Probability & Statistics

(Handwritten Classroom Study Material)

Submitted by
Sarojini Mohapatra
(MSc Math Student)
Central University of Jharkhand

No of Pages: 93

Your Note/Remarks

Submitted by
Sarojini Mohapatra
(MSc Math Student)
Central University of Jharkhand

Dt :- 08.01.2020

PROBABILITY & STATISTICS

Reference Books

1. Introduction to Probability & Statistics - S. Dharmaraja (6 chaps
2. Mathematical Statistics with Applications - Miller & Miller.
3. Probability & Statistics for Engineers & Scientists - S.M. Ross

40 + 60
↳ 3 sessionals (Each 20 marks)

UNIT - I

BASIC CONCEPT OF PROBABILITY

For a coin → $\{H, T\}$

For a die → $\{1, 2, 3, 4, 5, 6\}$

Experiment → Result known

Random experiment → Result is not known.

* Random experiment - An experiment is said to be random experiment if its result can not determined before hand.

Ex :- (i) An ordinary unbiased dice is rolled once.

(ii) Toss of an ordinary coin.

* Sample space - The set Ω of all possible results of a random experiment is called a sample space.

An element $\omega \in \Omega$ is called an outcome or a sample point.

Ex:- 1) Flipping of a coin, then sample space,

$\Omega = \{H, T\}$

2) When an ordinary die is rolled once,

$\Omega = \{1, 2, 3, 4, 5, 6\}$

3) When an ordinary die is rolled two times,

$$\Omega = \{(1,1), (1,2), \ldots, (1,6), \\ (2,1), (2,2), \ldots, (2,6), \\ \ldots \\ \ldots \\ (6,1), (6,2), \ldots, (6,6)\}$$

Types of Sample space

Discrete sample space – A sample space Ω is called discrete if it is either finite or countable

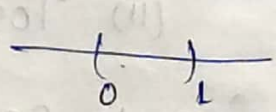
ex:- ① Toss of a coin.

② Arrival of patients in a hospital (countably infinite).

Continuous sample space:-

A sample space Ω is called continuous if it is infinite (uncountable).

ex:- Select a number between 0 & 1.



Here $\Omega = (0, 1)$.

σ - Algebra :-

Let $\Omega \neq \phi$, A collection F of subsets of Ω is called a σ - Algebra (σ - Field) over Ω if

(i) $\phi, \Omega \in F$

(ii) If $A \in F$, then $A^c \in F$

(iii) If $A_1, A_2, \dots \in F$, then $\bigcup_{i=1}^{\infty} A_i \in F$

$\left(\bigcup_{i=1}^{\infty} A_i\right)^c \in F$

$\Rightarrow \bigcap_{i=1}^{\infty} A_i^c \in F$

$\Rightarrow \bigcap_{i=1}^{\infty} B_i \in F$, where $B_i = A_i^c$

Ex:- Let $\Omega = \{H, T\}$

$F = \{\phi, \Omega\}$ and $F = P(\Omega)$ are trivial σ - algebra of Ω.

Ex:- Let $\Omega = \{HH, HT, TH, TT\}$

$F = \{\phi, \{HH, HT\}, \{TH, TT\}, \Omega\}$ is a σ - algebra.

(Ω, F) is called measurable space.

Dt :- 10.01.2020

Theorem :-

If $\Omega \neq \phi$ and $F_1, F_2, \ldots$ are σ-algebra over Ω, then $\bigcap_{i=1}^{\infty} F_i$ also a σ-algebra over Ω.

Proof :- Let $F = \bigcap_{i=1}^{\infty} F_i$

i) $\phi, \Omega \in F$, since each F_i is a σ-algebra.

ii) Let $A \in F$, then $A \in F_i \ \forall i \in \mathbb{N}$, since each F_i is a σ-algebra $\Rightarrow A^c \in F_i \ \forall i \in \mathbb{N}$

$\Rightarrow A^c \in \bigcap_{i=1}^{\infty} F_i$

$\Rightarrow A^c \in F$.

(iii) Let $A_1, A_2, \ldots \in F$.

Since each F_i is a σ-algebra

$\Rightarrow \bigcup_{i=1}^{\infty} A_i \in F_i \ \forall i$

$\Rightarrow \bigcup_{i=1}^{\infty} A_i \in \bigcap_{i=1}^{\infty} F_i$

$\Rightarrow \bigcup_{i=1}^{\infty} A_i \in F$.

$\therefore F = \bigcap_{i=1}^{\infty} F_i$ is a σ-algebra over Ω.

H.W $F_1 \cup F_2$ need not be a σ-algebra.

Borrel σ- algebrra :-

The smallest σ-algebra over $\mathbb{R}$ containing all intervals of the form $(-\infty, a]$ with $a \in \mathbb{R}$ is called the Borrel σ-algebra. It is written as $\mathcal{B}$.

↳ If $A \in \mathcal{B}$, then A is called a borel subset of $\mathbb{R}$.

$(\mathbb{R}, \mathcal{B})$, since $\mathcal{B}$ is a σ-algebra

$$(-\infty, a] \in \mathcal{B} \Rightarrow (a, \infty) \in \mathcal{B} \quad [\text{by 2nd property}]$$

Let $a, b \in \mathbb{R}$ with $a < b$, then we have some results

$$(a, b] = (-\infty, b] \cap (a, \infty)$$

$$(-\infty, a) = \bigcup_{i=1}^{\infty} (-\infty, a - \tfrac{1}{n})$$

$$(a, b) = (a, \infty) \cap (-\infty, b)$$

$$[a, b] = \mathbb{R} \setminus \{(-\infty, a) \cup (b, \infty)\}$$

Measurrable Space :-

Let $\Omega \neq \phi$ and F be a σ-algebra over Ω. The pair (Ω, F) is called a measurrable space.

Event :- If $A \in F$, then A is called an event, or an event is a subset of samplespace.

Incident :-

Mutually Exclusive Event :-

Two events A and B are said to be mutually exclusive if $A \cap B = \phi$.

Ex:- 1) A coin is flipped once.

A: the result obtained is head

B: the result obtained is tail

$\Omega = \{H, T\}$ $A = \{H\}$, $B = \{T\}$

$A \cap B = \phi$

$\therefore$ Events A and B are mutually exclusive.

2) A die is rolled once

Let A: the result obtained an odd number.

B: the result obtained an even number.

$\Omega = \{1, 2, 3, 4, 5, 6\}$

$A = \{1, 3, 5\}$

$B = \{2, 4, 6\}$

$\therefore A \cap B = \phi$

$\Rightarrow$ A and B are mutually exclusive events.

* **Probability Space :-**

Let (Ω, F) be a measurable space. A real valued function P defined on F satisfying the following conditions:

(i) $P(A) \geq 0 \;\forall\; A \in F$ (non-negative property)

(ii) $P(\Omega) = 1$ (Normed property)

(iii) If $A_1, A_2, \ldots$ are mutually exclusive events in F i.e. $A_i \cap A_j = \phi \;\forall\; i \neq j$, then

$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$ (Countable additivity) is called a probability measure over (Ω, F).

The triplet (Ω, F, P) is called a probability space.

$\boxed{0 \leq P \leq 1 \text{ always}}$

$\boxed{P \text{ is not a symbol, it is a function.}}$

13.01.20

Ex:-1 Let $\Omega = \{1, 2, 3\}$

$$F = \{\phi, \{1\}, \{2,3\}, \Omega\}$$

$$P(A) = \begin{cases} 1, & \text{if } 3 \in A \\ 0, & \text{if } 3 \notin A. \end{cases}$$

A:- Clearly F is a σ-algebra.

(i) $P(A) \geq 0 \ \forall A \subseteq \Omega$.

(ii) $P(\Omega) = 1 \quad (\because 3 \in \Omega)$

(iii) $A_1 = \{1\}$, $A_2 = \{2,3\}$, $A_3 = \phi$

$\Rightarrow A_1 \cup A_2 \cup A_3 = \Omega$

$P\left(\bigcup_{i=1}^{3} A_i\right) = 1$

$P(A_1) + P(A_2) + P(A_3) = 0 + 1 + 0 = 1$.

$\therefore P\left(\bigcup_{i=1}^{3} A_i\right) = \sum_{i=1}^{3} P(A_i)$

$\therefore P$ is a probability function.

Ex:-2 Let $\Omega = \{1,2\}$, $F = P(\Omega) = \{\phi, \{1\}, \{2\}, \{1,2\}\}$

$$P(A) = \begin{cases} 0, & \text{if } A = \phi \\ 1/3, & \text{if } A = \{1\} \\ 2/3, & \text{if } A = \{2\} \\ 1, & \text{if } A = \{1,2\} \end{cases}$$

Clearly F is a σ-algebra.

H.W Show that $P(A)$ is a probability function.

(i) $P(A) \geq 0 \quad \forall A \subseteq \Omega$

(ii) $P(\Omega) = 1$ (given)

(iii) Let $A_1 = \phi$, $A_2 = \{1\}$, $A_3 = \{2\}$

$A_1 \cap A_2 \cap A_3 = \phi$

$P(\bigcup_{i=1}^{3} A_i) = 1$

$P(\phi) = 0$

$P(\{1\}) = \frac{1}{3}$

$P(\{2\}) = 2/3$

$\therefore P(\phi) + P(\{1\}) + P(\{2\}) = 0 + \frac{1}{3} + \frac{2}{3} = 1$.

$\therefore P(\bigcup_{i=1}^{3} A_i) = \sum_{i=1}^{3} P(A_i)$.

$\therefore P(A)$ is a probability function.

Theorem - 1.2 :-

Let (Ω, F, P) be a probability space, then

(i) $P(\phi) = 0$

(ii) If $A, B \in F$ such that $A \cap B = \phi$, then
$P(A \cup B) = P(A) + P(B)$.

(iii) For any $A \in F$, $P(A^c) = 1 - P(A)$

(iv) For any $A, B \in F$, we have
$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Proof:-

(i) $\Omega = \Omega \cup \phi \cup \phi \cup \ldots$

$P(\Omega) = P(\Omega) + P(\phi) + P(\phi) + \ldots$

$\Rightarrow 1 = 1 + n P(\phi) + \ldots$

$\Rightarrow P(\phi) = 0$

(ii) $A \cup B = A \cup B \cup \phi \cup \phi \cup \ldots$

$\Rightarrow P(A \cup B) = P(A) + P(B) + P(\phi) + P(\phi) + \ldots$

$\Rightarrow P(A \cup B) = P(A) + P(B)$.

(iii) $\Omega = A \cup A^c$ & $A \cap A^c = \phi$

$\Rightarrow P(\Omega) = P(A \cup A^c)$

$\Rightarrow P(\Omega) = P(A) + P(A^c)$

$\Rightarrow 1 = P(A) + P(A^c)$

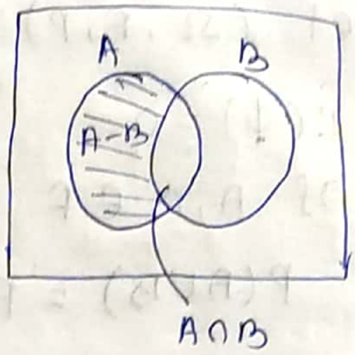
$\Rightarrow P(A^c) = 1 - P(A)$.

(iv) $A = (A \setminus B) \cup (A \cap B)$

$\Rightarrow P(A) = P(A \setminus B) + P(A \cap B)$ — (i)

$B = (B \setminus A) \cup (A \cap B)$

$\Rightarrow P(B) = P(B \setminus A) + P(A \cap B)$ — (ii)



Adding eq (i) & (ii) we get

$$P(A) + P(B) = \{P(A \setminus B) + P(B \setminus A) + P(A \cap B)\} + P(A \cap B)$$

$$= P(A \cup B) + P(A \cap B)$$

$$\Rightarrow P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

* If $A = \phi \Rightarrow P(A) = 0$

But if $P(A) = 0$, then A may or may not empty.

Ex:- $\Omega = [0,1]$

A = Choose a no. in $[0,1]$

$P(A) = \frac{1}{\infty} \to 0.$

* Conditional Probability :-

Let (Ω, F, P) be a probability space. If $A, B \in F$ with $P(A) > 0$, then the probability of the event B under the condition A is defined as follows

$$P(B \mid A) := \frac{P(A \cap B)}{P(A)}$$

Ex:- Two fair dice are rolled once. The probability that at least one of the results is 6 given that the results obtained are different equals $1/3$.

