8 If two students find the limit of a sequence $\langle a_n \rangle$ and both have different limit, then

(i) both are true on self position
(ii) one of them must be true
(iii) one of them must be false
(iv) both are wrong always

Pointwise limit / Convergence

$f_n : [0,1] \longrightarrow R \; : \; f_n(x) = x^n, \; x \in [0,1]$

Now limit

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ 1 & \text{if } x \text{ rational} \end{cases}$$

Let $\langle f_n \rangle$ be any sequence of function from $f_n : A \longrightarrow R$ and $f : A \longrightarrow R$

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$f_n \to f$ if $f_n(x) \to f(x)$ $\forall\, x \in A$ $(A \subseteq R)$
or $\lim\limits_{n\to\infty} f_n(x) = f(x)$ $\forall\, x \in A$

Eg $f_n : (0,1) \to R$ defined by $f_n(x) = \frac{n}{nx+1}$

No co

$\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} \frac{n}{nx+1}$

$= \lim\limits_{n\to\infty} \frac{1}{\left(x+\frac{1}{n}\right)}$

$= \frac{1}{x+\frac{1}{\infty}} = \frac{1}{x} = f(x)$

Q which of the following function on $(0,1)$ is/are uniformly continuous.

(i) $f(x) = x \sin\frac{1}{x}$ (ii) $f(x) = \frac{\sin x}{x}$

(iii) $f(x) = \frac{\cos x}{x}$ (iv) $f(x) = \frac{1}{1-x}$

(i) $\lim\limits_{x\to 0} x \sin\frac{1}{x} = 0 \sin\frac{1}{0} = 0k = 0$

Q Find the point wise limit of sequence of fuction $f_n\ R \to R$ such that

(i) $f_n(x) = \frac{1}{2^{n-1}} \cot\frac{x}{2^{n-1}}$

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(ii) $f_n(x) = \left(1+\frac{x}{n}\right)^n$

Sol^n (ii) Here $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$

$= \lim_{n\to\infty} e^{n\log\left(1+\frac{x}{n}\right)}$

$= \lim_{n\to\infty} e^{n\left(\frac{x}{n} - \left(\frac{x}{n}\right)^2\cdot\frac{1}{2} + \left(\frac{x}{n}\right)^3\frac{1}{3} - \dots\right)}$

$= \lim_{n\to\infty} e^{\left(x - \frac{x^2}{2n} + \frac{x^3}{3n^2} - \dots\right)}$

$= e^x$

which is point wise limit.

(i) $f_n(x) = \frac{1}{2^n}\cot\frac{x}{2^n}$

$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\frac{1}{2^n}\cot\frac{x}{2^n}$

Put $\frac{1}{2^n} = \theta$

$\Rightarrow$ As $n\to\infty$ $\theta\to 0$

$\therefore \lim_{n\to\infty}\frac{1}{2^n}\cot\frac{x}{2^n} = \lim_{\theta\to 0}\theta\cot\theta x$

$= \lim_{\theta\to 0}\theta\frac{\cos\theta x}{\sin\theta x}$

$= \lim_{\theta\to 0}\frac{\cos\theta x}{\left(\frac{\sin\theta x}{\theta x}\right)x}$

$= \frac{1}{x}$ pointwise

Asymtote touches the line at infinity

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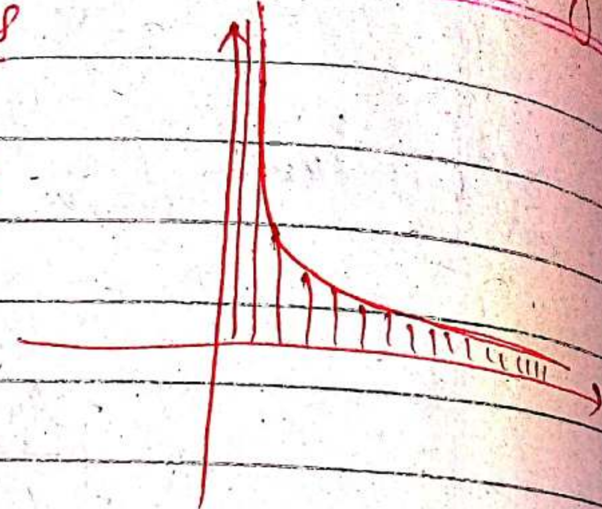
eg of uniform continuous

$f(x) = \frac{1}{x}$; $(0,1)$



$f(x) = \frac{1}{x}$ $(0,1)$

$= \frac{1}{0} = \infty$

It is not uniform convergent.

