IISc Banglore
PhD Maths Entrance Test

[Prev. Yr. Que. Papers (2000-2010)]

Contents:

No. of Pages: 169

ENTRANCE TEST FOR ADMISSION 2000

Integrated Ph.D

Mathematical Sciences

Day & Date	:	Sunday 14th May 2000
Time	:	1.30 p.m. to 4.30 p.m.

INDIAN INSTITUTE OF SCIENCE
BANGALORE

INSTRUCTIONS

- The question paper is in two parts: Part A and Part B. Part A carries 30 marks and Part B carries 70 marks.
- Part A comprises 30 multiple choice questions each carrying 1 mark. Four possible answers are provided for each question. Select the correct answer by marking ($\surd$) against (a), (b), (c) or (d) on the answer script exactly as given below.
  For example, Question: $2 + 2 =$ Answer: (a) 0 (b) 2 (c) 4 (d) 8.
  Answer all questions from Part A.
- Part B comprises 8 questions. Answer any 5 questions. Each question carries 14 marks.
- All answers must be written in the answer book and *not on the question paper.*

MATHEMATICAL SCIENCES

Part A

1. If $x \quad 1 :: 1 \quad 1 + x$ then x is

   (a) a rational number

   (b) an imaginary number

   (c) an irrational number

   (d) none of the above

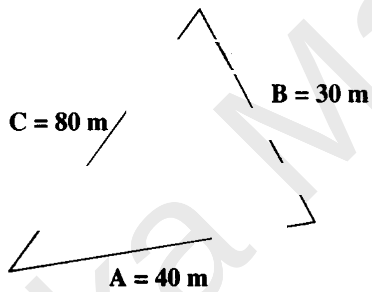

FIG. 1.

2. A surveyor measures the dimensions of a triangular piece of land (assumed to be flat) and claims that they are as shown in Figure 1. This means that the surveyor has most likely

   (a) underestimated side C

   (b) overestimated sides A and B

   (c) underestimated sides A and B, or overestimated C, or both

   (d) measured the sides correctly.

3. If x, y are nonzero real numbers, then $x^2 + xy + y^2$ is

(a) always positive

(b) always negative

(c) zero

(d) sometimes positive, sometimes negative.

4. If $\tan\theta + \cos\theta = m$ and $\tan\theta - \cos\theta = n$, then the value of $\sin 2\theta$ is

(a) $\frac{mn}{4}$

(b) $\frac{m^2n^2}{4}$

(c) $\frac{(m+n)^2(m-n)}{4}$

(d) $\frac{(m-n)^2(m+n)}{4}$

5. The area of the smallest region bounded by the curves $y = |x|$ and $x^2 + y^2 = 4$ is

(a) π

(b) 2π

(c) $\frac{3\pi}{4}$

(d) $\frac{3\pi}{2}$

6. Let $\hat{i}$, $\hat{j}$, and $\hat{k}$ be the unit vectors along the usual x, y, and z axes. A unit vector perpendicular to $\hat{i} + \hat{j} + \hat{k}$ and lying in the xy- plane is

(a) $\hat{j} - \hat{k}$

(b) $\frac{1}{\sqrt{2}}(\hat{j} - \hat{k})$

(c) $\hat{i} - \hat{j}$

(d) $\frac{1}{\sqrt{2}}(\hat{i} - \hat{j})$

7. Two spherical planets A and B have the same density, but the acceleration due to gravity at the surface of A is 1/6 of that at the surface of B. This means that the ratio of the radius of A to that of B is

(a) 1

(b) 1/6

(c) 1/36

(d) /216

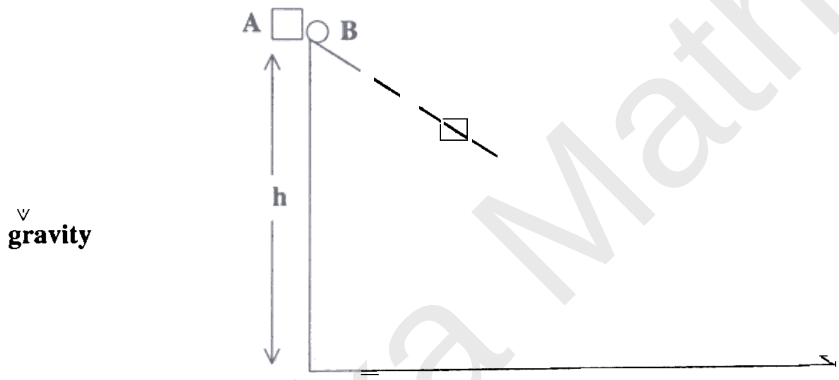

FIG. 2.

8. In Figure 2, object A is dropped vertically downwards with initial velocity 0, while object B rolls without slipping down the inclined plane. The masses of A and B are equal, and both start from the top of the inclined plane with initial velocity zero. Ignore rolling friction and air resistance. Which of the following is correct?

(a) B has higher total kinetic energy than A when they reach the bottom, because B rotates *and* translates;

(b) A will have higher total kinetic energy than B when they reach the bottom

(c) The final kinetic energy of each of the objects depends on its shape;

(d) none of the above.

9. A large plastic balloon has a volume of 300 m^3 when completely filled. Approximately how many cubic metres of helium gas, at temperature 27° C and standard atmospheric pressure, should it be filled with if it is to be completely full when it reaches its designed altitude where the pressure is 1/3 of an atmosphere, and the temperature is −53° C?
(a) 140 m^3
(b) 19600 m^3
(c) 14 m^3
(d) 1.4 m^3

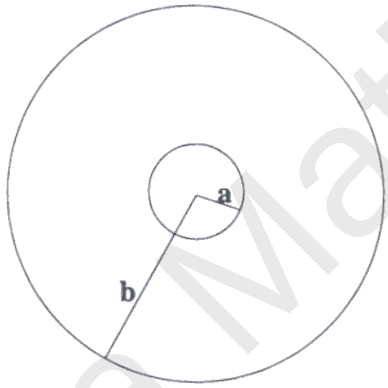

FIG. 3.

10. An iron washer (Figure 3) has an outer radius b and an inner radius a. If heated,
(a) a increases and b decreases
(b) b increases and a decreases
(c) Both a and b increase
(d) None of the above.

11 1 cc of oil is spread on a surface to form a circular film of uniform thickness with no holes. Based on what you know about the sizes of molecules, which of the following is a reasonable estimate for the maximum possible radius of the film?

(a) ∞

(b) 1 cm to 10 cm

(c) 1000 m to 3000 m

(d) 10 m to 60 m

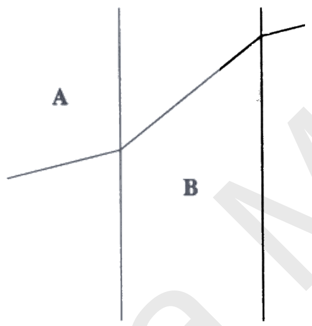

FIG. 4.

12. Figure 4 shows a ray of light passing from a medium A through a medium B and back into A. Which of the following is consistent with the figure?

(a) A is air, B is glass;

(b) A is vacuum, B is diamond;

(c) A is air, B is water;

(d) A is glass, B is air.

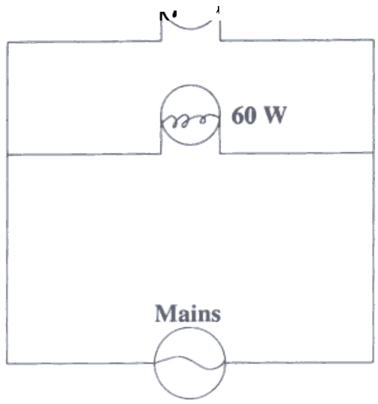

FIG. 5.

13. Figure 5 shows a 40 W and a 60 W light bulb connected to the mains (voltage fixed

(a) glows more brightly

(b) glows less brightly

(c) does not glow

(d) glows with unchanged brightness.

14. Two metal spheres of the same radius, with initial charges Q_1 and Q_2 attract each other. After they come into contact, it is observed that they repel each other. This means

(a) $Q_1 \times Q_2 < 0$, $Q_1 + Q_2 \neq 0$

(b) $Q_1 \times Q_2 > 0$, $Q_1 + Q_2 \neq 0$

(c) $Q_1 \times Q_2 > 0$, $Q_1 + Q_2 = 0$

(d) $Q_1 \times Q_2 < 0$, $Q_1 + Q_2 = 0$

15. How many unique types of hydrogen atoms are present in hexa-2,4-diene?

(a) 2;

(b) 3

(c) 4

(d) 5

16. During the sodium fusion test, nitrogen in an organic compound gets converted to

(a) cyanide;

(b) cyanate

(c) nitrogen gas;

(d) ammonia.

17. The entropy change ΔS associated with a spontaneous endothermic process satisfies

(a) $\Delta S = 0$;

(b) $\Delta S < 0$

(c) $\Delta S > 0$;

(d) $\Delta S > \Delta H$

18. In a first order chemical reaction, the concentration of the reactant decreases from 1.0 mol lit^{-1} to 0.25 mol lit^{-1} in 100 hours. The half-life of the reaction is

(a) 50 hours;

(b) 100 hours;

(c) 200 hours;

(d) 75 hours.

19. Although F is more electronegative than Cl, HF is a weaker acid than HCl because

(a) HF dimerizes;

(b) F is larger than Cl;

(c) the hydrogen bonding is stronger in HCl

(d) Cl has the higher electron affinity.

20. For a chemical reaction at equilibrium:

(a) a catalyst would shift the position of the equilibrium;

(b) a catalyst would increase the rates of forward and backward reactions;

(c) a catalyst would affect only the forward reaction;

(d) a catalyst would slow down the backward reaction.

21. Which of the following complexes will show paramagnetic behaviour?

(a) $Ni(CO)_4$;

(b) $K_2Cr_2O_7$;

(c) $KMnO_4$;

(d) $K_3[Fe(CN)_6]$.

22. The coordination numbers of calcium and fluorine in CaF_2 (Fluorite) structure are respectively

(a) 8 and 8;

(b) 6 and 6;

(c) 8 and 4;

(d) 4 and 8.

23. If there are 5 different bases in DNA and the genetic code consists of 4 bases per codon, the number of codons possible will be

(a) 125

(b) 256

(c) 625

(d) 1024

24. The concentration of carbon dioxide has been increasing steadily in recent times due to human activities. How will this affect plant productivity?

(a) Productivity will decrease because of CO_2 pollution

(b) Productivity will increase because of higher CO_2 levels.

(c) CO_2 will not change productivity.

(d) Difficult to predict the outcome.

25. Humans and apes are similar at the DNA sequence level to the extent of

(a) 50%

(b) 75%

(c) 90%

(d) $> 90\%$

26. The major protein in hair is

(a) keratin

(b) actin

(c) collagen

(d) fibrin

27. Plant cells can be distinguished from animal cells based on the fact that

(a) Plant cells have a cell membrane which is absent in animal cells

(b) Animal cells have mitochondria which are absent in plant cells.

(c Plant cells have chloroplasts that animal cells do not have.

(d) Plant cells do not have a nucleus.

28. In a double stranded DNA molecule,

(a) A+G C+T

b) A=T within each single strand

(c) G=C within each single strand

(d) All four bases are found in equal proportions

29. The strongest reason for believing that all life forms of today shared a common ancestor in the distant past is that

(a) We all have ATP

(b) The genetic code is nearly universal

(c) Life can come only from life

(d) The alternative will be absurd.

30. Glycogen belongs to the category of compounds known as

(a) Carbohydrate.

(b) Fat.

(c) Protein.

(d) nucleic acid.

MATHEMATICAL SCIENCES

PART B

1 Given a positive integer $m > 2$, show that there exist positive integers p and q such that $p < q$ and

$$\frac{1}{-} = \sum_{j=p+1}^{q} \frac{}{j(j+1)}$$

2. Find the angles α, β, γ of a triangle if they satisfy the relation

$$\sin(\tfrac{\alpha-\beta}{2}) + \sin(\tfrac{\alpha-\gamma}{2}) + \sin(\tfrac{3\alpha}{2}) \quad \tfrac{3}{2}$$

3. Find all integers a for which the cubic equation

$$x^3 - x + a \quad 0$$

has three integer roots.

4. Prove that if $p > 1, x > 0$, $\quad x^p - 1 \geq p(x - 1$

5. Show that for any $x > 0$. $\int_0^{x} \frac{\sin t}{1+t} dt > 0$.

6. Find the radius of the circle which is obtained as a section of the sphere $x^2+y^2+z^2 = 9$ by the plane $x + y + z = 3$. Also find the equation of the cone with its vertex at $(0, 0, 0)$ and containing the above circle.

7. Find all the integers x in the set $\{1, 2, 3, \quad 100\}$ such that $x^2 \equiv x \pmod{100}$

8. Solve the following equation for x:

$$\begin{vmatrix} x & p & q & 1 \\ a & x & r & 1 \\ a & b & x & 1 \\ a & b & c & 1 \end{vmatrix} = 0.$$

ENTRANCE TEST FOR ADMISSION 2001

Integrated Ph.D

Mathematical Sciences

Day & Date	:	Sunday 29th April 2001
Time	:	1.30 p.m. to 4.30 p.m.

INDIAN INSTITUTE OF SCIENCE
BANGALORE

INSTRUCTIONS

- The question paper is in two parts Part A and Part B. Part A carries 30 marks and Part B carries 70 marks.
- Part A comprises 30 multiple choice questions each carrying 1 mark. Four possible answers are provided for each question. Select the correct answer by marking (√) against (a), (b), (c) or (d) on the answer script exactly as given below.
  For example, Question: $2 + 2 =$ Answer: (a) 0 (b) 2 (√) 4 (d) 8.
  Answer all questions from Part A.
- Part B comprises 10 questions Answer any 7 questions. Each question carries 10 marks.
- All answers must be written in the answer book and *not on the question paper.*

MATHEMATICAL SCIENCES

Part A

1. The numbers $2^{800}, 3^{600}, 5^{400}, 6^{200}$ listed in the increasing order are
   (a) $2^{800}, 3^{600}, 5^{400}, 6^{200}$
   (b) $6^{200}, 2^{800}, 3^{600}, 5^{400}$
   (c) $6^{200}, 2^{800}, 5^{400}, 3^{600}$
   (d) $2^{800}, 5^{400}, 3^{600}, 6^{200}$

2. The point $(3, 4)$ in the xy-plane is reflected w.r.t the x-axis and then rotated through 90 degrees in the clockwise direction in the plane about the origin. The final position of the point is
   (a) $(3, -4)$
   (b) $(4, -3)$
   (c) $(-3, -4)$
   (d) $(-4, -3)$.

3. The maximum value of

$$10 - \sqrt{3\cos\theta - 4\sin\theta + 9}$$

   for $0 \leq \theta \leq 2\pi$ is
   (a) 8
   (b) 7
   (c) 10
   (d) $10 - \sqrt{14}$.

4. The derivative w.r.t. x of the product

$$+ x)(1 \quad x^2)(1 + x^4)(1 + x^8) \quad \cdot (1 + x^{2^n}$$

   at $x = 0$ is
   (a) 0
   (b) 1
   (c) n
   (d) $n +$

5. If z is a complex number for which $|z-3-4i| \leq 2$ then the maximum value of $|z|$ is

(a) 2

(b) 5

(c) 7

(d) 9.

6. If $I = \int_0^1 e^x dx$, then which of the following is true?

(a) $I < 1$

(b) $1 < I < 2$

(c) $2 < I < e$

(d) $I > e$.

7. Let f be the real function defined by

$$f(x) = \begin{cases} ax+b & \text{if } x < -1; \\ x^2+1 & \text{if } -1 \leq x \leq 1 \\ -ax+b & \text{if } x > 1, \end{cases}$$

where a, b are real numbers. If f is continuous on the real line then the product ab is equal to

(a) 2

(b) -4

(c) -2

(d) 0.

8. A heavy ball tied to a string spins around in a circle. While the ball is spinning, the length of the string is slowly halved. The angular frequency of rotation of the ball is

a) halved

b) doubled

c) quadrupled

d) unchanged

9. Unpolarized light passes through three polarizing filters. The axis of the second one is at an angle of +30° with respect to the first, and the axis of the third is at an angle +30° with respect to the second. The fraction of the original intensity that emerges from the third polarizer is

a) 9/32

b) 3/8

c) 2/9

d) 1/8

A violin string that is 22cm long and weighs 0.8g has a fundamental frequency of 960Hz. The speed of sound in air is 320m/s. The wavelength of the sound waves (in air) emitted when the string vibrates at its fundamental frequency is

a) 22cm

b) 33cm

c) 44cm

d) 88cm

11. Two large metal spheres, A and B, are near each other. The electrostatic force between them is attractive. Of the three possibilities:

i) the two spheres are oppositely charged

ii) one sphere is charged and the other is uncharged

iii) both spheres are uncharged

a) Only case i) is possible

b) Cases i) and ii) are possible, but not iii)

c) All three cases are possible

d) It depends on the size of the spheres compared to their separation.

Figure 1:

12. An object is placed between two mirrors at right angles to each other as shown. How many images are formed by the mirrors in each case?

a) 3 and 2

b) 3 and 3

c) 2 and 2

d) 3 and 0

13. A resistor, inductor and a capacitor are connected in series to an ac voltage source $v(t) = V\cos[2\pi\nu t]$. The peak voltages across the three elements are V_R, V_L and V_C.

a) V_R, V_L and V_C must be less than V.

b) V_R must be less than V, but V_L and V_C need not.

c) At any instant, the voltage across the resistor and the voltage from the source must have the same sign.

d) At any instant, the voltage across the resistor must be smaller in magnitude than the voltage from the source.

14. A slab of ice at 0°C is placed in a beaker of water at 0°C. (Take the melting point of ice to be 0°C.) Ignore heat exchange with the surroundings (air, etc.).

a) Some of the ice will melt to water if there is more water.

b) Some of the ice will melt and some of the water will also freeze.

c) Both the water and the ice will remain unchanged.

d) There is not enough information to decide between these.

15. Two spheres of radius r_1 and r_2, and at temperatures T_1 and T_2, are placed in vacuum. The first sphere is a blackbody. The second sphere may absorb more heat from the first than it radiates out if

a) $T_1 = T_2$, but r_1 is sufficiently large compared to r_2.

b) $T_1 = T_2$, but the second sphere is painted, with a colour matching the peak of the radiation from the first.

c) $T_1 > T_2$.

d) None of the above.

16. The pH of 10^{-10} molar solution of HCl is:

a) 10

b) 7

c) 4

d) 1

17. The molecular weight of $MgCl_2$ determined from elevation of boiling point experiment is (atomic masses of Mg and Cl are 24 and 35.5 respectively):

a) 47.5

b) 95.0

c) 63.4

d) 31.7

18. In a monoatomic body-centered cubic lattice with lattice constant a, the closest distance of approach between the atoms is:
   a) a
   b) $a\sqrt{2}$
   c) $a\sqrt{3}/2$
   d) $a/2$

19. The maximum number of electrons in an atom that can possess a principal quantum number of 4 is:
   a) 8
   b) 14
   c) 18
   d) 32

20. The empirical formula of the inorganic compound whose molecular structure most resembles that of benzene:
   a) HBS
   b) $PNCl_2$
   c) SN
   d) BNH_2

21. Aldol condensation is carried out under:
   a) acidic conditions
   b) basic conditions
   c) neutral conditions
   d) pyrolytic conditions

22. Enolisation involves:
   a) resonance
   b) complexation
   c) tautomerisation
   d) aromatisation

23. In DNA, the G-C base pairs are stronger than A-T base pairs because of
   a) their partial double bond character
   b) the presence of an additional hydrogen bond
   c) hydrophobic effect
   d) their covalent nature

24. Erythrocytes when placed in a hypotonic solution will

a) shrink

b) burst

c) first shrink and then burst

d) not show any effect

25. A protein has 3 glutamic acid and 4 lysine residues. It has no other charged residues. The pI of the protein is likely to be

a) 3

b) 4

c) 7

d) 8

26. The sequence of which of the following is used to establish phylogenetic relationships between organisms ?

a) DNA Polymerase protein

b) Actin gene

c) Ribosomal gene

d) Hexokinase gene

27. PKU is one of the best known hereditary disorders in amino acid metabolism. The defect is attributed to a lesion in one of the following enzymatic activities

a) Phenylalanine ammonia lyase

b) Phenylalanine hydroxylase

c) Tyrosine hydroxylase

d) Phenylalanine transaminase

28. Which of the following have the highest basal metabolic rate ?

a) Blue Whale

b) Cheetah

c) Humming Bird

d) Eagle

29. The place where an organism lives is known as its

a) home range

b) biome

c) habitat

d) community

30. Analysis of paleoclimatological data indicate that environments during the last 100,000 years

a) have essentially the same as they are now

b) have been consistently warming

c) have been consistently cooling

d) have fluctuated repeatedly from warm to cold

MATHEMATICAL SCIENCES

PART B

1. a) Show that the real function $f(x) = x|x|$ is differentiable everywhere on the real line.

   b) Let a, b be two non-zero complex numbers. If $az + b\bar{z} = 0$ represents a straight line in the plane then show that $|a| = |b|$. (Here $z = x + iy$ in the plane.)

2. Let a, b, c be three complex numbers such that

$$a^2 + b^2 + c^2 = a^3 + b^3 + c^3 = a^4 + b^4 + c^4 = 0.$$

   Show that $a = b = c = 0$.

3. a) Let ρ be a non-trivial relation on a non-empty set A (i.e., there exist $a, b \in A$ such that $a \rho b$ holds). If ρ is symmetric and transitive then show that there exists a non-empty set $B \subseteq A$ such that ρ is an equivalence relation on B.

   b) Let A be a non-empty finite set. If $f : A \to A$ is a bijection (i.e., one-to-one and onto) and $a \in A$ then show that there exists $n \geq 1$ such that $f^{(n)}(a) = a$. [Here $f^{(1)} \equiv f$ and for $n \geq 2$, $f^{(n)}(x) = f(f^{(n-1)}(x))$.]

4. Let '$*$' be a binary operation on a non-empty set S. If

$$x * y = y^n * x,$$

   for some positive integer n (≥ 2), then show that

   (i) $x^n = x^{n^2}$ for all $x \in S$ and

   (ii) $x * y = y * x$ for all $x, y \in S$.

5. Let

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix}$$

   be two real matrices. For $1 \leq i \leq 3$, let P_i be the plane given by $a_i x + b_i y + c_i z + d_i = 0$. Show that $P_1 \cap P_2 \cap P_3$ is a line if and only if A and B have the same rank and this common rank is equal to 2.

6. Let F, F' be the foci of an ellipse and P a point on the ellipse. Show that PF and PF' are equally inclined to the tangent at P to the ellipse.

7. Evaluate

$$\int_0^1 x f''(x)dx,$$

if

$$f(x) \quad \int_0^x t\, e^{-t^2} dt.$$

8. Find the solution of the system

$$\frac{dy_1}{dt} \quad 1 - \frac{1}{y_2}$$

$$\frac{dy_2}{dt} - \frac{1}{y_1 - t},$$

$$y_1(0) = y_2(0) = 1$$

9. Compute approximately the value of π using Simpson's rule (with four equal subintervals of the interval $[0, 1]$) on the integral

$$\int_0^1 \frac{dx}{1+x^2}.$$

10. a) Find the number of positive integers n such that $1 \le n \le 2000$ and $\gcd(2000, n) = 40$.

b) Find the number of positive integers m such that $1 \le m \le 2000$ and $\operatorname{lcm}(250, m) = 2000$.

INDIAN INSTITUTE OF SCIENCE
BANGALORE - 560012

ENTRANCE TEST FOR ADMISSIONS - 2002

Program Integrated Ph.D

Entrance Paper Mathematical Sciences

Day & Date
SUNDAY 28TH APRIL 2002

Time
1.30 P.M. TO 4.30 P.M.

MATHEMATICAL SCIENCES

General Instructions

- The question paper is in two parts: Part A and Part B
- Part A carries 30 marks and Part B carries 70 marks.
- There is no negative marking.
- All answers must be written in the answer book and *not on the question paper*

Notations: The set of natural numbers, integers, rational numbers, real numbers and complex numbers are denoted by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ respectively.

Part A

Part A consists of 30 multiple choice questions each carrying mark.

Answer all questions from Part A.

- Four possible answers are provided for each question ick √) the correc answer against A, B, C or D on page 3 of the answer book.

If $(a, b) \neq (0,0)$ then the real polynomial $x + ax + b$ must have

A. only real zeros.

B. only non-real complex zeros.

C. a real zero.

D. a non-real complex zero.

For any integer $n \geq 3$, the value of $\frac{1}{n+1} + \frac{1}{n+2} +$ $\frac{1}{2n}$ is always

A

B.

C. n

D $\frac{2}{3n}$

3. Let ρ be a non-trivial relation on a set X If is ymmetric and antisymm then ρ is

A reflexive

B. transitive

C. an equivalence relation.

D. the diagonal relation (i.e $\rho y \Leftrightarrow x$ y)

Let $f: \mathbb{Z}$ $\mathbb{R}$ be defined by $f(x)$ x^3 $3x$ 1 Then f is

A not a function

B surjective (onto) function

an injective (one-to-one) function.

function but neither injectiv nor surjectiv

Suppose f $\mathbb{R}$ $\mathbb{R}$ is defined by .2001 hen

A does not hav inverse over whole of $\mathbb{R}$.

B. has no inverse outside a finite open subset of $\mathbb{R}$.

C. has no inverse outside a finite closed subset of $\mathbb{R}$

D. has inverse over the whole of $\mathbb{R}$.

6. The set $\{5, 15, 25, 35\}$ is a group under multiplication modulo 40 The identity element of this group is

A. 5.

B. 15

C. 25

D. 35

Order of the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 6 & 1 & 2 & 3 & 4 & 5 \end{pmatrix}$ is

A. 3.

B. 4.

C. 7.

D. 12.

8. Let $\mathbb{Z}_n$ be the additive group of integers modulo n The number of homomorphisms from $\mathbb{Z}_n$ to itself is

A. 0.

B. 1.

C. n.

D. n^2

9. The number of non-isomorphic abelian group(s) of order 15 is

A. 1

B. 2

C. 3

D. 4

10. Let R be a commutative ring. An element $x \in R$ is said to be nilpotent if $x^n = 0$ for some positive integer n. If x and y in R are such that x and $x + y$ are nilpotents then y must be

A. the additive identity of R.

B. the multiplicative identity of R

C. x^m, for some integer m

D. nilpotent.

1 The characteristic polynomial of the matrix $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ is

A. $x(x^2+1)$.

B. $x(x-1)^2$

C. $x(x+1)^2$.

D. $x(x^2-1)$

12. Let $v = (1,1)$ and $w = (1,-1) \in \mathbb{R}^2$ Then a vector $u = (a,b) \in \mathbb{R}^2$ is in the $\mathbb{R}$-linear span of v and w

A. only when $a = b$.

B. always.

C. for exactly one value of (a,b).

D. for at most finitely many values of (a,b)

13. The dimension of the vector space $\{(x,y,z,w) \in \mathbb{R}^4 \quad w, x+z = y = z+w\}$ is

A 0.

B.

C. 2

D.

