METRIC SPACE

(Handwritten Classroom Study Material)

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Your Note/Remarks

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Metric :-

X : A non-empty set

$d : X \times X \longrightarrow \mathbb{R}^+ \cup \{0\}$ a mapping so that

(1) $d(x,y) \geq 0 \;\; \forall \; x, y \in X$ &

$d(x,y) = 0$ iff $x = y$.

(2) $d(x,y) = d(y,x)$

(3) $d(x,y) \leq d(x,z) + d(z,y)$ [Triangle inequality]

such a d is called a metric on X and (X, d) is called a metric space.

Example :- 1 $(\mathbb{R}, d^{abs})$

$$d^{abs}(x,y) = |x-y|$$

$$|x| = \begin{cases} x & \text{if } x \in \mathbb{R}^+ \cup \{0\} \\ -x & \text{, otherwise} \end{cases}$$

$$|\cdot| : \mathbb{R} \longrightarrow \mathbb{R}^+ \cup \{0\}$$

$$d^{abs} : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}^+ \cup \{0\}$$

Q:- Show that $(\mathbb{R}, d^{abs})$ is a metric space.

A:- (1) $d^{abs}(x,y) = |x-y| \geq 0 \;\; \forall \; x, y \in X$

$d^{abs}(x,y) = 0$

$\Leftrightarrow |x-y| = 0$

$\Leftrightarrow x = y$

$\therefore d^{abs}(x,y) = 0$ iff $x = y$.

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(2) $d^{abs}(x,y) = |x-y|$
$= |y-x|$
$= d^{abs}(y,x)$

(3) $d^{abs}(x,y) = |x-y|$
$= |x-z+z-y|$
$= |(x-z)+(z-y)|$
$\leq |x-z| + |z-y| = d^{abs}(x,z) + d^{abs}(z,y)$

$\therefore d^{abs}(x,y) \leq d^{abs}(x,z) + d^{abs}(z,y)$.

Since d^{abs} satisfies all the three conditions above so that d^{abs} is a metric on $\mathbb{R}$ and $(\mathbb{R}, d^{abs})$ is a metric space.

Example:-2

$d^{discrete} : \mathbb{R} \times \mathbb{R} \longrightarrow \{0,1\}$

$$d^{disc}(x,y) = \begin{cases} 0, & x=y \\ 1, & \text{otherwise} \end{cases}$$

Q:- Show that $(\mathbb{R}, d^{disc})$ is a metric space.

A:- (1) $d^{disc}(x,y) = \begin{cases} 0, & \text{if } x=y \\ 1, & \text{otherwise} \end{cases}$

$\therefore d^{disc}(x,y) \geq 0 \quad \forall\, x,y \in \mathbb{R}$.

Now, $d^{disc}(x,y) = 0$
$\Rightarrow x = y$.

(2) $d^{disc}(x,y) = 0$ if $x = y$

$\Rightarrow d^{disc}(y,x) = 0$ if $y = x$

$\therefore d^{disc}(x,y) = d^{dis}(y,x) = 0$, when $x = y$

$d^{disc}(x,y) = 1$ if $x \neq y$

$d^{disc}(y,x) = 1$ if $y \neq x$

$\therefore d^{disc}(x,y) = d^{disc}(y,x) = 1$, when $x \neq y$.

$\therefore d^{disc}(x,y) = d^{disc}(y,x)$

(3) <u>When $x = y$</u>

$d^{disc}(x,y) = 0$

Let $z \in R$ such that $x = z$, $z = y$

then, $d^{disc}(x,z) = 0$

$d^{disc}(z,y) = 0$

$\therefore d^{disc}(x,y) = d^{disc}(x,z) + d^{disc}(z,y)$

<u>When $x \neq y$</u>

<u>case-I</u> :- Let $x = z$, $z \neq y$

then, $d^{disc}(x,z) = 0$

$d^{disc}(z,y) = 1$

$\therefore d^{disc}(x,y) = 1 = d^{disc}(x,z) + d^{disc}(z,y)$.

Case-2:- Let $x \neq z$, $z = y$.

$\therefore d^{disc}(x,z) = 1$

$d^{disc}(z,y) = 0$

$\therefore d^{disc}(x,y) = 1 = d^{disc}(x,z) + d^{disc}(z,y)$.

Case-3:- Let $x \neq z$, $z \neq y$.

$\therefore d^{disc}(x,z) = 1$

$d^{disc}(z,y) = 1$

$\therefore d^{disc}(x,y) = 1 \leq d^{disc}(x,z) + d^{disc}(z,y) = 2$.

$\therefore$ From the above three cases we get,

$$d^{disc}(x,y) \leq d^{disc}(x,z) + d^{disc}(z,y).$$

Since d^{disc} ~~(x,y)~~ satisfy all the above properties, so d^{disc} is a metric on $\mathbb{R}$ and $(\mathbb{R}, d^{disc})$ is a metric space. □

* Basic outcome of metric is to find open sets.
* Collection of metric space $\subseteq$ Collection of topological space.
* Not all topological spaces are metric space But all metric spaces are topological space.

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References :-

1. J.R. Mumkens - Topology
2. Real Analysis - N.L. Carothers -(3 - 8)
3. Real Analysis - Robert G. Bartle

40 — 3 — 15 (Sessional)
— 10 (Assignment)

METRIC SPACE & NORMS

Metric Space :-

Real numbers

$(\mathbb{R}, +, *) \rightarrow$ Complete ordered field.

Continuous function :-

$|f(x) - f(a)| < \epsilon$ whenever $|x - a| < \delta$.

$x \in X \rightarrow$ Abstract set
$\hookrightarrow$ Abstract element.

$X \rightarrow$ open set
$\rightarrow$ closed set
$\rightarrow$ cont. funⁿ
$\rightarrow$ convg. seq.

Defⁿ of metric space :-

Let M be a non-empty set. A funⁿ d on $M \times M$ satisfying the following properties is called a metric on M

(i) $0 \leq d(x,y) < \infty \quad \forall\, x, y \in M$ (Non-negative property)

(ii) $d(x,y) = 0 \Leftrightarrow x = y$.

(iii) $d(x,y) = d(y,x) \quad \forall\, x, y \in M$ (Symmetric Property)

(iv) $d(x,y) \leq d(x,z) + d(z,y) \quad \forall\ x,y,z \in M$

(triangular property)

The couple (M,d) consisting of a set M together with a metric d defined on M is called a metric space.

Ex:-1 Every non-empty set has at least one metric

$$d(x,y) = \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases}$$ ← discrete metric space.

Ex:-2 $(\mathbb{R},d)$, where $d(x,y) = |x-y|$

↳ usual metric on $\mathbb{R}$.
↳ absolute metric on $\mathbb{R}$.

(M,d), $A \subseteq M$

$d : M \times M \longrightarrow [0,\infty)$

$d/_A : A \times A \longrightarrow [0,\infty)$

Ex:- 1, 2, 3, 5, 6, 12.

Bounded Set :-

DI:- 09.01.2020

We say that a subset A of a metric space (M,d) is bounded if there is some $x_0 \in M$ and some constant $c > 0$ such that $d(y, x_0) \leq c \ \forall\ y \in A$.

Ex:- Show that a finite union of bounded set is again bounded.

Proof:- Let $A_1, A_2, \cdots, A_n$ be the collection of finite sets.

$$A = \bigcup_{i=1}^{n} A_i$$

Given that A_i's , $1 \le i \le n$ are bounded sets.

For each A_i , $\exists\ c_i > 0$ & x_i such that

$$d(y, x_i) \le c_i \quad \forall\ y \in A_i$$

$$c = \max_{1 \le i \le n} \{c_1, c_2, \ldots, c_n\}$$

$$d(z, x_0) \le c \quad \forall\ z \in A.$$

$\therefore$ $A = \bigcup_{i=1}^{n} A_i$ is a bounded set.

* Diameter of a non-empty set :-

$$\text{diam}(A) = \sup\{d(a,b) : a, b \in A\}$$

Ex:- Show that A is bounded iff diam(A) is finite.

Proof:- A is bounded $\Leftrightarrow$ $\exists\ x_0$ and $c > 0$ such that

$$d(y, x_0) \le c \quad \forall\ y \in A.$$

Suppose $x_0 \in A$, then $d(y, x_0) \le c \quad \forall\ y \in A$

$\Leftrightarrow \sup\{d(y, x_0), \forall y \in A\} \le c$

$\Leftrightarrow \text{diam}(A) \le c$

$\Leftrightarrow \text{diam}(A)$ is finite. $\square$

Dt:- 13.01.20[illegible]

Normed Vector Space :-

Vector Space :-

Let V be a non-empty set & F be a field then $V(\mathbb{F})$ is said to be a vector space if it satisfies following properties:

(i) $(V, '+')$ is an abelian group.

(ii) $\alpha \cdot v \in V \ \forall \alpha \in \mathbb{F}$ and $v \in V$

(iii) $(\alpha+\beta)\cdot v = \alpha v + \beta v$, $\forall \alpha, \beta \in \mathbb{F}$ & $v \in V$.

(iv) $\alpha \cdot (v_1 + v_2) = \alpha \cdot v_1 + \alpha \cdot v_2$, $\forall \alpha \in \mathbb{F}$ and $v_1, v_2 \in V$

(v) $\alpha(xy) = (\alpha x)y$

(vi) $1 \cdot v = v \ \forall v \in V$.

Norm :- (Norm is used to measure the magnitude of a vector)

↳ A norm on a vectorspace V is a function $\|\cdot\| : V \longrightarrow [0, \infty)$ satisfying

(i) $0 \leq \|x\| < \infty \quad \forall x \in V$.

(ii) $\|\alpha x\| = |\alpha| \|x\| \quad \forall \alpha \in \mathbb{F}$ & $x \in V$.

(iii) $\|x\| = 0 \iff x = 0$.

(iv) $\|x+y\| \leq \|x\| + \|y\| \quad \forall x, y \in V$.

↳ $(V, \|\cdot\|)$ is said to be a normed vector space.

Ex:-1 $f : \mathbb{R} \longrightarrow [0, \infty)$ such that $f(x) = |x|$.