$A = \{(a,b) \mid \text{Either } a=6 \text{ or } b=6\}$

$|A| = 11$

$B = \{(a,b) \mid a \neq b\}$

$|B| = 30$

$$\Rightarrow P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

$|A \cap B| = 10$

$$P(A \mid B) = \frac{10/36}{30/36} = \frac{1}{3}$$

Dt:- 14.01.2020

Theorem :-

Let (Ω, F, P) be a probability space and let $A \in F$ with $P(A) > 0$, then

1) $P(A|A)$ is a probability measure over Ω centred on A.

$$P(A|A) = \frac{P(A \cap A)}{P(A)} = \frac{P(A)}{P(A)} = 1.$$

2) If $A \cap B = \phi$, then $P(B|A) = 0$

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(\phi)}{P(A)} = \frac{0}{P(A)} = 0$$

3) $P(B \cap C | A) = P(B | A \cap C) \cdot P(C|A)$ if $P(A \cap C) > 0$

$$LHS = P(B \cap C | A) = \frac{P(A \cap B \cap C)}{P(A)}$$

$$RHS = P(B | A \cap C) \cdot P(C|A)$$

$$= \frac{P(A \cap B \cap C)}{P(A \cap C)} \cdot \frac{P(C \cap A)}{P(A)} = \frac{P(A \cap B \cap C)}{P(A)}$$

$\therefore$ LHS = RHS. $\square$

4) If $A_1, A_2, \ldots, A_n \in F$ with $P(A_1 \cap A_2 \cap \ldots \cap A_n) > 0$,

then $P(A_1 \cap A_2 \cap \ldots \cap A_n) = P(A_1) \cdot P(A_2|A_1) \cdots P(A_n|A_1 \cap A_2 \cap \ldots \cap A_{n-1})$

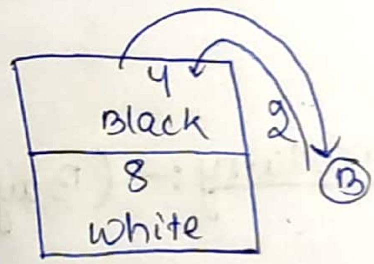
Proof:- $P(A_1 \cap A_2 \cap \ldots \cap A_n) = P(A_n|(A_1 \cap A_2 \cap \ldots \cap A_{n-1})) \cdot P(A_1 \cap \ldots \cap A_{n-1})$

$= P(A_n|A_1 \cap A_2 \cap \ldots \cap A_{n-1}) \cdot P(A_{n-1}|A_1 \cap \ldots \cap A_{n-2}) \cdot P(A_1 \cap \ldots \cap A_{n-2})$

continuing in this manner we get

$= P(A_n|A_1 \cap A_2 \cap \ldots \cap A_{n-1}) \cdot P(A_{n-1}|A_1 \cap \ldots \cap A_{n-2}) \cdots P(A_2|A_1) \cdot P(A_1)$

Ex:- $A_i \rightarrow$ drawn ball's color is black at ith drawn



$P(A_1 \cap A_2 \cap A_3) = P(A_1) \cdot P(A_2|A_1) \cdot P(A_3|A_1 \cap A_2)$

$= \frac{4}{12} \cdot \frac{6}{14} \cdot \frac{8}{16}$

$= \frac{1}{3} \cdot \frac{3}{7} \cdot \frac{1}{2} = \frac{1}{14}$.

Total Probability Theorem :-

Let $A_1, A_2, \ldots$ be a finite or countable partition of Ω i.e. $A_i \cap A_j = \phi \ \forall \ i \neq j$ and $\bigcup_{i=1}^{\infty} A_i = \Omega$, such that $P(A_i) > 0$, $\forall A_i \in F$, Then for any $B \in F$

$$P(B) = \sum_{i} P(B|A_i) \cdot P(A_i)$$

Proof:- By the observation, $B = B \cap \Omega$

$$= B \cap \left(\bigcup_{i=1}^{\infty} A_i\right)$$

$$= \bigcup_{i=1}^{\infty} (B \cap A_i)$$

$$\Rightarrow P(B) = \sum_{i=1}^{\infty} P(B \cap A_i) \quad \left[\because B \cap A_i = \phi \text{ As } (B \cap A_i) \cap (B \cap A_j) = B \cap (A_i \cap A_j) = B \cap \phi = \phi\right]$$

$$= \sum_{i=1}^{\infty} P(B|A_i) \cdot P(A_i)$$

Corollary:- (Bay's Rule)

Let $A_1, A_2, \ldots$ be a finite or countable partition of Ω with $P(A_i) > 0$, $\forall i$, Then for any $B \in F$ with $P(B) > 0$

$$P(A_i|B) = \frac{P(A_i) \cdot P(B|A_i)}{P(B)}$$

Proof:-

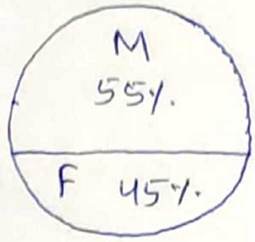
$$P(A_i|B) = \frac{P(A_i \cap B)}{P(B)} = \frac{P(B|A_i) \cdot P(A_i)}{P(B)}$$

$$= \frac{P(A_i) \cdot P(B|A_i)}{P(B)}$$

Ex:- It is known that the population of a certain city consists of 45% females and 55% males. Suppose that 70% of the male and 10% of the female smoke. Find the probability that ~~a smoker is male~~ (i) a person is smoker

(ii) a smoker is male

Soln:-



$P(M) = 0.55$

$P(F) = 0.45$

$P(S|M) = 0.70$

$P(S|F) = 0.10$

Let "S" denote a person is smoker

"M" " " " " male.

"F" " " " " female.

(i) $P(S) = P(S \cap M) + P(S \cap F)$

$= P(S|M) \cdot P(M) + P(S|F) \cdot P(F)$

$= (0.70)(0.55) + (0.10)(0.45)$

$= 0.3850 + 0.0450$

$= 0.4300$

$\therefore \quad P(S) = 0.43$

(ii) $P(M|S) = \dfrac{P(M) \cdot P(S|M)}{P(S)}$

$= \dfrac{(0.55)(0.70)}{0.43} = \dfrac{0.3850}{0.43} = 0.895$

(iii) Probability of a smoker is female.

$$P(F|S) = \frac{P(F)\cdot P(S|F)}{P(S)}$$

$$= \frac{(0.45)(0.10)}{0.43} = \frac{0.045}{0.43} = 0.104$$

Dt:- 20.01.2020

Ex:- Vivek knows that there is a chance of 40% that the company he works with will open a branch office in Delhi. If that happens the prob. that he will be as the manager in that branch office is 80%. If not prob. that he will be promoted as a manager to another office is only 10%. Find the prob. that Vivek will be appointed as a manager of a Branch office from his company.

Sol^n:- M = Vivek will become a manager

N = The company will open a new branch office in Delhi

$$P(M) = P(M\cap N) + P(M\cap N^c)$$

$$= P(M|N)\cdot P(N) + P(M|N^c)\cdot P(N^c)$$

$$= 0.8 \times 0.4 + 0.10 \times 0.6$$

$$= 0.32 + 0.06$$

$$= 0.38$$

Q:- What is the prob. that the company will open a new office in Delhi given that Vivek is appointed as the branch manager ?

$$P(N|M) = \frac{P(N \cap M)}{P(M)} = \frac{P(M|N) \cdot P(N)}{P(M)}$$

$$= \frac{0.32}{0.38} = 0.84211$$

Q:- A signal can be green or red with prob. $\frac{4}{5}$ or $\frac{1}{5}$ respectively. The probability that it received correctly by a station is $3/4$ of the two stations A and B, the signal is first received by station A and passes the signal to station B. If the signal received at the station B is green, then find the probability that the $\underbrace{\text{original signal}}_{\text{signal rcvd at A}}$ was green.

Sol^n:- $\boxed{S} \longrightarrow \boxed{A} \longrightarrow \boxed{B}$

$B_G \rightarrow$ Signal received at the station B is green.
$B_R \rightarrow$ " " " " B is red.
$A_G \rightarrow$ " " " " A is green.
$A_R \rightarrow$ " " " " A is red.

$$P(A_G|B_G) = \frac{P(A_G \cap B_G)}{P(B_G)}$$

$$P(A_G \cap B_G) = \frac{4}{5} \cdot \frac{3}{4} \cdot \frac{3}{4} + \frac{1}{5} \cdot \frac{1}{4} \cdot \frac{3}{4}$$

$$P(B_G) = \frac{4}{5} \cdot \frac{3}{4} \cdot \frac{3}{4} + \frac{1}{5} \cdot \frac{1}{4} \cdot \frac{3}{4} + \frac{4}{5} \cdot \frac{1}{4} \cdot \frac{1}{4} + \frac{1}{5} \cdot \frac{3}{4} \cdot \frac{1}{4}$$

* Independent Events :-

Two events A and B are said to be independent if $P(A \cap B) = P(A) \cdot P(B)$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$\Rightarrow P(A \cap B) = P(A|B) \cdot P(B)$$

$$= P(A) \cdot P(B)$$

$$P(A|B) = P(A).$$

Ex:-1 Suppose a fair die is rolled two times

A = "The sum of the results obtained is an even number".

B = "The result from the second roll is even number"

$A = \{(1,1), (1,3), (1,5), (2,2), (2,4), (2,6), (3,1), (3,3), (3,5), (4,2), (4,4), (4,6), (5,1), (5,3), (5,5), (6,2), (6,4), (6,6)\}$

$P(A) = \frac{18}{36} = \frac{1}{2}$

$|B| = 18$

$P(B) = \frac{1}{2}$

$|A \cap B| = 9$

$P(A \cap B) = \frac{9}{36} = \frac{1}{4}$

$P(A \cap B) = \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = P(A) \cdot P(B)$

$\therefore$ Events A and B are independent.

<u>Ex:-</u> A coin is flipped two times

A = "Tail is obtain on the first place"

B = " " " second place".

$A = \{(T,T), (T,H)\}$

$B = \{H, T, (T,T)\}$

$P(A) = \frac{1}{2}$, $P(B) = \frac{1}{2}$

$P(A \cap B) = \frac{1}{4} = P(A) \cdot P(B)$

$\therefore$ A and B are independent events.

<u>Remark:-</u> If two events are mutually exclusive, then these events need not be independent.

Ex:- A coin is flipped once and we consider

A = result obtained is head

B = " " is tail

$A = \{H\}$

$B = \{T\}$

$A \cap B = \phi$

$P(A) = P(B) = \frac{1}{2}$

$P(A \cap B) = 0$

$\therefore P(A \cap B) \neq P(A) \cdot P(B)$

$\therefore$ A and B are not independent events.

Theorem:-

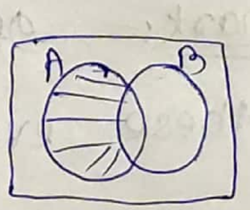
Let A and B be independent events, then

(1) A and B^c are two independent events.

(2) A^c and B are two independent events

(3) A^c and B^c are two independent events

Proof:-



(1) $P(A \cap B^c) = P(A \setminus A \cap B)$

$= P(A) - P(A \cap B)$

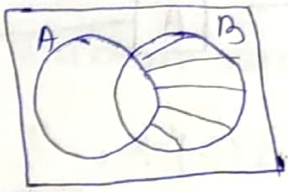
$= P(A) - P(A) \cdot P(B)$

$= P(A)[1 - P(B)] = P(A) \cdot P(B^c)$.

$\Rightarrow P(A \cap B^c) = P(A) \cdot P(B^c)$

$\therefore$ A and B^c are independent events

(2) $P(A^c \cap B) = P(B \setminus A \cap B)$



$$= P(B) - P(A \cap B)$$
$$= P(B) - P(A) \cdot P(B)$$
$$= [1 - P(A)]\, P(B)$$
$$= P(A^c) \cdot P(B)$$

$\therefore$ A^c and B are independent events.

(3) $P(A^c \cap B^c) = P\{(A \cup B)^c\}$

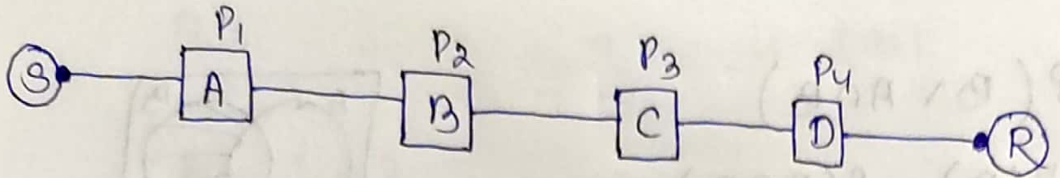
$$= P\{E - (A \cup B)\}$$
$$= P(E) - P(A \cup B)$$
$$= 1 - P(A \cup B)$$
$$= 1 - P(A) - P(B) + P(A \cap B)$$
$$= 1 - P(A) - P(B) + P(A) \cdot P(B)$$
$$= 1 - P(A) - P(B)(1 - P(A))$$
$$= \cancel{P(A^c)\ \text{or}\ P}\{1 - P(A)\}\{1 - P(B)\}$$
$$= P(A^c) \cdot P(B^c)$$

$\therefore$ A^c and B^c are independent events.

21.01.2020

* Theory of Reliability :-

Reliable → dependence.



Series system

$$P(A \cap B \cap C \cap D)$$
$$= P(A) \cdot P(B) \cdot P(C) \cdot P(D)$$
$$= P_1 P_2 P_3 P_4 = \prod_{i=1}^{4} P_i$$

Parallel system

$$P(A \cap (B \cup C \cup D) \cap (E \cup F) \cap G)$$
$$= P(A) \cdot P(B \cup C \cup D) \cdot P(E \cup F) \cdot P(G)$$

CHAPTER - 2

Random variable :-

Let (Ω, F, P) be a probability space. A random variable is a mapping $X : \Omega \to \mathbb{R}$ such that $\forall A \in \mathcal{B}$, $X^{-1}(A) \in F$, where $\mathcal{B}$ is the Borel σ-algebra over $\mathbb{R}$ i.e.

$X : \Omega \to \mathbb{R}$ is r.v iff $\boxed{X^{-1}((-\infty, x]) \in F}$ $\forall x \in \mathbb{R}$.

Ex:- Let $\Omega = \{a, b, c\}$

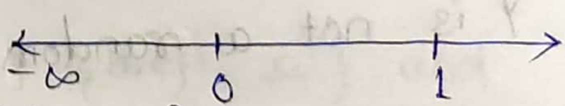
$F = \{\phi, \{a\}, \{b, c\}, \Omega\}$,

P be an arbitarily probability measure defined over F. Assume $X : \Omega \to \mathbb{R}$ given by

$$X(\omega) = \begin{cases} 0, & \text{if } \omega = a \\ 1, & \text{if } \omega = b \text{ or } c \end{cases}$$

Check this is r.v or not.

Sol^n:- If $x < 0$



$X^{-1}((-\infty, 0)) = \{\omega \in \Omega \mid X(\omega) \in (-\infty, 0)\}$

If $x = 0$ $\quad = \phi \in F$

$X^{-1}((-\infty, 0]) = \{\omega \in \Omega \mid X(\omega) \in (-\infty, 0]\}$

If $x = 1$ $\quad = \{a\} \in F$

$X^{-1}((-\infty, 1]) = \{\omega \in \Omega \mid X(\omega) \in (-\infty, 1]\}$

$= \{a, b, c\} \in F$

~~∴ X is a random variable.~~

If $x > 1$

$$X^{-1}((-\infty, \infty)) = \Omega \in F$$

$$\therefore X^{-1}((-\infty, x)) \in F \ \forall \ x \in \mathbb{R}.$$

∴ X is a random variable.