14. Let A be a 3×3 real matrix. Suppose $A^4 = 0$. Then A has

A. exactly two distinct real eigenvalues.

B. exactly one non-zero real eigenvalue.

C. exactly 3 distinct real eigenvalues.

D. no non-zero real eigenvalue.

15. Let a, b, c, d be real numbers and let $f : \mathbb{C} \to \mathbb{C}$ be the map defined by $f(x+iy)$ $(ax+by) + i(cx+dy)$. Then f is linear over $\mathbb{C}$ if and only if

A. $a = d$ and $b = c$.

B. $a = d$ and $b = -c$

C. $a = -d$ and $b = c$

D. a d and b c

16. Let C_1 and C_2 be two distinct ellipses in the plane. If C_1 and C_2 have a common tangent at a common point P then the number of distinct common points of C_1 and C_2 must be

A. 1.

B. 1 or 2.

C. 1, 2 or 3.

D. 1, 2, 3 or 4.

17. Let P $\{(x, y, z) \in \mathbb{R}^3 : x = 0\}$, $Q = \{(x, y, z) \in \mathbb{R}^3 : y$ $0\}$, R = $\{(x, y, z) \in \mathbb{R}^3$ $x + y$ $1\}$ be three planes in $\mathbb{R}^3$. Let $l = P \cap R$ and m $Q \cap R$. Then l and m

A. are two skew lines.

B. are two parallel lines.

C. intersect at the origin.

D. are perpendicular to each other

18. Let S be unit sphere with center $(0, 0, 1$ in $\mathbb{R}^3$ and P be the plane z Then the equation of $S \cap P$ is

A. $x^2 + y^2$ $\frac{3}{4}$, $z = \frac{1}{2}$

B. $x^2 + y^2$ 1, $z = \frac{1}{2}$

C. $x^2 + y^2$ $2x = 1$, z $\frac{1}{2}$

D. $x^2 + y^2$ $2y = \frac{3}{4}$, z $\frac{1}{2}$

19. The three lines $ax + a^2y$ $bx + b^2y$ $cx + c^2y$ in $\mathbb{R}^2$ are concurrent if and only if

A. $a = b = c$.

B. two of a, b, c are equal

C. a, b, c are all distinct.

D. $a = c^2$ and $b = c^3$.

20. The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x)$ $\max\{1 - |x$ $0\}$ is differentiable

A. at all points.

B. at all except one point.

C. at all except three points.

D. nowhere.

/21 Let $f: [0, 1] \to \mathbb{R}$ be a continuous function with $f(0) = f(1)$. If f is differentiable on $(0, 1)$ and the derivative f' is continuous on $(0, 1)$ then f' is

A. strictly positive in $(0, 1)$.

B. strictly negative in $(0, 1)$.

C. identically zero in $(0, 1)$.

D. zero at some point in $(0, 1)$

22. The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent for

A. all $p > 0$.

B. for only $p = 1$.

C. for all $p > 1$.

D. for all integer values of p.

23. Let $\mathbf{V}$ $\mathbb{R}^3 \to \mathbb{R}^3$ be the vector field defined by

$$\mathbf{V}(x_1, x_2, x_3) := (x_1^2 + x_2^2, x_1x_2 + x_2x_3, x_2^2 + x_1x_3)$$

The divergence of $\mathbf{V}$ is

A. $4x_1 + x_3$.

B. 0.

C. $x_1^2 + x_2^2 + 2x_1x_3$

D. $(2x_1, x_1 + x_3, x_1$

/ 24 A unit normal vector to the curve $\mathbf{C} := \{(x, x^2) \ x \in \mathbb{R}\}$ in the plane $\mathbb{R}^2$ at the point $(0, 0)$ is given by

A. $(0, -1)$.

B. $(-1, 0)$.

C. $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$

D. $(1, 0)$.

25. The number of zeros of the function $f(x) = \sin x \cos x$ in $(0, n\pi)$ is

A. $n + 1$.

B. $2n - 1$

C. $2n$.

D. $2n + 1$

26. The function $f : [-1, 1] \to \mathbb{R}$ defined by $f(x) \quad 1 - x^2$ has

A. no local maxima or minima in

B. has exactly one local maximum and two local minima in $(-1, 1)$

C. has exactly one local maximum in $(-1, 1)$.

D. has exactly one local minimum in $(-1, 1)$.

27. If $f(x, y) \quad x^7 + 100x^5y^2 + 200xy^6 + 10y^7$ then $x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2\frac{\partial^2 f}{\partial y^2}$ is =

A. $42x^7 + 4200x^5y^2 + 8400xy^6 + 420y^7$.

B. $42x^7 + 500x^5y^2 + 200xy^6 + 10y^7$.

C. $42x^7 + 1000x^5y^2 + 1200xy^6 + 420y^7$

D. $7x^7 + 700x^5y^2 + 1400xy^6 + 70y^7$.

28. A solution of the differential equation $\frac{dy}{dx} = y + 1$ is

A. $y = e^x - 1$.

B. $y = e^x + 1$.

C. $y = e^x + x$

D. $y = e^{x-1}$.

29. The differential equation $\frac{d^2y}{d^2x} + 4\frac{dy}{dx} + 4y = 0$ has general solution of the form

A. $A\cos 2x + B\sin 2x$

B. $Ae^{-2x} + Bxe^{-2x}$.

C. $Ae^{2x} + Bxe^{2x}$.

D. $Ae^{2x} + Be^{-2x}$.

30. The iteration $x_{n+1} = x_n^2 - 2$, $x_n \geq 0$ for $n \geq 1$ will converge to the solution $x \quad 2$ of the equation $x^2 - x - 2 = 0$ if and only if x_1 is

A. close to 2 from the left.

B. close to 2 from the right.

C. equal to 2.

D. equal to $\sqrt{2}$.

Part B

- Part B comprises 24 questions. Each question carries 5 marks
- Answer **any 14** full questions only.
- Only the **first 14 answered** questions will be evaluated
- Answer should be to the point.

---

1. Let a, b be real numbers and let $f, g\colon \mathbb{R} \to \mathbb{R}$ be the functions defined by

$$\begin{aligned} f(x & = ax + b \text{ and} \\ g(x & = x^2, \end{aligned}$$

respectively. Show that $f \circ g \quad g \circ f$ if and only if $(a, b) = (0, 0)$, $(0, 1)$ or $(1, 0)$

2. For an integer $n \geq 4$, compute the $n \times n$ determinant

$$\begin{array}{ccccc} & 2 & 3 & & n \\ 1 & 2^3 & 3^3 & & n^3 \\ . & & & & \\ 1 & 2^{2n-1} & 3^{2n-1} & & n^{2n} \end{array}$$

3. For all $n \in \mathbb{N}$ and for all positive real numbers x, y, show that

$$\left(1 + \frac{x}{y}\right) + \left(1 + \frac{y}{x}\right) \geq 2^{n+1}$$

4. Let ρ be a relation on a non-empty set X. For $Y \subseteq X$, let

$$\mathrm{N}(Y) := \{x \in X \mid \text{ there exists } y \in Y \text{ such that } y\rho x\}$$

Show that ρ is reflexive if and only if $Y \subseteq \mathrm{N}(Y$ for all $Y \subseteq X$

5. Let $f\colon \mathbb{Z} \to \mathbb{R}$ be a function. If $f(n) \quad f(n + 11) = f(n + 18)$ for all $n \in \mathbb{Z}$ then show that f is a constant function.

6. Let '+' and '·' be the operations on the set $C[0, 1]$ of continuous real valued functions on $[0, 1]$, defined by

$$\begin{aligned} (f + g)(x) & \quad f(x) + g(x) \\ (f \cdot g)(x) & = f(x) \cdot g(x), \end{aligned}$$

for all $x \in [0, 1]$. Show the following.
(a) $(C[0, 1], +, \cdot)$ is a ring. [3 marks]
(b) $(C[0, 1], +, \cdot)$ has a divisor of zero. [2 marks]

7. If

$$x_2 + x_3$$

$$x_{98} + x_{99} + x_{100} \quad 0.$$
$$x_{99} + x_{100} + x_1 \quad 0.$$
$$x_{100} + x_1 + x_2 \quad 0.$$

then show that $x_1 \quad x_2 \quad = x_{99} = x_{100} = 0.$

8. Let $f: \mathbb{R}^2 \to \mathbb{R}^3$ be defined by $f(x, y) := (x, x + y, y)$.
(a) Show that f is linear. [2 marks]
(b) Find the range and the kernel of f. [3 marks]

9. Let P, Q and R be three non-collinear points in the plane. Show that every point X in the plane can be uniquely written as $X = a_1 P + a_2 Q + a_3 R$, where a_1, a_2, a_3 are real numbers with $a_1 + a_2 + a_3 = 1$.

10. Find the volume of the largest (right circular) cone that can be inscribed in a sph of radius $R > 0$.

Let $f, g: \mathbb{R} \to \mathbb{R}$ be two continuous functions. Show that the function $h: \mathbb{R} \to \mathbb{R}$ defined by

$$h(x \quad \max\{f(x \quad g(x)\} \text{ for } x \in \mathbb{R}.$$

is continuous.

12 Let $f: [a, b] \to \mathbb{R}$ be a continuous function on the closed interval $[a, b]$ $\mathbb{R}$ with $f(a) = f(b)$. Show that there exists $c \in \left[a, \frac{a+b}{2}\right]$ such that $f\left(c + \frac{b-a}{2}\right) = f(c$

1 Let $f: [0, 1] \to \mathbb{R}$ be a continuous function. Suppose hat $f(r) = r^3 + 99r + 100$ for every rational number $r \in [0, 1]$. Prove that $f(x) = \ ^3 + 99x + 100$ for all $x \in [0, 1]$.

14. Does the series $\sum_{n=1}^{\infty} \frac{(n!)^2 5^n}{(2n)!}$ converge? Justify your answer.

15. If a sequence a_n, $n \in \mathbb{N}$ of real numbers is monotone decreasing and the series $\sum_{n=0}^{\infty} a_n$ is convergent, then show that the sequen $na_n \quad n \in \mathbb{N}$ converges to 0

16. Show that the function $f: \mathbb{R} \to \mathbb{R}$, $f(x) := x^3 + 2x +$ is strictly increasing and compute the derivative $(f^{-1})'(1)$ of the inverse function f^{-1} at the point $1 = f(-1)$.

17. Let $f: \mathbb{R} \to \mathbb{R}$ be a function which is 3-times differentiable in a neighbourhood of 0 and $f(0) = 0$. Show that the function $g: \mathbb{R} \to \mathbb{R}$, defined by

$$g(x \quad \begin{cases} f(x) & \text{if } x \neq 0 \\ f'(0) & \text{if } x \end{cases}$$

is differentiable at 0 and $g'(0) = \frac{1}{2}f''(0)$

18. For $n \in \mathbb{N}$, let

$$a_n := \int_0^{\pi/2} \sin^n t \, dt$$

Show the following

(a) $(n+1)a_{n+1} = na_{n-1}$ for $n \geq 1$ [2 marks]

(b) $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$ [3 marks]

19. Let $f: \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ be the map defined by

$$f(v, w) := v \times w \quad (\text{the vector product of } v \text{ and } w)$$

Show that f is surjective (onto)

20. Let $f: (0, \infty) \to \mathbb{R}$ be the function defined by $f(x) = x^x$. Find local maxima and minima of f.

21 Find out all the local maxima, local minima and points of inflection of the function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 3x^5 - 5x^3 + 15$.

22. Show that any solution y of the differential equation

$$\frac{dy}{dx} = \sin y$$

on an interval $[0, a]$ statisfies

$$|y(x) - y(0)| \leq x \quad \text{for all} \quad x \in [0, a]$$

23. Describe the Euler numerical scheme and the Runge-Kutta method of order 2 for solving the differential equation

$$\begin{aligned} \frac{dy}{dx} &= f(y) \quad x \in \mathbb{R} \\ y(0) &= y_0, \end{aligned}$$

where $f: \mathbb{R} \to \mathbb{R}$ is differentiable and the derivative f' is continuous on $\mathbb{R}$. Explain also why the Runge-Kutta method is preferred to the Euler method.

24. Compute the area of the region

$$R := \left\{(x, y) \in \mathbb{R}^2 \quad \max\{|x|, |y|\} \leq 1 \quad 4xy \leq 1\right\}$$

INDIAN INSTITUTE OF SCIENCE
BANGALORE - 560012

ENTRANCE TEST FOR ADMISSIONS - 2003

Program Integrated Ph.D

Entrance Paper Mathematical Sciences

Day & Date
SUNDAY 27th APRIL 2003

Time
1.30 P.M. TO 4.30 P.M.

General Instructions

- This question paper has two parts : Part A and Part B .
- Part A Carries 30 marks and Part B carries 70 marks.
- All answers must be written in the answer-book and **not on the question paper**.

Notation

The set of natural numbers, integers, rational numbers, real numbers and complex numbers are denoted by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ respectively.

Integrated Ph. D. / Mathematical Sciences

Part A

- Part A consists of 30 multiple choice questions, each carrying 1 mark.
- Answer **all** questions.
- Four possible answers are provided for each question. Tick ($\surd$) against correct answer, namely, A, B, C or D on the Page 3 of the answer book.

---

1. The number of reflexive relations on the set $\{1, 2, \ldots, n\}$ is

(A) $2^{n(n-2)}$.

(B) $2^{n(n-1)}$.

(C) 2^{n^2}.

(D) $2^{n(n+1)}$.

2. For any two natural numbers n and k, the number of all k-tuples $(a_1, \ldots, a_k) \in \mathbb{N}^k$ with $1 \leq a_1 \leq a_2 \leq \cdots \leq a_k \leq n$ is

(A) $\binom{n}{k}$.

(B) $\binom{n+k}{k}$.

(C) $\binom{n+k-1}{k}$.

(D) $\binom{n+k+1}{k}$.

3. The probability that a hand of 5 playing cards contains at least two aces is

(A) $\frac{\binom{4}{2}\cdot\binom{48}{3}}{\binom{52}{5}}$.

(B) $\frac{\binom{4}{2}+\binom{48}{3}}{\binom{52}{5}}$.

(C) $\frac{\binom{4}{2}\cdot\left[\binom{48}{3}+\binom{48}{2}+\binom{48}{1}\right]}{\binom{52}{5}}$.

(D) $\frac{\binom{4}{2}\cdot\binom{48}{3}+\binom{4}{3}\cdot\binom{48}{2}+\binom{4}{4}\cdot\binom{48}{1}}{\binom{52}{5}}$.

4. Let a, b, c, d be rational numbers with $ad - bc \neq 0$. Then the function $f : \mathbb{R} \setminus \mathbb{Q} \to \mathbb{R}$ defined by $f(x) := \frac{ax+b}{cx+d}$ is

(A) onto but not one-one.

(B) one-one but not onto.

(C) neither one-one nor onto.

(D) both one-one and onto.

A

5. The supremum of the set $\left\{\frac{n^2}{2^n} \quad n \in \mathbb{N}\right\}$

(A) is $\frac{9}{8}$.

(B) is 1

(C) is 0.

(D) does not exist.

6. Let $n \in \mathbb{N}$. Then the complex number $\left(\frac{1+i}{\sqrt{2}}\right)^n$ is purely imaginary if and only if

(A) $n \equiv 0 \pmod 4$.

(B) $n \equiv 1 \pmod 4$.

(C) $n \equiv 2 \pmod 4$.

(D) $n \equiv 3 \pmod 4$.

(For $a, b, m \in \mathbb{Z}, m > 1$, $a \equiv b \pmod m$ means that m divides $a - b$

7. The equation $\frac{1}{1+x^2} = \sqrt{x}$, $x \geq 0$ has

(A) no real solution.

(B) exactly one real solution.

(C) exactly 3 real solutions.

exactly 5 real solutions.

8. Let $p(X) := X^n + a_1 X^{n-1} + \cdots + a_{n-1} X + a_n$ be a real polynomial of degree $n \geq 1$. If n is even and a_n is negative, then

(A) f has at least one positive and one negative real zero.

(B) all real zeros of f are positive.

(C) all real zeros of f are negative.

f has no real zeros.

9. The points of intersection of the two plane curves defined by the equations $y^2 = a^2$ and $(y - bx)^2 = c^2$, $a, b, c \in \mathbb{R}$, $b \neq 0$ are vertices of

(A) an equilateral triangle.

(B) a square.

(C) a reactangle.

a paralleogram.

10. Let V be the $\mathbb{R}$-vector space of all functions $\mathbb{R} \to \mathbb{R}$ and let W be the $\mathbb{R}$-subspace of V generated by the functions $\sin t$, $\sin(t+1)$, $\sin(t+2)$. Then the dimension of W is

(A) 0
(B) 1
(C) 2
(D) 3

11. Let $n \in \mathbb{N}$, $n \geq 3$ and let $x_k := (kn+1, kn+2, \ldots, kn+n)$, $k = 0, 1, \ldots, n-1$. Then the maximal linearly independent subsequence of the sequence $x_0, x_1, \ldots, x_{n-1} \in \mathbb{R}^n$ has the length

(A) 1.
(B) 2.
(C) $n-1$
(D) n

12. For real numbers a, b, c, the following linear system of equations

$$\begin{aligned} x + y + z &= 1 \\ ax + by + cz &= 1 \\ a^2x + b^2y + c^2z &= 1 \end{aligned}$$

has a unique solution if and only if

(A) $b = c$ and $b \neq a$.
(B) $a = b$ and $a \neq c$.
(C) $a = c$ and $a \neq b$.
(D) $a \neq b$, $b \neq c$ and $a \neq c$

13. For $r, s \in \mathbb{N}$, the signature of the permutation

$$\sigma := \begin{pmatrix} 1 & 2 & & r-1 & r & r+1 & r+2 & & r+s \\ s+1 & s+2 & & s+r-1 & s+r & 1 & 2 & & s \end{pmatrix}$$

is

(A) $(-1)^{rs}$.
(B) $(-1)^{r+s}$
(C) $(-1)^{r}$.
(D) $(-1)^{s}$.

14. The number of subgroups in the cyclic group of order 360 is

(A) 6.

(B) 8.

(C) 12.

(D) 24.

15. Let m be an odd integer > 6. Then the multiplicative inverse of 2 in the ring $(\mathbb{Z}_m, +_m, \cdot_m)$ (where $+_m$ and $\cdot_m$ denote the addition and multiplication modulo m respectively.)

(A) does not exist.

(B) is $\frac{m-1}{2}$.

(C) is $\frac{m+1}{2}$.

(D) is $m-2$.

16. The power set $\mathfrak{P}(X)$ of a set X with the binary operations symmetric difference[1] Δ and intersection $\cap$ form a ring (the symmetric difference is the addition and the intersection is the multiplication) called the power set ring of the set X. If the set X has at least 3 elements, then in the power set ring $(\mathfrak{P}(X), \Delta, \cap)$ of X, every element is

(A) a unit.

(B) idempotent.

(C) nilpotent.

(D) a non-zero divisor.

17. The polynomial $f(X) := X^3 + aX + 1$ in $\mathbb{Z}_3[X]$ is

(A) irreducible in $\mathbb{Z}_3[X]$ if and only if $a = -1$.

(B) irreducible in $\mathbb{Z}_3[X]$ if and only if $a = 0$.

(C) irreducible in $\mathbb{Z}_3[X]$ if and only if $a = 1$.

(D) always reducible in $\mathbb{Z}_3[X]$.

18. Let x be a rational number which is not an integer. Then the sequence $a_n(x) := nx - [nx]$, $n \in \mathbb{N}$, (for any real number y, the bracket $[y]$ denote the largest integer $\leq y$) has

(A) infinitely many limit points.

(B) at least 2, but finitely many limit points.

(C) exactly one limit point.

(D) no limit point.

---

[1]For $A, B \in \mathfrak{P}(X)$, the subset $A \Delta B := (A \setminus B) \cup (B \setminus A)$ is called the symmetric difference of A and B

19. The sequence $\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots,$

(A) is a divergent sequence.

(B) is convergent and its limit is $\leq \sqrt{2}$.

(C) is convergent and its limit is $\geq 3/2$.

(D) is convergent and its limit is ≥ 4.

20. Let $f : (0, \infty) \to \mathbb{R}$ be the function defined by $f(x) := \dfrac{e^x}{x^x}$. Then the limit $\lim_{x \to \infty} f(x)$

(A) does not exist.

(B) exists and is 0.

(C) exists and is 1.

(D) exists and is e.

21. The series $\sum_{n=1}^{\infty} \dfrac{(-1)^n n!}{n^n}$ is

(A) absolutely convergent.

(B) conditionally convergent

(C) oscillatory.

(D) divergent.

22. Let $f : [-1, 1] \to \mathbb{R}$ be defined by

$$f(x) := \begin{cases} x, & \text{if } x \in \mathbb{Q},\ x > 0, \\ -x, & \text{if } x \in \mathbb{Q},\ x \leq 0, \\ 0, & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Then f is

(A) neither left continuous nor right continuous at 0.

(B) left continuous but not right continuous at 0.

(C) right continuous but not left continuous at 0.

(D) continuous at 0.

23. If tangent at the origin to the curve defined by the equation $y = ax + bx^2 + cx^3$ passes through the point (a, b), then

(A) $b = -a^2$.

(B) $b = a^2$.

(C) $b = -a$.

(D) $b = a$.

24. Let $x(t)$ and $y(t)$ be two non-constant differentiable real valued functions on $\mathbb{R}$ such that

$$\frac{dx(t)}{dt} = -y(t) \quad \text{and} \quad \frac{dy(t)}{dt} = x(t)$$

Then the plane curve $t \mapsto (x(t), y(t))$ is

(A) a constant curve.
(B) a straight line.
(C) a circle.
(D) a parabola.

25. The derivative of the function $\mathbb{R}^+ \to \mathbb{R}$, $x \mapsto x^x$ is

(A) $(\ln x + 1)\, x^x$
(B) $(\ln x + x)\, x^x$
(C) $\left(\ln x + \frac{1}{x}\right.$
(D) $x \cdot x^{x-1}$.

26. Let α, β be two real numbers and $\beta > 0$. The function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} 0, & \text{if } x \leq 0, \\ x^\alpha \sin(1/x^\beta), & \text{if } x > 0. \end{cases}$$

is differentiable at 0 if and only if

(A) $\alpha = \beta$
(B) $\alpha > \beta$
(C) $\alpha < \beta$
(D) $\alpha > 1$

27. Let $f : \mathbb{R} \to \mathbb{R}$ be a function which is differentiable at $a \in \mathbb{R}$ and $f(a) \neq 0$ Then the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) := |f(x)|$ is

(A) differentiable at a and $g'(a) = f'(a)$.
(B) differentiable at a and $g'(a) = -f'(a)$.
(C) differentiable at a and $g'(a) = \text{sign}(f(a)) \cdot f'(a)$
(D) not differentiable at a.

28. The function $f(x) := ax^2 + bx + c$, $a, b, c \in \mathbb{R}$, $a \neq 0$ with $\lim_{x \to \infty} f(x) = \infty$ has

(A) a unique point of minimum in $\mathbb{R}$.
(B) a unique point of maximum in $\mathbb{R}$.
(C) exactly two points of minimum in $\mathbb{R}$.
(D) exactly two points of maximum in $\mathbb{R}$

29. Let $F : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}^3$ be the vector field defined by

$$F(x) := \frac{x}{\|x\|},$$

where $x = (x_1, x_2, x_3) \in \mathbb{R}^3 \setminus \{0\}$ and $\|x\| := \sqrt{x_1^2 + x_2^2 + x_3^2}$. Then the divergence $\operatorname{div} F(x)$ of $F(x)$ is

(A) $\|x\|$.

(B) $1/\|x\|$.

(C) $2 \cdot \|x\|$.

(D) $2/\|x\|$.

30. For a partial differentiable vector field $v = (v_1, v_2, v_3) : U \to \mathbb{R}^3$ defined on an open subset $U \subseteq \mathbb{R}^3$, the vector product $\nabla \times v$ of ∇ and v is called the r o t a t i o n f i e l d of v, where $\nabla := \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right)$. For a two times continuously partially differentiable function $f : U \to \mathbb{R}$, the rotation field of the gradient field $\operatorname{grad} f$ of f is

(A) $\operatorname{grad} f$.

(B) $2 \cdot \operatorname{grad} f$

(C) $(0, 0, 0)$.

(D) $(1, 1, 1)$.

Part B

- Part B consists of 24 questions, each carrying 5 marks.
- Answer **any 14** questions.
- Only the **first 14 answered** questions will be evaluated.

---

1. For any natural number $n \geq 1$ prove the formula $\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$

2. For every real number $b > 1$, show that there exists a nautral number n_0 such that $b^n > n$ for all natural numbers $n \in \mathbb{N}$ with $n \geq n_0$.

3. Let $\leq$ denote the product order on $\mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$, i.e. for two tuples (x_1, x_2), $(y_1, y_2) \in \mathbb{N}^2$, $(x_1, x_2) \leq (y_1, y_2)$ if and only if $x_1 \leq y_1$ and $x_2 \leq y_2$. Show that every subset X of $(\mathbb{N}^2, \leq)$ has only finitely many minimal elements.

   (**Hint :** One may assume that if $x \in X$ and $x \leq y$, $y \in \mathbb{N}^2$, then $y \in X$.)

4. Let A be an uncountable subset of the set of all positive real numbers. For every real number r, show that there are finitely many distinct real numbers $a_1 \ldots, a_n \in A$ such that
$$a_1 + \cdots + a_n \geq r.$$

5. Let $a_1, \ldots a_n$ be distinct real numbers and let
$$F(x) := \frac{1}{x - a_1} + \frac{1}{x - a_2} + \cdots + \frac{1}{x - a_n}$$
   For any real number c, show that $F(x) = c$ has exactly $n - 1$ or n real solutions according as $c = 0$ or $c \neq 0$.

6. Let $n \geq 1$ and let $\mathbf{A}$ be a $n \times n$ real matrix of rank $n - 1$. Then show that the adjoint matrix $\mathrm{Adj}\,\mathbf{A}$ of $\mathbf{A}$ has rank 1.

7. Let $n \geq 1$ and let $\mathbf{A}$ be a $n \times n$ matrix with integer entries and let $a \in \mathbb{Q} \setminus \mathbb{Z}$. Show that the matrix $a\mathbf{I}_n + \mathbf{A}$ is invertible.

8. For every divisor d of $24 = 4!$, find the number $\alpha(d)$ of elements of order d in the permutation group $\mathfrak{S}_4$ on 4 symbols.

9. Let G be a group, e be the identity element in G and let $x \in G$ be an element of order 2. Show that $H := \{e, x\}$ is a subgroup of G. Further, show that H is normal if and only if x belongs to the center $Z(G) := \{y \in G \mid yz = zy \text{ for all } z \in G\}$.

10. Let a and b be real numbers and let $(a_n)_{n \in \mathbb{N}}$ be the sequence recursively defined by
$$a_0 := a, \; a_1 := b, \;\; a_n := \frac{1}{2}(a_{n-1} + a_{n-2}) \;\text{ for } n \geq 2.$$
    Is the sequence $(a_n)_{n \in \mathbb{N}}$ convergent ? If the answer is yes, then find its limit.