Ex:-2 Norms on $\mathbb{R}^n$:-

1) $\|x\|_1 = \sum_{i=1}^{n} |x_i|$ $\qquad x = (x_1, x_2, \dots, x_n)$

2) $\|x\|_\infty = \max_{1 \le i \le n} |x_i|$

3) $\|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2}$

4) $1 < p < \infty$, $\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$

l_p - spaces :-

For $1 \le p < \infty$, we define l_p to be the collection of all real sequences $x = (x_n)$ for which $\sum_{i=1}^{\infty} |x_i|^p < \infty$ and for $p = \infty$, we define

$$\|x\|_\infty = \sup_{n \in \mathbb{N}} |x_n| < \infty .$$

* $(l_p, \|\cdot\|_p)$, let $x \in l_p$

$\|x\|_p = \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{1/p}$, Prove that $\|x\|_p$ is a norm

Proof:- (i) $0 \le \|x\|_p < \infty$.

(ii) $\|\alpha x_p\|_p = \left(\sum_{i=1}^{\infty} |\alpha x_i|^p \right)^{1/p}$

$= |\alpha| \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{1/p}$

$= |\alpha| \, \|x\|_p$

(iii) $\|x\|_p = 0$

$\Rightarrow \sum_{i=1}^{\infty} |x_i|^p = 0$

$\Rightarrow x_i = 0 \;\; \forall i$.

(iv) $\|x+y\|_p \le \|x\|_p + \|y\|_p$

For this 1st we have to show that l_p is a vector space

Some important inequalities :-

Lemma - 3.5

Let $1 < p < \infty$ and $a, b \geq 0$. Then $(a+b)^p \le 2^p(a^p+b^p)$ consequently $x+y \in l_p$ wherever $x, y \in l_p$.

Proof:- $(a+b)^p \le (2\max\{a,b\})^p$

$$= 2^p \max\{a^p, b^p\}$$

$$\le 2^p(a^p+b^p)$$

* $x, y \in l_p$

$$\sum_{i=1}^{\infty} |x_i+y_i|^p \le \sum_{i=1}^{\infty} \big||x_i|+|y_i|\big|^p$$

$$\le \sum_{i=1}^{\infty} 2^p \big||x_i|^p + |y_i|^p\big|$$

$$= \sum 2^p |x_i|^p + \sum 2^p |y_i|^p$$

$$= 2^p M < \infty$$

H.W Show that l_p is a vector space.

(1) $(l_p, '+')$ is an abelian group.

(i) if $x, y \in l_p$ then $x+y \in l_p$.

↳ Since $x \in l_p \Rightarrow \sum_{i=1}^{\infty} |x_i|^p < \infty$

$y \in l_p \Rightarrow \sum_{i=1}^{\infty} |y_i|^p < \infty$

Now, $\sum_{i=1}^{\infty} |x_i + y_i|^p \le \sum_{i=1}^{\infty} \left| |x_i| + |y_i| \right|^p$

$$\le \cancel{\sum_{i=1}^{\infty}} 2^p \left(\sum_{i=1}^{\infty} |x_i|^p + \sum_{i=1}^{\infty} |y_i|^p \right)$$

$$< \infty .$$

$\therefore \; x + y \in l_p$

Dt :- 14.01.2020

Inequalities :-

(1) The Cauchy Schwartz Inequality :-

$$\sum_{i=1}^{\infty} |x_i y_i| \leq \|x\|_2 \|y\|_2 \text{ for any } x, y \in l_2 .$$

Proof :- If either $x = 0$ or $y = 0$, then this result trivially holds.

If $x, y \neq 0$ and let $t \in \mathbb{R}$, Now, we define

$$0 \leq \|x + ty\|_2^2$$

$\left\{ x, y \in \mathbb{R}^n \quad \langle x, y \rangle = \sum_{i=1}^{n} x_i y_i \quad \|x\| = \sqrt{\langle x, x \rangle} \right\}$

$$= \langle x + ty , x + ty \rangle$$

$$= \langle x, x \rangle + t \langle x, y \rangle + t \langle y, x \rangle + t^2 \langle y, y \rangle$$

$$= \|x\|^2 + 2t \langle x, y \rangle + t^2 \|y\|^2 .$$

$$\Rightarrow \|x\|^2 + 2t\langle x,y\rangle + t^2\|y\|^2 \geq 0$$

$$4|\langle x,y\rangle|^2 - 4\|x\|_2^2\|y\|_2^2 \leq 0$$

$ax^2+bx+c \geq 0$
$a>0$, $D \leq 0$
$D = \sqrt{b^2-4ac}$

$$\Rightarrow |\langle x,y\rangle| \leq \|x\|_2 \|y\|_2$$

$$\Rightarrow \left|\sum_{i=1}^{n} x_i y_i\right| \leq \|x\|_2 \|y\|_2$$

$$\Rightarrow \left|\sum_{i=1}^{n} |x_i||y_i|\right| \leq \||x|\|_2 \||y|\|_2$$

$$\Rightarrow \sum_{i=1}^{n} |x_i||y_i| \leq \|x\|_2 \|y\|_2$$

$$\Rightarrow \sum_{i=1}^{n} |x_i y_i| \leq \|x\|_2 \|y\|_2$$

Similarly, if $x, y \in l_2$, then

$$\sum_{i=1}^{\infty} |x_i y_i| \leq \|x\|_2 \|y\|_2 .$$

Proved

(2) Minkowski's Inequality :-

If $x, y \in l_2$, then $x+y \in l_2$ and

$$\|x+y\|_2 \leq \|x\|_2 + \|y\|_2$$

Proof:- Let us consider x and $y \in \mathbb{R}^n$

$$\|x+y\|_2^2 = \sum_{i=1}^{n} |x_i + y_i|^2$$

$$\leq \sum_{i=1}^{n} |x_i|^2 + 2\sum_{i=1}^{n} |x_i y_i| + \sum_{i=1}^{n} |y_i|^2$$

$$\leq \|x\|_2^2 + 2\|x\|_2\|y\|_2 + \|y\|_2^2$$ [From Cauchy Schwartz Inequality

$$\Rightarrow \|x+y\|_2^2 \le \left(\|x\|_2 + \|y\|_2\right)^2$$

$$\Rightarrow \|x+y\|_2 \le \|x\|_2 + \|y\|_2 .$$

H.W:- $(l_2, \|\cdot\|_2)$ is a normed space. (Prove this).

(3) Young's Inequality :-

Let $1 < p < \infty$ and let q be defined by, $\frac{1}{p} + \frac{1}{q} = 1$, then for any $a, b \ge 0$, we have

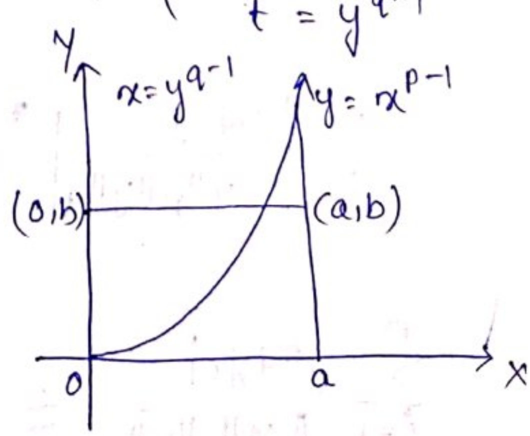
$$ab \le \frac{a^p}{p} + \frac{b^q}{q} .$$

Proof:- If either $a = 0$ or $b = 0$, then inequality trivially holds.

Suppose that $a, b > 0$, we have $q = \frac{p}{p-1}$ also satisfies $1 < q < \infty$ and $p-1 = \frac{p}{q} = \frac{1}{q-1}$. Thus the function $f(t) = t^{p-1}$ and $g(t) = t^{q-1}$ are inverse for $t \ge 0$.

$$\left\{ \begin{array}{l} y = t^{p-1} \\ t = y^{1/p-1} \\ t = y^{q-1} \end{array} \right.$$

$$ab \le \int_0^a x^{p-1}\,dx + \int_0^b y^{q-1}\,dy$$



$$\Rightarrow ab \le \left.\frac{x^p}{p}\right|_0^a + \left.\frac{y^q}{q}\right|_0^b$$

$$\Rightarrow ab \le \frac{a^p}{p} + \frac{b^q}{q} .$$

$$\therefore \quad ab \le \frac{a^p}{p} + \frac{b^q}{q} . \qquad \square$$

Dt:- 20.01.2?

Holder's Inequality :-

Let $1<p<\infty$ and let q be defined by $\frac{1}{p}+\frac{1}{q}=1$.

Given $x \in l_p$ and $y \in l_q$ we have $\sum_{i=1}^{\infty} |x_i y_i| \leq \|x\|_p \|y\|_q$.

Proof:- We may suppose that $\|x\|_p>0$ and $\|y\|_q>0$

Now, for $n \geq 1$, we use Young's inequality to estimate

$$\sum_{i=1}^{n} \left| \frac{x_i y_i}{\|x\|_p \|y\|_q} \right| \leq \frac{1}{p} \sum_{i=1}^{n} \left| \frac{x_i}{\|x\|_p} \right|^p + \frac{1}{q} \sum_{i=1}^{n} \left| \frac{y_i}{\|y\|_q} \right|^q$$