Ex:- $Y : \Omega \longrightarrow \mathbb{R}$, (Ω, F, P) same as above

$$Y(\omega) = \begin{cases} 0, & \text{if } \omega = b \\ 1 & \text{if } \omega = a \text{ or } c. \end{cases}$$

If $x < 0$

$$Y^{-1}((-\infty, x]) = \phi \in F$$

$x = 0$

$$Y^{-1}((-\infty, x]) = \{b\} \notin F$$

∴ Y is not a random variable.

Ex:- Let (Ω, F, P) be a probability space and $A \in F$ be fixed. The function $X_A : \Omega \to \mathbb{R}$

$$X_A(\omega) = \begin{cases} 1, & \text{if } \omega \in A \\ 0, & \text{if } \omega \notin A \end{cases}$$

Sol:- If $x<0$

$$X_A^{-1}((-\infty,x]) = \phi \in F$$

If $x=0$

$$X_A^{-1}((-\infty,x]) = A^c \in F$$

If $x=1$

$$X_A^{-1}((-\infty,x]) = \Omega \in F$$

If $x>1$

$$X_A^{-1}((-\infty,x]) = \Omega \in F$$

$\therefore$ X_A is a random variable.

Notation:-

↳ Let X be random variable defined over the prob. space (Ω, F, P). It is also defined set Notation form

$$\{X \in B\} = \{\omega \in \Omega \mid X(\omega) \in B\} \text{ with } B \in \mathcal{B}$$

and $P_X(B) = P(\{X \in B\}) = P(\{\omega \in \Omega \mid X(\omega) \in B\}) \ \forall \ B \in \mathcal{B}$

Ex:- Let $\Omega = \{a,b,c\}$, $F = \{\phi, \{a\}, \{b,c\}, \Omega\}$ and

$P(\phi) = 0$

$P(\{a\}) = \frac{1}{5}$

$P(\{b,c\}) = \frac{4}{5}$

$P(\Omega) = 1$

Let $X : \Omega \to \mathbb{R}$ be given by

$$X(\omega) = \begin{cases} 1, & \text{if } \omega = a \\ 2, & \text{if } \omega = b \text{ or } c \end{cases}$$

$P_X(\phi) = P(\phi) = 0 = P\{\omega \in \Omega \mid X(\omega) \in \phi\}$

$P_X(\{1\}) = P(\{a\}) = \frac{1}{5}$

$P_X(\{2\}) = P(\{b, c\}) = \frac{4}{5}$

$P_X(\mathbb{R}) = P(\Omega) = 1$.

Distribution Function :- (CDF)

Let X be a r.v. The function F_X defined over $\mathbb{R}$ as

$F_X(x) := P_X((-\infty, x]) = P(X \leq x)$ is called the distribution function of the RV X

$F_X : \mathbb{R} \to [0, 1]$

* Distribution funⁿ is unique regarding to r.v X.

Ex:- Let $\Omega = \{a, b, c\}$, $F = \{\phi, \{a\}, \{b, c\}, \Omega\}$ and P be given by,

$P(\phi) = 0$

$P(\{a\}) = \frac{1}{5}$

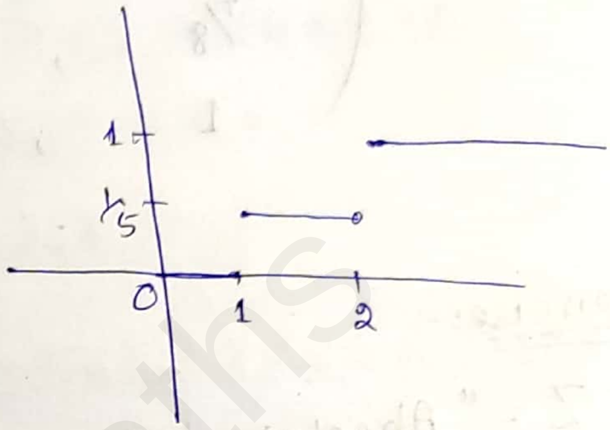
$P(\{b, c\}) = \frac{4}{5}$

$P(\Omega) = 1$

Let $X : \Omega \to \mathbb{R}$ st

$$X(\omega) = \begin{cases} 1, & \text{if } \omega = a \\ 2, & \text{if } \omega = b \text{ or } c. \end{cases}$$

$$F_X(x) = \begin{cases} 0 & , x < 1 \\ \frac{1}{5} & , 1 \le x < 2 \\ 1 & , x \ge 2 \end{cases}$$



Ex:- Consider the tossing of a fair coin three times and let X be a random variable defined by number of heads obtained.

X = "No. of heads obtained"

Soln:- $\Omega = \{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\}$

$X : \Omega \to \mathbb{R}$

$$X(\omega) = \begin{cases} 0, & \omega = \{TTT\} \\ 1, & \omega = \{TTH, HTT, THT\} \\ 2, & \omega = \{HHT, HTH, THH\} \\ 3, & \omega = \{HHH\} \end{cases}$$

$$F_X(x) = \begin{cases} 0 & , \quad x < 0 \\ \frac{1}{8} & , \quad 0 \le x < 1 \\ 4/8 & , \quad 1 \le x < 2 \\ 7/8 & , \quad 2 \le x < 3 \\ 1 & , \quad x \geq 3 \end{cases}$$

Exercise:-

Z = " Absolute diff of the result obtained"

* Properties of distribution function :-

Let X be a RV defined over (Ω, F, P). The distribution function F_X satisfies the following conditions

1) If $x \leq y$, then $F_X(x) \leq F_X(y)$

2) $F_X(x^+) = \lim_{h \to 0^+} F_X(x+h) = f_X(x) \quad \forall x \in \mathbb{R}$

3) $\lim_{x \to -\infty} F_X(x) = 0$

4) $\lim_{x \to \infty} F_X(x) = 1$.

Some important results:-

1) $F_X(x^-) = \lim_{h \to 0^+} F_X(x-h) = P(X < x)$

2) $P(a \le X \le b) = F_X(b) - F_X(a^-)$

3) $P(a < X \le b) = F_X(b) - F_X(a)$

4) $P(a \le X < b) = F_X(b^-) - F_X(a^-)$

5) $P(a < x < b) = F_X(b^-) - F_X(a)$

6) $P(X=a) = F_X(a) - F_X(a^-)$

7) If $P(a<X<b) = 0$, then F_X is constant interval (a,b).

Proof:- 2)

$\Omega = \{\omega \in \Omega \mid X(\omega) < a\} \cup \{\omega \in \Omega \mid a \le X(\omega) \le b\} \cup \{\omega \in \Omega \mid X(\omega) > b$

$1 = P(X<a) + P(a \le X \le b) + P(X > b)$

$\Rightarrow P(a \le X \le b) = 1 - P(X>b) - P(X<a)$

$= P(X \le b) - P(X<a)$

$= F_X(b) - F_X(a^-)$.

3) $\Omega = \{\omega \in \Omega \mid X(\omega) \le a\} \cup \{\omega \in \Omega \mid a < X(\omega) \le b\} \cup \{\omega \in \Omega \mid X(\omega) > b$

$1 = P(X \le a) + P(a < X \le b) + P(X > b)$

$\Rightarrow P(a < X \le b) = 1 - P(X > b) - P(X \le a)$

$= P(X \le b) - P(X \le a)$

$= F_X(b) - F_X(a)$.

4) $P(a \le X < b) = F_X(b^-) - F_X(a^-)$

Proof:- $\Omega = \{\omega \in \Omega \mid X(\omega) < a\} \cup \{\omega \in \Omega \mid a \le X(\omega) < b\} \cup \{\omega \in \Omega \mid X(\omega) \geq b\}$

$\Rightarrow 1 = P(X < a) + P(a \le X < b) + P(X \geq b)$

$\Rightarrow P(a \le X < b) = 1 - P(X \geq b) - P(X < a)$

$= P(X < b) - P(X < a)$

$= F_X(b^-) - F_X(a^-)$

5) $P(a < X < b) = F_X(b^-) - F_X(a)$

Proof:-

$\Omega = \{\omega \in \Omega \mid X(\omega) \le a\} \cup \{\omega \in \Omega \mid a < X(\omega) < b\} \cup \{\omega \in \Omega \mid X(\omega) \geq b\}$

$\Rightarrow 1 = P(X \le a) + P(a < X < b) + P(X \geq b)$

$\Rightarrow P(a < X < b) = 1 - P(X \geq b) - P(X \le a)$

$= P(X < b) - P(X \le a)$

$= F_X(b^-) - F_X(a)$

6) $P(X = a) = F_X(a) - F_X(a^-)$

Proof:-

$\Omega = \{\omega \in \Omega \mid X(\omega) < a\} \cup \{\omega \in \Omega \mid X(\omega) = a\} \cup \{\omega \in \Omega \mid X(\omega) > a\}$

$\Rightarrow 1 = P(X < a) + P(X = a) + P(X > a)$

$\Rightarrow P(X = a) = 1 - P(X > a) - P(X < a)$

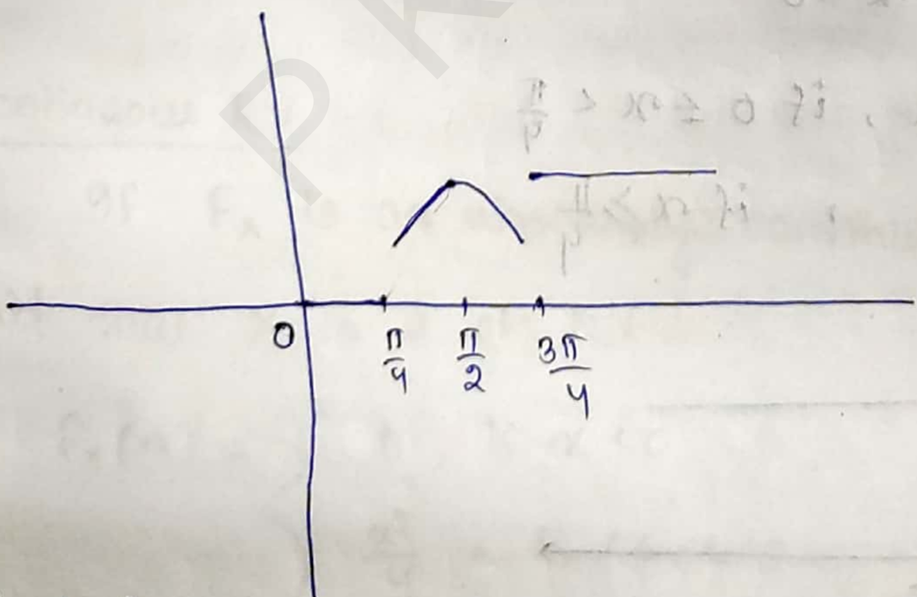
$= P(X \le a) - P(X < a)$

$= F_X(a) - F_X(a^-)$

22.01.20

Ex:- 7(a)

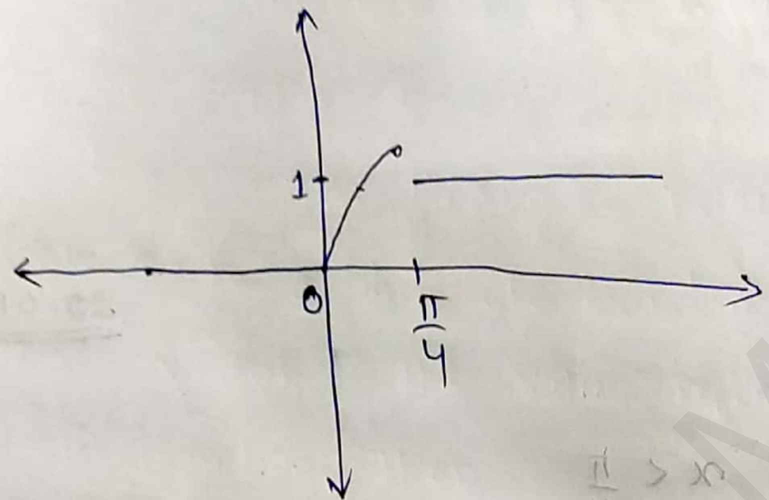
$$F(x) = \begin{cases} 0, & \text{if } x < \frac{\pi}{4} \\ \sin x, & \text{if } \frac{\pi}{4} \le x < \frac{3\pi}{4} \\ 1, & \text{if } x \ge \frac{3\pi}{4} \end{cases}$$



F(x) is not monotonic so F(x) is not a distribution function.

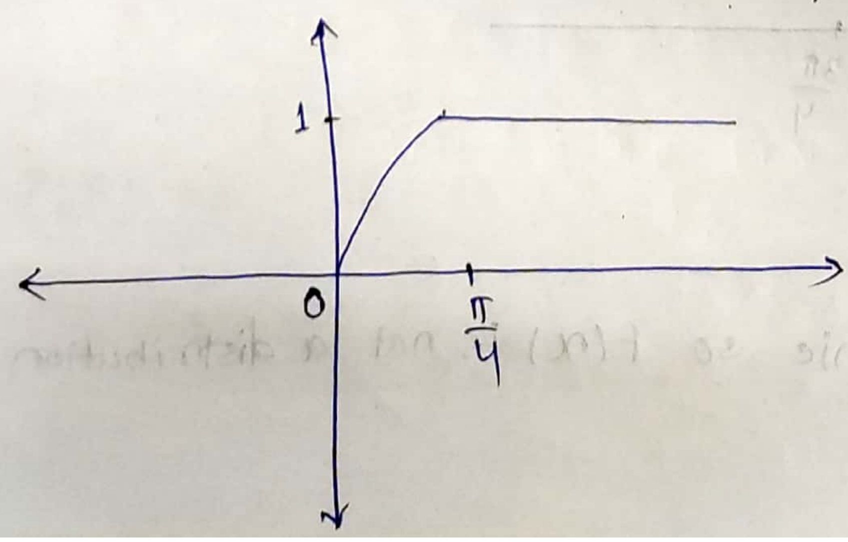
(b)

$$F(x) = \begin{cases} 0, & \text{if } x < 0 \\ 2\sin x, & 0 \le x < \frac{\pi}{4} \\ 1, & x \geq \frac{\pi}{4} \end{cases}$$



Since $F(x)$ is not monotonically increasing, so $F(x)$ is not a distribution function.