    (**Hint :** Note that $a_{k+1} - a_k = -\frac{1}{2}(a_k - a_{k-1})$ for all $k \geq 1$.)

11. Let $h_n := \sum_{k=1}^{n} \frac{1}{k}$, $n \in \mathbb{N}$, $n \geq 1$. Show that the series $\sum_{n=1}^{\infty} \frac{h_n}{2^n}$ is convergent and that

$$\sum_{n=1}^{\infty} \frac{h_n}{2^n} = 2 \cdot \sum_{n=1}^{\infty} \frac{1}{n2^n}.$$

12. Let $a \in \mathbb{R}$, $a > 0$ and let the sequences $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ are defined recursively by

$$x_0 := a, \quad x_{n+1} := \sqrt{x_n}, \quad y_n := 2^n(x_n - 1) \quad \text{for all } n \in \mathbb{N}.$$

Find $\lim_{n\to\infty} y_n$

13. Show that the series $\sum_{n=2}^{\infty} \ln\left|\left(1 - \frac{1}{n^2}\right)\right|$ is convergent and find its sum.

(**Hint** : First prove the formula $\prod_{n=1}^{N}\left(1 - \frac{1}{n^2}\right) = \frac{1}{2}\left(1 + \frac{1}{N}\right)$.)

14. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Show that f must be a multiplication by a fixed real number a. i.e. there exists $a \in \mathbb{R}$ such that $f(x) = ax$ for all $x \in \mathbb{R}$. (**Hint** : First prove that $f(x) = f(1) \cdot x$ for all $x \in \mathbb{Q}$.)

15. For $n \in \mathbb{N}$, let $f_n : \mathbb{R} \to \mathbb{R}$ be the function defined by $f_n(x) := \frac{nx}{1+|nx|}$. Show that all the functions f_n, $n \in \mathbb{N}$ are continuous. For which real numbers x, is the function $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto f(x) := \lim_{n\to\infty} f_n(x)$ defined ? and for which x is it continuous ?

16. A function $f : \mathbb{R} \to \mathbb{R}$ is called e v e n if $f(-x) = f(x)$ for all $x \in \mathbb{R}$ and is called o d d if $f(-x) = -f(x)$ for all $x \in \mathbb{R}$. Show that

(a) The derivative of a differentiable even (respectively odd) function is odd (respectively even). [3 marks]

(b) The polynomial function $f(x) := a_0 + a_1x + \cdots + a_nx^n$, $a_0, \ldots, a_n \in \mathbb{R}$, is even (respectively odd) if and only if $a_k = 0$ for all odd (respectively even) indices k. [2 marks]

17. Let $\tanh : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$\tanh(x) := \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Show that

(a) $\tanh$ is strictly monotone increasing. [1 mark]

(b) $\tanh$ maps $\mathbb{R}$ bijectively onto the open interval $(-1, 1)$. [$1\frac{1}{2}$ marks]

(c) The inverse function $\tanh^{-1} : (-1, 1) \to \mathbb{R}$ is differentiable. [$1\frac{1}{2}$ marks]

(d) Find the derivative of $\tanh^{-1}$. [1 mark]

18. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) := \begin{cases} 0, & \text{if } x \leq 0, \\ e^{-1/x^2}, & \text{if } x > 0. \end{cases}$$

Show that f is 3-times continuously differentiable and compute the k-derivative $f^{(k)}$ of f for all $k = 1, 2, 3$.

19. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. For any two real numbers a, b with $a < b$, show that there exists a real number $c \in (a, b)$ such that

$$\left| \frac{f(b) - f(a)}{b - a} - f'(a) \right| \leq |f'(c) - f'(a)|.$$

(**Hint** : Use mean value theorem.)

20. Let $n \geq 1$ be a natural number and let $f : (0, \infty) \to \mathbb{R}$ be the function defined by $f(x)$ $x^n e^{-x}$. Determine the maxima and minima of the function f.

21. Let $f, g : [a, b] \to \mathbb{R}$ be two continuous functions on the closed interval $[a, b] \subseteq \mathbb{R}$ such that $\int_a^b f(x)dx = \int_a^b g(x)dx$. Show that there exists a real number $x_0 \in [a, b]$ such that $f(x_0) = g(x_0)$.

22. Let t be a positive real number. Compute the area bounded by the hyperbola $y = \sqrt{x^2 - 1}$ and the two lines $y = (\tanh t) \cdot x$, $y = -(\tanh t) \cdot x$ passing through the points $(\cosh t, \sinh t)$, $(\cosh t, -\sinh t)$ respectively.

(**Hint** : Use the formula $\int_a^b \sqrt{x^2 - 1}\, dx = \frac{1}{2}\left[-\cosh^{-1}(x) + x\sqrt{x^2 - 1}\right]_a^b$.)

23. Show that the function $F : \left(\mathbb{R}^3 \setminus \{0\}\right) \times \mathbb{R} \to \mathbb{R}$ defined by

$$F(x, t) := \frac{\cos(\|x\| - ct)}{\|x\|}$$

where $x = (x_1, x_2, x_3) \in \mathbb{R}^3 \setminus \{0\}$, $t \in \mathbb{R}$ and $\|x\| := \sqrt{x_1^2 + x_2^2 + x_3^2}$ is a solution of the differential equation

$$\left(\Delta - \frac{1}{c^2} \frac{\partial}{\partial t^2}\right) F(x, t) = 0,$$

where $\Delta := \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$ is the Laplace operator in dimension 3

24. Let a be a positive real number and let $f, g : (-a, a) \to \mathbb{R}$ be two continuous functions. Suppose that f is an odd function and g is an even function, i.e.

$$f(-x) = -f(x), \quad \text{and} \quad g(-x) = g(x) \quad \text{for all } x \in (-a, a).$$

Show that the differential equation $y'' + f(x) \cdot y' + g(x) \cdot y = 0$ has two linearly independent solutions one of which is even and the other is odd.

INDIAN INSTITUTE OF SCIENCE
BANGALORE - 560012

ENTRANCE TEST FOR ADMISSIONS - 2004

Program Integrated Ph.D

Entrance Paper : Mathematical Sciences
Paper Code : MS

Day & Date
SUNDAY 25TH APRIL 2004

Time
1.30 P.M. TO 4.30 P.M.

Integrated Ph.D./Mathematical Sciences

General Instructions

(1) The question paper consists of two parts, Part A and Part B.

(2) Answers to Part A are to be marked in the OMR sheet provided.

(3) For each question darken the appropriate bubble to indicate your answer.

(4) Use only HB pencils for bubbling answers.

(5) Mark only one bubble per question. If you mark more than one bubble, the question be evaluated as incorrect.

(6) If you wish to change your answer, please erase the existing mark completely before marl the other bubble.

(7) Answers to Part B are to be written in the separate answer book provided.

(8) Candidates are asked to fill in the required fields on the sheet attached to the answer bc

Notation

The set of natural numbers, integers, rational numbers, real numbers, positive real number: complex numbers are denoted by $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{R}^+$ and $\mathbb{C}$ respectively.

Integrated Ph.D./Mathematical Sciences

Part A

- Part A contains 15 multiple choice questions.
- You will get 2 marks for each correct answer and -0.5 mark for each wrong answer.
- Four possible answers are provided for each question and only one of these is correct

(1) Define a relation ρ on the set of positive integers $\mathbb{Z}^+$ by $x\rho y$ if and only if g.c.d. of x and y is bigger than 1. Then the relation ρ is

(A) reflexive and symmetric but not transitive.

(B) symmetric and transitive but not reflexive.

(C) symmetric but neither reflexive nor transitive.

(D) an equivalence relation.

(2) Let f be a polynomial of degree n, say $f(x) = \sum_{k=0}^{n} c_k x^k$ such that the first and last coefficients c_0 and c_n have opposite signs. Then

(A) $f(x) = 0$ for at least one positive x.

(B) $f(x) = 0$ for at least one negative x.

(C) $f(x) = 0$ for at least one positive x and for at least one negative x.

(D) f need not vanish anywhere.

3) Let a, b and c be arbitrary real numbers. Let A be the matrix

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

Let I be the 3×3 identity matrix Then

(A) $A^2 - 3A + 3I = A^{-1}$.

(B) $A^2 + 3A + 3I = A^{-1}$.

(C) $A^2 + A + I = A^{-1}$.

(D) A is not invertible.

(4) Let A be the matrix $\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$ where $a_1\vec{i}+a_2\vec{j}+a_3\vec{k}$, $b_1\vec{i}+b_2\vec{j}+b_3\vec{k}$ and $c_1\vec{i}+c_2\vec{j}+c_3\vec{k}$ are three mutually orthogonal unit vectors in $\mathbb{R}^3$. Let A^t denote the transpose of A. Then

(A) $A = A^{-1}$

(B) $A^2 = A$.

(C) $A^t = A$.

(D) $A^t = A^{-1}$

(5) Let (x_1, y_1) and (x_2, y_2) be two points on the parabola $y^2 = 2mx$ with $x_1 \neq x_2$. If L_1 and L_2 are the tangents of the parabola at these two points respectively, then the point of intersection of L_1 and L_2 has coordinates

(A) $\left(\frac{y_1 y_2}{2m}, \frac{y_1 + y_2}{2}\right)$

(B) $\left(\frac{x_1 y_2 + x_2 y_1}{y_1 + y_2}, \frac{m(x_1 + x_2)}{y_1 + y_2}\right)$

(C) $\left(-\frac{m}{2}, 0\right)$.

(D) $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

(6) Let T be a tangent line to the ellipse $\frac{}{a^2} + \frac{y^2}{b^2} = 1$ where $a > b > 0$. If d_1 and d_2 are t distances from the foci of the ellipse to T, then the product $d_1 d_2$ is

(A) equal to a^2.

(B) equal to b^2.

(C) at least $\frac{a^2 + b^2}{2}$

(D) equal to ab.

(7) Let C be the circle with centre at the origin and radius $a > 0$. If P is a moving poi in the plane such that its distance from the nearest point on the circle is the same as i distance from the x-axis, then the locus of the point P is

(A) a parabola.

(B) a hyperbola.

(C) a pair of parabolas.

(D) a pair of straight lines.

(8)] $[-1, 1] \to \mathbb{R}$ is a differentiable function with $f'(x) = 1 - |x|$ and $f(0) = 2004$ then

(A) $f(x) = 1002 + \frac{x - x|x|}{2}$

(B) $f(x) = 2003 + \frac{x}{2} - x|x|$

(C) $f(x) = 2004 + x - \frac{x|x|}{2}$.

(D) $f(x)$ can not be determined

(9) Let $x = \pi - 0.01814$ and $y = e + 0.40517$. If $z = 3.123456789101112$ then

(A) $z = x$.

(B) $z = y$.

(C) z is rational.

(D) z is irrational, $z \neq x$ and $z \neq y$

(10) For a real number y, let $[y]$ denote the largest integer smaller than or equal to y. The value of the integral

$$\int_0^2 [x^2]dx$$

is equal to

(A) 1.

(B) $5 - \sqrt{2} \quad \sqrt{3}$.

(C) $3 - \sqrt{2}$.

(D) $8/3$.

(11 If f is an integrable function on the real line satisfying

$$\int_0^x tf(t)dt = \sin x - x\cos x - \frac{1}{2}x^2$$

for all real numbers x, then

(A) $f(x) = 1 - \cos x$.

(B) $f(x) = 1 + \sin x$.

(C) $f(x) = \sin x$.

(D) $f(x) = \sin x - 1$

12) Suppose $0 < p < 1$. Then

(A) $(\cos\theta)^p > \cos(p\,\theta)$ for all $\theta \in [0, \pi/2]$.

(B) $(\cos\theta)^p \leq \cos(p\,\theta)$ for all $\theta \in [0, \pi/2]$.

(C) $(\cos\theta)^p \leq p\,\cos(p\,\theta)$ for all $\theta \in [0, \pi/2]$.

(D) $(\cos\theta)^p$ and $\cos(p\,\theta)$ are not comparable in the interval $[0, \pi/2]$

(13) Let $A_1 > A_2 > \cdot > A_k > 0$ be k real numbers. Then

$$\lim_{n\to\infty}(A_1^n + A_2^n + \quad + A_k^n)^{1/n}$$

is equal to

(A) $(A_1 + A_2 + \cdots A_k)/k$.

(B) 0.

(C) A_k.

(D) A_1.

(14) The value of the limit

$$\lim_{x\to 0}\frac{5^x - 3^x}{3^x - 2^x}$$

is

(A) $\log_e(10/9)$.

(B) $\log_{\frac{3}{2}}(5/3)$.

(C) $\dfrac{\log_2 5}{\log_2 3}$

(D) $\log_2 5$.

(15) The function $f : \mathbb{R}^+ \to \mathbb{R}$ defined by

$$f(x) = e^{x^2/2} \int_0^x e^{-t^2/2}\, dt$$

is

(A) monotone increasing.

(B) monotone decreasing.

(C) constant.

(D) periodic.

Part B

- Part B consists of 25 questions, each carrying 5 marks
- Answer **any 14** questions.

---

(1) Let l and m be two positive integers. If the equation $1 + z^l + z^m = 0$ has a root z_0 on the unit circle, then show that z_0 is a root of unity.

(2) Let $Q(x, y)$ be a polynomial symmetric in x and y, i.e., $Q(x, y) = Q(y, x)$. If $x - y$ is a factor of $Q(x, y)$, then show that $(x - y)^2$ is also a factor of $Q(x, y)$.

(3) Let G and H be finite groups so that $(o(G), o(H)) = 1$, i.e., the order of G and the order of H are relatively prime. If K is a subgroup of the product group $G \times H$ and $(a, b) \in K$ then show that $(a, e) \in K$, where e is the identity element of H.

(4) Find all the real solutions of the system

$$x(x-1) + 2yz = y(y-1) + 2zx = z(z-1) + 2xy.$$

(5) Express the value of

$$\binom{2004}{0} \binom{2004}{3} \binom{2004}{6} +$$

in the form $\frac{a^b + c}{d}$ where a, b, c and d are positive integers.

(6) Find the minimum value of

$$\frac{(x + \frac{1}{x})^6 - (x^6 + \frac{1}{x^6}) - 2}{(x + \frac{1}{x})^3 + (x^3 + \frac{1}{x^3})}$$

for $x > 0$.

(7) Find the eigenvalues of the $n \times n$ real matrix

$$\begin{matrix} 0 & b & b & & b \\ b & 0 & b & & b \\ \\ \\ b & b & b & & 0 \end{matrix}$$

(8) Consider the system of equations in x_1, x_2, x_3 and x_4

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 &= b_3 \\ a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4 &= b_4 \end{aligned}$$

If (c_1, c_2, c_3, c_4) and (d_1, d_2, d_3, d_4) are two distinct solutions of the system, then show that the system has infinitely many solutions.

(9) Let $(\mathbb{C}\setminus\{0\},\cdot)$ be the multiplicative group of all non-zero complex numbers. If G is a finite subgroup of $(\mathbb{C}\setminus\{0\},\cdot)$, then show that G is cyclic.

(10) If $x+\frac{1}{x}=u$ and $x^3 = v$, then find polynomials $P(u,v)$ and $Q(u,v)$ such that $x^2 = P(u,v)/Q(u,v)$.

(11) (A) If P, Q, R and S are real polynomials, then find real polynomials T and U so that

$$((P(x))^2+(Q(x))^2)((R(x))^2+(S(x))^2) = (T(x))^2+(U(x))^2.$$

3 Marks

(B) Suppose $f(x) = ax^4+bx^3+cx^2+dx+e > 0$ for all $x \in \mathbb{R}$ where a, b, c, d, e are real constants. Show that $f(x) = (A(x))^2+(B(x))^2$ for some **real** polynomials A and B.

2 Marks

(12) (A) Find the area A of the region in the first quadrant bounded by the ellipse $x^2+9y^2=9$, the line $y = mx$ and the y-axis. 3 Marks

(B) Let B be the area in the first quadrant of the region bounded by the same ellipse, the line $y=2x$ and the x-axis. If $B = A$, then find m. 2 Marks

(13) Let (α,β) be a point on the hyperbola $x^2-y^2=a^2$ with $a>0$. Let L be the tangent line to the hyperbola at (α,β). Let Q be the foot of the perpendicular dropped on the line L from the origin.

(A) Find the equation to the locus of Q. $(x^2+y^2)^2 = a^2(x^2-y^2)$ / $r^2 = a^2\cos 2\theta$ 3 Marks

(B) Draw the locus of Q. 2 Marks

(14) (A) Show that $a(x-y)(y-z)+b(y-z)(z-x)+c(x-y)(z-x)=0$, where $(a,b,c)\neq(0,0,0)$, represents a pair of planes. 3 Marks

(B) When the planes are distinct, find the line of intersection. 2 Marks

(15) Let $\vec{a}\neq\vec{0}$ and $\vec{b}$ be two perpendicular vectors in $\mathbb{R}^3$ and let k be a real constant.

(A) Find a vector $\vec{x}$ such that $\vec{a}\cdot\vec{x}=k$ and $\vec{a}\times\vec{x}=\vec{b}$. 4 Marks

(B) Is the vector $\vec{x}$ unique? 1 Mark

(16) Verify Green's theorem for the line integral $\int_C x^2dx+xydy$, where C is the boundary of the region bounded by the x-axis, the line $x=y$ and the line $x+y=2$.

(17) (A) Prove that $\sinh x > x$ for all $x>0$. 2 Marks

(B) Prove that for $a,b>0$ and $a\neq b$,

$$\frac{a-b}{\log_e a-\log_e b} > \sqrt{ab}.$$

3 Marks

(18) Let X be a finite set and let $f: X\to X$ be a function. Let f^n denote $f\circ f\circ\cdots\circ f$ (n times).

(A) Let $a\in X$. If there exists an integer $n>1$ such that $f^n(x)=a$ for every $x\in X$, then show that $f(a)=a$. 2 Marks

(B) If for each $x \in X$, there is an n (depending on x) such that $f^n(x) = x$, the[n show] that f is a bijection.

(19) Evaluate

$$\int_2 \frac{\sqrt{\log_e(9-x)}}{\sqrt{\log_e(9-x)} + \sqrt{\log_e(x+3)}} dx.$$

(20) (A) Find a function f such that

$$\tan^{-1}\left(\frac{1}{x^2+x+1}\right) = f(x) - f(x+1)$$

for all real $x \geq 1$.

(B) Hence or otherwise find the sum

$$\sum_{n=0}^{\infty} \tan^{-1}\left(\frac{1}{n^2+n+1}\right)$$

(21) Suppose a particle is moving along the graph of $y = \log_e x$. Find a point on its tr which is closest to the point $(0, 1)$ and show that it is unique.

(22) Prove that the image of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ under the mapping $f(z) = z^2$, $z = x + iy$, is also a conic. Find its centre, eccentricity and foci.

(23) Let $f : (-1, 2) \to \mathbb{R}$ be a differentiable function such that $f(0) = 0$ and f' is a increasing function in the interval $[0, 1]$. Show that the function

$$g(x) = \frac{f(x)}{x} : (0, 1] \to \mathbb{R}$$

is also a strictly increasing function.

(24) (A) Let $\{x_n\}_{n\geq 1}$ and $\{y_n\}_{n\geq 1}$ be two real sequences having a common limit l. Pr the sequence

$$\{x_1, y_1, x_2, y_2, \ldots, x_n, y_n, \ldots\}$$

has the same limit l.

(B) Hence or otherwise prove that a function $f : \mathbb{R} \to \mathbb{R}$ which maps convergent s to convergent sequences, is continuous on $\mathbb{R}$.

(25) (A) Given that $y = e^x$ is a solution of the homogeneous equation

$$xy'' - (1+x)y' + y = 0,$$

find another linearly independent solution.

(B) Hence solve the inhomogeneous equation

$$xy'' - (1+x)y' + y = x^2e^x.$$

Integrated Ph.D./Mathematical Sciences

General Instructions

(1) The question paper has 50 multiple choice questions.

(2) Four possible answers are provided for each question and only one of these is correct.

(3) Each question carries 2 marks.

(4) There is no negative marking.

(5) Answers are to be marked in the OMR sheet provided.

(6) For each question darken the appropriate bubble to indicate your answer.

(7) Use only HB pencils for bubbling purpose.

(8) Mark only one bubble per question. If you mark more than one bubble, the question will be evaluated as incorrect.

(9) If you wish to change your answer, please erase the existing mark completely before marking the other bubble.

Notations : The set of natural numbers, integers, rational numbers, real numbers and complex numbers are denoted by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ respectively.

Integrated Ph.D./Mathematical Sciences

---

1. Let a, b be two real numbers such that $a > 0$ and $b > 0$. The number of real roots of the cubic $ax^3 + bx + 1 = 0$ is

   (A) 0
   (B) 1
   (C) 2
   (D) 3 .

2. Let α, β, γ be the roots of the cubic $x^3 + ax^2 + bx + c = 0$ where a, b, c are real. The expression $\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2$ is equal to

   (A) $b^2 - 2ac$
   (B) $b^2 - 4ac$
   (C) $b^2 + 2ac$
   (D) $b^2 + 4ac$.

3. The equation $x^{10} + 5x^3 + x - 15 = 0$ has

   (A) at least 2 positive real roots
   (B) at least 2 negative real roots
   (C) all real roots
   (D) at least 8 imaginary roots .

4. For real numbers $x > 1$ and $y > 1$, define P, Q as

$$P = \ln\sqrt{xy}, \quad Q = \sqrt{\ln x \ln y}\,.$$

   Which of the following is true for all $x > 1$ and $y > 1$?

   (A) $P \geq Q$
   (B) $P \leq Q$
   (C) $P = Q$
   (D) There is no relation between P, Q.

5. If $x \neq 0, y \neq 0$, then $x^2 + xy + y^2$ is

   (A) always negative
   (B) always positive
   (C) zero
   (D) sometimes positive, sometimes negative.

6. The sum $\sum_{k=0}^{n} \binom{n}{k} \binom{k}{n} x^k (1-x)^{n-k}$ is equal to

(A) x^n

(B) 1

(C) x^{2n}

(D) 0.

7. Let $z = x + iy$ be a complex number. Then $|z| = |x| + |y|$ holds if and only if

(A) $z = 0$

(B) z lies on the x-axis

(C) z lies on the y-axis

(D) z lies either on the x-axis or on the y-axis.

8. One of the values of $\arg(\sqrt{3} - \imath)^6$ is

(A) π

(B) $\pi/3$

(C) $2\pi/3$

(D) $5\pi/3$.

9. Let $f : \mathbb{C} \to \mathbb{C}$ be defined by $f(z) = \cos z$. Then

(A) $|f(z)| \leq 1$

(B) $|f(z)| \leq \pi$

(C) $|f(z)| \leq |z|$

(D) f is unbounded.

10. Let $f : \mathbb{C} \to \mathbb{C}$ be given by $f(z) = \bar{z}$. Then

(A) f is differentiable everywhere

(B) f is nowhere differentiable

(C) f is differentiable everywhere except at the origin

(D) f is an entire function.

11. The determinant $\begin{vmatrix} 1+x & 1 & 1 & 1 \\ 1 & 1-x & 1 & 1 \\ 1 & 1 & 1+y & 1 \\ 1 & 1 & 1 & 1-y \end{vmatrix}$ evaluates to

(A) xy

(B) $(xy)^2$

(C) $(1-x^2)(1-y^2)$

(D) $x^2 + y^2$.

12. The determinant $\begin{vmatrix} x_0 & x_1 & x_2 & x_3 & x_4 \\ x_0 & x & x_2 & x_3 & x_4 \\ x_0 & x_1 & x & x_3 & x_4 \\ x_0 & x_1 & x_2 & x & x_4 \\ x_0 & x_1 & x_2 & x_3 & x \end{vmatrix}$ evaluates to

(A) $[x_0(x-x_1)(x-x_2)(x-x_3)(x-x_4)]^4$

(B) $x_0(x-x_1)(x-x_2)(x-x_3)(x-x_4)$

(C) $x_0[(x-x_1)(x-x_2)(x-x_3)(x-x_4)]^4$

(D) $xx_0x_1x_2x_3x_4$.

13. The number of reflexive relations on a set of cardinality 3 is

(A) 64

(B) 32

(C) 8

(D) 4.

14. Up to isomorphism, the number of groups of cardinality 4 is

(A) one and it is abelian

(B) two – one is abelian and the other non-abelian

(C) two – both are abelian

(D) four – two abelian and two non-abelian.

15. Suppose G is a group with more than one element and no proper subgroup. Then the cardinality of G is

P a prime number.
Q a finite non prime number.
R infinite.

(A) P only

(B) P or Q, but not R

(C) P or R, but not Q

(D) any of P, Q or R.

16. The number of roots of the polynomial $x^3 - x$ in $\mathbb{Z}/6\mathbb{Z}$ is

(A) 1

(B) 2

(C) 3

(D) 6.

17. Let S and T be vector subspaces of a vector space V. Then $S \cup T$ is a subspace of V

(A) is never true

(B) if and only if one of S or T is trivial

(C) if and only if $S \subseteq T$ or $T \subseteq S$

(D) if and only if $S \cap T$ is a nonzero vector space.

18. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by $T(x, y, z) = (y + z, z + x, x + y)$. The matrix of T with respect to the basis $\{(1, 1, 1), (1, -1, 0), (1, 1, -2)\}$ is

(A) $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

(B) $\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & -2 \end{pmatrix}$

(C) $\begin{pmatrix} 2 & -1 & -1 \\ 2 & 1 & -1 \\ 2 & 0 & 2 \end{pmatrix}$

(D) $\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

19. Let V and W be vector spaces and $T : V \to W$ a linear transformation. Then T is a group homomorphism

(A) only if $\dim V \leq \dim W$

(B) only if $\dim V \geq \dim W$

(C) only if $\dim V = \dim W$

(D) is always true.

20. Minimum of dimension of the intersection of two seven dimensional vector subspaces in a twelve dimensional vector space is

(A) 0

(B) 2

(C) 5

(D) 7.

21. Dimension of kernel (i.e., null space) of the linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$ given by the matrix $\begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ with respect to the standard bases is

(A) 0
(B) 1
(C) 2
(D) 3.

22. Let P be an $n \times n$ matrix with real entries such that

$$P^2 + 2P + I = 0$$

where I denotes the $n \times n$ identity matrix. Which of the following is true?

(A) There does not exist a matrix P satisfying the given condition
(B) $P = -I$
(C) P exists and is invertible
(D) P exists but it may not always be invertible.