[* Let x & $y \in l_p^n$, $x=(x_1, x_2, \dots, x_n)$

$y=(y_1, y_2, \dots, y_n)$

$$a = \frac{|x_i|}{\|x_i\|_p}, \quad b = \frac{|y_i|}{\|y\|_q}$$

$$\Rightarrow \frac{|x_i||y_i|}{\|x\|_p \|y\|_q} \leq \frac{1}{p} \frac{|x_i|^p}{\|x\|_p^q} + \frac{1}{q} \frac{|y_i|^q}{\|y\|_q^q}$$

$$\Rightarrow \sum_{i=1}^{n} \left| \frac{x_i y_i}{\|x\|_p \|y\|_q} \right| \leq \frac{1}{p} \sum_{i=1}^{n} \left| \frac{x_i}{\|x\|_p} \right|^p + \frac{1}{q} \sum_{i=1}^{n} \left| \frac{y_i}{\|y\|_q} \right|^q \ ]$$

$$\sum_{i=1}^{n} \frac{|x_i y_i|}{\|x\|_p \|y\|_q} \leq \frac{1}{p} \frac{\|x\|_p^p}{\|x\|_p^p} + \frac{1}{q} \frac{\|y\|_q^q}{\|y\|_q^q}$$

$$= \frac{1}{p} + \frac{1}{q} = 1$$

$$\Rightarrow \sum_{i=1}^{n} |x_i y_i| \leq \|x\|_p \|y\|_q \quad \forall\, n \in \mathbb{N}$$

$$\Rightarrow \sum_{i=1}^{\infty} |x_i y_i| \leq \|x\|_p \|y\|_q .$$

$\square$

(5) Minkowski's Inequality :-

Let $1 < p < \infty$, If $x, y \in l_p$ then

$$\|x+y\|_p \leq \|x\|_p + \|y\|_p .$$

Remark:- If $x \in l_p$, then the sequence $\left(|x_n|^{p-1}\right)_{n \in \mathbb{N}} \in l_q$

$$\sum_{i=1}^{\infty} |x_i|^{(p-1)q} = \sum_{i=1}^{\infty} |x_i|^p < \infty \Rightarrow \left(|x_n|^{p-1}\right)_{n \in \mathbb{N}} \in l_q$$

$$\left[\because \frac{1}{p} + \frac{1}{q} = 1 \Rightarrow \frac{1}{q} = \frac{p-1}{p} \Rightarrow p = (p-1)q\right]$$

Proof:-

$$\sum_{i=1}^{\infty} |x_i + y_i|^p = \sum_{i=1}^{\infty} |x_i + y_i| \, |x_i + y_i|^{p-1}$$

$$\leq \sum_{i=1}^{\infty} |x_i| \, |x_i + y_i|^{p-1} + \sum_{i=1}^{\infty} |y_i| \, |x_i + y_i|^{p-1}$$

$$\leq \|x\|_p \, \| |x_n + y_n|^{p-1} \|_q + \|y\|_p \, \| |x_n + y_n|^{p-1} \|_q$$

[from Holder's Inequality]

$$\therefore \|x+y\|_p^p \leq \left(\|x\|_p + \|y\|_p\right)\left(\|(|x_n + y_n|^{p-1})\|_q\right)$$

$$= \left(\|x\|_p + \|y\|_p\right)\left(\|x+y\|_p^{p-1}\right)$$

$$\Rightarrow \|x+y\|_p \leq \|x\|_p + \|y\|_p$$

$\square$

As, $\|(|x_n+y_n|^{p-1})\|_q = \sum_{i=1}^{\infty}\left((|x_i+y_i|)^{(p-1)q}\right)^{1/q}$

$$= \sum_{i=1}^{\infty}\left(|x_i+y_i|^{p}\right)^{1/q}$$

$$= \sum_{i=1}^{\infty}\left(\left(|x_i+y_i|^{p}\right)^{1/p}\right)^{p/q}$$

$$= \|x+y\|_p^{p/q} = \|x+y\|_p^{p-1}$$

Q:- Prove that $(l_p, \|\cdot\|_p)$ is a normed space.

* Limits in Metric Space :-

Open ball:-

Given $x \in M$ and $r>0$, the set $B_r(x) = \{y \in M : d(x,y) < r\}$ is called open ball.

Closed ball:-

$B_r(x) = \{y \in M : d(x,y) \leq r\}$ is said to be closed ball.

Ex:-1 Open ball

$M = \mathbb{R}$

$B_r(x) = \{y \in \mathbb{R} : d(x,y) < r\}$

$\Rightarrow |x-y| < r$

$\Rightarrow y \in (x-r, x+r)$

Closed ball:-

$$B_r(x) = \{ y \in \mathbb{R} : d(x,y) \leq r \}$$
$$\Rightarrow |x-y| \leq r$$
$$\Rightarrow y \in [x-r, x+r].$$

Ex:-2 $(\mathbb{R}, \text{discrete})$

$$B_1(x) = \{ d(y,x) < 1 \} = \{x\}$$

Closed ball with radius 1

$$B_1(x) = \{ d(x,y) \leq 1 \} = \mathbb{R}.$$

Convergence in metric spaces:-

We say that a sequence of points (x_n) in M converges to a point $x \in M$ if any given $\epsilon > 0$, $\exists\, n_0 \in \mathbb{N}(\epsilon, x)$ st $d(x_n, x) < \epsilon \;\forall\, n \geq n_0$.

* **Remark:-** If it should happen that $\{x_n : n \geq N\} \subset A$ for some $N \in \mathbb{N}$, then we say that the sequence (x_n) eventually belongs to A.

So x_n converges to x iff any given $\epsilon > 0$, the sequence (x_n) is eventually in $B_\epsilon(x)$.

Ex:-1 $x_n = \frac{1}{n}$, $n \in \mathbb{N}$, $M = (0,1]$ with usual metric.

Soln:- $d(x_n, x_m) = \left| \frac{1}{n} - \frac{1}{m} \right| \leq 1 \quad x_n, x_m$

$\therefore$ The seq (x_n) is bounded.

$\lim_{n\to\infty} \frac{1}{n} = 0 \notin M.$

$\therefore$ Not cgt in M.

$d(x_n, x_m) = \left|\frac{1}{n} - \frac{1}{m}\right| \leq \frac{1}{n_0} < \epsilon \quad \forall\, n, m \geq n_0.$

$\therefore$ x_n is cauchy seq.

Dt:- 21.9.20

2) ($\mathbb{R}$, discrete)

sequence $(x_n) = n$, $n \in \mathbb{N}$

is a bounded sequence but has no cgt subsequence.

3) In discrete space a sequence (x_n) is cauchy iff it is eventually constant iff $x_n = x \ \forall\, n \geq n_0$.

$d(x_n, x_m) < \epsilon \quad \forall\, n \geq n_0$

$\Rightarrow d(x_n, x_m) = 0$

$\Rightarrow x_n = x_m \ \forall\, n \geq n_0.$

* Equivalent metrics:-

Two metrics d and ρ on a set M are said to be equivalent if they generate the same very convergent sequences

i.e. $d(x_n, x) \to 0$ as $n \to \infty$ iff $\rho(x_n, x) \to 0$ as $n \to \infty$.

Ex:- The usual metric on $\mathbb{N}$ is equivalent to the discrete metric $(\mathbb{N}, D)$, $(\mathbb{N}, U)$.

Soln:- Let x_n in discrete metric space cgt $\Rightarrow x_n = x \ \forall n \geq n_0$

$$U(x,y) = |x-y|$$

$$U(x_n, x) = |x_n - x| < \epsilon \ \forall n \geq n_0.$$

Now let $x_n \in (\mathbb{N}, U)$ is convergent.

$$U(x_n, x) < \epsilon, \forall n \geq n_0$$

$$\Rightarrow |x_n - x| < \epsilon, \forall n \geq n_0$$

$$\Rightarrow x_n \in (x-\epsilon, x+\epsilon), \forall n \geq n_0$$

Choose $\epsilon = \frac{1}{n}$ $\Rightarrow x_n \in (x - 1/n, x + 1/n), \forall n \geq n_0$

$$\Rightarrow x_n = x, \forall n \geq n_0.$$

$\Rightarrow$ The sequence (x_n) is eventually constant.

$\Rightarrow x_n$ is cgt in disc. metric space.

$\left[\{1, \underbrace{2, 3}_{1}, \dots\} \right.$ Let $\epsilon = 1/2$

$x_n \in (x - 1/2, x + 1/2)$

$\left. x_n = x \right]$

Exercise:- [(Ex-6) N.L. Carother]

ρ, σ and τ are equivalent.

Exercise:- The ~~matrices~~ metrics induced by $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$ on R^n are equivalent.

$$\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$$

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|$$

$$\|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2}$$

$$\|x\|_\infty = \max_{1\le i\le n} |x_i| \le \|x\|_1 \qquad x = \{x_1, x_2, \dots, x_n\}$$

$$\|x\|_2^2 = \sum_{i=1}^{n} |x_i|^2$$

$$\|x\|_\infty = |x_k| \le \sqrt{|x_k|^2 + \_\_}$$

$$\Rightarrow \|x\|_\infty \le \|x\|_2$$

$$\|x\|_1 = |x_1| + |x_2|$$

$$\|x\|_2 = \sqrt{x_1^2 + x_2^2}$$

$$\|x\|_1^2 = |x_1|^2 + |x_2|^2 + 2|x_1||x_2|$$

$$\|x\|_2^2 = x_1^2 + x_2^2 = |x_1|^2 + |x_2|^2$$

$$\Rightarrow \|x\|_1^2 - \|x\|_2^2 = 2|x_1||x_2| \ge 0$$

$$\Rightarrow \|x\|_2 \le \|x\|_1$$

$$\Rightarrow \boxed{\|x\|_\infty \le \|x\|_2 \le \|x\|_1}$$

Now, $\|x\|_1 = \sum_{i=1}^{n} |x_i| \le n|x_k| = n\|x\|_\infty$

$$\therefore \|x\|_1 \le n\|x\|_\infty$$

$$\|x\|_1 \le \sqrt{n}\,\|x\|_2$$

$$\|x\|_2 \le \sqrt{n}\,\|x\|_\infty$$

$\|x\|_1 \leq n\|x\|_\infty$ & $\|x\|_2 \leq \sqrt{n}\,\|x\|_\infty$.

$(\|x\|_\infty, \|x\|_2)$

$(\|x\|_\infty, \|x\|_1)$

$(\|x\|_2, \|x\|_1)$

CHAPTER – 4

Open Sets :-

Let (M, d) be a metric space. A set U in a metric space (M,d) is called an open set if given any $x \in U$, $\exists$ some $\epsilon > 0$ such that $B_\epsilon(x) \subset U$.

Ex-1 In a metric space, the whole space M and empty set ϕ are open sets.

Ex-2:- $(\mathbb{R}, \text{Usual})$, Any open interval is an open set.
(a,b), $(-\infty, b)$, (a, ∞)

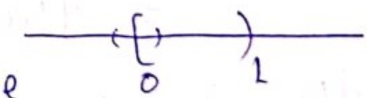
Sol^n:- Let $y \in (a,b)$

Let $\epsilon = \min\{y-a, b-y\}$

then $B_\epsilon(y) \subset (a,b) \ \forall \ y \in (a,b)$.