(c) $$F(x) = \begin{cases} 0, & \text{if } x < 0 \\ \sqrt{2}\sin x, & \text{if } 0 \le x < \frac{\pi}{4} \\ 1, & \text{if } x \geq \frac{\pi}{4} \end{cases}$$



Since $F(x)$ is cont. fun^n so prop-(2) satisfied.

(2) $F(x)$ is monotonically increasing fun^n

(3) $\lim_{x \to -\infty} F(x) = 0$

(4) $\lim_{x \to \infty} F(x) = 1$.

So $F(x)$ is a distribution function.

Random variables in terms of distribution function :-

Discrete rv :-

If the dist. fun^n F_X of the random variable X is a step fun^n then it is said that X is a Discrete r.V.

Ex:- $F_X(x) = \begin{cases} 0, & \text{if } x < -\sqrt{2} \\ 3/5, & \text{if } -\sqrt{2} \leq x < \pi \\ 1, & \text{if } x \geq \pi. \end{cases}$

Continuous RV :-

If F_X is a ~~absolutely~~ continuous function, then it is said that X is a cts. RV.

Ex:- $F_X(x) = \begin{cases} 0, & \text{if } x < 0 \\ \frac{x^2}{4}, & \text{if } 0 \leq x < 2 \\ 1, & \text{if } x \geq 2. \end{cases}$

Discrete RV :-

Def^n :- Let X be a RV and F_X be its DF. It is said that F_X presents a jump at a point $a \in \mathbb{R}$ if $F_X(a) - F_X(a^-) \neq 0$.

The difference $F_X(a) - F_X(a^-)$ is called the magnitude of the jump.

We define $P(X=a) = F_X(a) - F_X(a^-)$

Let X be a discrete random variable and suppose that X takes the values $x_1, x_2, \ldots$. Let x be a real number, then

$$F_X(x) = P(X \le x) = P\left(\bigcup_{x_i \le x} X = x_i\right) = \sum_{x_i \le x} P(X = x_i)$$

Probability Mass Function (PMF) :-

Def^n :- Let X be a discrete random variable with values $x_1, x_2, \ldots$. The function P_X defined in $\mathbb{R}$ through

$$P_X(x) = \begin{cases} P(X = x_i) & \text{if } x = x_1, x_2, \ldots, x_n, \ldots \\ 0 & , \text{otherwise} \end{cases}$$

is called a PMF of the DRV.

↳ The following properties hold for the PMF :

(i) $P(x_i) \geq 0 \ \forall \ i$

(ii) $\sum_i P(x_i) = 1$.

Ex:- Suppose that a fair die is rolled once and let X be a RV that indicates the result obtained. Then the PMF of X is given by

$\Omega = \{1, 2, 3, 4, 5, 6\}$

$$X = \begin{cases} 1, & \text{if } \Omega = 1 \\ 2, & \text{if } \Omega = 2 \\ 3, & \text{if } \Omega = 3 \\ 4, & \text{if } \Omega = 4 \\ 5, & \text{if } \Omega = 5 \\ 6, & \text{if } \Omega = 6 \end{cases}$$

$$P_X(x) = \begin{cases} \frac{1}{6}, & x = 1, 2, 3, 4, 5, 6 \\ 0, & \text{otherwise} \end{cases}$$

$$F_X(x) = \begin{cases} 0, & x < 1 \\ \frac{1}{6}, & 1 \leq x < 2 \\ \frac{2}{6}, & 2 \leq x < 3 \\ 3/6, & 3 \leq x < 4 \\ 4/6, & 4 \leq x < 5 \\ 5/6, & 5 \leq x < 6 \\ 1, & x \geq 6 \end{cases}$$

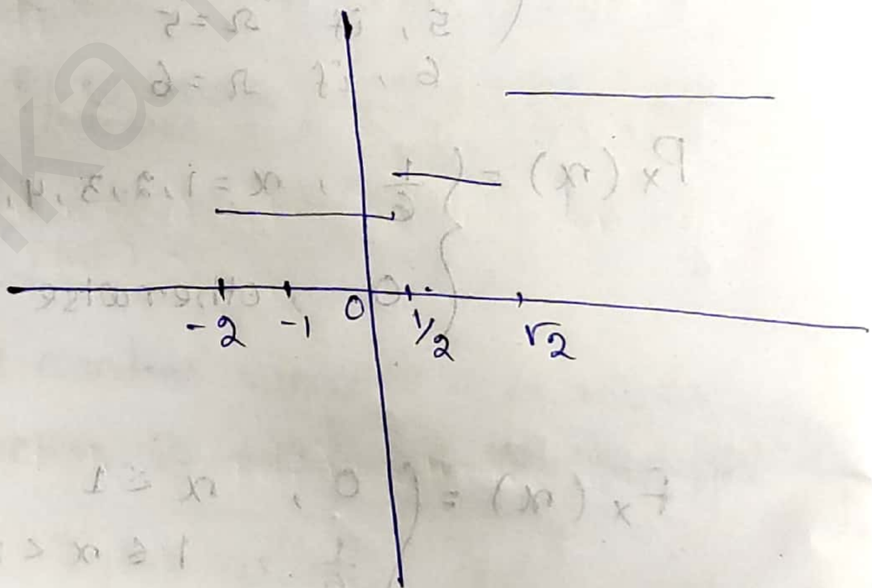
$$P(X \leq 3) = P(X=1) + P(X=2) + P(X=3)$$

$$= \frac{1}{6} + \frac{1}{6} + \frac{1}{6}$$

$$= \frac{3}{6}$$

Ex:-

$$F_X(x) = \begin{cases} 0, & \text{if } x < -2 \\ \frac{1}{7}, & \text{if } -2 \leq x < \frac{1}{2} \\ \frac{4}{7}, & \text{if } \frac{1}{2} \leq x < \sqrt{2} \\ 1, & \text{if } x \geq \sqrt{2} \end{cases}$$



$$P_X(x) = \begin{cases} \frac{1}{7}, & x = -2 \\ 3/7, & x = \frac{1}{2} \\ 3/7, & x = \sqrt{2} \\ 0, & \text{otherwise} \end{cases}$$

$$P_X(X=a) = F_X(a) - F_X(a^-)$$

$$\Rightarrow F_X(a) = P(X=a) + F_X(a^-)$$

$$\Rightarrow \boxed{F_X(a) = P(X=a) + P(X<a)}$$

24.01.2020

Ex:- Let X be a discrete RV with values $\{0, \pm 1, \pm 2\}$. Suppose $P(X=-2) = P(X=-1)$ and $P(X=1) = P(X=2)$ with the information that $P(X>0) = P(X<0) = P(X=0)$. Find the pmf and the distribution function of the random variable X.

A:-

$$P(X<0) = P(X=-1) + P(X=-2) = P(X=0)$$
$$= P(X=1) + P(X=2)$$

Suppose $P(X=-1) = \alpha$

$$\Rightarrow 2\alpha = P(X=0)$$

$$P(X=-2) + P(X=-1) + P(X=0) + P(X=1) + P(X=2) = 1$$

$$\Rightarrow 2\alpha + 2\alpha + 2\alpha = 1$$
$$\Rightarrow 6\alpha = 1$$
$$\Rightarrow \alpha = \frac{1}{6}$$

pmf, $P_X(x) = \begin{cases} \frac{1}{6}, & x = -2 \\ \frac{1}{6}, & x = -1 \\ 2/6, & x = 0 \\ 1/6, & x = 1 \\ 1/6, & x = 2 \end{cases}$

$F_X(x) = \begin{cases} 0, & x < -2 \\ 1/6, & -2 \le x < -1 \\ 2/6, & -1 \le x < 0 \\ 4/6, & 0 \le x < 1 \\ 5/6, & 1 \le x < 2 \\ 1, & x \ge 2 \end{cases}$

Ex:-

A rv X can take all non-negative integer values and $P(X=m)$ is proportional to α^m $(0<\alpha<1)$. Find $P(X=1)$.

* Continuous Random Variable:-

Def^n:- Let X be a real random variable defined over the probability space (Ω, F, P). It is said that X is a continuous random variable iff $\exists$ a non-negative and integrable real function f_X such that $\forall x \in \mathbb{R}$ it is satisfied

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\, dt .$$

The function f_X is known as Probability density function.

Remark:-

A probability density function (pdf) satisfies the following properties:-

(i) $f(x) \geq 0 \;\forall$ possible values of x.

(ii) $\int_{-\infty}^{\infty} f_X(x)\, dx = 1$.

Ex:- Let X be a random variable with distribution function given by,

$$F_X(x) := \begin{cases} 0 &, \text{if } x < 0 \\ x &, \text{if } 0 \le x \le 1 \\ 1 &, \text{if } x \ge 1. \end{cases}$$

$$f_X(x) := \begin{cases} 0 &, x < 0 \\ 1 &, 0 \le x \le 1 \\ 0 &, x > 1 \end{cases}$$

$$\therefore \; f_X(x) = \begin{cases} 1 &, 0 \le x \le 1 \\ 0 &, \text{otherwise} \end{cases}$$

Ex:- Let X be a cts RV with density function given by,

$$f(x) = \begin{cases} kx(1-x) &, \text{if } 0 \le x \le 1 \\ 0 &, \text{otherwise}. \end{cases}$$

then find (1) the value of k

(2) the dist. fun^n F_X

(3) $P\left(-1 \le X \le \frac{1}{2}\right)$.

A:- $\int_{-\infty}^{\infty} f_X(x)\,dx = \int_0^1 (kx - kx^2)\,dx = 1$

$= k\left.\frac{x^2}{2} - \frac{x^3}{3}\right|_0^1 = 1$

$= k\left(\frac{1}{2} - \frac{1}{3}\right) = 1$

$\Rightarrow \frac{k}{6} = 1$

$\Rightarrow k = 6.$

$$\therefore f_X(x) = \begin{cases} 6x(1-x) & , 0 \le x \le 1 \\ 0 & , \text{otherwise} \end{cases}$$

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt = \begin{cases} 0 & , \text{if } x < 0 \\ 6\left(\frac{x^2}{2} - \frac{x^3}{3}\right) & , \text{if } 0 \le x \le 1 \\ 1 & , \text{if } x \ge 1. \end{cases}$$

$P\left(-1 \le X \le \frac{1}{2}\right) = F\left(\frac{1}{2}\right) - F(-1^-)$

$= F\left(\frac{1}{2}\right) - F(-1)$ $[\because F \text{ is cts}]$

$= P\left(X \le \frac{1}{2}\right) - P(X \le -1)$

$= \int_{-\infty}^{1/2} f(t)\,dt - \int_{-\infty}^{-1} f(t)\,dt$

$$= \int_{-1}^{1/2} f(t)\,dt$$

$$= \int_{-1}^{0} f(t)\,dt + \int_{0}^{1/2} f(t)\,dt$$

$$= 0 + 6\int_{0}^{1/2} \left(\frac{t^2}{2} - \frac{t^3}{3}\right) dt$$

$$= 6\left[\frac{t^3}{6} - \frac{t^4}{12}\right]_0^{1/2}$$

$$= 6\left(\frac{1}{48} - \frac{1}{12 \times 16}\right)$$

$$= 6\left(\frac{64-1}{12 \times 16}\right) = \frac{63}{32}$$

$$= 0 + 6\int_{0}^{1/2} (t - t^2)\,dt$$

$$= 6\left[\frac{t^2}{2} - \frac{t^3}{3}\right]_0^{1/2}$$

$$= 6\left(\frac{1}{8} - \frac{1}{24}\right)$$

$$= 6 \cdot \frac{2}{24} = \frac{1}{2}.$$

<u>Ex:-</u> $f(x) = \begin{cases} kxe^{-\lambda x}, & \text{if } x \geq 0 \\ 0, & \text{otherwise} \end{cases}$ where $\lambda > 0$.

(i) Determine k

(ii) Find the Dist funⁿ of X

(iii) $P(1 \leq X \leq 2)$.

* Distribution function of a random variable.

Ex:- Let X be a random variable and Y be defined as
$Y = |X|$

* $F_Y(y) = P(Y \le y)$
$= P(|X| \le y)$
$= P(-y \le X \le y)$
$= \underbrace{F_X(y) - F_X(-y) + P(X = -y)}$ → If X is discrete RV.

* If X is cts RV

$$F_Y(y) = \begin{cases} F_X(y) - F_X(-y) & , y \ge 0 \\ 0 & , \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} f_X(y) + f_X(-y) & , \text{if } y \ge 0 \\ 0 & , \text{otherwise} \end{cases}$$

Q:-

$$f(x) = \begin{cases} kxe^{-\lambda x}, & \text{if } x \geq 0 \\ 0, & \text{otherwise where } \lambda > 0. \end{cases}$$

(i) Determine k.

A:- We know that $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\Rightarrow \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx = 1$$

$$\Rightarrow 0 + \int_{0}^{\infty} kxe^{-\lambda x}\,dx = 1$$

$$\Rightarrow k\left\{x\frac{e^{-\lambda x}}{-\lambda}\Big|_0^{\infty} + \int_0^{\infty}\frac{e^{-\lambda x}}{\lambda}\,dx\right\} = 1$$

$$\Rightarrow k\ \frac{e^{-\lambda x}}{-\lambda^2}\Big|_0^{\infty} = 1$$

$$\Rightarrow \frac{-k}{\lambda^2}(0-1) = 1$$

$$\Rightarrow \frac{k}{\lambda^2} = 1$$

$$\Rightarrow k = \lambda^2.$$

$$\therefore f(x) = \begin{cases} \lambda^2 x e^{-\lambda x}, & \text{if } x \geq 0 \\ 0, & \text{otherwise where } \lambda > 0 \end{cases}$$

(ii) Find the distribution function of x.