23. Let Δ be the triangle in $\mathbb{R}^2$ with vertices at $(0,0),(0,1),(1,0)$. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map given by $T(x,y) = (2x+3y, -x+4y)$. The ratio

$$\frac{\text{area } T(\Delta)}{\text{area } \Delta}$$

is equal to

(A) 11
(B) 12
(C) 13
(D) 14

24. Let $A = (3,-1,2)$ and $B = (0,2,-1)$. Then the locus of points $P = (x,y,z)$ that satisfy

$$\text{distance}(PA) = 2 \text{ distance}(PB)$$

is given by

(A) $(x+1)^2 + (y-3)^2 + (z+2)^2 = 12$
(B) $(x-1)^2 + (y+3)^2 + (z-2)^2 = 12$
(C) $(x+1)^2 + (y-3)^2 + (z-2)^2 = 12$
(D) $(x-1)^2 + (y-3)^2 + (z+2)^2 = 12.$

25. Let T be the graph of the function

$$f(x) = \begin{cases} 1+x, & -1 \le x \le 0; \\ 1-x, & 0 \le x \le 1. \end{cases}$$

Then the reflection of T in the line $y = 0$ is given by the graph of $g(x)$ where

(A) $g(x) = \begin{cases} -1-x, & -1 \le x \le 0 \\ -1+x, & 0 \le x \le 1 \end{cases}$

(B) $g(x) = \begin{cases} -1+x, & -1 \le x \le 0 \\ -1-x, & 0 \le x \le 1 \end{cases}$

(C) $g(x) = \begin{cases} -1-x, & -1 \le x \le 0 \\ 1-x, & 0 \le x \le 1 \end{cases}$

(D) $g(x) = \begin{cases} 1-x, & -1 \le x \le 0 \\ -1-x, & 0 \le x \le 1. \end{cases}$

26. In the Euclidean space $\mathbb{R}^3$, the nonempty intersection of a plane with the set $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$ is

P a circle.
Q an ellipse.
R a single straight line.
S a pair of parallel staright lines.

(A) P, Q, R but not S

(B) P, Q, S but not R

(C) P, R, S but not Q

(D) Any of P, Q, R or S.

27. Suppose there are two unit circles in the Euclidean plane such that center of one is a point of the circumference of the other. Distance between the points of intersection of the circles is

(A) 2 units

(B) $\sqrt{2}$ units

(C) $1/\sqrt{3}$ units

(D) $\sqrt{3}$ units.

28. The number of points in the Euclidean plane together with the three points $(1, -1)$, $(-5, 9)$ and $(7, -11)$ which form a parallelogram is

(A) 0

(B) 1

(C) 2

(D) infinite.

29. The equation of the tangent plane to the surface $x^2 - y^2 + xz = 2$ at the point $(1, 0, 1)$ is given by

(A) $3x - 2 - z = 0$
(B) $3x + 3 + z = 0$
(C) $3x - 4 + z = 0$
(D) $3x - 5 - z = 0$.

30. Let $u(x, y) = x^3 - 3xy^2$ and $v(x, y) = ax^2y + by^3$, where a, b are real constants. The family of curves given by $\{u(x, y) = \text{constant}\}$ and $\{v(x, y) = \text{constant}\}$ are orthogonal exactly when

(A) $a + 3b = 0$
(B) $a - 3b = 0$
(C) $3a + b = 0$
(D) $3a - b = 0$.

31. Let $\vec{X}, \vec{Y}, \vec{Z}$ be vectors in $\mathbb{R}^3$ such that

$$\vec{X} \times \vec{Y} = \vec{i} + 2\vec{j} - 3\vec{k}, \quad \vec{Z} = -\vec{i} - 2\vec{j} + \vec{k}.$$

The volume of the parallelepiped in $\mathbb{R}^3$ spanned by $\vec{X}, \vec{Y}, \vec{Z}$ is

(A) 5
(B) 6
(C) 7
(D) 8.

32. Let $\vec{v} = (2xyz)\,\vec{i} + (x^2z + y)\,\vec{j} + (x^2y + 3z^2)\,\vec{k}$. Then the magnitude of curl $\vec{v}$ at $(1, 1, 1)$ is

(A) not defined
(B) strictly greater than one
(C) equal to one
(D) equal to zero.

33. Let D be the square in $\mathbb{R}^2$ with vertices at $(0, 0), (1, 0), (0, 1), (1, 1)$. The integral

$$\int_{\partial D} x\, dy$$

where ∂D is the boundary of the square, is equal to

(A) 0
(B) 0.5
(C) 1
(D) 1.5.

34. The integral $\int_C (yz\,dx + (xz+1)\,dy + xy\,dz)$, where C is a simple closed curve, equals

(A) 0
(B) $3xyz + y$
(C) length of C
(D) area enclosed by C.

35. The value of

$$lim_{n\to\infty}\Big(\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n}\Big)$$

is

(A) 0
(B) $\ln 2$
(C) e
(D) e^2.

36. Let $S_k = \sum_{n=2}^{k} \frac{(-1)^n}{n\ \ln\ n}$. Then the sequence $\{S_k\}$

(A) converges to a finite number
(B) diverges to ∞
(C) diverges to $-\infty$
(D) oscillates.

37. The equation $x^2 = x\sin x + \cos x$ is true for

(A) no real value of x
(B) exactly one real value of x
(C) exactly two real values of x
(D) infinitely many real values of x.

38. Let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying $|f(x) - f(y)| \leq K|x-y|^{\frac{5}{3}}$, for all x, y, where K is a constant. Then

(A) f is a linear function
(B) f is a constant
(C) f is strictly increasing
(D) f is strictly decreasing.

39. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \min(|x|, x^2 - 1)$. Then f is

(A) a discontinuous function

(B) continuous and differentiable everywhere

(C) differentiable everywhere except at one point

(D) differentiable everywhere except at two points.

40. At $x = 2, f(x) = x^2 e^{-x}$ has a

(A) local minimum, but not global minimum

(B) local maximum, but not global maximum

(C) global minimum

(D) global maximum.

41. Let $f : [a, b] \to \mathbb{R}$ be continuous, $f(a) \geq b,\ f(b) \leq a$. Then there exists an $x \in [a, b]$ such that

(A) $f(x) = x$

(B) $f(x) = 0$

(C) $f'(x) = 0$

(D) $f''(x) = 0$.

42. Let $g(x) = \displaystyle\int_{x-\alpha}^{x+\alpha} \sin y^2 \, dy$. Then $g'(x)$ equals

(A) $\sin x^2$

(B) $\frac{\sin(x+\alpha)^2 + \sin(x-\alpha)^2}{2}$

(C) $\sin(x + \alpha)^2 - \sin(x - \alpha)^2$

(D) $\cos(x + \alpha)^2 - \cos(x - \alpha)^2$.

43. The double integral $\displaystyle\int_0^2 \int_{\frac{y}{2}}^1 e^{x^2} \, dx \, dy$ equals

(A) $e + 1$

(B) 1

(C) $e - 1$

(D) e^2.

44. If $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and $f(x) = \int_0^x f(y)\, dy$, then

(A) $f(x) = e^x$

(B) $f(x) = \ln x$

(C) f is identically zero

(D) f is identically equal to 1.

45. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $\lim_{x\to 0} f(x) = a$. Then $\lim_{x\to 0} \frac{1}{x} \int_0^x f(y)\, dy$

(A) equals 1

(B) equals a

(C) equals -1

(D) does not exist.

46. The partial derivative of $\int_0^{x+y} \sin^2(t+y)\, dt$ with respect to x is

(A) $\sin^2(x+2y)$

(B) $2\sin(x+y)$

(C) $2\sin(x+2y)$

(D) $2\cos(x+2y)$.

47. The initial value problem $\frac{dy}{dx} = 2y^{\frac{1}{3}}$, $y(0) = 0$, has

(A) no solution

(B) infinitely many solutions

(C) exactly one solution

(D) finitely many solutions.

48. Which of the following pair of functions is not a linearly independent pair of solutions of $y'' + 9y = 0$?

(A) $\sin 3x,\ \sin 3x - \cos 3x$

(B) $\sin 3x + \cos 3x,\ 3\sin x - 4\sin^3 x$

(C) $\sin 3x,\ \sin 3x \cos 3x$

(D) $\sin 3x + \cos 3x, 4\cos^3 x - 3\cos x$.

49. Determine the type of the following differential equation $\frac{d^2y}{d^2x} + \cos(x+y) = \sin x$.

(A) linear, homogeneous

(B) nonlinear nonhomogeneous

(C) linear, nonhomogeneous

(D) nonlinear, nonhomogeneous.

50. The solution of the first order ODE

$$xy' = xy + x + y + 1$$

is (in all the choices below, C is a constant)

(A) $y = Cx(e^x - 1)$

(B) $y = (Cxe^x) - 1$

(C) $y = (Ce^x) - x$

(D) $y = (Ce^x) - x - 1$.

INDIAN INSTITUTE OF SCIENCE
BANGALORE - 560012

ENTRANCE TEST FOR ADMISSIONS - 2006

Program : Integrated Ph.D

Entrance Paper : Mathematical Sciences
Paper Code : MS

Day & Date
SUNDAY, 30th APRIL 2006

Time
2.00 P.M. TO 5.00 P.M.

Integrated Ph.D. (Mathematical Sciences)

General Instructions

(1) The question paper has 50 multiple choice questions.

(2) Four possible answers are provided for each question and only one of these is correct.

(3) **Marking scheme:** Each correct answer will be awarded 2 marks, but **0.5 marks will be deducted** for each **incorrect** answer.

(4) Answers are to be marked in the OMR sheet provided.

(5) For each question darken the appropriate bubble to indicate your answer.

(6) Use only HB pencils for filling in the bubbles.

(7) Mark only one bubble per question. If you mark more than one bubble, your response will be evaluated as incorrect.

(8) If you wish to change your answer, please erase the existing mark completely before marking the other bubble.

Notations: The set of natural numbers, integers, rational numbers, real numbers and complex numbers are denoted by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ respectively.

Integrated Ph.D./Mathematical Sciences

1. The number of positive roots of $x^6 + 9x^5 + 2x^3 - x^2 - 2$ is

   (A) 0.
   (B) 1.
   (C) 3.
   (D) 5.

2. Let $f(x)$ be a polynomial with real coefficients and let $f'(x)$ denote its derivative. Then, between two consecutive roots of $f'(x) = 0$, there

   (A) never is a root of $f(x) = 0$.
   (B) always is a root of $f(x) = 0$.
   (C) is at most one root of $f(x) = 0$.
   (D) may be any number of roots of $f(x) = 0$.

3. Let $\alpha, \beta, \gamma, \delta$ be roots of the quartic $x^4 + px^3 + qx^2 + rx + s$ where p, q, r and s are real. Then $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$ equals

   (A) $p^2 + 4q$.
   (B) $p^2 - 4q$.
   (C) $p^2 + 2q$.
   (D) $p^2 - 2q$.

4. Let $z = x + \imath y$ be a complex number. Then $|\, z \,| = |\, x \,| - |\, y \,|$ holds if and only if

   (A) $z = 0$.
   (B) z is real.
   (C) z is purely imaginary.
   (D) z is real or purely imaginary.

5. Let $P = re^{\imath\theta}, Q = r$ and $R = P + Q$. If O is the origin, then OPQR is a square if and only if

(A) $r = 0$.

(B) $r = 1$.

(C) $\theta = \pm\frac{\pi}{2}$.

(D) $\theta = \pm\pi$.

6. The determinant $\begin{vmatrix} a+b & c+d & e & 1 \\ b+c & d+a & f & 1 \\ c+d & a+b & g & 1 \\ d+a & b+c & h & 1 \end{vmatrix}$ evaluates to

(A) 0.

(B) 1.

(C) $(a+b)(c+d) + e + f + g + h$.

(D) $(a+b+c+d)(e+f+g+h)$.

7. Let $M_2(\mathbb{C})$ be the vector space of 2×2 matrices over $\mathbb{C}$.

P: The determinant function from $M_2(\mathbb{C})$ to $\mathbb{C}$ is a linear transformation.

Q: The determinant function is a linear function of each row of the matrix when the other row is held fixed.

(A) Both P and Q are true.

(B) P is true, but Q is false.

(C) P is false, but Q is true.

(D) Both P and Q are false.

8. Let x, y and z be positive real numbers such that $xyz = 1$. Then

(A) $x + y + z \geq 3$ and $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq 3$.

(B) $x + y + z \geq 3$ and $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \leq 3$.

(C) $x + y + z \leq 3$ and $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq 3$.

(D) $x + y + z \leq 3$ and $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \leq 3$.

9. Let F_n be a finite set with n elements. The number of one-to-one maps from F_5 to F_7 is

(A) 35 .
(B) $\binom{7}{5}$.
(C) 5!
(D) $5!\binom{7}{5}$.

10. Consider the functions $f, g : \mathbb{Z} \to \mathbb{Z}$ defined by $f(n) = 3n + 2$ and $g(n) = n^2 - 5$.

(A) Neither f nor g is a one-to-one function.
(B) The function f is one-to-one, but not g.
(C) The function g is one-to-one, but not f.
(D) Both f and g are one-to-one functions.

11. The set of integers under subtraction is not a group because

(A) subtraction is not associative.
(B) there is no identity element for subtraction.
(C) every element does not have an inverse.
(D) subtraction is not commutative.

12. Consider $\mathbb{R} \setminus \{0\}$ as a multiplicative subgroup of $\mathbb{C} \setminus \{0\}$. Let $f : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ given by $f(z) = z^2$ and g the restriction of f to $\mathbb{R} \setminus \{0\}$. Then

(A) neither f nor g is a surjective (i.e. onto) homomorphism.
(B) f is a surjective homomorphism but g is not.
(C) g is a surjective homomorphism but f is not.
(D) both f and g are surjective homomorphisms.

13. The number of distinct homomorphisms from $\mathbb{Z}/5\mathbb{Z}$ to $\mathbb{Z}/7\mathbb{Z}$ is

(A) 0.
(B) 1.
(C) 7.
(D) 5.

14. Let S_3 denote the group of permutations on the set $\{1,2,3\}$ and $G = S_3 \times S_3$. The set $H := \{(\sigma, \tau) \mid \sigma(1) = \tau(1)\}$ is

(A) not a subgroup of G.

(B) a non-abelian subgroup of G

(C) an abelian subgroup of G.

(D) a normal subgroup of G.

15. The set $\{0, 2, 4\}$ under addition and multiplication modulo 6 is

(A) not a ring with unity (identity).

(B) a ring with 0 as unity (identity).

(C) a ring with 2 as unity (identity).

(D) a ring with 4 as unity (identity).

16. Suppose a and b are elements in R, a commutative ring with unity. Then the equation $ax = b$

(A) always has exactly one solution.

(B) has a solution only if a is a unit.

(C) has more than one solution only if $b = 0$.

(D) may have more than one solution.

17. For what values of r is the vector $(3, 2, r, 0)$ in $\mathbb{R}^4$ contained in the subspace generated by $(1, 0, 0, 0), (0, 1, 2, 0)$ and $(0, 1, 1, 1)$?

(A) For no value of r.

(B) For exactly one value of r.

(C) For more than one but finitely many values of r.

(D) For infinitely many values of r.

18. Denote by M a real 3×3 real matrix such that $M \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $M \begin{pmatrix} 4 \\ 5 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/4 \\ 1/5 \\ 0 \end{pmatrix}$, then $M \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix}$ is

(A) $\begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix}$

(B) $\begin{pmatrix} 1/6 \\ 0 \\ 0 \end{pmatrix}$

(C) $\begin{pmatrix} 2 + \frac{1}{4} \\ 0 \\ 0 \end{pmatrix}$

(D) not determined uniquely.

19. Suppose u, v and w are linearly dependent vectors and w is not expressible as a linear combination of u and v. Then which of the following statements is true?

(A) Such a situation is not possible.

(B) Vector w has to be zero.

(C) One of u or v has to be zero.

(D) Vector u is a multiple of v.

20. In a 13 dimensional vector space, the dimension of intersection of two 6 dimensional subspaces is

(A) at least 1.

(B) at most 1.

(C) at least 6.

(D) at most 6.

21. Which of the options is true about the following statement?

Let $T : \mathbb{R}^5 \to \mathbb{R}^7$ be a linear transformation whose kernel is of dimension 2 and whose image is a line.

(A) There does not exist any such linear transformation.

(B) There is exactly one such linear transformation.

(C) There are finitely many (but more than one) such linear transformations.

(D) There are infinitely many such linear transformations.

22. For maps from $\mathbb{R}^2$ to $\mathbb{R}^3$, match the following

P.	$f(x,y) = (x, x-y, y)$	1.	Not a linear transformation.
Q.	$g(x,y) = (x,x,x)$	2.	Rank two linear transformation.
R.	$h(x,y) = (1, x+y, x-y)$	3.	Rank one linear transformation.

(A) P-3, Q-2, R-1.

(B) P-3, Q-1, R-2.

(C) P-2, Q-1, R-3.

(D) P-2, Q-3, R-1.

23. Let $T : \mathbb{R}^5 \to \mathbb{R}^5$ be a linear transformation. Then T has

(A) no real eigenvalues.

(B) at least one real eigenvalue.

(C) at most one real eigenvalue.

(D) exactly one real eigenvalue.

24. A non-zero linear transformation T on $\mathbb{R}$

(A) may not have any eigenvector.

(B) has exactly one eigenvector.

(C) has more than one (but finitely many eigenvectors).

(D) has infinitely many eigenvectors.

25. Let X and Y denote spheres in $\mathbb{R}^3$ each with radius 2 and center at $(0,1,0)$ and $(0,-1,0)$ respectively. Then $X \cap Y =$

(A) $\{(x,y,z) \in \mathbb{R}^3 \mid x^2 + z^2 = 1\}$.
(B) $\{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1,\ z = 0\}$.
(C) $\{(x,y,z) \in \mathbb{R}^3 \mid y^2 + z^2 = 1,\ x = 0\}$.
(D) $\{(x,y,z) \in \mathbb{R}^3 \mid z^2 + x^2 = 1,\ y = 0\}$.

26. The set of points in the Euclidean plane satisfying the quadratic $x^2 + y^2 + x + y + 1 = 0$ is

(A) an empty set.
(B) a pair of straight lines.
(C) a circle.
(D) an ellipse.

27. Let $\boldsymbol{V}, \boldsymbol{W}$ and $\boldsymbol{X}$ be three vectors in $\mathbb{R}^3$, and let $.$ and $\times$ denote the usual dot product and cross product respectively. The product $\boldsymbol{X} \times (\boldsymbol{Y} \,.\, \boldsymbol{Z})$

(A) is a vector in $\mathbb{R}^3$.
(B) gives the volume of the parallelepiped spanned by $\boldsymbol{X}, \boldsymbol{Y}$ and $\boldsymbol{Z}$.
(C) always equals 0.
(D) cannot be defined.

28. The equation of the tangent plane to the surface $x^2 - 3xy + 2y^2 + z^2 = 1$ at the point $(1,1,1)$ is given by

(A) $x + y + 2z = 4$.
(B) $x - y - 2z = -2$.
(C) $x - 2y - 2z = -2$.
(D) $x + y + z = 3$.

29. Let X, Y and Z be vectors in $\mathbb{R}^3$ such that

$$X \times Y = 2i - 2j + 5k, \qquad Y = i + 3j.$$

The volume of the parallelepiped in $\mathbb{R}^3$ spanned by X, Y and Z is

(A) 4.
(B) 3.
(C) 2.
(D) 1.

30. Let x and y be any two real numbers satisfying $0 < x < 1 < y$. Then, the limit

$$\lim_{n \to \infty} \left(x + \frac{y}{n} \right)^n$$

is equal to

(A) $e^{y/x}$.
(B) $e^{x/y}$.
(C) 0.
(D) e.

31. The limit $\lim_{n \to \infty} (n^5 + 4n^3)^{1/5} - n$ equals

(A) 4.
(B) 4/5.
(C) 0.
(D) 5/4.

32. We are given a convergent series $\sum_{n=1}^{\infty} a_n$, where $a_n \geq 0$ for each n. Which of the following correctly describes the behaviour of the series

$$\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n^p}, \quad 1 \leq p \leq 2 \ ?$$

(A) Diverges when $p = 1$, but converges for $p \in (1, 2]$.
(B) Converges for every $p \in [1, 2]$.
(C) Diverges when $p \in [1, 5/4]$, but converges for $p \in (5/4, 2]$.
(D) Diverges for every $\in [1, 2]$.

33. Define the function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} ax^2 + b, & \text{if } x \leq 1, \\ cx + 1, & \text{if } x > 1. \end{cases}$$

We want to find appropriate values of a, b and c such that

I) f is increasing in the interval $(0, \infty)$; and

II) f' is continuous on $\mathbb{R}$.

Which of the following is the correct statement about the values of (a, b, c) for which both conditions (I) and (II) are satisfied ?

(A) $(3, 2, 6)$ is the only possible value.

(B) There are finitely many values of (a, b, c).

(C) $(-2, -3, -4)$ is one of the values.

(D) There are infinitely many values of (a, b, c).

34. The limit

$$\lim_{x \to 0} \frac{3^x - 2^x}{4^x - 3^x}$$

equals

(A) $3/4$.

(B) $\log_{4/3}(3/2)$.

(C) $\log_e(3/2)$.

(D) $2/3$.

35. The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \min(3x^3 + x, |x|)$ is

(A) continuous on $\mathbb{R}$, but not differentiable at $x = 0$.

(B) differentiable on $\mathbb{R}$, but f' is discontinuous at $x = 0$.

(C) differentiable on $\mathbb{R}$, and f' is continuous on $\mathbb{R}$.

(D) differentiable to any order on $\mathbb{R}$.

36. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, and define

$$g(x) = \int_0^{x^3+3x^2} f(t)dt.$$

The value of $g'(0)$

(A) equals 0.

(B) equals 1.

(C) is a positive real number.

(D) cannot be determined without knowing the value of $f(0)$.

37. Let $f(x)$ be a cubic polynomial $f(x) = ax^3 + bx^2 + cx + d$, where $a, b, c, d \in \mathbb{R}$ and $a \neq 0$. Suppose that the graph of f intersects the x-axis at exactly two distinct points and that $\lim_{x\to\infty} f(x) = -\infty$. Then:

(A) f has a unique point of global minimum in $\mathbb{R}$.

(B) f has a unique point of global maximum in $\mathbb{R}$.

(C) f has exactly two points of local maximum in $\mathbb{R}$.

(D) f has exactly one point of local maximum in $\mathbb{R}$.

38. For any $x \in \mathbb{R}$, let $[x]$ denote the greatest integer smaller than or equal to x. The value of the integral

$$\int_{1/2}^{1} [1/x^2]dx$$

equals

(A) $\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{2}} - \frac{1}{2}$.

(B) $\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{2}}$.

(C) $\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{2}} + \frac{1}{2}$.

(D) $\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{2}}$.

39. The value of the integral

$$\int_0^1 \int_{\sqrt{y}}^1 3\sqrt{x^3+1}\, dxdy$$

is

(A) $2^{5/2}/3$.
(B) $\frac{2}{3}(2^{3/2}-1)$.
(C) $\frac{2}{3}(\sqrt{2}-1)$.
(D) $2(\sqrt{2}-1)$.

40. Define

$$F(x,y) = \int_0^{x^2+y^2} \cos^2(t+x)dt, \quad (x,y) \in \mathbb{R}^2.$$

$\frac{\partial F}{\partial x}(0,y)$ equals

(A) 0.
(B) $\frac{\cos(y^2)}{2}$.
(C) $\frac{\cos(y^2)}{2} + \cos^2(y^2)$.
(D) $\cos^2(y^2) + \cos(y^2)$.

41. The vector field $\boldsymbol{V}(x,y) = ye^{2x}\boldsymbol{i} + e^{2x}\boldsymbol{j}$

(A) is a conservative vector field on $\mathbb{R}^2$ having the potential function $\phi(x,y) = ye^{2x}/2$.
(B) is a conservative vector field on $\mathbb{R}^2$.
(C) has the property that the work done by $\boldsymbol{V}$ along every path is 0.
(D) is not a conservative vector field.

42. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be twice continuously differentiable at each point in $\mathbb{R}^3$. Then $\mathbf{curl}[\text{grad}(f)]$

(A) is a vector field that is orthogonal to the level surfaces of f.
(B) is a conservative vector field.
(C) equals $\mathbf{0}$ (i.e. the zero vector) at each point in $\mathbb{R}^3$.
(D) is a non-constant vector field on $\mathbb{R}^3$.

43. Let $\triangle$ be the triangle in $\mathbb{R}^2$ with vertices $(0,0),(1,0)$ and $(0,1)$, and let $\boldsymbol{F}(x,y) = -xy\boldsymbol{i} + \sqrt{y^4+1}\boldsymbol{j}$. The line integral of the vector field $\boldsymbol{F}$:

$$\oint_{\partial\triangle} \boldsymbol{F} \cdot dr,$$

taking the anti-clockwise orientation on $\partial\triangle$ (here, $\partial\triangle$ denotes the boundary of $\triangle$), is

(A) $-1/6$.
(B) 0.
(C) $1/6$.
(D) 6.

44. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be defined as $f(x,y,z) = x^2 + 2xy + 5y^2 - z^4 - 1$. The unit vector $\boldsymbol{u}$ which gives the maximum value for the directional derivative $D_{\boldsymbol{u}}f$ at the point $(1,0,1)$ is

(A) $\boldsymbol{u} = (1,0,0)$.
(B) $\boldsymbol{u} = (0,0,1)$.
(C) $\boldsymbol{u} = -\frac{1}{\sqrt{2}}(1,0,1)$.
(D) $\boldsymbol{u} = \frac{1}{\sqrt{6}}(1,1,-2)$.

45. Let G be the tetrahedron in $\mathbb{R}^3$ with vertices $(0,0,0),(1,0,0),(0,1,0)$ and $(0,0,1)$. The outward flux of the vector field $\boldsymbol{V}(x,y,z) = 2z\cos(xy)\boldsymbol{i} - (z^2+1)\boldsymbol{j} + yz^2\sin(xy)\boldsymbol{k}$ across the boundary of G is

(A) $-1/6$.
(B) 0.
(C) $1/6$.
(D) 6.

46. The general solution of the first-order ODE

$$xy' + x^2y - y = 0$$

is (in all choices below, C denotes a constant)

(A) $y(x) = xe^{-x^2/2} + C$.
(B) $y(x) = e^{x^2/2}(x + C)$.
(C) $y(x) = e^{x^2/2} - Cx$.
(D) $y(x) = Cxe^{-x^2/2}$.

47. The general solution of the first-order ODE

$$xy' + 2x^2y - xe^{-x^2} = 0$$

is (in all choices below C denotes a constant)

(A) $y(x) = e^{x^2}(x + C)$.
(B) $y(x) = e^{-x^2}(x + C)$.
(C) $y(x) = xe^{x^2} + C$.
(D) $y(x) = x + C$.

48. Consider the functions $f(x) = x|x|$ and $g(x) = x^2$. Then

(A) $\{f, g\}$ is a linearly independent pair of functions on $(-\infty, 0)$.
(B) $\{f, g\}$ is a linearly independent pair of functions on $(0, \infty)$.
(C) $\{f, g\}$ is a linearly dependent pair of functions on $\mathbb{R}$.
(D) $\{f, g\}$ is a linearly independent pair of functions on $\mathbb{R}$.

49. Which of the following pair of functions is *NOT* a linearly independent pair of solutions for the second-order ODE $y'' + 9y = 0$?

(A) $(\sin 3x, \cos 3x)$
(B) $(3\cos 3x - 2\sin 3x, \sin 3x)$
(C) $(\cos^3 x - 3\sin^2 x \cos x, \cos 3x)$
(D) $(\cos 3x - 3\sin 3x, \sin 3x - 3\cos 3x)$

50. Consider the second-order ODE

$$x^2y'' + Axy' + y = 0.$$

This equation

(A) admits, for each $A > 0$, a linearly independent pair of solutions consisting of trigonometric functions.