Joint

$(0,1) \in]-\infty, \infty[$

* If any function is not uniform continuous then this is continuous but not uniform convergent

$(0,1)$ ~~is same as~~ $]-\infty, \infty[$

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Uniform convergence of sequence of function

Definition :-

let $\langle f_n \rangle$ be a sequence of function, $f_n : A \rightarrow R$ Such that $f_n(x) \longrightarrow f(x)$ as $n \longrightarrow \infty$

i.e. $f(x)$ is pointwise limit, then $f_n(x)$ is uniform cgt if for given $\varepsilon > 0$, $\exists$ a +ve integer $N(a)$ (only depend on ε not at point)

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Such that $|f_n(x) - f(x)| < \varepsilon \ \forall \ n \geq N(a)$
$\& \ x \in A.$

Q Show that $f_n(x) = x^n$ is uniform cgt on $[0,k]$ $k < 1$ but not uniform on $[0,1]$

Sol: Given that
$$f_n(x) = x^n$$

Now we have

$$\lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} x^n$$
$$= \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

~~This is~~ the pointwise limit in $[0,k]$, $k<1$ is 0

$$\therefore f(x) = 0$$

Then a given $\varepsilon > 0$ such that

$|f_n(x) - f(x)| < \varepsilon$

$\Rightarrow |x^n - 0| < \varepsilon$

$\Rightarrow |x^n| < \varepsilon$ let $a \in [0,k]$ be fixed

Then $n \log x < \log \varepsilon$

$\Rightarrow n < \dfrac{\log \varepsilon}{\log x}$ (select $x = a$)

or $n < \dfrac{\log \varepsilon}{\log a} = N(\varepsilon)$ say

Then we get

$|f_n(x) - 0| < \varepsilon \quad \forall\ n > N(\varepsilon)\ \&\ x \in [0,k]$

Hence $f_n(x)$ is uniform converges on $[0,k]$

~~But the function $f_n(x) = x^n$ is not~~ uniform cgt on $[0,1]$

because $\lim_{n\to\infty} x^n = \begin{cases} 0 & \text{if } 0 \le x < 1 \\ 1 & \text{if } x = 1 \end{cases}$

This show pointwise limit is not unique

So it do not more to ~~be~~ uniform cgt

Hence $f_n(x)$ is not uniform cgt at $[0,1]$

Q Show that the sequence of function $f_n(x) = \dfrac{nx}{1+n^2x^2}$ is not uniform convergent in any interval $[a,b]$ in which 0 is interior point.

Sol:- $f_n(x) = \dfrac{nx}{1+n^2x^2}$

Now pointwise limit

$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \dfrac{nx}{1+n^2x^2}, \quad x \in [a,b]$

$= \lim_{n\to\infty} \dfrac{\cancel{n}x}{n^{\cancel{2}}\left(x^2+\frac{1}{n^2}\right)}$

$= \frac{x}{\infty} = 0$

Let $\varepsilon = \frac{1}{10}$ and $x = \frac{1}{n}$

Then $\left| f_n(x) - f(x) \right| = \left| \frac{nx}{1+n^2x^2} - 0 \right|$

$= \left| \frac{n \cdot \frac{1}{n}}{1+n^2 \cdot \frac{1}{n^2}} \right| = \frac{1}{2} > \frac{1}{10} = \varepsilon$

$\Rightarrow \left| f_n(x) - f(x) \right| > \varepsilon$ for some n

Hence $f_n(x) = \frac{nx}{1+n^2x^2}$ is not Uniform cgt on $[a, b]$ where 0 is interior point.

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6/11/17 Cauchy's general principle of uniform convergence

Statement :- The necessary and sufficient Condition for sequence function $\langle f_n(x) \rangle$ defined over the interval I to be uniform cgt it is that for every $\varepsilon > 0$ $\exists$ +ve integer $N(\varepsilon)$ such that $|f_{n+p}(x) - f_n(x)| < \varepsilon$

$\forall\ n \geq N(\varepsilon)$

$p \geq 0$ & $\forall\ x \in I$

Proof :- Necessary condition :- Suppose $\langle f_n(x) \rangle$ is uniform Converges on the interval I.

Then for a given $\varepsilon > 0$ $\exists$ a +ve integer $N(\varepsilon)$ such that

$$|f_n(x) - f(x)| < \frac{\varepsilon}{2} \quad \forall \ n \geq N(\varepsilon) \quad \forall \ x \in I$$

let

$$p \geq 0$$

$$\Rightarrow n+p \geq N(\varepsilon) \ \& \ \forall \ x \in I.$$

$$\Rightarrow |f_{n+p}(x) - f(x)| < \frac{\varepsilon}{2} \quad \forall \ n \geq N(\varepsilon) \ \& \ p \geq 0 \ \forall \ x \in I$$

$$|f_{n+p}(x) - f_n(x)| = |f_{n+p}(x) - f(x) + f(x) - f_n(x)|$$

$$= |\{f_{n+p}(x) - f(x)\} - \{f_n(x) - f(x)\}|$$

$$\leq |f_{n+p}(x) - f(x)| + |f_n(x) - f(x)|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \quad \forall \ n \geq N(\varepsilon), \ p \geq 0 \ \forall \ x \in I$$