A:-

$$F_X(x) = \int_{-\infty}^{x} f(t)\,dt$$

$$= \int_0^{x} \lambda^2 t e^{-\lambda t}\,dt$$

$$= \lambda^2 \int_0^x t e^{-\lambda t}\, dt$$

$$= \lambda^2 \left\{ t \frac{e^{-\lambda t}}{-\lambda} \Big|_0^x + \int_0^x \frac{e^{-\lambda t}}{\lambda}\, dt \right\}$$

$$= \lambda^2 \left\{ \frac{x e^{-\lambda x}}{-\lambda} + \frac{e^{-\lambda t}}{-\lambda^2} \Big|_0^x \right\}$$

$$= -\lambda x e^{-\lambda x} - \left(e^{-\lambda x} - 1 \right)$$

$$= -e^{-\lambda x} (\lambda x + 1) + 1$$

$$\therefore \; F_X(x) = \begin{cases} -e^{-\lambda x}(\lambda x + 1) + 1, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

(iii) Find $P(1 \leq X \leq 2)$

$$P(1 \leq X \leq 2) = \int_1^2 f(x)\, dx$$

$$= \int_1^2 \lambda^2 x e^{-\lambda x}\, dx$$

$$\Rightarrow \lambda^2 \left[x \frac{e^{-\lambda x}}{-\lambda} + \frac{e^{-\lambda x}}{-\lambda^2} \right]_1^2$$

$$= \lambda^2 \left(\frac{2e^{-2\lambda}}{-\lambda} + \frac{e^{-\lambda}}{\lambda} + \frac{e^{-2\lambda}}{-\lambda^2} + \frac{e^{-\lambda}}{\lambda^2} \right)$$

$$= -2\lambda e^{-2\lambda} + \lambda e^{-\lambda} - e^{-2\lambda} + e^{-\lambda}$$

$$= (-2\lambda - 1) e^{-2\lambda} + (\lambda + 1) e^{-\lambda}.$$

27.01.2020

Distribution of a function of a random variable :-

$g, \quad y = g(x)$

$$y^{-1}((-\infty, x]) = g^{-1}(-\infty, x] \in \mathcal{F}.$$

Ex:- Let X be RRV with density function given by,

$$f_X(x) = \begin{cases} \frac{1}{2}, & \text{if } -1 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Findout the density fun^n for $Y = |X|$

$$F_Y(y) = \begin{cases} F_X(y) - F_X(-y), & \text{if } y \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

$$f_Y(y) = \begin{cases} f_X(y) + f_X(-y) & \text{if } y \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{1}{2} + \frac{1}{2} = 1, & \text{if } 0 \leq y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Thm:- Let X be a cts RV with density function $f_X(x)$ If h is a strictly monotonic and diff. fun^n then the probability density function of the RV $Y = h(X)$ is given by,

$$f_Y(y) = \begin{cases} f_X(h^{-1}(y)) \left| \frac{d}{dy} h^{-1}(y) \right| & \text{if } y = h(x) \text{ for some } x \\ 0 & \text{, if } y \neq h(x). \end{cases}$$

Ex:- Let X be a RV with distribution function $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, $-\infty < x < \infty$. Define $y = e^x$. Then find the pdf of y.

Sol^n:-

$h(x) = e^x$

* P. dist - fun^n → for cts X
* pmf → for disc. X

$$f_Y(y) = \begin{cases} f_X(\log_e y)\left(\frac{1}{y}\right) & \text{, if } y \in (0,\infty) \\ 0 & \text{, otherwise.} \end{cases}$$

$$= \begin{cases} \frac{1}{y\sqrt{2\pi}} e^{-\frac{1}{2}(\log_e y)^2} & \text{, if } y \in (0,\infty) \\ 0 & \text{, otherwise.} \end{cases}$$

Expected Value / Average Value / Expectation :-

Defⁿ :- Let X be a real RV defined over the probability space (Ω, F, P)

(1) If X is a disc. RV with values $x_1, x_2, \ldots$. It is said that X has an expected value if $\sum_{k=1}^{\infty} |x_k| P(X = x_k) < \infty$.

In such a way, the expected value $E(X)$ of X is defined as, $E(X) = \sum_{k=1}^{\infty} x_k P(X = x_k)$

(2) If X is a cts RV with pdf f_X, Then it is said that X has an expected value if

$\int_{-\infty}^{\infty} |x| f_X(x)\,dx < \infty$, then $E(X) = \int_{-\infty}^{\infty} x f_X(x)\,dx$.

Ex:- Suppose that a normal die is rolled once and X be a random variable that represents the result obtained. then find $E(X) = ?$

A:- $\Omega = \{1, 2, 3, 4, 5, 6\}$

$$X = \begin{cases} 1, & \Omega = \{1\} \\ 2, & \Omega = \{2\} \\ 3, & \Omega = \{3\} \\ 4, & \Omega = \{4\} \\ 5, & \Omega = \{5\} \\ 6, & \Omega = \{6\} \end{cases}$$

$$E(X) = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6} = \frac{21}{6} = \frac{7}{3}$$

Ex:- Let X be a Disc. R V with pmf

$$P(x) = \begin{cases} e^{-3}\dfrac{3^x}{x!} & , \text{if } x = 0, 1, 2, \ldots \\ 0 & , \text{otherwise} \end{cases}$$

$$E(X) = \sum_{k=0}^{\infty} x_k e^{-3} \frac{3^{x_k}}{x_k!}$$

$$= \sum_{k=0}^{\infty} k e^{-3} \frac{3^k}{k!}$$

$$= e^{-3} \sum_{k=0}^{\infty} \frac{k 3^k}{k!}$$

$$= e^{-3} \sum_{k=1}^{\infty} \frac{3^k}{(k-1)!}$$

$$= 3e^{-3} \sum_{k=1}^{\infty} \frac{3^{k-1}}{(k-1)!}$$

$$= 3e^{-3} \cdot e^{3} \qquad \left\{ e^3 = 1 + 3 + \frac{3^2}{2!} + \cdots \right\}$$

$$= 3.$$

Ex:- Let X be a cts RV with pdf given by,

$$f(x) = \frac{\alpha}{\pi(\alpha^2+x^2)}, \text{ if } x \in \mathbb{R} = (-\infty, \infty) \text{ and } \alpha > 0 \text{ is constant.}$$

Sol^n:-

$$\int_{-\infty}^{\infty} |x| \frac{\alpha}{\pi(\alpha^2+x^2)}\, dx = \frac{2\alpha}{\pi}\int_0^{\infty} \frac{x}{\alpha^2+x^2}\, dx$$

$$= \frac{\alpha}{\pi}\int_0^{\infty} \frac{dt}{t} \qquad \text{put } \alpha^2+x^2 = t \Rightarrow 2x\,dx = dt$$

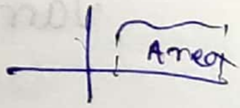
$$= \frac{\alpha}{\pi}\left[\log t\right]_0^{\infty} \not< \infty$$

$\Rightarrow E(X)$ doesn't exist.

* Some properties of Expectation

(1) If $P(X \geq 0) = 1$ and $E(X)$ exists, then $E(X) \geq 0$.

$$E(X) = \int_{-\infty}^{\infty} x f_X(x)\, dx = \int_0^{\infty} x f_X(x)\, dx \geq 0$$



(2) $E(\alpha) = \alpha$ for every constant α.

28.01.2020

Central moment around zero :-

Let X be a RV. The r^{th} central moment of X around zero denoted by μ'_r, $\mu'_r = E(X^r)$, whenever the expected value exists.

→ The central moment around any number a, is defined as $\mu''_r = E((x-a)^r)$

→ Central moment around expectation $E(X)$ is defined as,

$$\mu_r = E((X - E(X))^r)$$
$$= E((X-\mu)^r)$$

* Variance of a RV

Let X be a RV over the probability space (Ω, F, P). The variance is defined as,

$$var(X) = \sigma(X^2) = E((X - E(X))^2)$$

$$\Rightarrow var(X) = E(X^2 - 2XE(X) + (E(X))^2)$$
$$= E(X^2) - 2E(X)E(X) + (E(X))^2$$

$$\boxed{var(X) = E(X^2) - (E(X))^2}$$

Thm:- Let X be a RV whose expected value exists and $\alpha, \beta \in \mathbb{R}$ are constants, then

(1) $Var(X) \geq 0$

(2) $Var(\alpha) = 0$

(3) $Var(\alpha X) = \alpha^2 Var(X)$

(4) $Var(X+\beta) = Var(X)$

(5) $Var(X) = 0$ iff $P(X = E(X)) = 1$.

Ex:- Suppose that a die is rolled once and let X be a RV that represents the result obtained.

Find $E(X)$ and $Var(X)$.

A:- $X \to$ Disc. RV

$X = \{1, 2, 3, 4, 5, 6\}$

$$E(X) = 1\cdot\frac{1}{6} + 2\cdot\frac{1}{6} + 3\cdot\frac{1}{6} + 4\cdot\frac{1}{6} + 5\cdot\frac{1}{6} + 6\cdot\frac{1}{6} = \frac{21}{6} = \frac{7}{2} = 3.5$$

$$Var(X) = E(X^2) - (E(X))^2$$

$$E(X^2) = 1\cdot\frac{1}{6} + \frac{4}{6} + \frac{9}{6} + \frac{16}{6} + \frac{25}{6} + \frac{36}{6} = \frac{91}{6}$$

$$Var(X) = \frac{91}{6} - \frac{49}{4} = \frac{364 - 294}{24} = \frac{70}{24} = 2.91$$

$$(3.5 - 2.91,\ 3.5 + 2.91) = (0.59,\ 6.41)$$

Ex:- Let X be a cts RV with pdf given by,

$$f(x) = \begin{cases} 2x, & \text{if } 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$, then find

$E(X)$ and $Var(X)$.

A:- $E(X) = \int_{-\infty}^{\infty} x f_x(x)\, dx$

$$= \cancel{\int_{-\infty}^{0} x f(x)\, dx} + \int_{0}^{1} x f(x)\, dx + \cancel{\int_{1}^{\infty} x f(x)\, dx}$$

$$= \int_{0}^{1} 2x^2\, dx = \left.\frac{2x^3}{3}\right|_0^1 = \frac{2}{3}$$

$$E(X^2) = \int_0^1 x^2 \cdot 2x\, dx = \left.\frac{2x^4}{4}\right|_0^1 = \frac{1}{2}$$

$$\therefore Var(X) = E(X^2) - (E(X))^2$$

$$= \frac{1}{2} - \frac{4}{9} = \frac{9-8}{18} = \frac{1}{18}$$

* Moment generating function :-

Let X be a RV st $E(e^{tX})$ is finite for all $t \in (-\alpha, \alpha)$ with real positive α. The moment generating function (mgf) of X is denoted by $m_X(t)$, is defined as

$$m_X(t) = E(e^{tX}) \text{ with } t \in (-\alpha, \alpha)$$

$$= \begin{cases} \sum_k e^{tx_k} P_X(X = x_k) & \text{, if } X \text{ is disc. RV.} \\ \int_{-\infty}^{\infty} e^{tx} f(x)\,dx & \text{, if } x \text{ is cts RV} \end{cases}$$

Ex:- Let $X \to$ DRV

$$P_X(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & , x = 0, 1, 2, \dots, n \\ 0 & , \text{otherwise.} \end{cases}$$

$$m_X(t) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} e^{tk}$$

$$= \sum_{k=0}^{n} \binom{n}{k} (pe^t)^k q^{n-k} \quad , q = 1-p.$$

$$= (pe^t + q)^n \quad , t \in \mathbb{R}.$$

2) $$f(x) = \begin{cases} 2e^{-2x} & , \text{if } x > 0 \\ 0 & , \text{otherwise.} \end{cases}$$

$$m_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$

$$= \int_0^{\infty} e^{tx}\, 2e^{-2x}\,dx$$

$$= 2\int_0^{\infty} e^{-(2-t)x}\,dx$$

$$= 2\left.\frac{e^{-(2-t)x}}{-(2-t)}\right|_0^{\infty}$$

$$= \frac{2}{2-t} \quad , t < 2$$

Remark:- $$\left.\frac{d^r}{dt^r} m_X(t)\right|_{t=0} = E(X^r)$$

Proof:- $$m_X(t) = E(e^{tX})$$

$$= E\left(1 + tX + \frac{t^2X^2}{2!} + \cdots + \frac{t^rX^r}{r!} + \cdots\right)$$

$$= E(1) + tE(X) + \frac{t^2}{2!}E(X^2) + \cdots + \frac{t^r}{r!}E(X^r) + \cdots$$

$$\frac{d^r}{dt^r} m_X(t) = E(X^r) + t^1 E(X^{r+1})$$

$$\Rightarrow \left.\frac{d^r}{dt^r} m_X(t)\right|_{t=0} = E(X^r)$$

Thm:- Let X and Y be RV whose mgf exists. If $m_X(t) = m_Y(t)\ \forall t$, then X and Y have the same distribution.

* Characteristic function :-

$$m_X(t) = E(e^{tX}) \quad , t \in (-\alpha, \alpha) \; , \alpha > 0$$

$$E(X^r) = \frac{d^r}{dt^r} m_X(t) \Big|_{t=0}$$

Let X be a RV. The characteristic function of X is the function $\phi_X : \mathbb{R} \to \mathbb{C}$ defined by $\phi_X(t) = E(e^{itX})$

$$= E(\cos tX) + i E(\sin tX)$$

Ex:- Let X be a RV with $P(X=1) = P(X=-1) = \frac{1}{2}$, then find $\phi_X(t) = ?$

A:- $\phi_X(t) = E(e^{itX}) = e^{-it} \cdot \frac{1}{2} + e^{it} \cdot \frac{1}{2} = \cos t \ \forall t \in \mathbb{R}$

Ex:- Let X be a cts RV with pdf

$$f(x) = \begin{cases} 1, & \text{if } 0 < x < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Thm- 2.16

If X is a discrete or cts RV then $E(e^{itX})$ exists for all $t \in \mathbb{R}$.