$\Rightarrow (a,b)$ is an open set.

* $[0,1)$ is not an open set.



Because if we choose 0 as centre then we can't find any ϵ such that
$B_\epsilon(0) \subset [0,1)$

So $[0,1)$ is not an open set.

Ex-3:- In discrete space, $B_1(x) = \{x\}$ is an open set for any x.

As for ~~any~~ some $\epsilon > 0$, $B_\epsilon(x) = \{x\} \subset B_1(x)$.

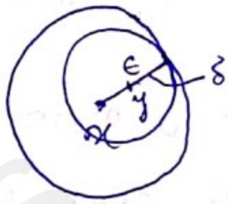
* Every subset of a discrete space is an open set.

Proposition:-

For any $x \in M$ and any $\epsilon > 0$, the open ball $B_\epsilon(x)$ is an open set.

Proof:- Claim :- $B_\epsilon(x)$ is an open set.

Let $y \in B_\epsilon(x)$



$\delta = \epsilon - d(x,y)$

$\Rightarrow d(x,y) < \epsilon$

Hence, $\delta = \epsilon - d(x,y) > 0$,

we will show that $B_\delta(y) \subset B_\epsilon(x)$

Let $z \in B_\delta(y)$

$\Rightarrow d(z,y) < \delta$

Now we have to show that $z \in B_\epsilon(x)$.

$$\begin{aligned} d(x,z) &\leq d(x,y) + d(y,z) \\ &< d(x,y) + \delta \\ &= d(x,y) + \epsilon - d(x,y) \end{aligned}$$

$\therefore d(x,z) < \epsilon$

$\Rightarrow z \in B_\epsilon(x)$

$\Rightarrow B_\delta(y) \subset B_\epsilon(x)$

$\Rightarrow B_\epsilon(x)$ is an open set.

Remark:- An open set must be a union of open balls.

~~i.e.~~ let U be an open set,

i.e. $U = \bigcup_{x \in U} B_{\epsilon_x}(x)$.

Remark:- Any arbitary union of open balls is again an open set.

Theorem-4.3

An arbitary union of open sets is again an open set.

Proof:- $A = \bigcup_{\alpha \in X} U_\alpha$

$U_\alpha = \bigcup_{i \in \lambda} B_i(x)$

Theorem-4.4

A finite intersection of open sets is an open set

i.e. $V = U_1 \cap U_2 \cap \ldots \cap U_n$ is open set.

Proof:-

Let $y \in V$

$\Rightarrow y \in U_i, \ 1 \leq i \leq n$.

$\Rightarrow B_{\epsilon_i}(y) \subset U_i$

$B_\epsilon(y), \ \epsilon = \min\{\epsilon_1, \ldots, \epsilon_n\}$

$B_\epsilon(y) \subset B_{\epsilon_i}(y)$

$\Rightarrow B_\epsilon(y) \subset V$

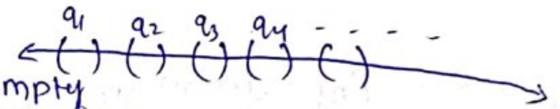
$\Rightarrow V$ is an open set.

Remark:- Arbitary intersection of open set need not be an open set.

Ex:- $V = \bigcap_{n=1}^{\infty} \left(\frac{-1}{n}, \frac{1}{n}\right) = \{0\}$ in $(\mathbb{R}$, Usual$)$.

~~Theorem-4.6:-~~

Remark:- Show that any collection of pairwise disjoint, non-empty open intervals in $(\mathbb{R}, \text{Usual})$ is at most countable.



Proof:- Let $U_1, U_2, \ldots\ldots$ are non-empty open intervals

and $U_1 \cap U_2 \cap \ldots\ldots = \phi$.

Regarding each open interval U_i there is a rational number which is different from other interval.

Since rational numbers are countable.

So $U_1, U_2, \ldots\ldots$ are countable.

Theorem-4.6 If U is an open subset of $\mathbb{R}$, then U may be written as a countable union of disjoint open intervals i.e. $U = \bigcup_{n=1}^{\infty} I_n$, where $I_n = (a_n, b_n)$ and $I_n \cap I_m = \phi$ for $n \neq m$.

Proof:- We first claim that for each $x \in U$ is contained in a maximal open interval $I_x \subset U$ in the sense that if $x \in I \subset U$, where I is an open interval then we must have $I \subset I_x$, Given that $x \in U$ and let

$a_x = \inf\{a : [a, x] \subset U\}$ and $b_x = \sup\{b : [x, b) \subset U\}$.

Then $I_x = (a_x, b_x)$ satisfies $x \in I_x \subset U$ and I_x is clearly maximal.

For any $x, y \in U$, we have either $I_x \cap I_y = \phi$ or $I_x = I_y$.

Let if $y \in I_x \cap I_y \neq \phi$, then $I_x \cup I_y$ is an open interval containing both I_x and I_y.

But by the maximality, we would then have $I_x = I_y$.

It follows that U is the union of disjoint intervals.

$$U = \bigcup_{x \in U} I_x$$

* <u>Closed Set</u> :- 23.01.2020

A set F in a metric space (M, d) is said to be a closed set if its compliment $F^c = M|F$ is an open set.

<u>Ex</u>:- (1) ϕ and M are trivially closed set.

(closed + open = clopen set)

(2) An arbitary intersection of closed set is closed.

$$V = \bigcap_{i \in \lambda} V_i \Rightarrow V^c = \left(\bigcup_{i \in \lambda} V_i^c \right) \Rightarrow V^c \text{ is an open set.}$$

(3) A finite union of closed set is again closed set.

(4) In $\mathbb{R}$ each of the interval $[a, b)$, $[a, \infty)$ and $(-\infty, b]$ is closed set.

Theorem-4.9

Given a set F in (M, d) then the following are equivalent

(1) F is closed i.e. $F^c = M \setminus F$ is an open set.

(2) If $B_\epsilon(x) \cap F \neq \phi$, for any $\epsilon > 0$, then $x \in F$.

(3) If a sequence $(x_n) \subset F$ converges to some point $x \in M$, then $x \in F$.

Proof:- $(1) \Rightarrow (2)$

F is closed $\Rightarrow F^c$ is an open set.

Let, for some $\epsilon>0$, $B_\epsilon(x) \cap F = \phi$ ~~$x \notin F$~~ $\Rightarrow B_\epsilon(x) \subset F^c$ for some $\epsilon > 0$.

$\Rightarrow x \in F^c$

But this is the same as saying F is closed iff $B_\epsilon(x) \cap F \neq \phi$

F is closed set and for some $\epsilon > 0$

Proof:- $(1) \Rightarrow (2)$

F is closed $\Rightarrow F^c$ is an open set

Suppose for some $\epsilon > 0$, $B_\epsilon(x) \cap F = \phi$

$\Rightarrow B_\epsilon(x) \subset F^c$ (is an open set)

$\Rightarrow x \in F^c$

$\therefore \forall \epsilon > 0$, $B_\epsilon(x) \cap F \neq \phi \Rightarrow x \in F$.

(2) ⇒ (1)

claim:- F is closed ⇒ F^c is an open set.

Let $y \in F^c$, then $\exists$ some $\epsilon > 0$ such that $B_\epsilon(y) \subset F^c \ \forall y \in F^c$

$\Rightarrow F^c$ is an open set.

$\Rightarrow F$ is a closed set.

(2) ⇒ (3)

Let $(x_n) \subset F$ be a sequence such that $x_n \to x$.

$\Rightarrow d(x_n, x) < \epsilon \quad \forall \ n \geq n_0$

which is same as $x_n \in B_\epsilon(x)$, $\forall \ n \geq n_0$.

$\Rightarrow B_\epsilon(x) \cap F \neq \phi \quad \forall \ \epsilon > 0$ (from (3))

$\Rightarrow x \in F \quad$ (from (2)).

(3) ⇒ (2)

Suppose $B_\epsilon(x) \cap F \neq \phi$, $\forall \ \epsilon > 0$.

$\forall \ n \in \mathbb{N}$ if $\epsilon = \frac{1}{n}$, $x_n \in B_\epsilon(x) \cap F \neq \phi$

Let $\epsilon > 0$, $n_0 \epsilon > 1 \Rightarrow \frac{1}{n_0} < \epsilon$

$\Rightarrow d(x_n, x) < \frac{1}{n_0} < \epsilon \quad \forall \ n \geq n_0$.

$\Rightarrow x \in F$

Interior of a set :-

Given a set A in (M,d), we define the interior of A denoted as $\text{int}(A)$ or A° to be the largest open set contained in A.

i.e. $\text{int}(A) = A^\circ = \cup \{U : U \text{ is an open set and } U \subset A\}$

$$= \cup \{B_\epsilon(x) : B_\epsilon(x) \subset A \text{ for some } \epsilon > 0\}$$

$$= \{x \in A : B_\epsilon(x) \subset A \text{ for some } \epsilon > 0\}$$

Ex:- A° of A is an open set.

27.01.2020

Closure of a set $(\bar{A})$:-

We define the closure of A, written as $Cl(A)$ or $\bar{A}$ to be the smallest closed set containing A i.e.

$Cl(A)$ or $\bar{A} = \cap \{F : F \text{ is closed and } A \subset F\}$

* $A \subseteq \bar{A}$ & $\text{int}(A) \subset A$

$\bar{A} = A \cup A'$ (set of limit points)

Ex:-(i) $A = (0,1)$

$$\bar{A} = (0,1) \cup [0,1] = [0,1]$$

* limit point,

x is a l.p if $\forall \epsilon > 0$, $B_\epsilon(x) \cap A \neq \phi$.

Ex-(ii) $A = [0,1]$

$\bar{A} = [0,1]$

Properties:-

$x \in \bar{A}$, iff for any given $\epsilon > 0$, $B_\epsilon(x) \cap A \neq \phi$

Proof:-

S.P:- Since $\bar{A}$ is a closed set.

let $\epsilon > 0$ be any number. and $B_\epsilon(x) \cap A \neq \phi$

Since $A \subseteq \bar{A}$

$\Rightarrow B_\epsilon(x) \cap \bar{A} \neq \phi$

From last proposition $x \in \bar{A}$.