(B) admits, for some values of $A > 0$, a linearly independent pair of solutions consisting of powers of x.

(C) does not admit any linearly independent pair of solutions consisting of powers of x for any $A > 0$.

(D) has no solutions.

⋆⋆⋆ **End of question paper** ⋆⋆⋆

Instructions

1. This question paper has fifty multiple choice questions.

2. Four possible answers are provided for each question and only one of these is correct.

3. Marking scheme: Each correct answer will be awarded **2** marks, but **0.5** marks will be **deducted** for each incorrect answer.

4. Answers are to be marked in the OMR sheet provided.

5. For each question darken the appropriate bubble to indicate your answer.

6. Use only HB pencils for bubbling answers.

7. Mark only one bubble per question. If you mark more than one bubble, the question will be evaluated as incorrect.

8. If you wish to change your answer, please erase the existing mark completely before marking the other bubble.

9. Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, integers, rational numbers, real numbers and complex numbers respectively.

Integrated Ph. D./ Mathematical Sciences

1. Let $f : [0, 1] \to [0, 1]$ be continuous and $f(0) = 0$, $f(1) = 1$. Then, f is necessarily
   (A) injective, but not surjective.
   (B) surjective, but not injective.
   (C) bijective.
   (D) surjective.

2. Let $f : (-1, 1) \to \mathbb{R}$ be a function such that $|f(x)| \leq |x|^2$. Then,
   (A) f need not be differentiable at the origin.
   (B) $f'(0) > 0$.
   (C) $f'(0) = 0$.
   (D) $f'(0) < 0$.

3. Let $\alpha_1, \cdots, \alpha_{2007}$ be the roots of the equation $1 + x^{2007} = 0$. Then, the value of the product $(1 + \alpha_1) \cdots (1 + \alpha_{2007})$ is
   (A) 0.
   (B) 1.
   (C) 2.
   (D) 2007.

4. Let $q > 1$ be a positive integer. Then, the set $\{(\cos \frac{\pi}{q} + i \sin \frac{\pi}{q})^n : n = 0, 1, 2, \cdots\}$, where $i = \sqrt{-1}$, is
   (A) a singleton.
   (B) a finite set, but not a singleton.
   (C) a countably infinite set.
   (D) dense on the unit circle.

5. Consider the second order ordinary differential equation $y'' + by' + cy = 0$, where b, c are real constants. You are given that $y = \exp(2x)$ is a solution. Then,

(A) $b^2 + 4c < 0$.

(B) $b^2 + 4c \geq 0$.

(C) $b^2 - 4c < 0$.

(D) $b^2 - 4c \geq 0$.

6. Consider the second order ordinary differential equation $y'' + 3y' + 2y = 0$. Then, $\lim_{n\to\infty} y(t)$ is

(A) a non-zero finite number.

(B) 0.

(C) $-\infty$.

(D) ∞.

7. Consider the system $x' = -y$, $y' = x$ with $x(0) = 1$, $y(0) = 1$. Then,

(A) $y = \sin t + \cos t$.

(B) $y = -\sin t + \cos t$.

(C) $y = t \exp t + \exp t$.

(D) y is not any of the above.

8. Consider the equation $x^{2007} - 1 + x^{-2007} = 0$. Let m be the number of distinct complex, non-real roots and n be the number of distinct real roots of the above equation. Then, $m - n$ is

(A) 0.

(B) 2006.

(C) 2007.

(D) 4014.

9. Let a, b, c be non-zero real numbers. Then, the minimum value of

$$a^2 + b^2 + c^2 + \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}$$

is

(A) 0.

(B) 6.

(C) 3^2.

(D) 6^2.

10. Consider the set $A = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x^2 + y^2 + 2x + 4y + 6 = 0\}$. Then, A is

(A) an infinite set.

(B) a finite set with more than one element.

(C) a singleton.

(D) an empty set.

11. Consider the sequence $\{l_n\}_{n \in \mathbb{N}}$ with $l_n = \frac{1}{n+1} + \cdots + \frac{1}{2n}$. This sequence

(A) is increasing and bounded.

(B) increases to ∞.

(C) decreases to 0.

(D) decreases to a positive number.

12. Let p be a polynomial of degree $2n + 1$ with real coefficients. We say that a real number a is a fixed point of p if $p(a) = a$. Then, p has

(A) exactly $2n + 1$ fixed points.

(B) at least one fixed point.

(C) at most one fixed point.

(D) n fixed points.

13. Let $f(x) = e^{(e^{-x})}$ and define $g(x) = f(x+1) - f(x)$. Then, as $x \to \infty$, the function $g(x)$ converges to

(A) 0.
(B) 1.
(C) e.
(D) e^e.

14. Let A, B be 2×2 matrices with real entries, and assume that $AB - BA = cI$ for some constant c, where I is the identity matrix. Then, c is

(A) 0.
(B) 1.
(C) 2.
(D) 4.

15. Let f and g be any two non-constant Riemann-integrable functions on an interval $[a, b]$. Then, $\int_a^b f(x)g(x)dx$

(A) is $(\int_a^b f(x)dx)(\int_a^b g(x)dx)$.
(B) is $f(a)(\int_a^b g(x)dx)$.
(C) is $f(a)(\int_a^b g(x)dx) + g(a)(\int_a^b f(x)dx)$.
(D) does not have a representation as above.

16. Let $A = \begin{bmatrix} a & \pi \\ \pi & 1/49 \end{bmatrix}$, where a is a real number. Then, A is invertible

(A) for all $a \neq 22^2$.
(B) for all $a \neq 180^2 \times 49$.
(C) for all $a \neq 22^2$ or $a \neq 180^2 \times 49$.
(D) for all rational a.

17. Let A be an $n \times n$ matrix with real entries and suppose that the system $Ax = 0$ has the unique solution $x = 0$. Then, the mapping $T : \mathbb{R}^n \to \mathbb{R}^n$ defined by $Tx = Ax$ is

(A) a bijection.

(B) one-one, but not onto.

(C) onto, but not one-one.

(D) neither one-one nor onto.

18. If A is an $n \times n$ matrix with real or complex entries and $A^3 = 0$, then

(A) $(I + A)^3 = 0$.

(B) $I + A$ is invertible.

(C) $I + A$ is not invertible.

(D) necessarily $A = 0$.

19. Let A be an $n \times n$ invertible matrix with integer entries and assume that A^{-1} also has only integer entries. Then,

(A) $\det A = n$.

(B) $\det A = \pm 1$.

(C) $\det A = n^2$.

(D) $\det A$ will depend on the entries of A and A^{-1}.

20. The eigenvalues of $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ are

(A) $\cos\theta$ and $\sin\theta$.

(B) $\tan\theta$ and $\cot\theta$.

(C) $e^{i\theta}$ and $e^{-i\theta}$.

(D) 1 and 2.

21. Let $A(t) = \begin{bmatrix} a(t) & b(t) \\ c(t) & d(t) \end{bmatrix}$, where $a(t)$, $b(t)$, $c(t)$ and $d(t)$ are differentiable on $\mathbb{R}$. Then, $\frac{d}{dt} \det A(t)$ is

(A) $\det \begin{bmatrix} a'(t) & b'(t) \\ c'(t) & d'(t) \end{bmatrix}$.

(B) $\det \begin{bmatrix} a(t) & b'(t) \\ c(t) & d'(t) \end{bmatrix}$.

(C) $\det \begin{bmatrix} a'(t) & b(t) \\ c'(t) & d(t) \end{bmatrix}$.

(D) $\det \begin{bmatrix} a'(t) & b'(t) \\ c(t) & d(t) \end{bmatrix} + \det \begin{bmatrix} a(t) & b(t) \\ c'(t) & d'(t) \end{bmatrix}$.

22. For $n > 1$, let $f(n)$ be the number of $n \times n$ real matrices A such that $A^2 + I = 0$. Then,

(A) $f \equiv 0$.

(B) $f(n) = 0$ if and only if n is even.

(C) $f(n) = 0$ if and only if n is odd.

(D) $f \equiv \infty$.

23. Let the sequence $\{x_n\}_{n \in \mathbb{N}}$ of real numbers converge to a non zero real number a and let $y_n = a - x_n$. Then $\max_{n \in \mathbb{N}} \{x_n, y_n\}$ converges to

(A) a always.

(B) 0 always.

(C) $\max\{a, 0\}$.

(D) $\min\{a, 0\}$.

24. Let $f(x) = \sum_{k=0}^{n} c_k x^k$ be a polynomial with real coefficients, where $c_0 > 0$ and $c_n < 0$. Then,

(A) $f(x) > 0$ for all $x > 0$.

(B) $f(x) < 0$ for all $x < 0$.

(C) $f(x) = 0$ for some $x > 0$.

(D) $f(x) = 0$ for infinitely many values of x.

25. Which of the following is an equivalence relation in $\mathbb{R}$:

(A) $x \leq y$ for all $x, y \in \mathbb{R}$.

(B) $x - y$ is an irrational number.

(C) $x - y$ is divisible by 3.

(D) $x - y$ is a perfect square.

26. Let X be anon-empty set. A relation $\sim$ on X is called *circular* if whenever $x \sim y$ and $y \sim z$, then $z \sim x$; and *triangular* if whenever $x \sim y$ and $x \sim z$, then $y \sim z$. An equivalence relation is

(A) circular and triangular.

(B) neither circular nor triangular.

(C) circular, but not triangular.

(D) triangular, but not circular.

27. Let f be a real differentiable function defined on $[a, b]$, where the derivative is an increasing function and $x_0 \in [a, b]$. Then,

(A) f is always strictly increasing.

(B) f is always strictly decreasing.

(C) $f(x) \leq f(x_0) + (x - x_0)f'(x_0)$ for all $x \in [a, b]$.

(D) $f(x) \geq f(x_0) + (x - x_0)f'(x_0)$ for all $x \in [a, b]$.

28. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous odd function and define $g(x) = \displaystyle\int_0^{3x - \sin 2x} f(t)dt$. Then, the value of $g'(0)$ is

(A) 1.

(B) 0.

(C) 3.

(D) cannot be determined from the given data.

29. Let x, y and z be any 3 positive real numbers. Then, always:

(A) $\sqrt{xyz} \leq \frac{x+y+z}{3}$.

(B) $\sqrt{xyz} \geq \frac{x+y+z}{3}$.

(C) $\sqrt{xyz} \leq \left(\frac{x+y+z}{3}\right)^{3/2}$.

(D) $\sqrt{xyz} \geq \left(\frac{x+y+z}{3}\right)^{3/2}$.

30. Consider the two functions $f(x) = |x| \sin x$ and $g(x) = x \sin x$. Then, $\{f, g\}$ is

(A) linearly independent on $(-\infty, 0)$.

(B) linearly independent on $(0, \infty)$.

(C) linearly dependent on $\mathbb{R}$.

(D) linearly independent on $\mathbb{R}$.

31. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation. Assume that $T(x) = 0$ for all x such that $|x| = 1$. Then,

(A) $T \equiv 0$.

(B) T is onto.

(C) dimension of kernel of T is 1.

(D) dimension of range of T is 1.

32. Let A be a matrix of order 2 with real entries such that $AB = BA$ for all matrices B of order 2. Then,

(A) A is always the zero matrix.

(B) $A = \lambda I$ for some $\lambda \in \mathbb{R}$.

(C) A is always invertible.

(D) A is never invertible.

33. Consider the space $V = \{(x_1 + x_2 + x_3, x_1 + x_2, x_3) : (x_1, x_2, x_3) \in \mathbb{R}^3\}$. Then, the dimension of V is

(A) 0.
(B) 1.
(C) 2.
(D) 3.

34. Let $n > 2$ and for $1 \leq j \leq n$, define a_j to be the vector in $\mathbb{R}^n$ with j^{th} entry 0 and the remaining entries 1. Then, $\{a_1, \cdots, a_n\}$

(A) is a linearly dependent set.
(B) is an orthogonal system.
(C) spans a proper subspace of $\mathbb{R}^n$.
(D) is a basis for $\mathbb{R}^n$.

35. Let V be a 25 dimensional vector space. Then, the dimension of the intersection of two 13 dimensional subspaces of V

(A) is always 1.
(B) can be any integer between (and including) 0 and 13.
(C) can be any integer between (and including) 1 and 13.
(D) is none of the above.

36. Let S_4 denote the symmetry group of 4 letters and $\mathbb{R}^\star$ be the multiplicative group of non-zero real numbers. If $f : S_4 \to \mathbb{R}^\star$ is a homomorphism, then the set $\{x \in S_4 : f(x) = 1\}$ has

(A) at least 12 elements.
(B) exactly 24 elements.
(C) at most 12 elements.
(D) exactly 4 elements.

37. For positive integers n and m, where $n, m > 1$, suppose that $n\mathbb{Z}$ and $m\mathbb{Z}$ are isomorphic as rings. Then,

(A) there is no restriction on n and m.
(B) $n = m$.
(C) $g.c.d(n, m) = 1$.
(D) necessarily $n|m$ or $m|n$, but not both.

38. Let $\mathbb{Z}_n$ denote the additive group of integers modulo n. Suppose $\mathbb{Z}_n \times \mathbb{Z}_m \simeq \mathbb{Z}_{mn}$. Then,

(A) $g.c.d(n, m) = 1$.
(B) $n = m = 1$.
(C) $n|m$.
(D) $mn = m + n$.

39. Let S_n be the symmetry group of n letters and assume that it is abelian. Then,

(A) $n = 1$ or $n = 2$.
(B) n is a prime greater than 2.
(C) n is an even number greater than 2.
(D) n is an odd number greater than 2.

40. Let a and b be two non-zero vectors in $\mathbb{R}^3$ such that $|a \times b| = |a|\,|b|$. Then,

(A) a and b are orthogonal.
(B) a and b are parallel.
(C) the angle between a and b is $\pi/4$.
(D) a conclusion is not possible with the given data.

41. Let a, b and c be three vectors in $\mathbb{R}^3$, Then, $(a \times b) \cdot ((b \times c) \times (c \times a))$ is

(A) $((a \times b) \cdot c)^2$.

(B) $(a \cdot (b \times c))^2$.

(C) $a \cdot (b \times c) + (a \times b) \cdot c$.

(D) is always 0.

42. Consider the two space curves given by the parametric equations $\gamma_1(t) := (t, t^2, t^3)$, for all $t \in \mathbb{R}$ and $\gamma_2(s) := (s-1, s^2+s+4, 7s-13)$ for all $s \in \mathbb{R}$. Then, they

(A) never intersect.

(B) intersect exactly at 1 point.

(C) intersect exactly at 2 points.

(D) intersect exactly at 3 points.

43. For the surface $x^2 + 9y^2 - z^2 = 16$, the tangent plane at $(4, 1, 3)$ is given by

(A) $8x + 18y - 3z = 41$.

(B) $4x + 9y - 3z = 16$.

(C) $x + 9y - z = 10$.

(D) $4x + y - 3z = 8$.

44. Let $\sigma : (-1, 1) \to \mathbb{R}^3$ be a differentiable curve such that $\sigma'(t) \cdot \sigma'(t) = 1$ for all $t \in (-1, 1)$. Then,

(A) $\sigma''(t)$ is perpendicular to $\sigma'(t)$ for all $t \in (-1, 1)$.

(B) $\sigma''(t)$ is parallel to $\sigma'(t)$ for all $t \in (-1, 1)$.

(C) $\sigma(t) = (t, 0, 0)$ for all $t \in (-1, 1)$.

(D) $\sigma(t) \cdot \sigma'(t) = t$ for all $t \in (-1, 1)$.

45. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be thrice differentiable and vanish on the boundary of the region $\Omega = (-1, 1) \times (-1, 1)$. Then,

$$\int_{-1}^{1} \int_{-1}^{1} \operatorname{div}(\operatorname{grad} f)(x, y) dx dy$$

is

(A) never 0.
(B) 1.
(C) 0.
(D) dependent on f.

46. Let X, Y, Z be three vectors in $\mathbb{R}^3$ such that $X = \hat{\mathbf{i}} + 2\hat{\mathbf{k}}$ and $Y \times Z = \hat{\mathbf{i}} - 2\hat{\mathbf{j}} - 6\hat{\mathbf{k}}$, where $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ are the standard unit vectors along the coordinate directions. Then, the volume of the parallelpiped spanned by X, Y, Z is

(A) 2.
(B) 4.
(C) 6.
(D) 8.

47. Let E be the ellipsoid $(x - 1)^2 + y^2 + \frac{1}{9}z^2 = 1$ and S be the sphere with center $(1, 0, 4)$ and radius $\sqrt{7}$. Then, $E \bigcap S$ is

(A) an ellipse, but not a circle.
(B) the set $\{(x, y, z) : (x - 1)^2 + y^2 = 3/4\}$.
(C) the set $\{(x, y, z) : (x - 1)^2 + y^2 = 3/4, z = 3/2\}$.
(D) the empty set.

48. Let S be the plane whose normal vector make angles $\pi/3, \pi/4, \pi/3$ with x, y, z axes respectively. If the point $(1, 1, 1)$ is in S, then, the equation of S is

(A) $\sqrt{2}x + y + z = 2 + \sqrt{2}$.
(B) $x + \sqrt{2}y + z = 2 + \sqrt{2}$.
(C) $x - \sqrt{2}y + z = 1 - \sqrt{2}$.
(D) $\sqrt{2}x + y + \sqrt{2}z = 2\sqrt{2} + 1$.

49. Let x be a real number with $\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{2}}$. Then, the quantity $\frac{x}{\sqrt{2}} + \frac{\sqrt{2}}{x}$ lies in

(A) $[1, \sqrt{2})$.
(B) $[\sqrt{2}, \sqrt{3})$.
(C) $[\sqrt{3}, 2)$.
(D) $[2, \infty)$.

50. Let a_1, a_2, a_3, a_4 be any 4 consecutive binomial coefficients in the expansion of $(x+y)^n$. Then, $\frac{a_1}{a_1+a_2} + \frac{a_3}{a_3+a_4}$ is

(A) $\dfrac{2a_1}{a_1+a_2}$.
(B) $\dfrac{2a_2}{a_2+a_3}$.
(C) $\dfrac{2a_3}{a_3+a_4}$.
(D) $\dfrac{2a_4}{a_4+a_1}$.

Instructions

1. This question paper has forty multiple choice questions.
2. Four possible answers are provided for each question and only one of these is correct.
3. Marking scheme: Each correct answer will be awarded **2.5** marks, but **0.5** marks will be **deducted** for each incorrect answer.
4. Answers are to be marked in the OMR sheet provided.
5. For each question darken the appropriate bubble to indicate your answer.
6. Use only HB pencils for bubbling answers.
7. Mark only one bubble per question. If you mark more than one bubble, the question will be evaluated as incorrect.
8. If you wish to change your answer, please erase the existing mark completely before marking the other bubble.
9. Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, integers, rational numbers, real numbers and complex numbers respectively.

Integrated Ph. D./ Mathematical Sciences

1. Let X be a set with 30 elements. Let A, B, C be subsets of X with 10 elements each such that $A \cap B \cap C$ has 4 elements. Suppose $A \cap B$ has 5 elements, $B \cap C$ has 6 elements, and $C \cap A$ has 7 elements, how many elements does $A \cup B \cup C$ have ?

   (A) 16.
   (B) 14.
   (C) 15.
   (D) 30.

2. If $\alpha_1, \alpha_2, \cdots, \alpha_6$ are roots of $x^6 + x^2 + 1 = 0$, then which of the following is the value of $(1 - 2\alpha_1)(1 - 2\alpha_2)\cdots(1 - 2\alpha_6)$?

   (A) 0.
   (B) 1.
   (C) 64.
   (D) 81.

3. If a, b are arbitrary positive real numbers, then the least possible value of $\dfrac{6a}{5b} + \dfrac{10b}{3a}$ is

   (A) 4.
   (B) $\dfrac{6}{5}$.
   (C) $\dfrac{10}{3}$.
   (D) $\dfrac{68}{15}$.

4. Let $p(x) = x^{10} + a_1x^9 + \cdots + a_{10}$ be a polynomial with real coefficients. Suppose $p(0) = -1$, $p(1) = 1$, $p(2) = -1$. Let R be the number of real zeros of $p(x)$. Which of the following must be true ?

(A) $R \geq 4$.
(B) $R = 3$.
(C) $R = 2$.
(D) $R = 1$.

5. Let $p(x)$ and $q(x)$ be non-zero polynomials with real coefficients such that degree$(p(x)) >$ degree$(q(x))$. If the graphs of $y = p(x)$ and $y = q(x)$ intersect in 3 points, which of the following must be true ?

(A) degree$(p(x)) \leq 2$.
(B) degree$(p(x)) \geq 3$.
(C) degree$(p(x)) = 2$.
(D) degree$(p(x)) = 6$.

6. Let $A = \begin{pmatrix} 12 & 24 & 5 \\ x & 6 & 2 \\ -1 & -2 & 3 \end{pmatrix}$. The value of x for which the matrix A is not invertible is

(A) 6.
(B) 12.
(C) 3.
(D) 2.

7. Let a, b be arbitrary real numbers satisfying $a^2 + b^2 = 10$. The largest possible value of $|a + 2b|$ is

(A) 7.
(B) 5.
(C) $3\sqrt{10}$.
(D) $\sqrt{50}$.

8. Let $A = \begin{pmatrix} \pi & p \\ q & r \end{pmatrix}$ where p, q, r are rational numbers. If $\det A = 0$ and $p \neq 0$, then the value of $q^2 + r^2$

   (A) is 2.

   (B) is 1.

   (C) is 0.

   (D) cannot be determined using the given information.

9. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a 2×2 real matrix with $\det(A) = 1$. If A has no real eigenvalues then

   (A) $(a + d)^2 < 4$.

   (B) $(a + d)^2 = 4$.

   (C) $(a + d)^2 > 4$.

   (D) $(a + d)^2 = 16$.

10. Let $P = \{(x, y, z) \in \mathbb{R}^3 \mid x + y - z = 0\}$. Suppose $A : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation satisfying $A(v) = \mathbf{0}$ for all $v \in P$ and also $A(0, 0, 1) = \mathbf{0}$ (here $\mathbf{0}$ denotes the vector $(0, 0, 0)$). Then

    (A) The dimension of the null space of A is 2.

    (B) A is the zero linear transformation.

    (C) Image $A = \mathbb{R}^3$.

    (D) The dimension of the image of A is 2.

11. Suppose $A : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation such that $A^3 = I$, where I is the identity transformation. Then

    (A) All eigenvalues of A have to be real.

    (B) The product of the eigenvalues of A must be 1.

    (C) Necessarily $A = I$.

    (D) A need not be an invertible matrix.

12. Let the group $G = \mathbb{R}$ under addition and the group $H =$ the set of all positive real numbers under multiplication. Then

    (A) H is a cyclic group.

    (B) G is a cyclic group.

    (C) G and H are isomorphic

    (D) G and H are not isomorphic.

13. A *generator* for a group G is an element $g \in G$ such that every element of G is equal to some power of g. Let G be a cyclic group of order 7. Then the number of generators of G is

    (A) 1.

    (B) 3.

    (C) 6.

    (D) 7.

14. Let G be the set of 2×2 real matrices which are invertible. Consider G with the binary operation $\circ$ of matrix multiplication. Then

    (A) $(G, \circ)$ is a finite group.

    (B) $(G, \circ)$ is an infinite group.

    (C) $(G, \circ)$ is an abelian group.

    (D) $(G, \circ)$ is not a group.

15. Define a relation $\sim$ on $\mathbb{R}$ as follows: given $x, y \in \mathbb{R}$, $x \sim y$ iff $x - y$ is a rational number. Then

    (A) Given x, there are only finitely many y such that $y \sim x$.

    (B) Given x, the set of y such that $y \sim x$ is a bounded subset of $\mathbb{R}$.

    (C) $\sim$ is not an equivalence relation.

    (D) $\sim$ is an equivalence relation.

16. Let S denote the set of unit vectors in $\mathbb{R}^3$ and W a vector subspace of $\mathbb{R}^3$. Let $V = W \cap S$. Then

(A) V is always a subspace of $\mathbb{R}^3$.

(B) V is a subspace of $\mathbb{R}^3$ iff W has dimension 1.

(C) V is a subspace of $\mathbb{R}^3$ iff W has dimension 3.

(D) V is never a subspace of $\mathbb{R}^3$.

17. Define a sequence s_n by

$$s_n = \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+k}}$$

Then the limit of s_n as n tends to infinity

(A) is 0.

(B) is 1.

(C) is ∞.

(D) doesn't exist.

18. If $\lim_{x\to 0}\left(\frac{1+cx}{1-cx}\right)^{\frac{1}{x}} = 4$, then $\lim_{x\to 0}\left(\frac{1+2cx}{1-2cx}\right)^{\frac{1}{x}}$ is

(A) 2.

(B) 4.

(C) 16.

(D) 64.

19. Let the limits of the sequences a_n and b_n, respectively, be k and k^3. If the sequence $a_1, b_1, a_2, b_2, \cdots, \cdots$ has a limit, then the value of this limit

(A) is 0 or 1 or -1.

(B) is 0 or 1.

(C) is $k + k^3$.

(D) is k^4.

20. Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Define $g : [a, b] \to \mathbb{R}$ by

$g(x) = \sup\{f(y) : y \in [a, x]\}$. Then $g(x)$

(A) must be differentiable.

(B) must be continuous and Riemann integrable.

(C) must be continuous, but not Riemann integrable.

(D) need not be continuous.

21. If p is a real polynomial with $p(0) = 1$ and $p'(x) > 0$ for all x then

(A) p has more than one real zero.

(B) p has exactly one positive zero.

(C) p has exactly one negative zero.

(D) p has no real zero.

22. If $y = f(x)$ satisfies the differential equation $y' = \cos y$, $y(0) = 0$ then

(A) $|f(x)| \leq x^2$.

(B) $|f(x)| \leq |x|$.

(C) $|f(x)| \leq |\sin x|$.

(D) $|f(x)| \leq |\cos x|$.

23. For a square matrix A, let $tr(A)$ denote the sum of its diagonal entries. Let I denote the identity matrix. If A and B are 2×2 matrices with real entries such that $\det(A) = \det(B) = 0$ and $tr(B) \neq 0$, then the limit of $\dfrac{\det(A + tI)}{\det(B + tI)}$ as $t \to 0$ is

(A) zero.

(B) infinity.

(C) $\dfrac{tr(A)}{tr(B)}$.

(D) $\det(A + B)$.

24. Let $p(x) = a_k x^k + a_{k-1}x^{k-1} + \cdots + a_0$ be a polynomial. Then $\lim_{n\to\infty} n \int_0^1 x^n p(x)dx$ equals

(A) $p(1)$.

(B) $p(0)$.

(C) $p(1) - p(0)$.

(D) ∞.

25. The function f defined by $f(x) = \begin{cases} e^{-1/x}, & x > 0 \\ 0, & x \leq 0 \end{cases}$

(A) is differentiable for all real values of x.

(B) is not differentiable at $x = 0$.

(C) is not differentiable for $x < 0$.

(D) is not differentiable for $x > 0$.