Proof:-

$$E(e^{itx}) = \int_{-\infty}^{\infty} |x| f(x)\,dx$$

$$= \int_{-\infty}^{\infty} |e^{itx}| f(x)\,dx$$

$$= \int_{-\infty}^{\infty} f(x)\,dx = 1 < \infty$$

Thm:-

Let X be a RV. The characteristic function $\phi_X(\cdot)$ of X satisfies

(1) $\phi_X(0) = 1$.

(2) $|\phi_X(t)| \leq 1 \quad \forall t$

(3) If $E(x^k)$ exists then $\frac{d^k}{dt^k} \phi_X(t) \Big|_{t=0} = i^k E(x^k)$

Proof :-

2) $|\phi_X(t)| = \left| \int_{-\infty}^{\infty} e^{itx} f(x)\,dx \right|$

$\Rightarrow |\phi_X(t)| \leq \int_{-\infty}^{\infty} |e^{itx}|\,|f(x)|\,dx = 1$

$\therefore |\phi_X(t)| \leq 1$.

Thm:-

If X and Y are random variables and $\phi_X(t) = \phi_Y(t) \; \forall \, t \in \mathbb{R}$

then X and Y have the same distribution.

Ex:- $\phi_X(t) = \frac{1}{7} + \frac{2}{7}e^{-it} + \frac{3}{7}e^{it} + \frac{1}{7}e^{i2t}$

Find the probability distribution of X.

$X = -1, 0, 1, 2$

$$P(x) = \begin{cases} \frac{2}{7} & \text{if } x = -1 \\ \frac{1}{7} & \text{if } x = 0 \\ 3/7 & \text{if } x = 1 \\ 1/7 & \text{if } x = 2 \end{cases}$$

$$F_X(x) = \begin{cases} 0 & , x < -1 \\ 2/7 & , -1 \le x < 0 \\ 3/7 & , 0 \le x < 1 \\ 6/7 & , 1 \le x < 2 \\ 1 & , x \ge 2 \end{cases}$$

CHAPTER - 3

Some discrete distribution

*1 Discrete uniform distribution :-

A random variable X has DUD with N points, where N is a positive integer with possible values X_i, $i = 1, 2, \ldots, N$ if its pmf is given by,

$$P(x) = \begin{cases} \frac{1}{N}, & \text{if } x = x_1, x_2, \ldots, x_N \\ 0, & \text{otherwise} \end{cases}$$

(a) $E(X) = \sum_{i=1}^{N} x_i P(X = x_i) = \frac{1}{N} \sum_{i=1}^{N} x_i$

(b) $E(X^r) = \sum_{i=1}^{N} x_i^r P(X = x_i)$

(3) $m_X(t) = \sum_{i=1}^{N} \frac{e^{tx_i}}{N}$

2) Bernoulli Distribution

A RV X is said to be Bernoulli Distribution if its outcome can be classified as either "success" or "failure" is performed & we assume when X = 1 then we will consider outcome is success and if X = 0, the we will define outcome is failure.

$$P(x) = \begin{cases} p & , \text{if } x = 1 \\ 1-p & \text{if } x = 0 \end{cases}$$

If we repeat a Bernoulli experiment n times independently then RV X represents the no. of success that occur in the n-trials. then X is said to be a binomial RV with parameter (n, p) and it is denoted by $X \sim B(n, p)$.

↳ The pmf of Binomial distribution is given as,

$$P(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & , \text{if } x = 0, 1, 2, \cdots, n \\ 0 & , \text{otherwise}. \end{cases}$$

where n is a positive integer $0 < p < 1$

Find out.

(i) $E(X)$, (ii) $Var(X)$ (iii) $m_X(t)$

(i) $E(X) = np$

(ii) $Var(X) = npq$

(iii) $m_X(t) = (pe^t + q)^n$

Ex:- A fair die is rolled five consecutive times. Let X be the RV representing the number of times that number 5 was obtained. Find the pmf of X.

Sol^n:- $n = 5$

$p = \frac{1}{6}$

$q = 1 - \frac{1}{6} = \frac{5}{6}$

$$P(x=k) = \begin{cases} \binom{5}{k} \frac{1}{6^k} \left(\frac{5}{6}\right)^{5-k} & , k = 0,1,2,3,4,5 \\ 0 & , \text{otherwise.} \end{cases}$$

$P(x=0) = \binom{5}{0} \frac{1}{6^0} \left(\frac{5}{6}\right)^5 = \left(\frac{5}{6}\right)^5$

$P(x=1) = \binom{5}{1} \frac{1}{6} \left(\frac{5}{6}\right)^4 = \left(\frac{5}{6}\right)^5$

$P(x=2) =$

$P(x=3) =$

$P(x=4) =$

$P(x=5) =$

* Recurrence Relation :-

Let X be a Binomial discrete distribution.

$$B(X=k) = \binom{n}{k} p^k q^{n-k} \quad , \quad \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$$= \frac{n-(k-1)}{k} \binom{n}{k-1}$$

Then for $k = 1, 2, \ldots, n$

$$B(k) = \frac{n-k+1}{k} \binom{n}{k-1} p \cdot p^{k-1} \cdot \frac{q^{n-k+1}}{q}$$

$$= \frac{n-k+1}{k} \frac{p}{q} \left\{ \binom{n}{k-1} p^{k-1} q^{n-(k-1)} \right\}$$

$$= \frac{n-k+1}{k} \frac{p}{q} B(X=k-1)$$

$$\boxed{B(k+1) = \frac{n-k}{k+1} \frac{p}{q} B(k)}$$

* Poisson Distribution :-

A RV X is said to be have a Poisson distribution with parameter $\lambda > 0$ if its pmf is defined as,

$$P(X) = \begin{cases} \dfrac{e^{-\lambda} \lambda^x}{x!} & , \text{if } x = 0, 1, 2, \ldots \\ 0 & , \text{otherwise.} \end{cases}$$

It is denoted by $X \overset{d}{\sim} P(\lambda)$

Some properties of Poisson Dist^n:-

(1) $E(X) = \lambda$ (2) $Var(X) = \lambda$

(3) $m_X(t) = e^{\lambda(e^t - 1)}$

Recurrence Relation:-

$$P(X=k) = \frac{e^{-\lambda}\lambda^k}{k!}$$

$$= \frac{\lambda}{k}\frac{e^{-\lambda}\lambda^{k-1}}{(k-1)!} = \frac{\lambda}{k}P(X=k-1)$$

Ex:- The number of patients who come daily to the emergency room of a certain hospital has a poisson distribution with mean 10. What is the prob. that during a normal day, the number of patient admitted in the emergency room of the hospital will be ≤ 3?

A:- $X \overset{d}{\sim} P(10)$

$$P(X \leq 3) = P(X=0) + P(X=1) + P(X=2) + P(X=3)$$

$$= e^{-10} + \frac{e^{-10}\,10}{1!} + \frac{e^{-10}\,100}{2!} + \frac{e^{-10}\,1000}{3!}$$

$$= 1.0366 \times 10^{-2}$$

Theorem-3.6

If $n \geq 100$ and $p \leq 0.01$ with $np \leq 20$. Then Binomial dist. $B(n,p)$ follows Poisson dist. i.e.

$$B(n,p) \longrightarrow P(np) \quad , \lambda = np.$$

Sol^n:- $P(X=k) = \dfrac{n!}{k!(n-k)!} p^k (1-p)^{n-k}$

$$= \frac{n!}{k!(n-k)!}\left(\frac{\lambda}{n}\right)^k \left(1-\frac{\lambda}{n}\right)^{n-k}$$

$$= \frac{n(n-1)\cdots\cdots(n-k+1)\left(\frac{\lambda}{n}\right)^k \left(1-\frac{\lambda}{n}\right)^n}{k!\left(1-\frac{\lambda}{n}\right)^k}$$

$$= \frac{n(n-1)\cdots(n-k+1)}{n^k} \cdot \frac{\lambda^k \left(1-\frac{\lambda}{n}\right)^n}{k!\left(1-\frac{\lambda}{n}\right)^k}$$

$$= 1 \cdot \frac{\lambda^k}{k!} e^{-\lambda} \quad \text{as } n \to \infty$$

$$\therefore P(X=k) \simeq \frac{e^{-\lambda}\lambda^k}{k!}$$

Ex:- There are 135 students inside a conference hall. The prob. that one of the students celebrates his/her birthday today equals to $\frac{1}{365}$. What is the prob. that two or more students from the same conference hall are celebrating their birthdays today ?

Sol:- $X \overset{d}{\sim} B(135, \frac{1}{365})$

$$np = \frac{135}{365} = 0.369 \leq 20$$

$$\lambda = np = \frac{27}{73}$$

$P(X \geq 2) = 1 - P(X=0) - P(X=1)$ (Calculate by using Poisson dist.)

$$= 1 - \frac{e^{-0.36} \cdot 1}{1} - \frac{e^{-0.36} \times 0.36}{1}$$

$$= 1 - e^{-0.36}(1 - 0.36)$$

$$= 1 - e^{-0.36}(0.64)$$

$$= 1 - 0.697(0.64)$$

$$= 1 - 0.446 = 0.554.$$

* Geometric and Negative Binomial Distribution :-

↳ A RV X is said to be a negative binomial distribution with parameter k and p if its pmf is given by

$$P(x) = \begin{cases} \binom{x-1}{k-1} p^k (1-p)^{x-k} & , \text{if } x = k, k+1, \ldots \\ 0 & , \text{otherwise.} \end{cases}$$

↳ It is denoted by $X \overset{d}{\sim} B_N(k, p)$

↳ In the special case where $k=1$. It is said that the RV has a geometric distribution with parameter p.

(1) $E(X) =$

(2) $Var(X) =$

(3) $m_X(t) =$

Q:- In a quality control dept. units coming from an assembly line are inspected. If the proportion of defective units is 0.03. What is the prob. that 20 units inspected is the third one found defective?

A:- $X = 20$, $k = 3$, $p = 0.03$

$$P(X=20) = \binom{19}{2}(0.03)^3(0.07)^{17} = 2.7509 \times 10^{-3}.$$

07.02.2020

Tutorial – 3 Ch – 1

1.1, 1.7, 1.16, 1.17, 1.18, 1.19, 1.28, 1.39, 1.57

Tutorial – 4 Ch – 2

2.1, 2.3, 2.4, 2.9, 2.10, 2.11, 2.12, 2.16, 2.31, 2.33, 2.34, 2.35, 2.36, 2.38, 2.51, 2.52, 2.59, 2.61, 2.62,

Tutorial – 5 Ch – 3

3.1, 3.3, 3.4, 3.7, 3.10, 3.14, 3.21, 3.28, 3.36, 3.49, 3.50, 3.52

Tutorial – 6 Ch – 4 :-

4.1, 4.2, 4.11, 4.12, 4.14, 4.19, 4.31, 4.33, 4.55.

Tuesday :- 2:00 – 03:30
Tut – 1 + 3

Thursday : 02:00 – 03:30
Tut – 2 + Tut 4

CHAPTER - 4

Some Continuous Distribution

(1) Uniform Distribution :-

It is said that a RV X is uniformly distributed over the interval $[a,b]$ with $a<b$ real numbers, if its density function is defined by,

$$f(x) = \begin{cases} \frac{1}{b-a}, & \text{if } a \le x \le b \\ 0, & \text{otherwise} \end{cases}$$

It is denoted by $X \xrightarrow{d} U[a,b]$

→ The distribution function of uniform RV $X \sim U[a,b]$ is defined as,

$$F(x) = \int_{-\infty}^{x} f(t)\,dt$$

$$= \begin{cases} 0, & \text{if } x < a \\ \frac{x-a}{b-a}, & \text{if } a \le x < b \\ 1, & \text{if } x \ge b \end{cases}$$

$$F(x) = \int_{-\infty}^{x} f_x(t)\,dt = \int_{-\infty}^{a} \cancel{f(t)}^{\,0}\,dt + \int_{a}^{x} f(t)\,dt$$

$$= \frac{1}{b-a}\, t\Big|_a^x = \frac{x-a}{b-a}$$

Ex:- Let $X \stackrel{d}{\sim} U[-3,2]$

(i) $P(X \geq 0)$ (ii) $P\left(-5 \leq X \leq \frac{1}{2}\right)$

$$f(x) = \begin{cases} \frac{1}{5}, & -3 \leq x \leq 2 \\ 0, & \text{otherwise.} \end{cases}$$

A:- (i) $P(X \geq 0) = 1 - P(X < 0)$

$$= 1 - F(0)$$

$$= 1 - \int_{-\infty}^{0} f(x)\,dx$$

$$= 1 - \int_{-\infty}^{-3} f(x)\,dx - \int_{-3}^{0} f(x)\,dx$$

$$= 1 - \int_{-3}^{0} f(x)\,dx$$

$$= 1 - \frac{x}{5}\Big|_{-3}^{0} = 1 - \left(0 + \frac{3}{5}\right) = \frac{2}{5}$$

(ii) $P\left(-5 \leq X \leq \frac{1}{2}\right) = F\left(\frac{1}{2}\right) - F(-5)$

$$= \int_{-\infty}^{1/2} f(x)\,dx - \int_{-\infty}^{-5} f(x)\,dx$$

$$= \cancel{\int_{-\infty}^{-5} f(x)\,dx} + \int_{-5}^{1/2} f(x)\,dx - \cancel{\int_{-\infty}^{-5} f(x)\,dx}$$

$$= \int_{-5}^{1/2} f(x)\,dx$$

$$= \int_{-5}^{-3} \overset{0}{\cancel{f(x)}}\,dx + \int_{-3}^{1/2} f(x)\,dx$$

$$= \int_{-3}^{1/2} \frac{1}{5}\,dx$$

$$= \left.\frac{x}{5}\right|_{-3}^{1/2}$$

$$= \frac{1}{10} + \frac{3}{5} = \frac{1+6}{10} = \frac{7}{10}$$

Q:- A number is randomly choosen in the interval $[1,3]$. What is the probability that the first digit to the right side of the decimal point is 5 ?