$$= \varepsilon$$

Sufficient Condition :- Suppose that for a sequence of function $\langle f_n(x) \rangle$ for a given $\varepsilon > 0$ $\exists$ +ve integer $N(\varepsilon)$ such that

$$|f_{n+p}(x) - f_n(x)| < \varepsilon \quad \forall \ n \geq N(\varepsilon), \ p \geq 0 \ \& \ x \in I$$

$$\Rightarrow f_n(x) - \varepsilon < f_n(x) \leq f_{n+p}(x) < f_n(x) + \varepsilon$$

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When $p \rightarrow \infty$

$\Rightarrow f_n(x) - \varepsilon < f_n(x) \leq f(x) < f_n(x) + \varepsilon$

$\Rightarrow |f_n(x) - f(x)| < \varepsilon \quad \forall\, n \geq N(\varepsilon) \;\forall\, x \in I$

Hence $\langle f_n(x) \rangle$ is uniform cgt to $f(x)$

Theorem

M_n test for uniform covergence of sequence of function.

Statement :- let $\langle f_n(x) \rangle$ be a sequence of function defined on a set I with pointwise limit $f(x)$ such that

$M_n = \sup \{ |f_n(x) - f(x)| : \forall\, x \in I \;\&\; \forall\, n \geq N(\varepsilon) \}$

(where $N(\varepsilon)$ is choosen integer)

Then $f_n(x)$ is uniform cgt at $f(x)$ iff

$M_n \rightarrow 0$ as $n \rightarrow \infty$

$$\lim_{n \to \infty} f_n(x) \longrightarrow f(x)$$

$$|f_n(x) - f(x)| \leq M_n$$

Proof :- Suppose $\langle f_n(x) \rangle$ cges unifom at $f(x)$, where $f(x)$ pointwis limit of $f_n(x)$

i.e. $\lim_{n \to \infty} f_n(x) = f(x)$

then for a given $\varepsilon > 0 \;\exists$ +ve integer $N(\varepsilon)$

Such that $|f_n(x) - f(x)| < \varepsilon \quad \forall\ n \geq N(\varepsilon)$ & $\forall\ x \in I$

$\Rightarrow \sup |f_n(x) - f(x)| < \varepsilon$

$\Rightarrow M_n < \varepsilon \quad \forall\ n \geq N(\varepsilon)$

$\Rightarrow M_n \to 0$ as $n \to \infty$

Converse

Suppose $M_n = \sup\{ |f_n(x) - f(x)| : \forall\ n \geq N(\varepsilon)\ \forall\ x \in I \}$

Such that $M_n \to 0$ as $n \to \infty$

$\Rightarrow |M_n - 0| < \varepsilon \quad \forall\ n \geq N(\varepsilon)$

$\Rightarrow \sup |f_n(x) - f(x)| < \varepsilon \quad \forall\ n \geq N(\varepsilon)$

$\Rightarrow |f_n(x) - f(x)| < \varepsilon \quad \forall\ n \geq N(\varepsilon)$

Hence $\langle f_n(x) \rangle$ cgs uniformly at $f(x)$

7/11/17

Q Show that the sequence of function $f_n : R \to R$ obtained by $f_n(x) = \frac{x}{1+nx^2}$ is uniformly cgt on R.

we have $f_n(x) = \frac{x}{1+nx^2}, \quad x \in R$

Now

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$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x}{1+nx^2} = \frac{x}{\infty} = 0 = f(x)$ (say)

Now $\left|f_n(x) - f(x)\right| = \left|\frac{x}{1+nx^2} - 0\right|$

$\frac{\frac{1}{n}}{1+\frac{1}{n}} \quad \frac{0}{1} = 0$

$= \frac{x}{1+nx^2}$

$= \frac{\sqrt{n}\, x}{\sqrt{n}(1+nx^2)}$ let $\sqrt{n}\, x = z$

$= \frac{z}{\sqrt{n}(1+z^2)}$

Clearly $\frac{z}{1+z^2} \le \frac{1}{2}$, $\forall z \in R$ as $(2z < 1+z^2$ $\forall z \in R)$

$\Rightarrow \left|f_n(x) - f(x)\right| \le \frac{1}{2\sqrt{n}} = M_n$ (say)

$\therefore M_n = \frac{1}{2\sqrt{n}} \longrightarrow 0$ as $n \to \infty$

Hence by M_n test the given sequence of function $\langle f_n(x) \rangle$ is uniformly cgt.