26. Let $\{a_n\}$ be a sequence of distinct real numbers which has no convergent subsequence. Then $\lim_{n\to\infty} |a_n|$

(A) is 0.

(B) is ∞.

(C) is 1.

(D) does not exist.

27. The largest term in the sequence $x_n = \frac{1000^n}{n!}$, $n = 1, 2, 3, \ldots$

(A) is x_{999}.

(B) is x_{1001}.

(C) is x_1.

(D) does not exist.

28. A curve in $\mathbb{R}^2$ whose normal at each point passes through $(0, 0)$ is a

(A) straight line.

(B) parabola.

(C) hyperbola.

(D) circle.

29. Let f be a continuous function on $[0, 1]$. Then $\lim_{n\to\infty} \sum_{j=0}^{n} \frac{1}{n} f(\frac{j}{n})$ is

(A) $\frac{1}{2}\int_0^{\frac{1}{2}} f(x)dx.$

(B) $\int_{\frac{1}{2}}^1 f(x)dx.$

(C) $\int_0^1 f(x)dx.$

(D) $\int_0^{\frac{1}{2}} f(x)dx.$

30. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that the partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ exist and are continuous. Let $D_u f(x, y)$ denote the directional derivative of f in the direction of $u \in \mathbb{R}^2$. If $D_{(1,1)}f(0, 0) = 0$ and $D_{(1,-1)}f(0, 0) = 0$, then

(A) $D_u f(0, 0) = 1$ for some $u \in \mathbb{R}^2$.

(B) $D_u f(0, 0) = -1$ for some $u \in \mathbb{R}^2$.

(C) $D_u f(0, 0) = 0$ for all $u \in \mathbb{R}^2$.

(D) $D_u f(0, 0)$ may not exist for some $u \in \mathbb{R}^2$.

31. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying $f \circ f = f$. Then

(A) f must be constant.

(B) $f(x) = x$ for all x in the range of f.

(C) f must be a non-constant polynomial.

(D) There is no such function.

32. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function such that $f(x) \geq 0$ for $x \in [0, 1]$.

If $f(x) \leq \int_0^x f(t)\,dt$ for all $0 \leq x \leq 1$, then

(A) $f(x) = 0$ for all $x \in [0, 1]$.

(B) $f(x) = x$ for all $x \in [0, 1]$.

(C) There is no such function.

(D) $f(x) = c$ for all $x \in [0, 1]$ and some $c > 0$.

33. Consider the ordinary differential equation

$$y'' + 4y = \sin 2t, \quad y(0) = 0.$$

Then the solution $y(t)$

(A) converges to 0 as $t \to \infty$ with no oscillations.
(B) converges to 0 as $t \to \infty$ and the solution is oscillating.
(C) is oscillating and bounded.
(D) is unbounded.

34. Let $y(t)$ be a solution to the differential equation $y' = y^2 + t$, then $y(t)$ is differentiable

(A) once but not twice.
(B) twice but not 3 times.
(C) 3 times but not 4 times.
(D) infinitely many times.

35. Which of the following is a solution to the differential equation $y' = |y|^{1/2}$, $y(0) = 0$, where square root means the positive square root ?

(A) $y(t) = t^2/4$.
(B) $y(t) = -t^2/4$.
(C) $y(t) = t|t|/4$.
(D) $y(t) = -t|t|/4$.

36. The number of independent solutions of the differential equation $y^{(4)} - 2y^{(2)} + y = 0$ (here $y^{(2)}$ and $y^{(4)}$ represent the second and fourth derivatives of y respectively) is

(A) 4.
(B) 3.
(C) 2.
(D) 1.

37. The number of non-trivial polynomial solutions of the differential equation $x^3 y'(x) = y(x^2)$ is

(A) zero.

(B) one.

(C) three.

(D) infinity.

38. Let $\vec{p} = 3\vec{i}+2\vec{j}+\vec{k}$ and $\vec{q} = \vec{i}+2\vec{j}+3\vec{k}$ be vectors in $\mathbb{R}^3$ (here $\vec{i}, \vec{j}, \vec{k}$ denote the unit vectors along the positive X, Y, Z axes respectively). Suppose $\vec{v} = a\vec{i} + b\vec{j} + c\vec{k}$ is a unit vector such that $\vec{v}\cdot\vec{p} = 0 = \vec{v}\cdot\vec{q}$. The value of $|a+b+c|$ is :

(A) 6.

(B) 3.

(C) 1.

(D) 0.

39. Let $\vec{a}$, $\vec{b}$, $\vec{c}$ be vectors in $\mathbb{R}^3$. If $\vec{a} \neq 0$ and $\vec{a}\times\vec{b} = \vec{a}\times\vec{c}$, then which of the following must certainly be true ?

(A) $\vec{a}\cdot\vec{b} = \vec{a}\cdot\vec{c}$

(B) $\vec{b} = \vec{c}$

(C) There is a real number λ such that $\vec{b} = \vec{c} + \lambda\,\vec{a}$

(D) $\vec{a}$ must be orthogonal to both $\vec{b}$ and $\vec{c}$

40. For a curve $\gamma : [a,b] \to \mathbb{R}^2$, let $\int_\gamma f$ denote the line integral of a function $f : U \subset \mathbb{R}^2 \to \mathbb{R}$ defined on some open set U containing $\{\gamma(t) : t \in [a,b]\}$. The value of $\int_{\mathbb{S}^1} f$, where $f(x,y) = \frac{y}{x^2+y^2}$ and $\mathbb{S}^1 = \{(\cos t, \sin t) : 0 \leq t \leq 2\pi\}$ (i.e, the circle of radius one centered at the origin) is

(A) 0.

(B) 1.

(C) π.

(D) 2π.

Instructions

1. This question paper has forty multiple choice questions.

2. Four possible answers are provided for each question and only one of these is correct.

3. Marking scheme: Each correct answer will be awarded **2.5** marks, but **0.5** marks will be **deducted** for each incorrect answer.

4. Answers are to be marked in the OMR sheet provided.

5. For each question, darken the appropriate bubble to indicate your answer.

6. Use only HB pencils for bubbling answers.

7. Mark only one bubble per question. If you mark more than one bubble, the question will be evaluated as incorrect.

8. If you wish to change your answer, please erase the existing mark completely before marking the other bubble.

9. Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, the set of integers, the set of rational numbers, the set of real numbers and the set of complex numbers respectively.

10. Let $[x]$ denote the greatest integer less than or equal to x for a real number x.

Integrated Ph. D./ Mathematical Sciences

1. Let T and S be linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$. Let T rotate each point counterclockwise through an angle θ about the origin and let S be the reflection about the line $y = x$. Then determinant of TS is

    (A) 1.
    (B) -1.
    (C) 0.
    (D) 2.

2. Let V be a 7 dimensional vector space. Let W and Z be subspaces of V with dimensions 4 and 5 respectively. Which of the following is not a possible value of $\dim(W \cap Z)$?

    (A) 1.
    (B) 2.
    (C) 3.
    (D) 4.

3. If $a, b \in \mathbb{R}$ satisfy $a^2 + 2ab + 2b^2 = 7$, then the largest possible value of $|a - b|$ is

    (A) $\sqrt{7}$.
    (B) $\sqrt{\frac{7}{2}}$.
    (C) $\sqrt{35}$.
    (D) 7.

4. Suppose a finite group G has an element a which is not the identity such that a^{20} is the identity. Which of the following can not be a possible value for the number of elements of G?

(A) 12.

(B) 9.

(C) 20.

(D) 15.

5. Let A be a 10×10 matrix in which each row has exactly one entry equal to 1, the remaining nine entries of the row being 0. Which of the following is not a possible value for the determinant of the matrix A?

(A) 0.

(B) -1.

(C) 10.

(D) 1.

6. A subset V of $\mathbb{R}^3$ consisting of vectors (x_1, x_2, x_3) satisfying $x_1^2 + x_2^2 + x_3^2 = k$ is a subspace of $\mathbb{R}^3$ if k is

(A) 0.

(B) 1.

(C) -1.

(D) none of the above.

7. Let $v_1 = (1, 0), v_2 = (1, -1)$ and $v_3 = (0, 1)$. How many linear transformations $T : \mathbb{R}^2 \to \mathbb{R}^2$ are there such that $Tv_1 = v_2, Tv_2 = v_3$ and $Tv_3 = v_1$?

(A) 3!.

(B) 3.

(C) 1.

(D) 0.

8. The equation $x^3 + 7x^2 + 1 + ixe^{-x} = 0$ has

   (A) no real solution.

   (B) exactly one real solution.

   (C) exactly two real solutions.

   (D) exactly three real solutions.

9. How many complex numbers $z = x + iy$ are there such that $x + y = 1$ and $\exp i(x^2 + y^2) = 1$?

   (A) Zero.

   (B) Non-zero but finitely many.

   (C) Countably infinite.

   (D) Uncountably infinite.

10. Let $G = \{g_1, g_2, \ldots, g_n\}$ be a finite group and suppose it is given that $g_i^2 =$ identity for $i = 1, 2, \ldots, n-1$. Then

    (A) g_n^2 is identity and G is abelian.

    (B) g_n^2 is identity, but G could be non-abelian.

    (C) g_n^2 may not be identity.

    (D) none of the above can be concluded from the given data.

11. Let $X = \{2, 3, 4, \ldots\}$ be the set of integers greater than or equal to 2. Consider the binary relation R on X given by the following: mRn if m and n have a common integer factor $r \neq 1$. Then R is

    (A) reflexive and transitive but not symmetric.

    (B) reflexive and symmetric but not transitive.

    (C) symmetric and transitive but not reflexive.

    (D) an equivalence relation.

12. If X and Y are two non-empty finite sets and $f : X \to Y$ and $g : Y \to X$ are mappings such that $g \circ f : X \to X$ is a surjective (i.e., onto) map, then

(A) f must be one-to-one.

(B) f must be onto.

(C) g must be one-to-one.

(D) X and Y must have the same number of elements.

13. Let X and Y be two non-empty sets and let $f\colon X \to Y$, $g\colon Y \to X$ be two mappings. If both f and g are injective (i.e., one-to-one) then

(A) X and Y must be infinite sets.

(B) $g = f^{-1}$ always.

(C) one of $f \circ g\colon Y \to X$ and $g \circ f\colon X \to Y$ is always bijective (one-to-one and onto).

(D) There exists a bijective mapping $h\colon X \to Y$.

14. Consider the system of linear equations

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1, \\ a_2x + b_2y + c_2z &= d_2, \\ a_3x + b_3y + c_3z &= d_3, \end{aligned}$$

where a_i, b_i, c_i, d_i are real numbers for $1 \leq i \leq 3$. If $\begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix} \neq 0$ then the above system has

(A) at most one solution.

(B) always exactly one solution.

(C) more than one but finitely many solutions.

(D) infinitely many solutions.

15. Consider the group $G = \mathbb{Z}_4 \times \mathbb{Z}_4$ of order 16, where the operation is component wise addition modulo 4. If G is a union of n subgroups of order 4 then the minimum value of n is

(A) 4.

(B) 5.

(C) 6.

(D) 7.

16. The altitude of a triangle is a line which passes through a vertex of the triangle and is perpendicular to the opposite side. The orthocenter is the point of intersection of the three altitudes. Let A be the triangle whose vertices are $(1,0)$, $(3,-1)$ and $(0,3)$. Then the orthocenter of A is

(A) $(4/3, 2/3)$.

(B) $(-3,-3)$.

(C) $(-1,1)$.

(D) $(3,5)$.

17. The area of the triangle formed by the straight lines $8x - 3y = 48$, $7y + 4x = 24$ and $5y - 2x = 22$ is

(A) 26.

(B) 30.

(C) 34.

(D) 36.

18. The equation $x^2 - y^2 + (a+b)x + (a-b)y + c = 0$ represents

(A) either a hyperbola or a pair of straight lines.

(B) always a hyperbola.

(C) always a pair of straight lines.

(D) always a parabola.

19. If the volume of the tetrahedron whose vertices are $(1,1,1)$, $(3,2,0)$, $(0,4,3)$ and $(5,0,k)$ is 6 then the value of k is

(A) $-16/7$.
(B) $-4/7$.
(C) $2/7$.
(D) 2.

20. Which one of the following curves intersects every plane in the 3-dimensional Euclidean space $\mathbb{R}^3$?

(A) $(x,y,z) = (t,t^2,t^3)$.
(B) $(x,y,z) = (t,t^3,t^4)$.
(C) $(x,y,z) = (t,t^3,t^5)$.
(D) $(x,y,z) = (t,t^2,t^5)$.

21. Let $Q = (0,0,b)$ and $R = (0,0,-b)$ be two points in the 3-dimensional Euclidean space $\mathbb{R}^3$. If the difference of the distances of a point P in $\mathbb{R}^3$ from Q and R is $2a$ (where $a \neq \pm b$) then the locus of P is

(A) $\dfrac{x^2}{b^2-a^2} + \dfrac{y^2}{b^2-a^2} - \dfrac{z^2}{a^2} - 1 = 0$.
(B) $\dfrac{x^2}{b^2-a^2} + \dfrac{y^2}{b^2-a^2} - \dfrac{z^2}{a^2} + 1 = 0$.
(C) $\dfrac{x^2}{a^2-b^2} + \dfrac{y^2}{a^2-b^2} - \dfrac{z^2}{a^2} + 1 = 0$.
(D) $\dfrac{x^2}{b^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{a^2} + 1 = 0$.

22. Define a function f on the real line by

$$f(x) = \begin{cases} x - [x] - \frac{1}{2} & \text{if } x \text{ is not an integer,} \\ 0 & \text{if } x \text{ is an integer} \end{cases}$$

Then which of the following is true:

(A) f is periodic with period 1, i.e., $f(x+1) = f(x)$ for all x.
(B) f is continuous.
(C) f is one-to-one.
(D) $\lim_{x\to a} f(x)$ exists for all $a \in \mathbb{R}$.

23. Let a, b and c be non-zero real numbers. Let

$$f(x) = \begin{cases} \sin x & \text{if } x \leq c \\ ax + b & \text{if } x > c \end{cases}$$

Suppose b and c are given. Then

(A) There is no value of a for which f is continuous at c.

(B) There is exactly one value of a for which f is continuous at c.

(C) There are infinitely many values of a for which f is continuous at c.

(D) Continuity of f at c can not be determined from what is given.

24. Let

$$f(x) = \begin{cases} 1 \text{ if } |x| \leq 1, \\ 0 \text{ if } |x| > 1 \end{cases} \quad \text{and } g(x) = 2 - x^2.$$

Let $h(x) = f(g(x))$. Then $h(x)$

(A) is continuous everywhere.

(B) has exactly one point of discontinuity.

(C) has exactly two points of discontinuity.

(D) has four points of discontinuity.

25. Let $0 < a < b$. Define a function $M(r)$ for $a \leq r \leq b$ by

$$M(r) = \max\{\frac{r}{a} - 1, 1 - \frac{r}{b}\}.$$

Then $\min\{M(r) : a \leq r \leq b\}$ is

(A) 0.

(B) $2ab/(a+b)$.

(C) $(b-a)/(b+a)$.

(D) $(b+a)/(b-a)$.

26. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$ and $f(x) = 1 + xg(x)$ where $\lim_{x\to 0} g(x) = 1$. Then the function $f(x)$ is

(A) e^x,

(B) 2^x,

(C) a non-constant polynomial,

(D) equal to 1 for all $x \in \mathbb{R}$.

27. Let $f(x)$ be a continuous function on $[0, a]$ such that $f(x)f(a-x) = 1$. Then

$$\int_0^a \frac{dx}{1+f(x)}$$

is

(A) 0,

(B) 1,

(C) a,

(D) $a/2$.

28. Let $f : [0, \infty) \to [0, \infty)$ satisfy

$$(f(x))^2 = 1 + 2\int_0^x f(t)dt.$$

Then $f(1)$ is

(A) $\log_e 2$,

(B) 1,

(C) 2,

(D) e.

29. Let

$$f(x) = \int_1^x \frac{e^t}{t} dt$$

for $x \geq 1$. Then $f(x) > \log_e x$

(A) for no value of x.

(B) only for $x > e$.

(C) for $1 \leq x \leq e$.

(D) for all $x > 1$.

30. Consider the first order ODE

$$\frac{dy}{dx} = F\left(\frac{ax + by + c}{Ax + By + C}\right)$$

where a, b, c, A, B and C are non-zero constants. Under what condition, does there exist a linear substitution that reduces the equation to one in which the variables are separable?

(A) Never.

(B) if $aB = bA$.

(C) if $bC = cB$.

(D) if $cA = aC$.

31. Let φ be a solution of the ODE

$$x^2y' + 2xy = 1 \text{ on } 0 < x < \infty.$$

Then the limit of $\varphi(x)$ as $x \to \infty$

(A) is zero.

(B) is one.

(C) is ∞.

(D) does not exist.

32. Let φ be the solution of $y' + iy = x$ such that $\varphi(0) = 2$. Then $\varphi(\pi)$ equals

(A) $i\pi$.

(B) $-i\pi$.

(C) π.

(D) $-\pi$.

33. Consider the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with real entries. Suppose it has repeated eigenvalues. Pick the correct statement:

(A) $bc = 0$.

(B) A is always a diagonal matrix.

(C) $\det(A) \geq 0$.

(D) $\det(A)$ can take any real value.

34. Let G denote the group of all 2×2 real matrices with non-zero determinant. Let H denote the subgroup of all matrices with determinant 1. Let G/H denote the set of left cosets of H. Then

(A) H is not a normal subgroup.

(B) G/H is isomorphic to the real numbers under addition.

(C) G/H is isomorphic to the non-zero real numbers under multiplication.

(D) G/H is a finite group.

35. Let $\vec{a}$ and $\vec{b}$ be two non-zero vectors in $\mathbb{R}^3$ such that $|\vec{a} \times \vec{b}| = \vec{a}.\vec{b}$. Then the smaller of the two angles subtended by $\vec{a}$ and $\vec{b}$ is

(A) zero.

(B) an acute angle.

(C) a right angle.

(D) an obtuse angle.

36. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that f' is continuous. Define the function

$$G(x, y) = f(\sqrt{x^2 + y^2}) \text{ for all } (x, y) \in \mathbb{R}^2.$$

Then

(A) $\frac{\partial G}{\partial x}$ and $\frac{\partial G}{\partial y}$ are always continuous at each $(x, y) \in \mathbb{R}^2$.

(B) $\frac{\partial G}{\partial x}$ and $\frac{\partial G}{\partial y}$ always exist but are not continuous at some point.

(C) G is always continuous on $\mathbb{R}^2$.

(D) The continuity of G depends on the choice of f.

37. The value of the integral

$$\int_0^1 \int_y^1 y\sqrt{1+x^3}dxdy$$

is

(A) $2\sqrt{2}$.

(B) $(2\sqrt{2}-1)/2$.

(C) $(2\sqrt{2}-1)/8$.

(D) $(2\sqrt{2}-1)/9$.

38. Consider the pair of first order ordinary differential equations

$$\frac{dx}{dt} = Ax + By, \;\; \frac{dy}{dt} = x,$$

where $B < -1 < A < 0$. Let $(x(t), y(t))$ be the solution of the above that satisfies $(x(0), y(0)) = (0, 1)$. Pick the correct statement:

(A) $(x(t), y(t)) = (0, 1)$ for all $t \in \mathbb{R}$.

(B) $x(t)$ is bounded on $\mathbb{R}$.

(C) $x(t)$ is bounded on $[0, \infty)$.

(D) $y(t)$ is bounded on $\mathbb{R}$.

39. Let $f(x)$ be a non-constant second degree polynomial such that $f(2) = f(-2)$. If the real numbers a, b and c are in arithmetic progression, then $f'(a), f'(b)$ and $f'(c)$ are

(A) in arithmetic progression.

(B) in geometric progression.

(C) in harmonic progression.

(D) equal.

40. Let $P(x)$ be a non-constant polynomial such that $P(n) = P(-n)$ for all $n \in \mathbb{N}$. Then $P'(0)$

(A) equals 1.

(B) equals 0.

(C) equals -1.

(D) can not be determined from the given data.

Instructions

1. This question paper has forty multiple choice questions.

2. Four possible answers are provided for each question and only one of these is correct.

3. Marking scheme: Each correct answer will be awarded **2.5** marks, but **0.5** marks will be **deducted** for each incorrect answer.

4. Answers are to be marked in the OMR sheet provided.

5. For each question, darken the appropriate bubble to indicate your answer.

6. Use only HB pencils for bubbling answers.

7. Mark only one bubble per question. If you mark more than one bubble, the question will be evaluated as incorrect.

8. If you wish to change your answer, please erase the existing mark completely before marking the other bubble.

9. Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, the set of integers, the set of rational numbers, the set of real numbers and the set of complex numbers respectively.

10. Let $[x]$ denote the greatest integer less than or equal to x for a real number x.

Integrated Ph. D. Mathematical Sciences

1. Let A be an $n \times n$ matrix with real entries such that $A^2 + I = 0$. Then

   (A) n is an odd integer.

   (B) n is an even integer.

   (C) n has to be 2.

   (D) n could be any positive integer.

2. Consider the group

$$G = \left\{ \begin{pmatrix} \lambda & a \\ 0 & \mu \end{pmatrix} : a \in \mathbb{C} \text{ and } \lambda, \mu \in \mathbb{C} \setminus \{0\} \right\}.$$

   Then the subset

$$H = \left\{ \begin{pmatrix} 1 & a \\ 0 & \mu \end{pmatrix} : a \in \mathbb{C} \text{ and } \mu \in \mathbb{C} \setminus \{0\} \right\}$$

   is

   (A) a normal subgroup

   (B) a subgroup but not a normal subgroup.

   (C) not a subgroup in general.

   (D) an abelian subgroup.

3. Let k be a positive integer. Let $n_1, n_2, \ldots, n_k$ and n be integers, each greater than one. Suppose they satisfy

$$\sum_{i=1}^{k} (1 - \frac{1}{n_i}) = 2 - \frac{2}{n}.$$

   Then the only possible values of k are

(A) any integer.

(B) 1 and 2.

(C) 2 and 3.

(D) 3 and 4.

4. Let S_4 be the group of all permutations of 4 symbols. Let H be the following subset of S_4:

$$H = \{e, (12)(34), (13)(24), (14)(23)\},$$

where e stands for the identity permutation. Then

(A) H is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

(B) H is isomorphic to $\mathbb{Z}_4$.

(C) H is not a subgroup.

(D) H is isomorphic to $\mathbb{A}_4$.

5. Suppose f is a continuous real-valued function. Let $I = \int_0^1 f(x)x^2dx$. Then it is necessarily true that I equals

(A) $\frac{f(1)}{3} - \frac{f(0)}{3}$.

(B) $\frac{f(c)}{3}$ for some $c \in [0, 1]$.

(C) $f(\frac{1}{3}) - f(0)$.

(D) $f(c)$ for some $c \in [0, 1]$.

6. Let G be a finite group of odd order. Let $f : G \to G$ be the function defined by $f(g) = g^2$. Then f is

(A) always an isomorphism.

(B) always a bijection, but not necessarily an isomorphism.

(C) never an isomorphism.

(D) not always a bijection.

7. If V is a ten dimensional vector space, then the dimension of the intersection of two six dimensional subspaces

(A) is always 6.

(B) can be any integer between 0 and 6, both inclusive.

(C) can be any integer between 2 and 6, both inclusive.

(D) can be any integer between 4 and 6, both inclusive.

8. Let S_3 denote the permutation group on 3 symbols and let $\mathbb{R}^*$ denote the multiplicative group of non-zero real numbers. Suppose

$$h : S_3 \to \mathbb{R}^*$$

is a homomorphism. Then kernel of h has

(A) always at most 2 elements.
(B) always at most 3 elements.
(C) always at least 3 elements.
(D) always exactly 6 elements.

9. Let $y(x)$ be a solution of the ODE

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + By = 0,$$

where $0 < B < 1$. Then $\lim_{x\to\infty} y(x)$ equals

(A) 0.
(B) $+\infty$.
(C) $-\infty$.
(D) $B/2$.

10. Consider the sequence $\{a_n\}$ defined by

$$a_n = \frac{1}{(n+1)^{3/2}} + \cdots + \frac{1}{(2n)^{3/2}}.$$

As $n \to \infty$, the sequence a_n

(A) converges to 0.
(B) diverges to ∞.
(C) is bounded but does not converge.
(D) converges to a positive number.

11. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable even function. Let

$$G(x) = \int_0^{f(x)} \sqrt{\tan\theta}\, d\theta.$$

Then the value of $G'(0)$

(A) equals -1.

(B) equals 0.

(C) equals 1.

(D) cannot be determined from the given data.

12. Let X be a non-empty set and let $f, g : X \to X$ be functions. Suppose $f \circ g \circ f$ equals the identity function on X. Then

(A) g is one-one but not necessarily onto.

(B) g is onto but not necessarily one-one.

(C) g is one-one and onto.

(D) g is necessarily the identity function on X.

13. Let G be the additive group of integers modulo 12. The number of different isomorphisms of G onto itself is

(A) 3.

(B) 4.

(C) 12.

(D) 24.

14. Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be

$$f(x, y, z) = xye^{-z} - xze^{-y} + yze^{-x}.$$

The unit vector $\mathbf{u}$ that maximizes the directional derivative of f in the direction of $\mathbf{u}$ at the point $(1, 0, 0)$ is

(A) $\frac{1}{\sqrt{2}}(1, -1, 0)$.

(B) $\frac{1}{\sqrt{2}}(0, 1, -1)$.

(C) $\frac{1}{\sqrt{2}}(-1, 0, 1)$.

(D) $\frac{1}{\sqrt{3}}(1, -1, 1)$.

15. Consider the second order ODE

$$\frac{d^2y}{dx^2} + A\frac{dy}{dx} + B = 0$$

where A and B are positive real numbers. The equation

(A) always admits a linearly independent pair of solutions that are trigonometric functions.

(B) always admits a linearly independent pair of solutions that are products of exponential and trigonometric functions.

(C) need not admit a linearly independent pair of solutions that are products of exponential and trigonometric functions.

(D) need not admit any solution.

16. Consider the following subsets of $\mathbb{R}^3$:

$$X_1 = \{(x, y, z) \in \mathbb{R}^3 : z^2 - x^2 + 16x - y^2 + 9y = 25\},$$
$$X_2 = \{(x, y, z) \in \mathbb{R}^3 : x + z = 9\}.$$

Then $X_1 \cap X_2$ is

(A) a pair of lines,

(B) an ellipse lying in some plane in $\mathbb{R}^3$,

(C) a parabola lying in some plane in $\mathbb{R}^3$,

(D) a hyperbola lying in some plane in $\mathbb{R}^3$.

17. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that

$$f(f(x)) = x \text{ for all } x.$$

Then

(A) f is monotone.

(B) f has to be the identity.

(C) f need not be monotone.

(D) $f(x) = \sqrt{x}$.

18. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that

$$|f(x) - f(y)| \leq |x - y|^2 \text{ for all } x, y \in \mathbb{R}.$$

Then

(A) f has to be a linear function.

(B) $f(x) = x^2$.

(C) f has to be a constant.

(D) f has to be the identity function.

19. Let A be an $n \times n$ real non-zero matrix of rank less than n. Then

(A) there exists an $n \times n$ real non-zero matrix B such that $BA = 0$.