N.P Let $x \in \bar{A}$ and let for some $\epsilon > 0$ suppose $B_\epsilon(x) \cap A = \phi$

$\Rightarrow \bar{A} \subset (B_\epsilon(x))^c$ & $(B_\epsilon(x))^c$ is a closed set.

$\Rightarrow \bar{A} \subseteq (B_\epsilon(x))^c$

$\therefore x \in \bar{A} \Rightarrow x \in (B_\epsilon(x))^c$

which is the contradiction.

$\Rightarrow B_\epsilon(x) \cap A \neq \phi$.

(No proof)

Corollary:- $x \in \bar{A}$ iff there is a sequence $(x_n) \subset A$ with $x_n \to x$

Ex:-1 (i) $[0,1]^\circ = (0,1)$

(ii) $A = [0,1] \Rightarrow \bar{A} = [0,1]$

Ex:- 2) $A = \{\frac{1}{n} : n \in \mathbb{N}\}$

$A^\circ = \phi$

$\bar{A} = A \cup \{0\} \quad \therefore A \subset \bar{A}$

Ex:-3

$\text{Int}(\mathbb{Q}) = \phi$

$\overline{\mathbb{Q}} = \mathbb{R}$

$\overline{\mathbb{Q}^c} = \mathbb{R}$

$\text{Int}(\mathbb{Q}^c) = \phi$

Ex:-4

$\text{Int}(\text{Cantor set } \Delta) = \phi$

$\overline{\Delta} = \Delta$

Ex:-5

Set of natural number $= \mathbb{N}$

$\overline{\mathbb{N}} = \mathbb{N} \cup \mathbb{N}'$

$= \mathbb{N}$

* A set A is closed iff $A = \overline{A}$
* " " " open iff $A = A^{\circ}$
* $\mathbb{Q}$ is neither open nor closed.

Def^n:- Perfect Set

A set P is called a perfect set if $A = \overline{A} = A \cup A'$

Ex-i. $A = [0,1]$

$A = \overline{A} = A \cup A'$

(ii) Cantor set

(iii) $\mathbb{R}$

(iv) $\mathbb{N}$

(v) $\mathbb{Z}$ all are perfect set.

* Check that $A = [0,1)$ is perfect or not.

$\bar{A} = [0,1] \neq A$

* $\mathbb{Q}, \mathbb{Q}^c$, open interval, $(0,\infty)$ are not perfect sets.

CHAPTER - 5

= CONTINUITY =

Def?:- Let (M,d) and (N,ρ) be two arbitrary metric spaces and $f: M \to N$ is continuous at a point $x \in M$ if for any given $\epsilon > 0$, $\exists\ \delta > 0\ (x, \epsilon)$ such that $\rho(f(x), f(y)) < \epsilon$, whenever $y \in M$ st $d(x,y) < \delta$.

or

f is cont. at x if for any $\epsilon > 0$, $\exists\ \delta > 0$ such that

$$f(B^d_\delta(x)) \subset B^\rho_\epsilon(f(x)).$$

↳ If f is cts at every point of M, we simply say that f is continuous on M.

28.01.2020

Ex:- 1 $f(x) = x$ on $\mathbb{R}$ is a cts funⁿ.

2) A funⁿ $f: M \to N$ between metric spaces is called an isometry if f preserves distance $\rho(f(x), f(y)) = d(x,y)\ \forall\ x, y \in M$. Show that an isometry is a cont. funⁿ.

* Isometry funⁿ is one-one uniform cts. funⁿ.

Theorem:-

2) Proof :-

Let $\epsilon > 0$ and let δ be st $d(x,y) < \delta$

Now, $\rho(f(x), f(y)) = d(x,y) < \delta = \epsilon$

$\Rightarrow \rho(f(x), f(y)) < \epsilon \quad \forall\, y \in M$ st $d(x,y) < \delta$

$\therefore$ Isometry is a cts funⁿ.

Theorem :- 5.1

Given $f : (M, d) \rightarrow (N, \rho)$, then following statements are equivalent

(i) f is cont. on M (by the ϵ-δ defⁿ)

(ii) for any $x \in M$ if $x_n \rightarrow x$ in M, then $f(x_n) \rightarrow f(x)$ in N.

(iii) If E is closed in N, then $f^{-1}(E)$ is closed in M.

(iv) If V is an open set in N, then $f^{-1}(V)$ is an open set in M.

Proof:- (i) $\longrightarrow$ (ii)

Let $x \in M$ and a seq. $(x_n) \in M$ st $x_n \longrightarrow x$ in M

i.e. for any given $\delta > 0$, $\exists\, n_0 \in \mathbb{N}$ st

$d(x_n, x) < \delta \quad \forall \quad n \geq n_0$

Let $\epsilon > 0$ be any number and since f is a cts funⁿ, then by the defⁿ of cts funⁿ,

$\rho(f(x_n), f(x)) < \epsilon \quad \forall\, n \geq n_0$

$\Rightarrow f(x_n) \longrightarrow f(x)$

(ii) $\longrightarrow$ (iii)

Let E be a closed set in (N, ρ). Let $(x_n) \in f^{-1}(E)$ such that $x_n \xrightarrow{d} x$ in M. We have to show that $f^{-1}(E)$ is a closed set.

From (ii) $f(x_n) \longrightarrow f(x)$ and since $f(x_n) \in E$. since E is a closed set and $f(x_n) \to f(x)$ by the property of closed set $f(x) \in E \Rightarrow x \in f^{-1}(E)$,

So $x_n \in f^{-1}(E)$ such that $x_n \to x$ and $x \in f^{-1}(E)$ which implies $f^{-1}(E)$ is closed set.

(iii) $\longrightarrow$ (iv)

Since $f^{-1}(A^c) = \left(f^{-1}(A)\right)^c$

Let V be an open set in N, then we have to show that $f^{-1}(V)$ is an open set in M.

Since V is an open set then V^c is closed set in N

Now, $f^{-1}(V^c) = \left(f^{-1}(V)\right)^c$ is closed in M.

$\Rightarrow f^{-1}(V)$ is an open set in M.

(iv) $\longrightarrow$ (i)

Given $x \in M$ and $\epsilon > 0$, the set $B^{\rho}_{\epsilon}(f(x))$ is open in (N, ρ)

By the (iv) the set $f^{-1}\left(B^{\rho}_{\epsilon}(f(x))\right)$ is an open set in (M, d).

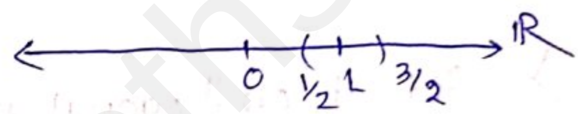
But then $B^d_\delta(x) \subset f^{-1}\left(B^\rho_\epsilon(f(x))\right)$ for some $\delta > 0$, because $x \in f^{-1}\left(B^\rho_\epsilon(f(x))\right)$

$$\Rightarrow f\left(B^d_\delta(x)\right) \subset B^\rho_\epsilon(f(x))$$

Ex:-1

(i) $\chi_Q : \mathbb{R} \to \mathbb{R}$, $\chi_Q(x) = \begin{cases} 1, & x \in Q \\ 0, & x \notin Q \end{cases}$

~~Cont.~~



$\left(\frac{1}{2}, \frac{3}{2}\right)$ is an open set

$$\chi_Q^{-1}\left(\tfrac{1}{2}, \tfrac{3}{2}\right) = \left\{x \in \mathbb{R} \mid f(x) \in \left(\tfrac{1}{2}, \tfrac{3}{2}\right)\right\}$$

$= Q$, which is not open

$\therefore$ χ_Q is not a cont. funⁿ.

2) Show that $(0, \infty)$ is an open set in $\mathbb{R}$.

A:- Define a cts. funⁿ $f(x) = x$.

$f^{-1}(0, \infty) = (0, \infty) \to$ open interval

(Every open interval is an open set)

$\therefore$ $(0, \infty)$ is an open set.

3) $M_n(\mathbb{R})$, $GL_n(\mathbb{R})$ is an open set.

A:- $f : M_n(\mathbb{R}) \to \mathbb{R}$

$f(M) = \det(M)$

if $M \in GL_n(\mathbb{R})$, then $f(M) = \det(M) \neq 0$.

$M_n(\mathbb{R}) \setminus GL_n(\mathbb{R})$

$f^{-1}(\{0\}) = (GL_n(\mathbb{R}))^c \to$ closed set

$\Rightarrow GL_n(\mathbb{R}) \to$ open set

As $\{0\}$ is closed $\Rightarrow f^{-1}(\{0\})$ is closed.

* Show that $SL_n(\mathbb{R})$ is closed set.

5) Set of orthogonal matrix $O(n) = \{A \in M_n(\mathbb{R}) \mid AA^T = A^TA = I\}$

A:- Define $f : M_n(\mathbb{R}) \to M_n(\mathbb{R}) \approx \mathbb{R}^{n^2} \approx \mathbb{R}^{n \times n}$

$f(A) = AA^T - A^TA$

$f^{-1}(\{0\}) = O(n)$

$\uparrow$ closed $\Rightarrow$ $\uparrow$ closed

29.01.2020

Ex:- $S = \{A \in M_2(\mathbb{R}) \mid Eig(A) \in \mathbb{C} \setminus \mathbb{R}\}$

$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$ $\quad \lambda^2 - (a_1 + a_4)\lambda + (a_1a_4 - a_2a_3) = 0$

$f(A) \longrightarrow D(A) < 0 \quad (-\infty, 0)$

$f^{-1}(-\infty, 0) =$ open set

$ax^2 + bx + c$ $\quad x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $\quad b^2 - 4ac < 0$ for $\mathbb{C}$ root

2) Let $f: \mathbb{N} \to \mathbb{R}$ be any function, then f is continuous.

$f^{-1}((a,b)) = \cup \{x\}$

$\uparrow$

open in disc. metric space.

So $f^{-1}((a,b))$ is open set.

3) Let $f: \overset{d}{\mathbb{R}} \to \overset{\rho}{\mathbb{N}}$ is cts iff f is constant.