Q Examine the uniform cgt of sequence of function $\langle f_n(x) \rangle$ defined by

$f_n(x) = nx(1-x)^n$ on $[0,1]$

we have

$f_n(x) = nx(1-x)^n, \quad x \in [0,1]$

Now $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} nx(1-x)^n$

$\frac{\infty}{\infty}$ form ∴ differentiating

$$= \lim_{n\to\infty} \frac{nx}{(1-x)^{-n}} = \lim_{n\to\infty} \frac{x}{-(1-x)^{-n}\log(1-x)}$$

$\because x \in [0,1]$

$$= \lim_{n\to\infty} \frac{x(1-x)^n}{-\log(1-x)} = 0 = f(x) \text{ (say)}$$

Now $|f_n(x) - f(x)| = |nx(1-x)^n - 0|$

$$= nx(1-x)^n = y \text{ (say)}$$

$\therefore y = nx(1-x)^n$

Diff w.r.t. x

$$y_1 = nx(-n)(1-x)^{n-1} + (1-x)^n n$$

$$= n(1-x)^{n-1}[-nx + 1 - x]$$

$y_1 = 0$

$$\Rightarrow n(1-x)^{n-1}[-nx + 1 - x] = 0$$

$$\Rightarrow -(n+1)x + 1 = 0 \quad \text{as } n \neq 0$$

$$x = \frac{1}{n+1}$$

$$y_2 = n(1-x)^{n-1}(-n-1) + n(-nx+1-x)(n-1)(1-x)^{n-2}$$

$$= n(1-x)^{n-1}\left[-n-1 \quad \cancel{\ldots}\right.$$

$$(y_2)_{x = \frac{1}{n+1}} < 0$$

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$\therefore$ $x = \frac{1}{n+1}$ is max^{m} point

$$y = n \cdot \frac{1}{n+1}\left(1-\frac{1}{n+1}\right)^n$$

$$= \frac{n+1-1}{n+1}\left(1-\frac{1}{n+1}\right)^n$$

$$= \left(1-\frac{1}{n+1}\right)\left(1-\frac{1}{n+1}\right)^n = \left(1-\frac{1}{1+n}\right)^{n+1} = M_n$$

(Say)

$$\therefore M_n = \sup\left\{|f_n(x)-f(x)| : x \in [0,1]\right\}$$

$$= \left(1-\frac{1}{n+1}\right)^{n+1}$$

$$= e^{-1} \text{ as } n \to \infty$$

$$= \frac{1}{e} \neq 0$$

Then by M_n test the sequence of function $\langle f_n(x)\rangle$ is not uniformly cgt on $[0,1]$

(Q1) Show that the sequence of function $\langle f_n(x)\rangle$ defined by $f_n(x) = \frac{n}{nx+1}$ is not uniformly cgt on $(0,1)$ but uniformly on $[a,1]$ as $a > 0$

$\lim_{n\to\infty} f_n(x) = \frac{1}{x} = f(x)$ is not ~~unt~~ uniformly cont because unbounded at 0

(Q2) Examin

$$f_n(x) = \frac{\sin nx}{\sqrt{n}}, \quad \forall\ x \in (0,\pi)$$

Q.1 $f_n(x) = \dfrac{n}{nx+1}$

$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \dfrac{n}{nx+1}$

$= \lim_{n\to\infty} \dfrac{\cancel{n}}{\cancel{n}(x+\frac{1}{n})} = \lim_{n\to\infty} \dfrac{1}{(x+\frac{1}{n})}$

$= \dfrac{1}{x} = f(x)$ say.

which is not uniformly continuous on $(0,1)$. Hence the pointwise limit is not uniform.

Means $f(x) = \frac{1}{x}$ is unbounded on $[0,1]$

which $\Rightarrow$ the seqn of fun $f_n(x) = \dfrac{n}{nn+1}$ not uniformly on $[0,1)$.

(ii) let $a > 0$

$f_n(x) = \dfrac{n}{nx+1}$; $x \in [a,1)$

Then we have to show $f_n(x)$ is uniformly cgt on $[a,1)$

Clearly $\lim_{n\to\infty} f_n(x) = \dfrac{1}{x} = f(x)$

Now $|f_n(x) - f(x)| = \left|\dfrac{n}{nx+1} - \dfrac{1}{x}\right|$

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$$= \left| \frac{\cancel{nx} - \cancel{nx} - 1}{x(nx+1)} \right| = \frac{1}{x(nx+1)} < \frac{1}{nx^2} \quad x \in [a,1]$$

$$\leq \frac{1}{na^2} < \varepsilon$$

$$\Rightarrow |f_n(x) - f(x)| < \varepsilon \quad \forall \ n > \frac{1}{\varepsilon a^2} = N(\varepsilon) \text{ say}$$

$$\Rightarrow |f_n(x) - f(x)| < \varepsilon \quad \forall \ n \geq N(\varepsilon)$$

Hence $\langle f_n(x) \rangle$ cges uniformly on $[a, 1]$