(B) there may not always exist an $n \times n$ real non-zero matrix B such that $BA = 0$.

(C) there exists an $n \times n$ real non-zero matrix B such that $BA = I$.

(D) if B is such that $BA = 0$, then $AB = 0$.

20. Let T be a 4×4 matrix with real entries. Suppose $T^5 = 0$. Then which of the following is necessarily true?

(A) T is the zero matrix.

(B) T need not be the zero matrix, but T^2 is the zero matrix.

(C) T^2 need not be the zero matrix, but T^3 is the zero matrix.

(D) T^3 need not be the zero matrix, but T^4 is the zero matrix.

21. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable with $f(0) = 0$ and $|f'(x)| \leq 1$ for all $x \in \mathbb{R}$. Then there exists c in $\mathbb{R}$ such that

(A) $|f(x)| \leq c\sqrt{|x|}$ for all x with $|x| \geq 1$.

(B) $|f(x)| \leq c|x|^2$ for all x with $|x| \geq 1$.

(C) $f(x) = x + c$ for all $x \in \mathbb{R}$.

(D) $f(x) = 0$ for all $x \in \mathbb{R}$.

22. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function such that $\nabla f(x) \times \mathbf{u} = 0$ for all $x \in \mathbb{R}^3$ where $\mathbf{u}$ is the vector $(1, 0, 0)$. Then it must be that

(A) $f(x_1, y_1, z) = f(x_2, y_2, z)$ for all x_1, y_1, x_2, y_2 and z.

(B) $f(x_1, y, z_1) = f(x_2, y, z_2)$ for all x_1, z_1, x_2, z_2 and y.

(C) $f(x, y_1, z_1) = f(x, y_2, z_2)$ for all y_1, z_1, y_2, z_2 and x.

(D) f is a constant function.

23. Let l be a line segment realizing the distance between a circle C and an ellipse E in the plane. Then

(A) l must meet C orthogonally, but need not meet E orthogonally.

(B) l need not meet C or E orthogonally.

(C) l must meet E orthogonally, but need not meet C orthogonally.

(D) l must meet both C and E orthogonally.

24. Let $\mathbf{u}$ and $\mathbf{v}$ be two non-zero vectors in $\mathbb{R}^3$. Then,

(A) there is a unique $\mathbf{y}$ in $\mathbb{R}^3$ such that $\mathbf{u} \times \mathbf{y} = \mathbf{v}$.

(B) there is a $\mathbf{y}$ in $\mathbb{R}^3$ such that $\mathbf{u} \times \mathbf{y} = \mathbf{v}$, but this need not be unique.

(C) there may not exist any $\mathbf{y}$ in $\mathbb{R}^3$ such that $\mathbf{u} \times \mathbf{y} = \mathbf{v}$.

(D) there is a unit vector $\mathbf{y}$ in $\mathbb{R}^3$ such that $\mathbf{u} \times \mathbf{y} = \mathbf{v}$ if and only if $\|\mathbf{u}\| \geq \|\mathbf{v}\|$.

25. Let $f : G_1 \to G_2$ be a homomorphism of the group G_1 into the group G_2. Let H be a subgroup of G_2. Then which of the following is true?

(A) If H is abelian, then $f^{-1}(H)$ is an abelian subgroup of G_1.

(B) If H is normal, then $f^{-1}(H)$ is a normal subgroup of G_1.

(C) $f^{-1}(H)$ need not be a subgroup of G_1.

(D) $f^{-1}(H)$ must be contained in the kernel of f.

26. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a non-zero linear transformation such that $T\mathbf{v} = 0$ for all $\mathbf{v} \in S = \{(x, y, z) : x^2 + y^2 + z^2 = 1, x + y + z = 0\}$. Then the dimension of the kernel of T has to be

(A) 0.

(B) 1.

(C) 2.

(D) 0 or 1.

27. Let R be the set of matrices of the form

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$$

where $a, b, c \in \mathbb{R}$. Consider R with usual addition and multiplication of matrices. Which of the following is true?

(A) R is a ring without zero-divisors.

(B) R is a ring with zero-divisors.

(C) R is a commutative ring.

(D) Every non-zero element in R has a multiplicative inverse.

28. Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be non-coplanar unit vectors such that

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \frac{\mathbf{v} + \mathbf{w}}{2}.$$

Let the angle between $\mathbf{u}$ and $\mathbf{v}$ be α and let the angle between $\mathbf{u}$ and $\mathbf{w}$ be β with $0 \leq \alpha, \beta \leq \pi$. Then (α, β) equals

(A) $(\frac{\pi}{3}, \frac{\pi}{3})$.

(B) $(\frac{2\pi}{3}, \frac{\pi}{3})$.

(C) $(\frac{\pi}{3}, \frac{2\pi}{3})$.

(D) $(\frac{2\pi}{3}, \frac{2\pi}{3})$.

29. Let

$$\mathbf{u} = \mathbf{i} + 4x\mathbf{j} + (x - 6)\mathbf{k} \text{ and } \mathbf{v} = y^2\mathbf{i} + y\mathbf{j} + 4\mathbf{k}$$

where $x, y \in \mathbb{R}$. If the angle between $\mathbf{u}$ and $\mathbf{v}$ is acute for all $y \in \mathbb{R}$, then

(A) $x < 2$.

(B) $x > 3$.

(C) $x < -3$ or $x > 2$.

(D) $-2 < x < 3$.

30. Consider the two parabolas $y^2 = 4ax$ and $x^2 = 4by$. Suppose, given any point in the plane, the tangents to the first parabola from that point are normal to the second. Then

(A) $a = \pm b$.

(B) $ab = 4$.

(C) $a^2 > 8b^2$.

(D) $a^2 < 8b^2$.

31. Let α, β, a and b be real constants with a and b non-zero. Let θ be a real number. Let $P(\theta) = (a\tan(\theta + \alpha), b\tan(\theta + \beta))$. Then $P(\theta)$ lies on

(A) a hyperbola.

(B) a parabola.

(C) an ellipse.

(D) a straight line.

32. A tangent to the parabola $x^2 = 4y$ meets the hyperbola $xy = 1$ in P and Q. If the tangent varies, then the locus of the mid-point of P and Q is

(A) a straight line,

(B) a hyperbola,

(C) an ellipse,

(D) a parabola.

33. Let A be a subset of $\mathbb{R}^3$ such that

$$tx + (1-t)y \in A$$

for all x and y in A and for all t in $\mathbb{R}$. Then

(A) the set A is a straight line,

(B) for any $u \in A$, the set $A_u = \{v - u : v \in A\}$ is a vector subspace of $\mathbb{R}^3$,

(C) the set A is a vector subspace of $\mathbb{R}^3$,

(D) the set A is a bounded convex set.

34. Which of the following need not be true for an $n \times n$ real matrix A?

(A) If columns of A span $\mathbb{R}^n$, then rows of A span $\mathbb{R}^n$.

(B) If columns of A are linearly independent, then rows of A are linearly independent.

(C) If columns of A are orthogonal, then rows of A are orthogonal.

(D) If columns of A are orthonormal, then rows of A are orthonormal.

35. Let $\mathcal{S}$ be a collection of non-empty subsets of $\{1, 2, \ldots, 10\}$ such that if $A, B \in \mathcal{S}$, then either $A \subset B$ or $B \subset A$. The maximum possible cardinality of $\mathcal{S}$ is

(A) 10.

(B) $\binom{10}{2}$.

(C) $\binom{10}{5}$.

(D) $\binom{10}{6}$.

36. The value of

$$\lim_{n\to\infty} (1 + \frac{1}{n})^{n^2} e^{-n}$$

is

(A) 1.

(B) $e^{-1/2}$.

(C) e.

(D) e^2.

37. The value of

$$\frac{1}{1.2} - \frac{1}{2.3} + \frac{1}{3.4} - \frac{1}{4.5} + \cdots$$

is

(A) between 0 and 1/4.

(B) between 1/4 and 1/3.

(C) between 1/3 and 1/2.

(D) between 1/2 and 1.

38. Which of the following is true?

(A) $e^{\frac{1}{2}(x+y)} \geq \frac{1}{2}(e^x + e^y)$,

(B) $\log \frac{x+y}{2} \geq \frac{1}{2}(\log x + \log y)$,

(C) $\frac{1}{2}(x^{\frac{3}{2}} + y^{\frac{3}{2}}) \geq \left(\frac{x+y}{2}\right)^{3/2}$,

(D) $\frac{1}{2}(xe^{-x} + ye^{-y}) \leq \frac{1}{2}((x+y)e^{-\frac{x+y}{2}})$.

39. If $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, which of the statements implies that $f(0) = 0$?

(A) $\int_0^1 f(x)^n dx \to 0$ as $n \to \infty$.

(B) $\int_0^1 f(\frac{x}{n}) dx \to 0$ as $n \to \infty$.

(C) $\int_0^1 f(nx) dx \to 0$ as $n \to \infty$.

(D) $\int_0^1 f(x+n) dx \to 0$ as $n \to \infty$.

40. The image of a circle under a non-constant linear transformation can be

(A) a rectangle,

(B) a parabola,

(C) an ellipse,

(D) Any of the above.

Instructions

1. This question paper has forty multiple choice questions.

2. Four possible answers are provided for each question and only one of these is correct.

3. Marking scheme: Each correct answer will be awarded **2.5** marks, but **0.5** marks will be **deducted** for each incorrect answer.

4. Answers are to be marked in the OMR sheet provided.

5. For each question, darken the appropriate bubble to indicate your answer.

6. Use only HB pencils for bubbling answers.

7. Mark only one bubble per question. If you mark more than one bubble, the question will be evaluated as incorrect.

8. If you wish to change your answer, please erase the existing mark completely before marking the other bubble.

9. Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, the set of integers, the set of rational numbers, the set of real numbers and the set of complex numbers respectively.
10. Let S_n denote the group of permutations of $\{1, 2, \cdots, n\}$ and $\mathbb{Z}_n$ the group $\mathbb{Z}/n\mathbb{Z}$.
11. Let $f : X \to Y$ be a function. For $A \subset X$, $f(A)$ denotes the image of A under f.

Integrated Ph. D. Mathematical Sciences

1. Consider the following system of linear equations.

$$\begin{aligned} x+y+z+w &= b_1. \\ x-y+2z+3w &= b_2. \\ x-3y+3z+5w &= b_3. \\ x+3y-w &= b_4. \end{aligned}$$

For which of the following choices of b_1, b_2, b_3, b_4 does the above system have a solution?

(A) $b_1 = 1, b_2 = 0, b_3 = -1, b_4 = 2$.
(B) $b_1 = 2, b_2 = 3, b_3 = 5, b_4 = -1$.
(A) $b_1 = 2, b_2 = 2, b_3 = 3, b_4 = 0$.
(A) $b_1 = 2, b_2 = -1, b_3 = -3, b_4 = 3$.

2. Let $y : [0,1] \to \mathbb{R}$ be a twice continuously differentiable function such that,

$$\frac{d^2y}{dx^2}(x) - y(x) < 0, \text{ for all } x \in (0,1), \text{and } y(0) = y(1) = 0.$$

Then,

(A) y has at least two zeros in $(0,1)$.
(B) y has at least on zero in $(0,1)$.
(C) $y(x) > 0$ for all $x \in (0,1)$.
(D) $y(x) < 0$ for all $x \in (0,1)$.

3. Which one of the following boundary value problems has more than one solution?

(A) $y'' + y = 1,\ y(0) = 1,\ y(\pi/2) = 0$.
(B) $y'' + y = 1,\ y(0) = 0,\ y(2\pi) = 0$.
(C) $y'' - y = 1,\ y(0) = 0,\ y(\pi/2) = 0$.
(D) $y'' - y = 1,\ y(0) = 0,\ y(\pi) = 0$.

4. Let A be an $n \times n$ nonsingular matrix such that the elements of A and A^{-1} are all integers. Then,

(A) $\det A$ must be a positive integer.

(B) $\det A$ must be a negative integer.

(C) $\det A$ can be $+1$ or -1.

(D) $\det A$ must be $+1$.

5. Let Q be a polynomial of degree 23 such that $Q(x) = -Q(-x)$ for all $x \in \mathbb{R}$ with $|x| \geq 10$. If $\int_{-1}^{1} (Q(x) + c)\, dx = 4$ then c equals

(A) 0.

(B) 1.

(C) 2.

(D) 4.

6. Let $b > 0$ and $x_1 > 0$ be real numbers. Then the sequence $\{x_n\}_{n=1}^{\infty}$ defined by

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{b}{x_n}\right)$$

(A) diverges.

(B) converges to $\sqrt{x_1}$.

(C) converges to $\sqrt{(b + x_1)}$.

(D) converges to $\sqrt{b}$.

7. Let $f(x) = \begin{cases} \frac{3x}{4} & \text{if } x \in \mathbb{Q}. \\ \sin x & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$

Then the number of points where f is continuous equals

(A) 1.

(B) 2.

(C) 3.

(D) ∞.

8. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying, $f(x) = 5\int_0^x f(t)\, dt + 1, \quad \forall x \in \mathbb{R}$. Then $f(1)$ equals

(A) e^5.

(B) 5.

(C) $5e$.

(D) 1.

9. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and let $g(x) = \int_0^{x^2+3x+2} f(t)\, dt$. Then, $g'(0)$ equals

(A) $3f(2)$.

(B) $f(2)$.

(C) $3f(0)$.

(D) $f(0)$.

10. Let $x_n > 0$ be such that $\sum_{n=1}^{\infty} x_n$ diverges and $\sum_{n=1}^{\infty} x_n^2$ converges. Then x_n **cannot** be

(A) $\frac{n}{n^2+1}$.

(B) $\frac{\log n}{n}$.

(C) $\frac{1}{n\sqrt{\log n}}$.

(D) $\frac{1}{n(\log n)^2}$.

11. If B is a subset of $\mathbb{R}^3$ and $u \in \mathbb{R}^3$, define $B - u = \{w - u : \ w \in B\}$. Let $A \subset \mathbb{R}^3$, be such that $tu + (1-t)v \in A$ whenever $u, v \in A$ and $t \in \mathbb{R}$. Then,

(A) A must be a straight line.

(B) A must be a line segment.

(C) $A - u_0$ is a subspace for a unique $u_0 \in A$.

(D) $A - u$ is a subspace for all $u \in A$.

12. Minimum value of $|z+1| + |z-1| + |z-i|$ for $z \in \mathbb{C}$ is

(A) 2.

(B) $2\sqrt{2}$.

(C) $1 + \sqrt{3}$.

(D) $\sqrt{5}$.

13. The minimum value of $|z-w|$ where $z, w \in \mathbb{C}$ such that $|z| = 11$, and $|w+4+3i| = 5$ is

(A) 1.

(B) 2.

(C) 5.

(D) 6.

14. Let $\mathcal{P}$ be the vector space of polynomials with real coefficients. Let T and S be two linear maps from $\mathcal{P}$ to itself such that $T \circ S$ is the identity map. Then,

    (A) $S \circ T$ may not be the identity map.
    (B) $S \circ T$ must be the identity map, but T and S need not be the identity maps.
    (C) T and S must both be the identity map.
    (D) There is a scalar α such that $T(p) = \alpha p$ for all $p \in \mathcal{P}$.

15. Let ℓ_1 and ℓ_2 be two perpendicular lines in $\mathbb{R}^2$. Let P be a point such that the sum of the distances of P from ℓ_1 and ℓ_2 equals 1. Then the locus of P is

    (A) a square.
    (B) a circle.
    (C) a straight line.
    (D) a set of four points.

16. Let $0 < b < a$. A line segment AB of length b moves on the plane such that A lies on the circle $x^2 + y^2 = a^2$. Then the locus of B is

    (A) a circle.
    (B) union of two circles.
    (C) a region bounded by two concentric circles.
    (D) an ellipse, but not a circle.

17. Let u, v and w be three vectors in $\mathbb{R}^3$. It is given that $u \cdot u = 4$, $v \cdot v = 9$, $w \cdot w = 1$, $u \cdot v = 6$, $u \cdot w = 0$ and $v \cdot w = 0$. Then the dimension of the subspace spanned by $\{u, v, w\}$ is

    (A) 1.
    (B) 2.
    (C) 3.
    (D) cannot be determined.

18. Let a_n be the number of ways of arranging n identical black balls and $2n$ identical white balls in a line so that no two black balls are next to each other. Then a_n equals

    (A) $3n$.

(B) $\binom{2n+1}{n}$.

(C) $\binom{2n}{n}$.

(D) $\binom{2n-1}{n(2n+1)}$.

19. Maximal area of a triangle whose vertices are on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

(A) $\frac{3\sqrt{3}}{4}ab$.

(B) $\frac{3\sqrt{3}}{4}\frac{(a^2+b^2)}{2}$.

(C) $\frac{3\sqrt{3}}{4}\frac{2}{\frac{1}{a^2}+\frac{1}{b^2}}$.

(D) $\frac{3\sqrt{3}}{4}$.

20. Let $a_k = \frac{1}{2^{2k}}\binom{2k}{k}$, $k = 1, 2, 3, \cdots$. Then

(A) a_k is increasing.

(B) a_k is decreasing.

(C) a_k decreases for first few terms and then increases.

(D) none of the above.

21. What is the limit of $(2^n + 3^n + 4^n)^{\frac{1}{n}}$ as $n \to \infty$?

(A) 0.

(B) 1.

(C) 3.

(D) 4.

22. What is the limit of $e^{-2n} \sum_{k=0}^{n} \frac{(2n)^k}{k!}$ as $n \to \infty$?

(A) 0.

(B) 1.

(C) $1/e$.

(D) e.

23. Let $f, g : [-1, 1] \to \mathbb{R}$ be **odd** functions whose derivatives are continuous. You are given that $|g(x)| < 1$ for all $x \in [-1, 1]$, $f(-1) = -1$, $f(1) = 1$ and that $f'(0) < g'(0)$. Then the minimum possible number of solutions to the equation $f(x) = g(x)$ in the interval $[-1, 1]$ is

(A) 1.

(B) 3.

(C) 5.

(D) 7.

24. Let $f : S_3 \to \mathbb{Z}_6$ be a group homomorphism. Then the number of elements in $f(S_3)$ is

(A) 1.

(B) 1 or 2.

(C) 1 or 3.

(D) 1 or 2 or 3.

25. Consider the multiplicative group $S = \{z : \ |z| = 1\} \subset \mathbb{C}$. Let G and H be subgroups of order 8 and 10 respectively. If n is the order of $G \cap H$ then

(A) $n = 1$.

(B) $n = 2$.

(C) $3 \le n \le 5$.

(D) $n \ge 6$.

26. Let G be a finite abelian group. Let H_1 and H_2 be two distinct subgroups of G of index 3 each. Then the index of $H_1 \cap H_2$ in G is

(A) 3.

(B) 6.

(C) 9.

(D) Cannot be computed from the given data.

27. A particle follows the path $c : [-\frac{\pi}{2}, \frac{\pi}{2}] \to \mathbb{R}^3$, $c(t) = (\cos t, 0, |\sin t|)$. Then the distance travelled by the particle is

(A) $\frac{3\pi}{2}$.

(B) π.

(C) 2π.

(D) 1.

28. Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be the map given by

$$T(x_1, x_2, x_3) = (x_1 - 2x_2, x_2 - 2x_3, x_3 - 2x_1, x_1 - 2x_3).$$

Then the dimension of $T(\mathbb{R}^3)$ equals

(A) 1.
(B) 2.
(C) 3.
(D) 4.

29. The tangent plane to the surface $z^2 - x^2 + \sin(y^2) = 0$ at $(1, 0, -1)$ is

(A) $x - y + z = 0$.
(B) $x + 2y + z = 0$.
(C) $x + y - 1 = 0$.
(D) $x + z = 0$.

30. Let A and B be two 3×3 matrices with real entries such that $\text{rank}(A) = \text{rank}(B) = 1$. Let $N(A)$ and $R(A)$ stand for the null space and range space of A. Define $N(B)$ and $R(B)$ similarly. Then which of the following is necessarily true ?

(A) $\dim(N(A) \cap N(B)) \geq 1$.
(B) $\dim(N(A) \cap R(A)) \geq 1$.
(C) $\dim(R(A) \cap R(B)) \geq 1$.
(D) $\dim(N(A) \cap R(A)) \geq 1$.

31. For a permutation π of $\{1, 2, \cdots, n\}$, we say that k is a fixed point if $\pi(k) = k$. Number of permutations in S_5 having exactly one fixed point is

(A) 24.
(B) 45.
(C) 60.
(D) 96.

32. Let $A = \{1, 2, \cdots, 10\}$. If S is a subset of A, let $|S|$ denote the number of elements in S. Then

$$\sum_{S\subset A, S\neq\phi} (-1)^{|S|}$$

equals

(A) -1.

(B) 0.

(C) 1.

(D) 10.

33. Let $\mathcal{P}_m$ be the vector space of polynomials with real coefficients of degree less than or equal to m. Define $T : \mathcal{P}_m \to \mathcal{P}_m$ by $T(f) = f' + f$. Then the dimension of range(T) equals

(A) 1

(B) $(m - 1)$.

(C) m.

(D) $(m + 1)$.

34. Let A and B be two finite sets of cardinality 5 and 3 respectively. Let G be the collection of all mappings f from A into B such that the cardinality of $f(A)$ is 2. Then, cardinality of G equals

(A) $3 \cdot 2^5 - 6$.

(B) $3 \cdot 2^5$.

(C) $3 \cdot 5^2$.

(D) $\frac{1}{2}(3^5 - 3)$.

35. Let G be the group $\mathbb{Z}_2 \times \mathbb{Z}_2$ and let H be the collection of all isomorphisms from G onto itself. Then the cardinality of H is

(A) 2.

(B) 4.

(C) 6.

(D) 8.

36. A line L in the XY-plane has intercepts a and b on X-axis and Y-axis respectively. When the axes are rotated through an angle θ (keeping the origin fixed), L makes equal intercepts with the axes. Then $\tan\theta$ equals

(A) $\frac{a-b}{a+b}$.

(B) $\frac{a-b}{2(a+b)}$.

(C) $\frac{a+b}{a-b}$.

(D) $\frac{a^2-b^2}{a^2+b^2}$.

37. Let B_1, B_2 and B_3 be three distinct points on the parabola $y^2 = 4x$. The tangents at B_1, B_2 and B_3 to the parabola (taken in pairs) intersect at C_1, C_2 and C_3. If a and A are the areas of the triangles $B_1B_2B_3$ and $C_1C_2C_3$ respectively, then

(A) $a = A$.

(B) $a = 2A$.

(C) $2a = A$.

(D) $a = \sqrt{2}A$.

38. Let P be a 3×2 matrix, Q be a 2×2 matrix and R be a 2×3 matrix such that PQR is equal to the identity matrix. Then,

(A) rank of $P = 2$.

(B) Q is nonsingular.

(C) Both (A) and (B) are true.

(D) There are no such matrices P, Q and R.

39. The number of elements of order 3 in the group $\mathbb{Z}_{15} \times \mathbb{Z}_{15}$ is

(A) 3.

(B) 8.

(C) 9.

(D) 15.

40. The number of surjective group homomorphisms from $\mathbb{Z}$ to $\mathbb{Z}_3$ equals

(A) 1.

(B) 2.

(C) 3.

(D) ∞.

INDIAN INSTITUTE OF SCIENCE
BANGALORE - 560012

ENTRANCE TEST FOR ADMISSIONS - 2008

Program : Research
Entrance Paper : Mathematics
Paper Code : MA

Day & Date
SUNDAY, 27TH APRIL 2008

Time
9.00 A.M. TO 12.00 NOON

Instructions

(1) This question paper consists of two parts: Part A and Part B and carries a total of 100 Marks.

(2) There is no negative marking.

(3) Candidates are asked to fill in the required fields on the sheet attached to the answer book.

(4) Part A carries 20 multiple choice questions of 2 marks each. Answer all questions in Part A.

(5) Answers to Part A are to be marked in the OMR sheet provided.

(6) For each question, darken the appropriate bubble to indicate your answer.

(7) Use only HB pencils for bubbling answers.

(8) Mark only one bubble per question. If you mark more than one bubble, the question will be evaluated as incorrect.

(9) If you wish to change your answer, please erase the existing mark completely before marking the other bubble.

(10) Part B has 12 questions. Answer any 6 in this part. Each question in this part carries 10 marks.

(11) Answers to Part B are to be written in the separate answer book provided.

(12) **Answer to each question in Part B should begin on a new page.**

(13) Let $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{C}$ ($\mathbb{Z}_+$, $\mathbb{R}_+$, $\mathbb{Q}_+$ and $\mathbb{C}_+$) denote the set of (respectively positive) integers, real numbers, rational numbers and complex numbers respectively.

(14) For $n \geq 1$, the norm given by $\|(x_1, x_2, \dots, x_n)\| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}$ denote the standard norm on $\mathbb{R}^n$. The metric given by $d(x, y) = \|x - y\|$ is called the standard metric on $\mathbb{R}^n$.

MATHEMATICS

PART A

(1) The limit, $\lim_{R\to\infty} \dfrac{\int_R^\infty r^n e^{-\frac{r^2}{2}}\, dr}{R^{n-1}e^{-\frac{R^2}{2}}}$, $n \in \mathbb{Z}_+$, equals

(A) -1.
(B) 0.
(C) 1.
(D) ∞.

(2) The function $f : \mathbb{R} \to \mathbb{R}$ satisfies $|f(x) - f(y)| \le c|x - y|$ for all $x, y \in \mathbb{R}$ and some constant $c \in \mathbb{R}_+$. Then,

(A) f must be bounded.
(B) f must be continuous but may not be uniformly continuous.
(C) f must be uniformly continuous but may not be differentiable.
(D) f must be differentiable.

(3) Consider the sequence $\{x_n\}_{n\ge1}$ defined recursively as $x_1 = 1$,

$$x_n := \sup\left\{x \in [0, x_{n-1}) : \sin\left(\frac{1}{x}\right) = 0\right\}, \qquad n \ge 2.$$

Then, $\limsup_{n\to\infty} x_n$ equals

(A) $-\infty$.
(B) 0.
(C) 1.
(D) ∞.

(4) Let $f : [0, 1] \to \mathbb{R}$ be continuous. Then $\int_0^1 f(x)e^{-x}dx$ equals

(A) $f'(0) - f'(1)e^{-1}$.
(B) $\int_0^1 e^{-x}(\int_0^x f(y)\, dy)\, dx$.
(C) $f(c)(1 - e^{-1})$, for some $c \in [0, 1]$.
(D) $e^{-c}\int_0^1 f(x)\, dx$, for some $c \in [0, 1]$.

(5) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous with $f(0) = f(1) = 0$. Which of the following is **not** possible?

(A) $f([0,1]) = \{0\}$.

(B) $f([0,1]) = [0,1)$.

(C) $f([0,1]) = [0,1]$.

(D) $f([0,1]) = [-\frac{1}{2}, \frac{1}{2}]$.

(6) The series $\sum_{n=1}^{\infty} \frac{e^{i\sqrt{n}x}}{n^2}$ converges

(A) only at $x = 0$.

(B) only for $|x| \leq 1$.