Absolute metric. disc. metric

A:- Let f is cts. i.e. $\forall \epsilon > 0, \exists \delta > 0$ st

if $d(x,y) < \delta$, then $\rho(f(x), f(y)) < \epsilon$

Let $\epsilon = \frac{1}{2}$

$\rho(f(x), f(y)) < \frac{1}{2}$

$\therefore \forall$ $y \in (x-\delta, x+\delta)$, $f(x) = f(y)$.

4) A function $f: (M,d) \to (N,\rho)$ is called Lipschitz funⁿ if there is a constant $k < \infty$ st $\rho(f(x), f(y)) \leq k\, d(x,y)$ $\forall x, y \in M$. Then show that a Lipschitz mapping is cts.

* This is also unif. cont. funⁿ

* Homeomorphism

Homeo $\to$ similar

(M,d) and (M,ρ) are said to be similar

iff they generate same cgt. seq.

" " " " open set.

" " " " cts. funⁿ.

(M, d), (M, ρ) similar iff $i: M \to M$ is cts &
$i^{-1}: M \to M$ is cts.

N.P. let $x_n \to x$

$\rho(i(x_n), i(x)) \to 0$

$\Rightarrow \rho(x_n, x) \to 0$.

$\therefore$

* Homeomorphism:-

Two metric spaces (M, d) and (N, ρ) are homeomorphic (similar shape) if there is a funⁿ $f: (M, d) \to (N, \rho)$ such that

(i) f is one-one

(ii) f is onto funⁿ

(iii) f and f^{-1} both are cts. funⁿ.

then f is said to be a homeomorphism.

Note:- f is a homeomorphism from M onto N iff f^{-1} is also a homeomorphism.

Ex-1:- If d, ρ are equivalent metrics on M, then (M,d) and (M,ρ) are homeomorphic.

A:- $i : (M,d) \longrightarrow (M,\rho)$
and $i^{-1} : (M,\rho) \longrightarrow (M,d)$

Let $(x_n) \in M$ st $x_n \xrightarrow{d} x \Rightarrow d(x_n, x) \to 0$ as $n \to \infty$.

$\Rightarrow \rho(x_n, x) \to 0$ as $n \to \infty$.

$\rho(i(x_n), i(x)) = \rho(x_n, x) \longrightarrow 0$

$\therefore i(x_n) \xrightarrow{\rho} i(x)$

$\therefore i$ is cts.

Similarly we can prove i^{-1} is cts.

then (M,d) & (M,ρ) are h.m.m.

Ex:-2 The rel^n "is homeomorphic to" is an equivalence rel

Ex:-3 $(\mathbb{R}$, Usual$)$ is not homeomorphic to $(\mathbb{R}$, Disc$)$

Ex:-4 All three of the spaces $(\mathbb{R}^n, \|\cdot\|_1)$, $(\mathbb{R}^n, \|\cdot\|_2)$ and $(\mathbb{R}^n, \|\cdot\|_\infty)$ are homeomorphic.

Proof:- $i : (\mathbb{R}^n, \|\cdot\|_1) \longrightarrow (\mathbb{R}^n, \|\cdot\|_2)$

$\|x_n - x\|_1 \longrightarrow 0$

$\|i(x_n) - i(x)\|_2 = \|x_n - x\|_2 \longrightarrow 0$

i.e. $i(x_n) \longrightarrow i(x)$

4) We say that $f : M \to N$ is an isometry if f is an onto map and satisfying $\delta(f(x), f(y)) = d(x,y)$ $\forall\ x, y \in M$ ~~such~~ how that f is a homeomorphism.

* $f : \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$f(x) = \tan^{-1} x$.

30.01.2020

Homeomorphism:-

5) In $\mathbb{R}$, any two intervals that look like same are homeomorphic. So $(0,1)$ and (a,b) are homeomorphic.

* (a, ∞) homeomorphic to (b, ∞)

$[a, \infty)$ " " $[b, \infty)$

For $(0,1)$ and (a,b)

$f : (a,b) \longrightarrow (0,1)$

$$f(x) = \frac{x-a}{b-a}$$

*i) $f : \mathbb{R} \longrightarrow \left(\frac{-\pi}{2}, \frac{\pi}{2}\right) \approx (a,b) \approx (0,1)$

$f(x) = \tan^{-1} x$

$\mathbb{R} \not\approx [a,b]$.

*(ii) $f : (0,1) \longrightarrow (-1, 2)$

$$f(x) = 2 - 3x$$

$(0,1) \approx (-1, 2)$

6) Any two interval that look like different are different.

E.g:- $[0,1] \not\approx (0,1)$ and $[0,1) \not\approx (a,b]$

Theorem:-

Let $(M,d) \longrightarrow (N,\rho)$ be a bijective map. Then following are equivalent.

(i) f is a homeomorphism

(ii) $x_n \xrightarrow{d} x$ iff $f(x_n) \xrightarrow{\rho} f(x)$

(iii) V is open in M iff $f(V)$ is open in N.

(iv) E is closed in M iff $f(E)$ is closed in N.

(Proof do yourself)

Theorem:-

Let $f : (M,d) \longrightarrow (N,\rho)$, T.F.A.E

(i) f is a homeomorphism

(ii) $\hat{d}(x,y) = \rho(f(x), f(y))$ defines a metric on M equivalent on d.

CHAPTER-6

CONNECTEDNESS

↳ Let (M,d) be a metric space. We say that a metric space M is disconnected if M can be split into the union of two non-trivial open sets i.e. M is disconnected if $\exists$ two disjoint open sets in M, A and B such that, $A \neq \phi$, $B \neq \phi$, $A \cap M \neq \phi$, $B \cap M \neq \phi$, $M = A \cup B$, $A \cap B = \phi$

↳ The pair of open sets A and B is called a disconnection of M.

↳ We say that M is connected if no such disconnection can be found.

Theorem-6.1:-

M is connected iff M contains no-nontrivial clopen sets.

Ex:- $\mathbb{R}$ is connected with usual metric. i.e $(\mathbb{R}, U)$ is connected

A:- Let $\mathbb{R}$ is disconnected.

then $\exists$ A & B disjoint clopen sets st

$\mathbb{R} = A \cup B$

But in $\mathbb{R}$ only ϕ and $\mathbb{R}$ are clopen.

i.e. $\mathbb{R}$ does n't contain no- nontrivial clopen sets.

So $\mathbb{R}$ is connected.

2) A discrete space containing two or more points is disconnected.

(A, d)

$A = \{a, b, c\}$

$A = \{a\} \cup \{b, c\}$

$= \{a\} \cup \{b\} \cup \{c\}$ clopen set in disc metric.

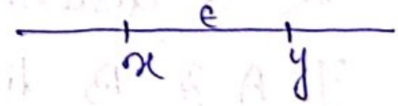
* (M, d) is said to be totally disconnected if only singleton sets are connected sets.

ex:- $(\mathbb{N}, d)$, $(\mathbb{Z}, \text{disc})$

Ex:- $(\mathbb{Q}$, Absolute$)$

Let $x \neq y \in \mathbb{Q}$

$d(x, y) = |x - y| = \epsilon$



$S = \{x, y\}$

$A_1 = B_{\epsilon/2}(x) \cap S = \{x\}$

$A_2 = B_{\epsilon/4}(y) \cap S = \{y\}$

$\therefore S = A_1 \cup A_2$

$\therefore$ $\mathbb{Q}$ is totally disconnected.

3) The empty set and any one point space are connected.

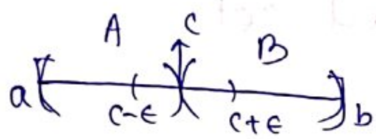
4) Canto set Δ is disconnected.

$A_1 = (-\infty, z) \cap \Delta$

$A_2 = (z, \infty) \cap \Delta$

$\therefore \Delta = A_1 \cup A_2$.

5) $[a,b]$ is a connected set.



06.02.2020

A:- We have to show that $[a,b]$ is a connected set. We will prove it by contradiction.

Let us assume that $[a,b]$ is disconnected. By the def^n of disconnected set, we can write $[a,b] = A \cup B$.

We might assume that $b \in B$, so $(b-\epsilon, b] \subset B$ for some $\epsilon > 0$. Because B is an open set.

Now let $c = \sup A$,

clearly $a \le c \le b$, but note that, since A and B are open sets in $[a,b]$.

Then we have actually $a < c < b$.

Now, from the def^n of supremum $(c-\epsilon, c) \cap A \neq \phi$ and $(c, c+\epsilon) \cap B \neq \phi$.

for any given $\epsilon > 0$.

Then $(c, b) \subset B$ which implies $c \in \bar{A}$ and $c \in \bar{B}$,

But then $c \in \bar{A} \cap \bar{B} \Rightarrow c \in A \cap B$.

[$\because x \in \bar{A}$ iff $B_\epsilon(x) \cap A \neq \phi$]

But we have $A \cap B = \phi$

So $c \in A \cap B$ which is contradiction.

Hence $[a, b]$ is connected set.

Lemma- 6.3 :-

Let E be a subset of a metric space M, if U and V are disjoint open sets in E, then there are disjoint open set A and B in M such that $U = A \cap E$ and $V = B \cap E$.

Proof:-

Since U is an open set, then for each $x \in U$, there is an $\epsilon_x > 0$ such that

$$E \cap B_{\epsilon_x}(x) \text{ (in } M) \subset U \quad (\because U \text{ is open in } E)$$

Same as for each $y \in V$, there exists a $\delta_y > 0$ such that

$E \cap B_{\delta_y} \subset V$, since $U \cap V = \phi$, we also get

$$E \cap B_{\epsilon_x}(x) \cap B_{\delta_y}(y) = \phi.$$

claim:- $B_{\epsilon_{x/2}}(x) \cap B_{\delta_{y/2}}(y) = \phi$, for every $x \in U$ and $y \in V$

Thus $A = \cup\{B_{\epsilon x/2}(x) : x \in U\}$ and

$B = \cup\{B_{\delta y/2}(y) : y \in V\}$

clearly, $U = A \cap E$ and $V = B \cap E$.