Sol^n:-

$$f(x) = \begin{cases} \frac{1}{2}, & \text{if } 1 \le x \le 3 \\ 0, & \text{otherwise} \end{cases}$$

P(first digit to the right side of the decimal point of x is 5

$$= P(1.5 \le X < 1.6) + P(2.5 \le X < 2.6)$$

(ii) What is the probability that the second digit to the right of the decimal point is 2.

A:- P(   )

$= P(1.02 \le X \le 1.03) + P(1.12 \le X < 1.13) + \dots$

$\dots + P(1.92 \le X < 1.93) + P(2.02 \le X < 2.03) + \dots$

$\dots + P(2.92 \le X < 2.93)$

$= 10 \times 0.01 \times \frac{1}{2} + 10 \times 0.01 \times \frac{1}{2}$

$=$

Ex:- A point X is choosen at random in the interva[l] $[-1, 3)$. Find the pdf of $Y = X^2$.

A:-

$$F_Y(y) = P(Y \le y) = P(X^2 \le y) = P(|X| \le \sqrt{y})$$

$$f(x) = \begin{cases} \frac{1}{4}, & \text{if } -1 \le x \le 3 \\ 0, & \text{otherwise} \end{cases}$$

$$F_Y(y) = P(-\sqrt{y} \le X \le \sqrt{y})$$

$$= \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{4}\, dx$$

$$= \frac{2}{4}\sqrt{y}$$

$$\cancel{F_Y(y) = \frac{2}{4}\sqrt{y}}$$

$$F_Y(y) = \begin{cases} 0 & , y < 0 \\ \frac{1}{2}\sqrt{y} & , \text{if } 0 \le y < 1 \\ \frac{\sqrt{y}+1}{4} & , \text{if } 1 \le y < 9 \\ 1 & , y \ge 9 \end{cases}$$

For $1 \le y < 9$

$$\int_{-1}^{1} \frac{1}{4}\, dy + \int_{1}^{\sqrt{y}} \frac{1}{4}\, dy = \frac{1}{2} + \frac{\sqrt{y}-1}{4} = \frac{2+\sqrt{y}-1}{4} = \frac{\sqrt{y}+1}{4}$$

Properties:-

↳ If X is a RV with uniform distribution over the interval $[a,b]$, then

(i) $E(X) = \frac{a+b}{2}$

(ii) $Var(X) = \frac{(b-a)^2}{12}$

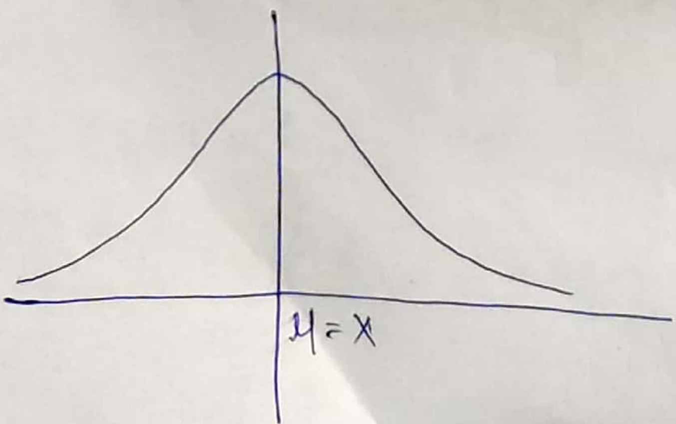
(iii) $m_X(t) = \frac{e^{bt} - e^{at}}{t(b-a)}$

Normal Distribution / Gaussian Distribution :- 10.02.2020

It is said that a RV X has normal distribution with parameters μ and σ, where μ is a real number and σ is a positive real number, if its density function is given by,

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

, $-\infty < x < \infty$, $-\infty < \mu < \infty$ and $\sigma > 0$



Exercise:- Prove that $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\left[\Gamma(n) = \int_0^{\infty} x^{n-1} e^{-x}\,dx\right.$$

Standard Normal Distribution:-

If $X \overset{d}{\sim} N(0,1)$, then it is said that X has a standard normal distribution.

* Let $X \overset{d}{\sim} N(\mu, \sigma^2)$, then we defined standard normal distribution, $Z = \frac{X-\mu}{\sigma}$.

$$F_Z(z) = P(Z \le z)$$
$$= P\left(\frac{X-\mu}{\sigma} \le z\right)$$
$$= P(X \le \mu + \sigma z)$$
$$= F_X(\mu + \sigma z)$$

$$f_Z(z) = f_X(\mu + \sigma z)\cdot \sigma = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}, \quad -\infty < z < \infty.$$

Remark:- The pdf of a std normal distribution is symmetric with respect to the Y-axis, therefore for all $z < 0$, it is satisfied that $\phi(z) = 1 - \phi(-z)$, where $\phi(z) = \int_{-\infty}^{z} f_Z(z)\,dz$

Ex:- Let $X \overset{d}{\sim} N(1, 4)$, then calculate

(i) $P(0 \le X \le 1)$ (ii) $P(X^2 > 4)$

A:- $Z = \dfrac{X-\mu}{\sigma} = \dfrac{X-1}{2}$

$$P(0 \le X \le 1) = P(-1 \le X-1 \le 0)$$

$$= P\left(\frac{-1}{2} \le \frac{X-1}{2} \le 0\right)$$

$$= P\left(\frac{-1}{2} \le Z \le 0\right)$$

$$= \phi(0) - \phi\left(-\frac{1}{2}\right)$$

$$= \phi(0) - 1 + \phi\left(\frac{1}{2}\right) \quad \left[\because \phi\left(\frac{-1}{2}\right) = 1 - \phi\left(\frac{1}{2}\right)\right]$$

$$= \phi(0) + \phi(0.5) - 1$$

$$= 0.5 + 0.6915 - 1$$

$$= 1.1915 - 1$$

$$= 0.1915$$

(ii) $$P(X^2 > 4) = 1 - P(X^2 \le 4)$$

$$= 1 - P(-2 \le X \le 2)$$

$$= 1 - P\left(-\frac{3}{2} \le \frac{X-1}{2} \le \frac{1}{2}\right)$$

$$= 1 - \left\{\phi\left(\frac{1}{2}\right) - \phi\left(\frac{-3}{2}\right)\right\}$$

$$= 1 - \phi\left(\frac{1}{2}\right) + 1 - \phi\left(\frac{3}{2}\right)$$

$= -\phi\left(\frac{3}{2}\right) - \phi\left(\frac{1}{2}\right) + 2$

$= -\phi(1.5) - \phi(0.5) + 2$

$= -0.9332 - 0.6915 + 2$

$= 2 - 1.6247$

$= 0.3753$

Ex:- Let $X \overset{d}{\sim} N(12, 4)$. Find the value of c st

$P(X > c) = 0.10$

$\Rightarrow P(X \leq c) = 0.90$

$P\left(\frac{X-12}{2} \leq \frac{c-12}{2}\right) = 0.90$

$\Rightarrow \phi\left(\frac{c-12}{2}\right) = 0.90$

$\Rightarrow \frac{c-12}{2} = \frac{1.28 + 1.29}{2}$

$\Rightarrow c - 12 = 2.57$

$\Rightarrow c = 2.57 + 12 = 14.57$

Thm:- Let $X \overset{d}{\sim} N(\mu, \sigma^2)$, then

(i) $E(X) = \mu$

(ii) $var(X) = \sigma^2$

(iii) $m_X(t) = \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)$

iii) $$m_x(t) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\left(tx - \frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right)} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{(x-\mu-\sigma^2 t)^2}{2\sigma^2} + \mu t + \frac{\sigma^2 t^2}{2}\right) dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right) \int_{-\infty}^{\infty} \exp\left(-\frac{(x-\mu-\sigma^2 t)^2}{2\sigma^2}\right)$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right) \int_{-\infty}^{\infty} \exp\left(\frac{-1}{2}\left(\frac{x-(\mu+\sigma^2 t)}{\sigma}\right)^2\right)$$

$$= \cancel{\frac{1}{\sigma\sqrt{2\pi}}} \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right) \cdot 1$$

$$= e^{\mu t + \frac{\sigma^2 t^2}{2}}$$

For std. normal dist. when $\mu = 0$; $\sigma = 1$

$$m_x(t) = e^{t^2/2}$$

<u>Remark-1</u>

Characteristic, $\phi_x(t) = e^{\left(i\mu t - \frac{\sigma^2 t^2}{2}\right)}$

For std. norm. dist

$$\phi_x(t) = e^{-t^2/2}$$

Remark-α.

The normal dist. is another form of the binomial dist. If $n \to \infty$ and neither p nor $1-p$ is very small.

i.e. $X \overset{d}{\sim} B(n,p)$ and when $n \to \infty$ & $p, 1-p$ not very small, then $X \overset{d}{\sim} N(np, npq)$.

$$Z = \frac{X - np}{\sqrt{np(1-p)}}$$

Q:- A normal die is tossed 1000 consecutive times. Calculate the prob. that the number 6 shows up between 150 and 200 times. What is the probability that the number 6 appears exactly 150 times.

A:- $X \overset{d}{\sim} B\left(1000, \frac{1}{6}\right)$

$P(150 \le X \le 200)$

$\mu = np = 1000 \times \frac{1}{6} = \frac{500}{3}$

$\sigma^2 = npq = 1000 \times \frac{1}{6} \times \frac{5}{6} = \frac{1250}{9}$

$$Z = \frac{X - \frac{500}{3}}{\sqrt{\frac{1250}{9}}}$$

$$P\left(\frac{150 - \frac{500}{3}}{\sqrt{\frac{1250}{9}}} \le Z \le \frac{200 - \frac{500}{3}}{\sqrt{1250/9}}\right) = P(-1.1412 \le Z \le 2.8284)$$

$= \phi(2.8284) - \phi(-1.1412)$

$= \phi(2.8284) - 1 + \phi(1.1412)$

$= 0.9976 - 1 + 0.8729$

17.02.2020

(3) Exponential Distribution

A cont. RV X is said to be an exponential RV if its pdf is defined as,

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{, otherwise} \end{cases}$$

, where λ is a parameter, $\lambda > 0$.

(i) $E(X) = \frac{1}{\lambda}$.

(ii) $Var(X)$

(iii) $m_X(t)$

* Memoryless Property :-

The key property of an exponential random variable is that it is memoryless, where we say that a non-negative RV X is memoryless if

$$P(X > s+t \mid X > t) = P(X > s) \text{ For all } t, s \geq 0$$

$$\Rightarrow \frac{P(X > s+t \cap X > t)}{P(X > t)} = P(X > s)$$

$$\frac{P(X > s+t)}{P(X > t)} = \frac{\int_{t+s}^{\infty} f_X(t)\, dt}{\int_{t}^{\infty} f_X(t)\, dt} = \frac{\int_{t+s}^{\infty} \lambda e^{-\lambda t}\, dt}{\int_{t}^{\infty} \lambda e^{-\lambda t}\, dt}$$

$$= \frac{e^{-\lambda(t+s)}}{e^{-\lambda t}} = e^{-\lambda s} = P(X > s)$$

Ex:- Suppose that a number of miles that a car can run before its battery wears out is exponentially distributed with an average value of 10,000 miles. If a person desires to take a 5,000 mile trip. What is the prob. that she will be able to complete her trip without having to replace her car battery? What can be said when dist. is not exponential?

Sol^n:- $E(X) = \frac{1}{\lambda} = 10,000$ miles

$$P(T > 5000) = \int_{5000}^{\infty} f_X(t)\,dt = e^{-\frac{1}{10,000}(5000)} = e^{-1/2}$$

$$P(T > t + 5000 \mid T > 5000) = \frac{P(T > t + 5000)}{P(T > 5000)}$$

$$= \frac{1 - F(t+5000)}{1 - F(5000)}$$

(4) Gamma Distribution

It is said that the RV X has Gamma distribution with parameters $r > 0$ and $\lambda > 0$ if its density function (pdf) is defined as,

$$f(x) = \begin{cases} \frac{\lambda}{\Gamma(r)} (\lambda x)^{r-1} e^{-\lambda x}, & \text{if } x > 0 \\ 0, & \text{otherwise} \end{cases}$$

$$\Gamma(r) = \int_0^{\infty} t^{r-1} e^{-t}\,dt.$$

It is denoted by $X \overset{d}{\sim} \Gamma(r, \lambda)$

Thm:-

If $X \overset{d}{\sim} \Gamma(r, \lambda)$, then

(1) $E(X) = \frac{r}{\lambda}$

(2) $var(X) = \frac{r}{\lambda^2}$

(3) $m_X(t) = \left(\frac{\lambda}{\lambda - t}\right)^r$, if $t < \lambda$.

Particular case:-

(1) If $r = 1$ and $\lambda > 0$, then Gamma dist. follows as an exponential distribution.

(2) If $\lambda = \frac{1}{2}$ and $r = \frac{k}{2}$ with positive integer, then Gamma dist. follows chi-square dist.

$$f(x) = \begin{cases} \frac{1}{2^{k/2}\,\Gamma(k/2)}\, x^{\frac{k-2}{2}}\, e^{-x/2} & \text{, if } x > 0 \text{, where } k\text{-degree of freedom} \\ 0 & \text{, otherwise ;} \end{cases}$$

It is denoted as $X \overset{d}{\sim} \chi^2_{(k)}$.

Ex:- The length of life time T, in hours of a certain device has an exponential dist. with mean 100 hrs. Calculate the reliability at time $t = 200$ hrs.
↳ means device will work at least 200 hrs.

A:- $P(T \geq 200) = \int_{200}^{\infty} \frac{1}{100} e^{-\frac{1}{100}t} \, dt = e^{-\frac{200}{100}} = e^{-2}$

$\lambda = \frac{1}{100}$ $= 0.13534$.

