Complex Analysis

[For NET/SET/GATE/PSC/CUCET/MSc...etc.]

Complex

[Quick revision Notes]

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→ Every time Question based on bilinear map & analyticity has been asked on CSIR-NET.

→ Questions based on Complex Integration, Residue, Singularity, meromorphic f^n, Analytic f^n, ... etc are very very important for NET & GATE.

→ Mostly time in GATE, Que. based on Bilinear map Cauchy Principle Value, Conformal map, .. etc comes.

Important Topics for NET & GATE

- Inequality (ML, Cauchy, ..)
- Power Series
- Radius of Convergence
- Complex Integration
- Singularities
- Analyticity / Analytic
- Max./Min. modulus Principle
- Schwarz Lemma
- Residue thm
- Mobius transf
- Bilinear map.
- Rouche's Theorem

- Marks in CSIR-NET (28 ~ 34.50 marks)

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Complex Analysis

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* **Continuity:** A complex f^n $f(z) = u(x,y) + iv(x,y)$ is continuous at $(x_0, y_0) = z_0$ if $u(x,y)$ & $v(x,y)$ are continuous at (x_0, y_0).

* **Limit**

(1). $\lim_{z \to z_0} f(z) = \infty \iff \lim_{z \to z_0} \frac{1}{f(z)} = 0$.

(2).

Differentiable f^n:

A complex valued f^n $f(z)$ is s.t.b differentiable at $z = z_0$ if —

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{h \to 0} \frac{f(z+h) - f(z_0)}{h}$$

exists and is denoted by $f'(z_0)$, called derivative of f at $z = z_0$.

· Eg. (1) $f(z) = z^n$ at $z = z_0 \in \mathbb{C}$

$\rightarrow f'(z) = n z^{n-1}$ ← (H.W)

Result

(1). Differentiable $\Rightarrow$ Continuity (But $\Leftarrow$ ~~)~~ $\not\Leftarrow$)

Eg. $f(z) = \bar{z}$

(2). Re f & Img f are differentiable $\not\Rightarrow$ f is differentiable

(3) Necessary Cond^n for differentiable → CR-Equations

→ f is diff^n $\Rightarrow$ $u_x = v_y$ & $u_y = -v_x$

But if $u_x = v_y$ & $u_y = -v_x$ at $z = z_0$ $\not\Rightarrow$ f diff^n

(i.e CR. equation is NOT sufficient for diff).

(4) $f_x = u_x + i v_x$, $f_x = i f_y$

VVI

Results for Constant f^n

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Let $f(z) =$ ~~u+iv~~ $u(x,y)+iv(x,y)$ be an analytic f^n on a domain D. Then —

(1) $\boxed{\text{If } u=0 \text{, then } f \text{ is Constant function.}}$

Hint: $\because u=0 \Rightarrow u_x = 0 = v_y$ & $u_y = 0 = -v_x$ (By CR-Eq^n)

$\Rightarrow v$ is constant

$\Rightarrow f$ is Constant f^n

(2). $\boxed{\text{If } v=0 \text{, then } f \text{ is constant function.}}$

Hint: Pf. similar to (1).

(3) $\boxed{\text{If } u \text{ is Constant} \Rightarrow f(z) \text{ is constant function.}}$

Hint: $\because u =$ Constant $\Rightarrow u_x = 0 = v_y$ & $u_y = 0 = -v_x$ (By CR-Eq^n)

$\Rightarrow v$ is also constant

$\Rightarrow f = u+iv$ is constant f^n.

(4). $\boxed{\text{If } v \text{ is constant} \Rightarrow f(z) \text{ is constant function.}}$

Hint: Pf. similar to (3).

(5). $\boxed{\text{If } f \text{ is real valued } f^n \Rightarrow f(z) \text{ is constant}}$

Hint: $\because f$ is real valued f^n

$\Rightarrow v = 0 \Rightarrow$ case (2).

(6). $\boxed{\text{If } f \text{ is imaginary valued } f^n \Rightarrow f(z) \text{ is constant}}$

Hint: If f is imaginary valued $\Rightarrow u=0$

$\because u=0 \Rightarrow$ case (1).

(7). $\boxed{\text{If } \bar{f} \text{ is analytic } f^n \Rightarrow f \text{ is constant } f^n}$

Hint

$f+\bar{f} =$ analytic + Real $\rightarrow$ Constant

$f-\bar{f} =$ analytic + Imag $\rightarrow$ Constant

Add $\quad 2f =$ constant

$\Rightarrow f$ is constant

Result, That will directly Applied

(1) ~~ML- Inequality (A bounding thm):~~

(I) **ML- Inequality (A bounding thm):—**

Let f be an analytic f^n on a contour C with length L, & $|f(z)| \le M$ $\forall$ $z \in C$ then—

$$\left|\int_C f(z)\,dz\right| \le ML$$

M: bound of f^n
L: Length of the Contour

(2). **Cauchy's Inequality**

f be analytic on & inside $C: |z-z_0| = R$ with $|f(z)| \le M$ on ∂C ($|z-z_0| = R$) then—

$$|f^n(z_0)| \le \frac{M\,n!}{R^n}$$

(4) **Liouville's thm:** Non-constant entire f^n is unb.dd f^n.

- Entire + non-constant = Unb.dd f^n
- Entire + b.dd = Constant f^n
- Eg. $\sin z, \cos z, e^z, e^{\sin z}, \sin e^z, p(z)$ --- etc are non-constant entire f^n $\exists$ unb.dd.

* Picard Little thm:

- Non-constant entire f^n can omit at-most one complex no. [i.e. If a entire f^n omits more than 1-pt, then it is constant]
- A non-constant meromorphic f^ns can omit at-most 2-pts

(5) Lucas theorem: If all zeros of $p(z)$ lies in upper half plane, so all zeros of $p'(z)$ lies ~~on~~ also ~~on~~ same half plane.

* **Schwarz's Lemma**

Let $f: \Delta \to \overline{\Delta}$ be analytic & having a zero of order n at the origin. ie $f(0)=0$

$$\Delta = \{z \in \mathbb{C} : |z| < 1\}$$

Then —

(1). $|f(z)| \le |z|^n \quad \forall z \in \Delta$

$(\le |z|)$

(2). $|f^n(0)| \le n!$

* **Corollary (Schwarz lemma)**

:— If f is analytic & $|f(z)| \le M$ in $D(a,R)$ and $f(a)=0$, then —

1. $|f(z)| \le \dfrac{M(z-a)}{R} \quad \forall z \in D(a,R)$

2. $|f'(a)| \le \dfrac{M}{R}$

* **Schwarz's Pick lemma** (P-)

Let f be analytic on the unit disc. D and satisfies —

1. $|f(z)| \le 1$ on D (ie $\forall z \in D$)
2. $f(a) = b$ for some a, b in D

Then — $|f'(a)| \le \dfrac{1-|f(a)|^2}{1-|a|^2}$

and ~~also~~ i.e $\boxed{|f'(z)| \le \dfrac{1-|f(z)|^2}{1-|z|^2}}$

and also $\boxed{|f(z)| \le \dfrac{|f(0)|+|z|}{1+|f(0)|\cdot|z|}}$