* Let $y \in A \cap E$

$\Rightarrow y = \cup\{B_{\epsilon x/2}(x) \cap E$

$= \bigcup_{x \in U} (B_{\epsilon x/2} \cap E)$

$\in U$

$\Rightarrow A \cap E \subseteq U$

Let $y \in U$

$\Rightarrow B_{\epsilon y/2}(y) \cap E \subset U$ [$\because U$ is open set]

$\Rightarrow B_{\epsilon y/2} \cap E \subset \left\{\bigcup_{y \in U} B_{\epsilon y/2}(y) \cap E\right\} = A \cap E$

$\therefore y \in A \cap E$

$\therefore U \subseteq A \cap E$

$\therefore U = A \cap E$

Suppose $z \in B_{\epsilon x/2}(x) \cap B_{\delta y/2}(y)$

$\Rightarrow z \in U \wedge z \in V$

which is contradiction.

$\therefore B_{\epsilon x/2}(x) \cap B_{\delta y/2}(y) = \phi$.

* Disconnection of a subset of a metric space :-

Let (M, d) be a metric space and A be a subset of M. We will call a pair of disjoint open sets A and B a disconnection of E if $A \cap E \neq \phi$, $B \cap E \neq \phi$ and $E \subset A \cup B$.

Theorem - 6.4

A subset E of $\mathbb{R}$ containing more than one point is connected iff whenever $x, y \in E$ with $x < y$, we also have $[x, y] \subset E$. i.e. connected subset of $\mathbb{R}$ containing more than one point are precisely intervals.

Proof :- $\Rightarrow$ Let E is a connected subset of $\mathbb{R}$ and let $x, y \in E$ with $x < y$.

Now, we have to show that $[x, y] \subset E$.

We will prove it by contradiction.

Let us assume $[x, y] \not\subset E$. Let $z \in [x, y]$ with $x < z < y$ and $z \notin E$.

Let $A = (-\infty, z) \cap E$ and $B = (z, \infty) \cap E$,

Clearly $A \cap B = \phi$

and $E \subseteq A \cup B$

$\Rightarrow E$ is a disconnected set. which is contradiction.

because E is a connected set.

So $[x, y] \subset E$.

$\Leftarrow$ We have to show that E is connected.
We will prove it by contradiction.
Let us assume E is a disconnected set.
By the def^n of disconnected set $E \subseteq A \cup B$ where $A, B \subseteq M$

Let $a \in A \cap E$ and $b \in E \cap B$ and let $a < b$.
$\Rightarrow [a, b] \subset E \subseteq A \cup B$
$\Rightarrow [a, b] \subseteq A \cup B$
i.e. $[a, b]$ is disconnected set
which is a contradiction because $[a, b]$ is a connected set $\Rightarrow E$ is a connected set.

Finally suppose that E satisfies $[x, y] \subset E$, whenever $x, y \in E$ with $x < y$.
We have to show that E is an interval. But it follows from the condition that E contains open interval $(\inf E, \sup E)$. Where if $E = \mathbb{R}$, then we can include the possibilities $\inf E = -\infty$ and $\sup E = \infty$.
Thus E must be an interval.

Continuity & Connectedness :-

Lemma:- M is disconnected iff there exists a continuous map from M onto $\{a, b\}$ (The two-point discrete space)

Proof:- Let M be a disconnected set.

By the def^n of disconnected set, $M = A \cup B$

$f : M = A \cup B \longrightarrow \{a, b\}$

$$f(x) = \begin{cases} a, & \text{if } x \in A \\ b, & \text{if } x \in B \end{cases}$$

$\Rightarrow$ f is a cts and onto map [By criteria of open set i.e. inverse of ~~image~~ open set is open set then f is cts]

$\Leftarrow$ $A = f^{-1}\{a\}$

$B = f^{-1}\{b\}$

$M = A \cup B$ and $A \cap B = \phi$

$\Rightarrow$ M is disconnected.

Q:- The set of orthogonal matrices is disconnected.

A:- $AA^T = A^T A = I$, $\det(A) = \pm 1$

$f : M \longrightarrow \{-1, 1\}$

$f(A) = \det(A)$ which is cts and onto,

So set of orthogonal matrices is disconnected.

Theorem:-

Let $f:(M,d) \longrightarrow (N,\rho)$ be continuous and let E be a subset of M, if E is connected, then $f(E)$ is also connected.

Proof:- Let $f(E)$ is disconnected.

By the prev. lemma $\exists$ a cts function,

$g: f(E) \longrightarrow \{0,1\}$ and g is an onto map.

Define, $f: E \longrightarrow f(E)$ and $g: f(E) \longrightarrow \{0,1\}$

Then $gof : E \longrightarrow \{0,1\}$

gof is cts and onto map

$\therefore E$ is disconnected which is a contradiction, because E is connected.

$\Rightarrow f(E)$ is connected set. $\square$

Q:- $f: \mathbb{R} \longrightarrow \mathbb{N}$ → constant / → non-constant

define a funⁿ.

It is not cont. and not onto.

But $f: \mathbb{N} \longrightarrow \mathbb{R}$ is cont. funⁿ.

as $f^{-1}(a,b) = \bigcup_{n \in \mathbb{N}} \{n\}$ is open

Corrollary :- (Mean Value Theorem)

If I is an interval in $\mathbb{R}$ & $f : I \rightarrow \mathbb{R}$ is a non-constant cont. funⁿ, then $f(I)$ is an open interval.

12.02.2020

Totally disconnected :-

We say that A is totaly disconnected if singletons are only connected set.

Ex :- (i)

$(M, \text{discrete})$

$A = \{x_1, x_2\} \subseteq M$

$A = B^1_{1/2}(x_1) \cup B^2_{1/2}(x_2)$ and $B^1_{1/2}(x_1) \cap B^2_{1/2}(x_2) = \phi$.

(ii) Cantor Set (Δ) is totally disconnected set.

$\Delta = \{x_1, x_2, \ldots\}$

$A = \{x_1, x_2\}$, $x_1 < z < x_2$

$A_1 = (-\infty, z) \cap A$

$A_2 = (z, \infty) \cap A$

$A_1 \cap A_2 = \phi$, $A = A_1 \cup A_2$

(iii) The set Q and Q^c

2) If E is connected subset of M and if A and B are disjoint open sets in M with $E \subset A \cup B$. Prove that $E \subset A$ or $E \subset B$.

A:- Since $E \subset A \cup B$

$\Rightarrow E$ is disconnected

But E is connected.

So either $E \subset A$ or $E \subset B$.

3) If E and F are connected subset of M with $E \cap F \neq \phi$, then $E \cup F$ is also connected.

A:- Let $E \cup F$ is disconnected.

$E \subset A \cup B \quad , \quad A \cap B = \phi$

$F \subset C \cup D \quad , \quad C \cap D = \phi$

$\therefore$ ~~$E \cap F \neq \phi$~~

$E \cup F \subset (A \cup B) \cup (C \cup D)$

Lemma:- 6.9

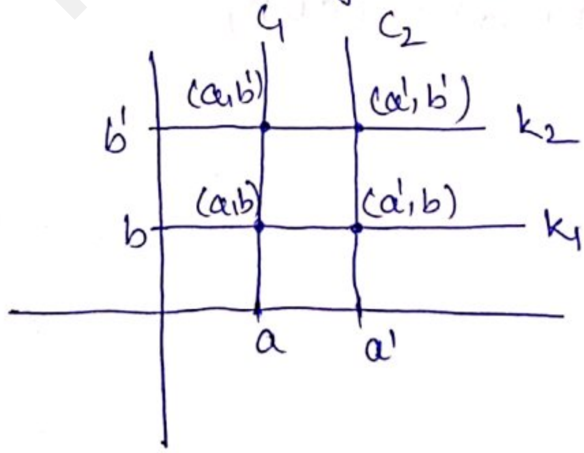
If A and B are connected sets, then $A \times B$ is connected

Proof:- Given that A and B are connected sets, then we have to show that $A \times B$ is a connected set.

Suppose $f : A \times B \to \{0,1\}$ is continuous, we have to show that f is a constant.

Given any $a \in A$ and $b' \in B$, Each of the function $f(a,\cdot) : B \to \{0,1\}$ and $f(\cdot,b') : A \to \{0,1\}$ is continuous consequently, since A and B are connected each of these new map must be constant.

This means that f is constant along "horizontal" and "vertical" lines in $A \times B$. Thus $f(a,b) = f(a',b')$ because $f(a,\cdot)$ and $f(\cdot,b')$ are constant and the two functions must agree at (a,b') i.e. f is constant



$k_1 = c_1$

$k_2 = c_1$ $\quad \therefore k_1 = k_2 = c_1 = c_2$

$k_1 = c_2$

$k_2 = c_2$

Some Important Examples :-

1) Prove that $[a,b]$, $(a,b]$ and (a,b) can't be homeomorphic by the help of connectedness.

Sol^n:- Suppose $(a,b]$ and (a,b) are homeomorphic.

$$f : (a,b] \longrightarrow (a,b)$$

$f(b) = c$ where $a < c < b$.

$$f_1 : (a,b) \longrightarrow (a,b] \setminus \{c\} = (a,c) \cup (c,b)$$

$$f_1((a,b)) = (a,c) \cup (c,b) = A$$

But A is disconnected.

So that $(a,b]$ and (a,b) are not homeomorphic.

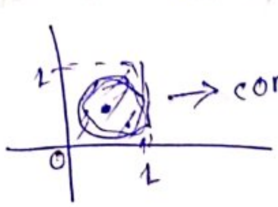
$$\begin{array}{lcl} f : [a,b] & \longrightarrow & (a,b] \\ \quad \big\downarrow -b & & \\ \quad [a,b) & \longrightarrow & (a,b) \\ \quad \big\downarrow -a & & \\ \quad (a,b) & \longrightarrow & (a,c) \cup (c,b) \end{array}$$

Ex:-2 $[0,1] \times [0,1)$ can not be homeomorphic to $[0,1)$.

$$A \times B \setminus (\underset{\substack{\downarrow \\ \text{count.}}}{M_1} \times \underset{\substack{\downarrow \\ \text{count.}}}{M_2})$$

$\downarrow$

connected.



Ex:-3 $\mathbb{R}^2$ can't be homeomorphic to $\mathbb{R}$. In general $\mathbb{R}^n, n>1$ is not homeomorphic to $\mathbb{R}$.