* Cauchy Distribution :-

It is said that a RV X has a Cauchy distribution with parameters θ and β, $\theta \in \mathbb{R}$ and $\beta > 0$, if its pdf is defined as,

$$f(x) = \frac{1}{\pi\beta} \frac{1}{1+\left(\frac{x-\theta}{\beta}\right)^2}, \quad x \in \mathbb{R}.$$

→ When $\theta = 0$ & $\beta = 1$, then $f(x) = \frac{1}{\pi(1+x^2)}$ is known as the std. Cauchy dist.

→ $\boxed{\text{Its expectation doesn't exist.}}$

03.03.2020

Def^n:- (n - dimensional Random vector) :-

Let $X_1, X_2, \ldots, X_n$ be n real random variables defined over the same probability space (Ω, F, P). The function $X: \Omega \longrightarrow \mathbb{R}^n$ defined by $X(\omega) = (X_1(\omega), \ldots, X_n(\omega))$ is called an n-dimensional random vector.

Distribution of a Random Vector :-

Let X be an n-dimensional random vector. Then probability measure defined by $P_X(B) := P(X \in B)$; $B \in \mathcal{B}_n$ is called the distribution of the random vector X.

* $\mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2 \times \ldots \times \mathcal{B}_n$

Joint Probability mass function :-

Let $X = (X_1, X_2, \ldots, X_n)$ be an n-dimensional random vector. If the R.Vs X_i with $i = 1, 2, \ldots, n$ are all discrete, It is said that the random vector X is discrete. In this case the pmf of X also called the joint distribution function of the random variables $X_1, X_2, \ldots, X_n$ is defined by,

$$P_X(x) = \begin{cases} P(X = x) & \text{, if } x \text{ belongs to the image of } X,\ x \in \mathbb{R}^n \\ 0 & \text{, otherwise} \end{cases}$$

Note :-

Let X_1 and X_2 be discrete random variables, then

$$P(X_1 = x) = P\left((X_1 = x) \cap \bigcup_y (X_2 = y)\right)$$

$$= P\left(\bigcup_y (X_1 = x ; X_2 = y)\right)$$

$$= \sum_y P(X_1 = x, X_2 = y)$$

In general, we have

Thm - 5.1

let $X = (X_1, X_2, \ldots, X_n)$ be a discrete n-dimensional random vector. Then for $j = 1, 2, \ldots, n$, we have

$$P(X_j = x) = \sum_{x_1} \cdots \sum_{x_{j-1}} \sum_{x_{j+1}} \cdots \sum_{x_n} P(X_1 = x_1, \ldots, X_{j-1} = x_{j-1}, X_{j+1} = x_{j+1}, \ldots, X_n = x_n)$$

Then the function,

$$P_{X_j}(x) = \begin{cases} P(X_j = x) & \text{, if } x \text{ belongs to the image of } X \\ 0 & \text{, otherwise} \end{cases}$$

Ex:- Suppose that a fair coin is flipped three consecutive times and let X and Y be the RVs defined as follows

X = "Number of heads obtained"

Y = Flip number where a head was first obtained (If there are none, we define Y = 0)

(1) Joint dist. of X and Y

(2) Marginal dist. of X and Y

(3) $P(X \le 2, Y = 1)$, $P(X \le 2, Y \le 1)$, and $P(X \le 2 \text{ or } Y \le 1)$

A:-

$$X = \begin{cases} 0, & \text{if } \omega = \{TTT\} \\ 1, & \text{if } \omega = \{HTT, THT, TTH\} \\ 2, & \text{if } \omega = \{THH, HHT, HTH\} \\ 3, & \text{if } \omega = \{HHH\} \end{cases}$$

$$Y = \begin{cases} 0, & \omega = \{TTT\} \\ 1, & \omega = \{HTT, HHT, HHH, HTH\} \\ 2, & \omega = \{THT, THH\} \\ 3, & \omega = \{TTH\} \end{cases}$$

X|Y	0	1	2	3
0	$\frac{1}{8}$	0	0	0
1	0	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
2	0	$2/8$	$1/8$	0
3	0	$1/8$	0	0

$P(X=0) = P\{\omega \in \Omega \mid X(\omega) = 0\}$

$P(X=0, Y=0) = P(X=0) \cap P(Y=0)$

$= P(\{TTT\} \cap \{TTT\})$

$= P\{TTT\}$

$= \frac{1}{8}$

(ii) Marginal distribution of x and y

$$P_x(x) = \begin{cases} \frac{1}{8}, & x=0 \\ \frac{3}{8}, & x=1 \\ \frac{3}{8}, & x=2 \\ \frac{1}{8}, & x=3 \end{cases}$$

$$P_y(x) = \begin{cases} \frac{1}{8}, & y=0 \\ \frac{4}{8}, & y=1 \\ \frac{2}{8}, & y=2 \\ \frac{1}{8}, & y=3 \end{cases}$$

(3) $P(x \le 2 \text{ or } y \le 1) = P(x \le 2) + P(y \le 1) - P(x \le 2, y \le 1$

04.03.2020

Joint Commulative Distribution Function

Let $X = (X_1, X_2, \ldots, X_n)$ be an n-dimensional random vector. The function defined by,

$F(x_1, x_2, \ldots, x_n) := P(X_1 \leq x_1, X_2 \leq x_2, \ldots, X_n \leq x_n)$. For all $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ is called the joint cummulative Distribution function of the random variables $X_1, X_2, \ldots, X_n$ or simply distribution function of n-dimensional random vector X.

Note-1:-

Let X_1 and X_2 be random variables JCDF F. Then

$$F_{X_1}(x) = P(X_1 \leq x) = P\left((X_1 \leq x) \cap \bigcup_y (X_2 \leq y)\right)$$

$$= P\left(\bigcup_y (X_1 \leq x, X_2 \leq y)\right)$$

$$= \lim_{y \to \infty} P(X_1 \leq x, X_2 \leq y)$$

$$= \lim_{y \to \infty} F(x, y)$$

$$F_{X_2}(y) = \lim_{x \to \infty} F(x, y)$$

Theorem-5.1

Let $X = (X_1, X_2, \ldots, X_n)$ be a discrete n-dimensional random vector, then for all $j = 1, 2, \ldots, n$ we have

$$P(X_j = x) = \sum_{x_1} \cdots \sum_{x_{j-1}} \sum_{x_{j+1}} \cdots \sum_{x_n} P(X_1 = x_1, \ldots, X_{j-1} = x_{j-1}, X_{j+1} = x_{j+1}, \ldots, X_n = x_n)$$

Thm :-

let $X = (X_1, X_2, \ldots, X_n)$ be an n-dimensional RV with joint cummulative distribution function F, for each $j = 1, 2, \ldots, n$, the CDF of the random variable X_j is given by,

$$F_{X_j}(x) = \lim_{x_1 \to \infty} \cdots \lim_{x_{j-1} \to \infty} \lim_{x_{j+1} \to \infty} \cdots \lim_{x_n \to \infty} F(x_1, x_2, \ldots, x_n)$$

→ The dist. function F_{X_j} is called the marginal distributio function of the RV X_j.

Thm :-

Let $X = (X, Y)$ be a two dimensional random vector. The JCDF F of the random variables X, Y has the following properties.

(i) $\Delta_a^b F = F(b_1, b_2) + F(a_1, a_2) - F(a_1, b_2) - F(b_1, a_2) \geq 0$,

where $a = (a_1, a_2), b = (b_1, b_2) \in \mathbb{R}^2$ with $a_1 \leq b_1$, $a_2 \leq b_2$

(ii) $\lim_{x \to x_0} F(x, y) = F(x_0, y)$ and $\lim_{y \to y_0} F(x, y) = F(x, y_0)$

(iii) $\lim_{x \to -\infty} F(x, y) = 0$ and $\lim_{y \to -\infty} F(x, y) = 0$

(iv) $\lim_{(x,y) \to (\infty, \infty)} F(x, y) = 1$.

Ex:- Check whether the following functions are joint distributions.

(1) $F(x,y) = \begin{cases} e^{-(x+y)} & , \text{ if } 0<x<\infty, 0<y<\infty \\ 0 & , \text{ otherwise} \end{cases}$

S:-

(iv) $\lim\limits_{(x,y)\to(\infty,\infty)} F(x,y) = \lim\limits_{(x,y)\to(\infty,\infty)} e^{-(x+y)} \to 0$

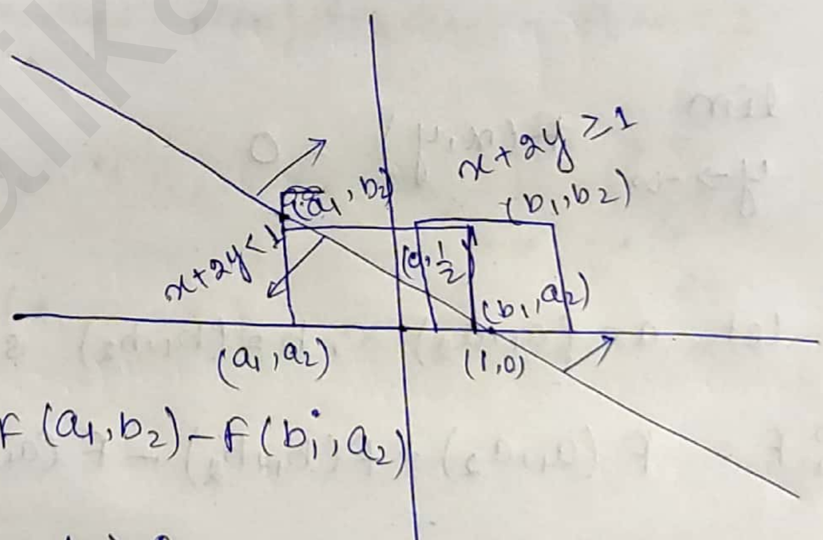
$\therefore F(x,y)$ is not JCDF.

(2) $F(x,y) = \begin{cases} 1 , & \text{if } x+2y \geq 1 \\ 0 , & \text{if } x+2y < 1 \end{cases}$

$\Rightarrow y = \frac{1-x}{2}$

$x+2y=1$

$\Rightarrow x = 1-2y$



$F(b_1,b_2) + F(a_1,a_2) - F(a_1,b_2) - F(b_1,a_2)$

$= 1+0-0-0 = 1 \geq 0$

$1+0-1-1 = -1 < 0$

$\therefore F(x,y)$ is not a JCDF.

(3) $F(x,y) = \begin{cases} 1-e^{-x}-e^{-y}+e^{-(x+y)} & \text{, if } x>0, y>0 \\ 0 & \text{, otherwise} \end{cases}$

(iv) $\lim_{(x,y)\to(\infty,\infty)} F(x,y) = \lim_{(x,y)\to(\infty,\infty)} (1-e^{-x}-e^{-y}+e^{-(x+y)}$

$= 1$

(iii) $\lim_{x\to-\infty} F(x,y) = \lim_{x\to\infty} \left(\cancel{1-e^{-x}}-e^{-y}+e^{-(x+y)}\right)$

$= 0$

$\lim_{y\to-\infty} F(x,y) = 0$

(i) Let $a=(a_1,a_2)$, $b=(b_1,b_2)$ st $a_1\le b_1$, $a_2\le b_2$

$\Delta_a^b F = F(a_1,a_2)+F(b_1,b_2)-F(a_1,b_2)-F(a_2,b_1)$

$= 1-e^{-}$

(ii) Since F is cts it satisfy (ii)

$\therefore$ F is JCDF.

Jointly Continuous Random Variables:-

Let $X_1, X_2, \ldots, X_n$ be n- real valued RVs defined over the same probability space. It is said that the random variables are jointly cts, if there is an integrable function $f: \mathbb{R}^n \to [0,\infty)$ such that for every Borel set C of $\mathbb{R}^n$

$$P((X_1, X_2, \ldots, X_n) \in C) = \int_C \cdots \int f(x_1, x_2, \ldots, x_n)\, dx_1\, dx_2 \cdots dx_n.$$

$\hookrightarrow$ The function f is called the joint probability density function of the RVs $X_1, X_2, \ldots, X_n$

Remark (i) $\displaystyle\int_{\mathbb{R}^n} \cdots \int f(x_1, x_2, \ldots, x_n)\, dx_1\, dx_2 \cdots dx_n = 1$

(ii) $\displaystyle P(X_1 \le x_1, X_2 \le x_2, \ldots, X_n \le x_n) = \int_{-\infty}^{x_n} \cdots \int_{-\infty}^{x_2} \int_{-\infty}^{x_1} f(t_1, \ldots, t_n)\, dt_1 \ldots dt_n$

$\Rightarrow F_X(x) \mid X = (X_1, \ldots, X_n)$, $x = (x_1, x_2, \ldots, x_n)$.

Submitted by
Sarojini Mohapatra
(MSc Math Student)
Central University of Jharkhand

Some Useful Links:

1. Free Maths Study Materials (https://pkalika.in/2020/04/06/free-maths-study-materials/**)**

2. BSc/MSc Free Study Materials (https://pkalika.in/2019/10/14/study-material/**)**

3. MSc Entrance Exam Que. Paper: (https://pkalika.in/2020/04/03/msc-entrance-exam-paper/**)**
[JAM(MA), JAM(MS), BHU, CUCET, ...etc]

4. PhD Entrance Exam Que. Paper: (https://pkalika.in/que-papers-collection/**)**
[CSIR-NET, GATE(MA), BHU, CUCET,IIT, NBHM, ...etc]

5. CSIR-NET Maths Que. Paper: (https://pkalika.in/2020/03/30/csir-net-previous-yr-papers/**)**
[Upto 2019 Dec]

6. Practice Que. Paper: (https://pkalika.in/2019/02/10/practice-set-for-net-gate-set-jam/**)**
[Topic-wise/Subject-wise]

Submitted by
Sarojini Mohapatra
(MSc Math Student)
Central University of Jharkhand

(Provide your Feedbacks/Comments at maths.whisperer@gmail.com)