Ex:-4 $\mathbb{R}^n$ is not homeomorphic to $\mathbb{R}^m$, for $n \neq m$.

$$\left.\begin{array}{l} f : \mathbb{R}^n \to \mathbb{R}^m \Rightarrow n \le m \\ f^{-1} : \mathbb{R}^m \to \mathbb{R}^n \Rightarrow m \le n \end{array}\right\} m = n.$$

Ex:-5

The set of orthogonal matrices is a disconnected.

Ex:-6

Prove that $GL_n(\mathbb{R})$ is an open set, but disconnected.

$$f : GL_n(\mathbb{R}) \to \mathbb{R}$$

$$f(A) = \det(A)$$

$$\Rightarrow GL_n(\mathbb{R}) = \overbrace{f^{-1}(-\infty, 0)}^{\text{open}} \cup \overbrace{f^{-1}(0, \infty)}^{\text{open}}$$

$$= \overbrace{A_1}^{\text{open}} \cup \underbrace{A_2}_{\text{open}}, \quad A_1 \cap A_2 = \phi$$

$\therefore GL_n(\mathbb{R})$ is disconnected.

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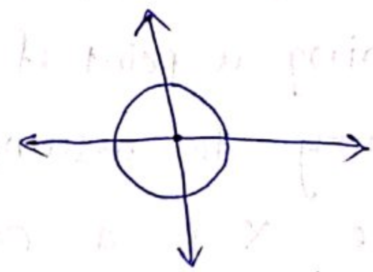
17.02.2020

Ex:- $GL_n(\mathbb{C})$ is open set ~~but~~ and connected.

A:- $f : GL_n(\mathbb{C}) \longrightarrow \mathbb{C}$

$f(A) = \det A$



One open set is $\mathbb{C} \setminus \{0\}$

But we can't find another non-trivial open set.

So $GL_n(\mathbb{C})$ is connected.

Definition:-

Given points x and y of the space X, a path in X from x to y is a continuous map $f : [a, b] \longrightarrow X$ of some closed interval in the real line into X, such that $f(a) = x$ and $f(b) = y$.

→ A space X is said to be path connected if every pair of points in X can be joined by a path in X.

Remark:- 1

Show that a path connected space X is connected.

Proof:- Let X be a path connected set and X is not a connected set.

So by the defn of disconnectedness $\exists$ two non-empty open sets A and B such that $X = A \cup B$.

Let $f : [a, b] \longrightarrow X$ be any path in X.

Being the cont. image of a connected set, the set $f([a,b])$ is connected. So that it lies entirely in either A or B. Therefore there is no path in X joining a point of A to a, point of B to b. That is contrary to assumption that X is a path connected. Hence X is a connected set.

Remark-2

A connected set need not be a path connected.

ex:- The ordered square $I_0 \times I_0$, $I_0 = [0,1]$.

Convex Set

Def^n:- Let $S \subseteq \mathbb{R}^n$ is said to be a convex set if $\forall\, x, y \in S$ and $0 \le \theta \le 1$ such that $\theta x + (1-\theta) y \in S$.

$$f : [0,1] \longrightarrow X$$

x ——— y

$\theta(x) + (1-\theta)y$

$0 \le \theta \le 1$

$$f(t) = x(1-t) + ty$$

$$f(0) = x \quad f(1) = y.$$

↳ A convex set is a path connected.

Proof:- Let $x, y \in X$

$$f : [0,1] \longrightarrow X \text{ st } f(t) = (1-t)x + ty \in X.$$

Ex:- Show that set $S = \{A \in M_n(\mathbb{R}) \mid tr(A) = 0\}$ is connected.

Proof:- Let $A, B \in S$

$\Rightarrow tr(A) = tr(B) = 0$

Consider, $(1-t)A + tB$

Now, $tr[(1-t)A + tB] = (1-t)\,tr(A) + t\,tr(B)$

$= 0$

$\therefore (1-t)A + tB \in S$.

$\therefore S$ is convex set $\Rightarrow S$ is path connected set $\Rightarrow S$ is connected set.

Q:- A unit ball $D_1(X) = \{X \in \mathbb{R}^n \mid \|X\| \leq 1\}$

A:-

Consider $X, Y \in D_1(X)$

and $(1-t)X + tY$

Now $\|(1-t)X + tY\| \leq |1-t|\,\|X\| + |t|\,\|Y\|$

$\leq |1-t| + |t|$

$\leq$

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Q:- $S = \{ f \in C^1[0,1] \mid f'(\tfrac{1}{2}) = 0 \}$

Q:- The set of positive semidefinite matrices

$$S = \{ A_{n\times n} \mid x^T A x \geq 0 \,,\ x^T \in \mathbb{R}^n \}$$

05 • March Sessional 02:30 – 04:30 set-1 → 3 set-2 → 3

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CHAPTER - 7

12.03.2020

COMPLETENESS

Totally bounded set :-

A set A in a metric space (M, d) is said to be totally bounded if given any $\epsilon > 0$, $\exists$ finitely many points $x_1, x_2, \dots, x_n \in M$ such that $A \subset \bigcup_{i=1}^{n} B_\epsilon(x_i)$

Ex:- $M = \{x_1, x_2, \dots, x_n\}$, $(M, \text{discrete})$

as $M \subseteq B_\epsilon(x_i)$, $i = 1, 2, \dots, n$

Lemma - 7.1

A is totally bounded iff any $\epsilon > 0$, there are finitely many sets $A_1, A_2, \dots, A_n \subset A$, with $\operatorname{diam}(A_i) < \epsilon$ for all i, such that $A \subseteq \bigcup_{i=1}^{n} A_i$

Proof:- First suppose that A is totally bounded.

Given any $\epsilon > 0$, $\exists$ finitely many x_i's $x_1, x_2, \dots, x_n \in M$ such that $A \subset \bigcup_{i=1}^{n} B_{\epsilon/2}(x_i)$

Let $A_i = A \cap B_{\epsilon/2}(x_i)$

$$\Rightarrow \bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} \left(A \cap B_{\epsilon/2}(x_i)\right)$$

$$= A \cap \left(\bigcup_{i=1}^{n} B_{\epsilon/2}(x_i)\right)$$

$$= A$$

$$\therefore A \subseteq \bigcup_{i=1}^{n} A_i$$

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$\Leftarrow$ let $\epsilon > 0$ be any number

Suppose that for any $\epsilon > 0$, there are finitely many sets $A_1, A_2, \dots, A_n \subset A$ with $\operatorname{diam}(A_i) < \epsilon$ for all i such that $A \subset \bigcup_{i=1}^{n} A_i$

Given, $x_i \in A_i$, we have $A_i \subset B_{2\epsilon}(x_i)$

For each i, $A \subset \bigcup_{i=1}^{n} B_{2\epsilon}(x_i)$

So A is totally bounded.

Ex:-2 If M is finite, (M, D) is totally bounded.

Ex:-1 $(\mathbb{N}, D) \rightarrow$ is bdd

$\mathbb{N} = \{1, 2, 3, 4, \dots\}$

Let $\epsilon = \frac{1}{2}$

Let $\mathbb{N}$ is totally bounded

i.e. $\mathbb{N} \subseteq \bigcup_{i=1}^{n} B_{1/2}(x_i)$

$B_{1/2}\{1\} = \{1\}$

which is a contradiction.

$B_{1/2}(2) = \{2\}$

So $\mathbb{N}$ is not totally bounded.

$B_{1/2}(3) = \{3\}$

$\vdots$

Ex:-3 $(\mathbb{R}, \overset{\text{Discrete}}{D}) \rightarrow$ bounded set but not totally bounded.

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Ex-4:- A totally bounded set must be bounded set.

Proof:- Let (X, d) is a totally bounded set i.e. for any $\epsilon > 0$ $\exists$ $x_1, x_2, \ldots, x_n$ such that $M \subset \bigcup_{i=1}^{n} B_\epsilon(x_i)$

Now, we have to show that M is a bounded set.

i.e. $\operatorname{diam}(M) < \infty$

$\Rightarrow \sup\{d(x,y) \mid x, y \in M\} < \infty$

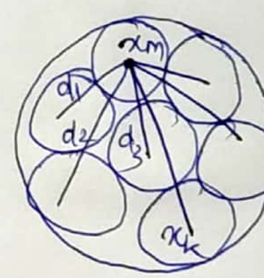
$\Rightarrow d(x,y) < \infty \ \forall\, x, y \in M$

Let $x, y \in M \Rightarrow \exists\ B(x_m, \epsilon)$ and $B(x_k, \epsilon)$

such that $x \in B(x_m, \epsilon)$ and $y \in B(x_k, \epsilon)$ for $1 \leq k, m \leq n$.

$$d(x,y) \leq d(x, x_m) + d(x_m, x_k) + d(x_k, y)$$

$$\leq 2\epsilon + d(x_m, x_k)$$



$$m = \max_{1 \leq i \leq n} \{d(x_m, x_k)\}$$

$\therefore\ d(x,y) \leq 2\epsilon + m < \infty$

$\Rightarrow d(x,y) < \infty \ \forall\, x, y \in M$

$\Rightarrow$ M is a bdd set.

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Some Useful Links:

1. Free Maths Study Materials (https://pkalika.in/2020/04/06/free-maths-study-materials/**)**

2. BSc/MSc Free Study Materials (https://pkalika.in/2019/10/14/study-material/**)**

3. MSc Entrance Exam Que. Paper: (https://pkalika.in/2020/04/03/msc-entrance-exam-paper/**)**
[JAM(MA), JAM(MS), BHU, CUCET, ...etc]

4. PhD Entrance Exam Que. Paper: (https://pkalika.in/que-papers-collection/**)**
[CSIR-NET, GATE(MA), BHU, CUCET,IIT, NBHM, ...etc]

5. CSIR-NET Maths Que. Paper: (https://pkalika.in/2020/03/30/csir-net-previous-yr-papers/**)**
[Upto 2019 Dec]

6. Practice Que. Paper: (https://pkalika.in/2019/02/10/practice-set-for-net-gate-set-jam/**)**
[Topic-wise/Subject-wise]

(Provide your Feedbacks/Comments at maths.whisperer@gmail.com)