(C) converges pointwise for all $x \in \mathbb{R}$, but not uniformly.

(D) converges uniformly on $\mathbb{R}$.

(7) Let $P(x)$ be a non-zero polynomial of degree N. The radius of convergence of the power series

$$\sum_{n=0}^{\infty} P(n)x^n$$

(A) depends on N.

(B) is 1 for all N.

(C) is 0 for all N.

(D) is ∞ for all N.

(8) The function $f(z) = \exp\left(\left(\frac{\cos z - 1}{z^2}\right)^2\right)$

(A) has a removable singularity at $z = 0$.

(B) has a pole of order 2 at $z = 0$.

(C) has a pole of order 4 at $z = 0$.

(D) has an essential singularity at $z = 0$.

(9) The region described by $\left|\frac{z-i}{z+i}\right| < 1$, where $z = x + iy \in \mathbb{C}$ is

(A) $\{z \in \mathbb{C} : x < 0\}$.

(B) $\{z \in \mathbb{C} : x > 0\}$.

(C) $\{z \in \mathbb{C} : y < 0\}$.

(D) $\{z \in \mathbb{C} : y > 0\}$.

(10) Let $A \neq \mathbb{R}$ be a dense subset of $\mathbb{R}$. If $U \subseteq \mathbb{R}$ is a non-empty open subset then

(A) $U \subseteq \overline{A \cap U}$.

(B) $\overline{A \cap U} = \emptyset$.

(C) $\overline{A \cap U} \subseteq U$.

(D) $\overline{A \cap U} = A \cap \overline{U}$.

(11) Let $X = \{1/n : n \in \mathbb{Z}, n \geq 1\}$ and let $\bar{X}$ be its closure. Then

(A) $\bar{X} \setminus X$ is a single point.

(B) $\bar{X} \setminus X$ is open in $\mathbb{R}$.

(C) $\bar{X} \setminus X$ is infinite but not open in $\mathbb{R}$.

(D) $\bar{X} \setminus X = \phi$.

(12) Let X be a subset of $\mathbb{R}$ homeomorpic to $(0, \pi)$. Then

(A) X must be bounded.

(B) X must be compact.

(C) The closure $\bar{X}$ of X must be unbounded.

(D) X must be open.

(13) Let $A \subseteq \mathbb{R}$ be an open set. If $(0,1) \cup A$ is connected, then

(A) A must be connected.

(B) A must have one or two componets.

(C) $A \setminus (0,1)$ has at most two components.

(D) A must be a Cantor set.

(14) Let A be an $n \times n$ matrix with entries 0 and 1 and $n > 1$. If there is exactly one non zero entry in each row and each column of A, then the determinant of A must be

(A) ± 1.

(B) 0.

(C) n.

(D) 1.

(15) Let A be an $n \times n$ matrix over real numbers such that $AB = BA$ for all $n \times n$ matrices B. Then

(A) A must be 0.

(B) A must be the identity.

(C) A must be a diagonal matrix.

(D) A must be either 0 or the identity.

(16) Let A be a matrix such that $A^3 = -I$. Then which of the following numbers can be an eigenvalue of A?

(A) i.

(B) 1.

(C) -1.

(D) $\frac{1}{2} + \frac{\sqrt{3}}{2}i$.

(17) Let V be the vector space of all polynomials whose degree is less than or equal to n. Let $D : V \to V$ be the differentiation operator on V, that is, $DP(x) = P'(x)$. Then the trace of D, $\mathrm{tr}(D)$ equals

(A) 0.

(B) 1.

(C) n.

(D) n^2.

(18) Let A be a 3×3 matrix over real numbers satisfying $A^{-1} = I - 2A$. Then the determinant of A, $\det(A)$ equals

(A) $-\frac{1}{2}$.

(B) $\frac{1}{2}$.

(C) 1 .

(D) 2 .

(19) For which of the following integers n is every group of order n abelian?

(A) $n = 6$.

(B) $n = 9$.

(C) $n = 12$.

(D) $n = 18$.

(20) Let G be the cyclic subgroup of order 18. The number of subgroups of G, including G and the trivial group, is

(A) 4.

(B) 6.

(C) 9.

(D) 18.

PART B

(1) Let $F_n : \mathbb{R} \to [0,1]$, $n \geq 0$, be continuous functions satisfying

(i) $F_n(x) \leq F_n(y)$ for all $x \leq y$,

(ii) $\lim_{x \to -\infty} F_n(x) = 0$, and

(iii) $\lim_{x \to \infty} F_n(x) = 1$.

Suppose that F_n converges pointwise to F_0 on $\mathbb{R}$, that is $F_n(x) \to F_0(x)$ for all $x \in \mathbb{R}$, as $n \to \infty$. Show that F_n converges uniformly to F_0 on $\mathbb{R}$, as $n \to \infty$.

(2) Let $f : [0,1] \to \mathbb{R}$ be a continuous function of bounded variation. Then for any $p \in (1, \infty)$, show that

$$\lim_{n \to \infty} \sum_{k=1}^{n} \left| f\left(\frac{k}{n}\right) - f\left(\frac{k-1}{n}\right) \right|^p = 0.$$

(3) Given n points $z_1, z_2, \cdots, z_n$ on the unit circle $\{z \in \mathbb{C} : |z| = 1\}$, prove that there exists a point z on the unit circle such that $\prod_{i=1}^{n} |z - z_i| \geq 1$.

(4) Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. Let $\Delta(0, R) = \{z = x + iy \in \mathbb{C} : |z| < R\}$, denote the open disc in the complex plane around the origin of radius $R > 0$. If $n - 1 \in \mathbb{Z}_+$, show that

$$\int_{\Delta(0,R)} \overline{z}^{\,n-1} f(z)\, dx\, dy = \frac{\pi R^{2n}}{n!} f^{(n-1)}(0),$$

where $f^{(k)}(0)$ denotes the k-th derivative of f at the origin.

(5) Let $f : \Delta(0,1) \to \mathbb{C}$ be analytic. Show that it is not possible for $f(z)$ to satisfy

$$f\left(\frac{1}{n}\right) = f\left(-\frac{1}{n}\right) = \frac{n}{n+1},$$

for all $n \geq 2$.

(6) For $A, B \subseteq \mathbb{R}^2$, define the distance $d(A, B) := \inf\{\|x - y\| \;:\; x \in A,\; x \in B\}$. Let $C, D \subseteq \mathbb{R}^2$ be two closed subsets. If $C \cap D = \emptyset$ and $d(C, D) = 0$ then show that both C and D are unbounded.

(7) Let $L : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible linear transformation and let $V \subset \mathbb{R}^n$ be a subspace such that $L(V) \subseteq V$. Show that $L(V) = V$. Here $L(V) = \{L(v) : v \in V\}$.

(8) Let A be a 2×2 invertible matrix over real numbers such that for some 2×2 invertible matrix P, $PAP^{-1} = A^2$. Show that either $A^3 = I$ or $QAQ^{-1} = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, for some invertible 2×2 matrix Q.

(9) Suppose V is an n-dimensional vector space over a field F. Let $W \subseteq V$ be a subspace of dimension $r < n$. Show that $W = \cap\{U : U \text{ is an } (n-1)\text{-dimensional subspace of } V \text{ and } W \subseteq U\}$.

(10) Suppose G is a finite group. Show that every element x of G can be expressed as $x = y^2$ for some $y \in G$ if and only if the order of G is odd.

(11) Let G be a group with identity e. Let N_1, N_2, N_3 be three normal subgroups of G. If $N_i \cap N_j = \{e\}$ and $N_iN_j = G$ for $1 \leq i \neq j \leq 3$ then show the following:

(i) $xy = yx$ for $x \in N_i$, $y \in N_j$, $1 \leq i \neq j \leq 3$.

(ii) $yz = zy$ for $y, z \in N_i$, $1 \leq i \leq 3$.

(iii) G is commutative. [4 + 4 + 2]

(12) Let $y : \mathbb{R} \to \mathbb{R}$ be an infinitely many times differentiable function which satisfies

$$y'' + y' - y \geq 0, \quad y(0) = y(1) = 0.$$

If $y(x) \geq 0$ for all $x \in [0, 1]$, prove that y is identically zero in $[0, 1]$.

INDIAN INSTITUTE OF SCIENCE
BANGALORE - 560012

ENTRANCE TEST FOR ADMISSIONS - 2009

Program : Research
Entrance Paper : Mathematics
Paper Code : MA

Day & Date
SUNDAY, 26TH APRIL 2009

Time
9.00 A.M. TO 12.00 NOON

Instructions

(1) This question paper consists of two parts: Part A and Part B and carries a total of 100 Marks.

(2) There is no negative marking.

(3) Candidates are asked to fill in the required fields on the sheet attached to the answer book.

(4) Part A carries 20 multiple choice questions of 2 marks each. Answer all questions in Part A.

(5) Answers to Part A are to be marked in the OMR sheet provided.

(6) For each question, darken the appropriate bubble to indicate your answer.

(7) Use only HB pencils for bubbling answers.

(8) Mark only one bubble per question. If you mark more than one bubble, the question will be evaluated as incorrect.

(9) If you wish to change your answer, please erase the existing mark completely before marking the other bubble.

(10) Part B has 8 questions. Answer any 6 in this part. Each question in this part carries 10 marks.

(11) Answers to Part B are to be written in the separate answer book provided.

(12) **Answer to each question in Part B should begin on a new page.**

(13) Let $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{C}$ ($\mathbb{Z}_+$, $\mathbb{R}_+$, $\mathbb{Q}_+$ and $\mathbb{C}_+$) denote the set of (respectively positive) integers, real numbers, rational numbers and complex numbers respectively.

(14) For $n \geq 1$, the norm given by $\|(x_1, x_2, \ldots, x_n)\| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}$ denote the standard norm on $\mathbb{R}^n$. The metric given by $d(x, y) = \|x - y\|$ is called the standard metric on $\mathbb{R}^n$.

MATHEMATICS

PART A

(1) Let $f : [-1, 1] \to \mathbb{R}$ be continuous. Assume that $\int_{-1}^{1} f(t)dt = 2$. Then

$$\lim_{n\to\infty} \int_{-1}^{1} f(t)sin^2(nt)dt$$

(A) equals 0.

(B) equals 1.

(C) equals $f(1) - f(-1)$.

(D) does not exist.

(2) The radius of convergence R of the power series

$$\sum_{n=1}^{\infty} \left(\frac{a^n}{n} + \frac{b^n}{n^2}\right) x^n$$

where $a > 0$, $b > 0$ and $a \neq b$, is

(A) $R = 0$.

(B) $R = \infty$.

(C) $R = min(1/a, 1/b)$.

(D) $R = max(1/a, 1/b)$.

(3) For $z = x + iy \in \mathbb{C}$,

$$\left|e^{z^2}\right| = e^{|z|^2}$$

holds

(A) for all $z \in \mathbb{C}$.

(B) if and only if $y = 0$.

(C) if and only if $x = 0$.

(D) only when $z = 0$.

(4) Let C be the circle $\{|z| = 1\}$ in the complex plane described counterclockwise. Then

$$\int_C \frac{1+z}{(2-z)z}dz$$

equals

(A) πi.

(B) $-\pi i$.

(C) $2\pi i$.

(D) $-2\pi i$.

(5) Suppose the function $f : \mathbb{R} \to \mathbb{R}$ has left and right derivatives at 0. Then, at $x = 0$,

(A) f must be continuous but may not be differentiable.

(B) f need not be continuous but must be left continuous or right continuous.

(C) f must be differentiable.

(D) If f is continuous then f must be differentiable.

(6) Let $\{x_n\}_{n\geq 1}$ be a sequence of real numbers. Suppose that for each $\epsilon > 0$, there is a subsequence $\{x_{n_k}\}_{k\geq 1}$ so that $x_{n_k} \leq x + \epsilon$, for all $k \geq 1$. Then we must have

(A) $\limsup_{n\to\infty} x_n \leq x$.

(B) $\limsup_{n\to\infty} x_n \geq x$.

(C) $\liminf_{n\to\infty} x_n \leq x$.

(D) $\liminf_{n\to\infty} x_n \geq x$.

(7) Let the sequence of functions $f_n : [0,1] \to \mathbb{R}$, $n \geq 2$, be given by

$$f_n = \begin{cases} n^2 x, & 0 \leq x \leq 1/n, \\ 2n - n^2 x, & 1/n < x < 2/n, \\ 0, & 2/n \leq x \leq 1. \end{cases}$$

Then,

(A) f_n converges pointwise but not uniformly as $n \to \infty$.

(B) f_n converges uniformly as $n \to \infty$.

(C) The functions $\{f_n\}_{n\geq 1}$ are equicontinuous.

(D) $\int_0^1 f_n(x)dx$ converges to 0 as $n \to \infty$.

(8) The function $f(x) = e^{-|x|}$ is

(A) continuous but not uniformly continuous.

(B) uniformly continuous but not differentiable.

(C) differentiable but not uniformly continuous.

(D) differentiable and uniformly continuous.

(9) Let A_n be the sequence of intervals

$$A_n = \left(1 + \frac{(-1)^n}{n}, 1 + \frac{2}{n}\right)$$

for $n \geq 1$. Then

$$\liminf_{n\to\infty} A_n = \bigcup_{n=1}^{\infty}(\bigcap_{k=n}^{\infty} A_n)$$

is

(A) the empty set.

(B) $\{1\}$.

(C) $(0,3)$.

(D) $(0,1)$.

(10) Let $\mathbf{v}$ and $\mathbf{w}$ be 3×1 row vectors. If $\mathbf{w}^T$ denotes the transpose of $\mathbf{w}$, then for the matrix $\mathbf{v}\mathbf{w}^T$

(A) 0 is not an eigenvalue.

(B) 0 is an eigenvalue with multiplicity 1.

(C) 0 is an eigenvalue with multiplicity 2.

(D) 0 is an eigenvalue with multiplicity 3.

(11) Let A be the matrix

$$A = \begin{pmatrix} a & c \\ 0 & a \end{pmatrix}$$

with $a, c \in \mathbb{R}$ and $c \neq 0$. Then there is a 2×2 matrix P such that PAP^{-1} is diagonal

(A) for all values of a.

(B) for no value of a.

(C) if and only if $a = c$.

(D) if and only if $a = 0$.

(12) Let A be a 3×3 matrix over $\mathbb{R}$ such that $AB = BA$ for all 3×3 matrices B over $\mathbb{R}$. Then

(A) A must be I or 0.

(B) A must be diagonal.

(C) A must be orthogonal.

(D) A must have 3 distinct eigenvalues.

(13) Let $V, W \subset \mathbb{R}^5$ be subspaces with $dim(V) = dim(W) = 3$. Let

$$V + W = \{v + w : v \in V, w \in W\}$$

(A) We always have $V + W = \mathbb{R}^5$.

(B) We never have $V + W = \mathbb{R}^5$.

(C) We must have $\dim(V \cap W) \geq 1$.

(D) If $V + W = \mathbb{R}^5$, then $\dim(V \cap W) = 2$.

(14) Suppose A is a 2×2 matrix over real numbers with eigenvalues i and $-i$. Then

(A) A cannot be orthogonal.

(B) A cannot be symmetric.

(C) A cannot be skew-symmetric.

(D) A cannot be invertible.

(15) Let G be the group $G = \mathbb{Z}_2 \times \mathbb{Z}_3$. Then

(A) G is isomorphic to S_3.

(B) G is isomorphic to a subgroup of S_4.

(C) G is isomorphic to a proper subgroup of S_5.

(D) G is not isomorphic to a subgroup of S_n for all $n \geq 3$.

(16) Let G be a group of order 121. Then

(A) G must be cyclic.

(B) G must have an element of order 11.

(C) G must have an element of order 121.

(D) G cannot have an element of order 11.

(17) For which of the following values of n does there exist a field of order n.

(A) $n = 6$.

(B) $n = 81$.

(C) $n = 21$.

(D) $n = 36$.

(18) The number of group homomorphisms $\varphi : \mathbb{Z} \to \mathbb{Z}$ is

(A) one

(B) two

(C) three

(D) infinity

(19) The set $[0, 1] \times (0, 1) \subset \mathbb{R}^2$ is

(A) open

(B) closed

(C) compact

(D) connected

(20) Which of the following sets is homeomorphic to

$$D = \{z \in \mathbb{C} : |z| \leq 1\}$$

(A) $\{z \in \mathbb{C} : |z| < 2\}$.

(B) $[0, 1] \times (0, 1)$.

(C) $\{z \in \mathbb{C} : |z| \leq 2, Re(z) \leq 1\}$.

(D) $(0, 1) \times (0, 1)$.

PART B

(1) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function and a a constant with $0 < a < 1$ so that $0 < f'(x) < a$ for all $x \in \mathbb{R}$. Define the sequence $\{x_n\}_{n\geq 0}$ by $x_0 = 0$ and $x_n = f(x_{n-1})$ for $n \geq 1$. Show that $|x_{n+1} - x_n| < a|x_n - x_{n-1}|$ for $n \geq 1$.

(2) Let $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and a a constant with $0 < a < 1$. Suppose the the sequence $\{x_n\}_{n\geq 0}$ defined by $x_0 = 0$ and $x_n = f(x_{n-1})$ for $n \geq 1$ satisfies $|x_{n+1} - x_n| < a|x_n - x_{n-1}|$ for $n \geq 1$. Show that x_n converges and that $x = \lim_{n\to\infty} x_n$ satisfies $f(x) = x$.

(3) Let $f(z)$ be a complex analytic function on $\mathbb{C} \setminus S$, where $S = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$. Suppose that there is an integer $k \geq 1$ such that

$$|f(z)| \leq |z|^k$$

for all $z \in \mathbb{C} \setminus S$. Show that all the singularities of f are removable.

(4) Let $f(z)$ be a complex analytic function of $\mathbb{C}$ satisfying, for some integer k,

$$|f(z)| \leq |z|^k$$

for all $z \in \mathbb{C}$. Show that there exists a constant $c \in \mathbb{C}$ such that $f(z) = cz^k$.

(5) Let V and W be subspaces of $\mathbb{R}^n$ with $\dim(V) = \dim(W)$. Show that there is an isomorphism $T : \mathbb{R}^n \to \mathbb{R}^n$ such that $T(V) = W$. Here $T(V) = \{T(v) : v \in V\}$.

(6) Let V and W be subspaces of $\mathbb{R}^n$. Show that there is a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ such that $T(V) = W$ if and only if $dim(V) \geq dim(W)$.

(7) Let G be a cyclic group such that G has exactly three subgroups, $\{1\}$, G and a proper subgroup H. Show that the order of G is p^2 for some prime p.

(8) Let G be a finite group such that G has exactly three subgroups, $\{1\}$, G and a proper subgroup H. Show that G is cyclic.

End of question paper

INDIAN INSTITUTE OF SCIENCE
BANGALORE - 560012

ENTRANCE TEST FOR ADMISSIONS - 2010

Program : Research
Entrance Paper : Mathematics
Paper Code : MA

Day & Date
SUNDAY, 25TH APRIL 2010

Time
9.00 A.M. TO 12.00 NOON

Instructions

(1) This question paper consists of two parts: Part A and Part B, and carries a total of 100 Marks.

(2) There is no negative marking.

(3) Candidates are asked to fill in the required fields on the sheet attached to the answer book.

(4) Part A carries 20 multiple choice questions carrying 2 marks each. Answer all questions in Part A.

(5) Answers to Part A are to be marked in the OMR sheet provided.

(6) For each question, darken the appropriate bubble to indicate your answer.

(7) Use only HB pencils for bubbling answers.

(8) Mark only one bubble per question. If you mark more than one bubble, the question will be evaluated as incorrect.

(9) If you wish to change your answer, please erase the existing mark completely before marking the other bubble.

(10) Part B has 8 questions. Answer any 6 in this part. Each question in this part carries 10 marks.

(11) Answers to Part B are to be written in the separate answer book provided.

(12) **Answer to each question in Part B should begin on a new page.**

(13) Let $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{C}$ ($\mathbb{Z}_+$, $\mathbb{R}_+$, $\mathbb{Q}_+$ and $\mathbb{C}_+$) denote the set of (respectively positive) integers, real numbers, rational numbers and complex numbers respectively.

(14) For $n \geq 1$, the norm given by $\|(x_1, x_2, \dots, x_n)\| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}$ denotes the standard norm on $\mathbb{R}^n$. The metric given by $d(x,y) = \|x - y\|$ is called the standard metric on $\mathbb{R}^n$.

MATHEMATICS

PART A

(1) Let $\{x_n\}$ be an unbounded sequence of non-zero real numbers. Then,

(A) $\{x_n\}$ must have a convergent subsequence.

(B) $\{x_n\}$ cannot have a convergent subsequence.

(C) $\{1/x_n\}$ must have a convergent subsequence.

(D) $\{1/x_n\}$ cannot have a convergent subsequence.

(2) Let $f_n : [0,1] \to \mathbb{R}$ be the sequence of functions

$$f_n(x) = \begin{cases} \sin(n\pi x) & \text{if } x \in [0, 1/n], \\ 0 & \text{if } x \in (1/n, 1]. \end{cases}$$

Then,

(A) The sequence $\{f_n\}$ does not converge pointwise.

(B) The sequence $\{f_n\}$ converges pointwise but the limit is not continuous.

(C) The sequence $\{f_n\}$ converges pointwise but not uniformly.

(D) The sequence $\{f_n\}$ converges uniformly.

(3) Suppose $f : [0,1] \to \mathbb{R}$ is a function satisfying $|f(x) - f(y)| \le (x-y)^2$ for all $x, y \in [0,1]$. Then,

(A) f is necessarily continuous but need not be differentiable.

(B) f may be strictly decreasing.

(C) f is necessarily constant.

(D) no such function f exists.

(4) The number of symmetric, positive definite 8×8 matrices having trace equal to 8 and determinant equal to 1 is

(A) 0.

(B) 1.

(C) greater than 1 but finite.

(D) infinite.

(5) Suppose $K \subset \mathbb{R}^2$ is a connected set such that for all points $x \in K$. $K \setminus \{x\}$ (the complement of x in K) is not connected. Then,

(A) K must be homeomorphic to an interval of $\mathbb{R}$.

(B) K must have empty interior.

(C) K must be open.

(D) K must be closed.

(6) Let S be a collection of pairwise disjoint open sets in the plane $\mathbb{R}^2$. Then,

(A) S cannot be finite.

(B) S cannot be countably infinite.

(C) S cannot be uncountably infinite.

(D) S must be empty.

(7) Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation. Then,

(A) T must be continuous but is not necessarily uniformly continuous.

(B) T must be uniformly continuous.

(C) T is continuous if and only if T is onto.

(D) T is uniformly continuous if and only if T is onto.

(8) Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation such that $\langle Tx, x\rangle = 0$ for all $x \in \mathbb{R}^n$. Then, it is necessarily true that

(A) $\text{trace}(T) = 0$.

(B) $\det(T) = 0$.

(C) all eigenvalues of T are real.

(D) $T = 0$.

(9) Let $G \subset (\mathbb{C}^*, \cdot)$ be a finite subgroup of the group $\mathbb{C}^*$ of non-zero complex numbers with multiplication as the group operation. Then we must have

(A) $\sum_{z\in G} z = 0$.

(B) $\sum_{z\in G} z = 1$.

(C) $\prod_{z\in G} z = 0$.

(D) $\prod_{z\in G} z = 1$.

(10) Suppose A is a 2×2 matrix over real numbers with trace$(A) = 0$ and $\det(A) = 2$. Then A may be

(A) orthogonal.

(B) symmetric.

(C) skew-symmetric.

(D) diagonal.

(11) Let $\{x_n\}$ be a sequence of real numbers so that $\sum_{n=1}^{\infty} |x_n - x| = c$, with c finite. Then

(A) $\{x_n\}$ may not be bounded.

(B) $\{x_n\}$ must converge to x.

(C) $\{x_n\}$ must converge to $x + c$.

(D) $\{x_n\}$ is bounded but not necessarily convergent.

(12) For which of the following values of n is every abelian group of order n cyclic?

(A) $n = 12$.

(B) $n = 45$.

(C) $n = 8$.

(D) $n = 21$.

(13) Assume $a > 1$. Then, $\lim_{n \to \infty} n^{-2} e^{(\log(n))^a}$ is

(A) 0 for all $a > 1$.

(B) 0 if and only if $1 < a < 2$.

(C) ∞ for all $a > 1$.

(D) ∞ if and only if $1 < a < 2$.

(14) Let $f\colon [0,1] \to [0,1]$ be a continuous function such that $x < f(x) \leq 1/2$ if $0 \leq x < 1/2$ and $1/2 \leq f(x) < x$ if $1/2 < x \leq 1$. Let $a \in [0,1]$ and define x_n inductively by $x_1 = a$ and $x_{n+1} = f(x_n)$ for $n \geq 1$. Then $\lim_{n \to \infty} x_n$ is

(A) $1/2$ for all $a \in [0,1]$.

(B) 0 if and only if $0 \leq a < 1/2$.

(C) 0 if and only if $1/2 < a \leq 1$.

(D) 1 if and only if $1/2 < a \leq 1$.

(15) Let $f(x) = x^4 + ax^2 + bx + c$, where a, b, and c are real numbers. Then as a polynomial over $\mathbb{R}$,

(A) $f(x)$ is irreducible if and only if $b^2 - 4ac > 0$.

(B) $f(x)$ is irreducible if and only if $b^2 - 4ac < 0$.

(C) $f(x)$ is always irreducible.

(D) $f(x)$ is always reducible.

(16) Consider the permutation group S_6 on 6 letters and let $H \subset S_6$ be a subgroup with 9 elements. It is necessarily true that

(A) H is abelian but not cyclic.

(B) H is cyclic.

(C) H is not abelian.

(D) if H is abelian then H is cyclic.

(17) Consider a set S of unit vectors in $\mathbb{R}^2$ such that $\langle x, y\rangle = -1/2$ if $x, y \in S$, $x \neq y$. Then, it is necessarily true that

(A) the set S is linearly independent.

(B) the set S generates $\mathbb{R}^2$.

(C) the set S is either linearly independent or generates $\mathbb{R}^2$.

(D) if the set S is linearly independent, then S generates $\mathbb{R}^2$.

(18) The radius of convergence of $\sum_{n=0}^{\infty} z^{n!}$ is

(A) 0.

(B) 1.

(C) 2.

(D) ∞.

(19) The function $f(z) = e^{e^{1/z}}$

(A) is analytic at $z = 0$.

(B) has a removable singularity at $z = 0$.

(C) has a pole at $z = 0$.

(D) has an essential singularity at $z = 0$.

(20) Let $f : \mathbb{C} \to \mathbb{C}$ be a function. Then the region $\Omega = \{z \in \mathbb{C} : |e^{-f(z)}| < 2\}$ can be described as

(A) $\Omega = \{z \in \mathbb{C} : \text{Re}f(z) > -\log(2)\}$.

(B) $\Omega = \{z \in \mathbb{C} : \text{Re}f(z) < -\log(2)\}$.

(C) $\Omega = \{z \in \mathbb{C} : \text{Im}f(z) > -\log(2)\}$.

(D) $\Omega = \{z \in \mathbb{C} : \text{Im}f(z) < -\log(2)\}$.

PART B

(1) Let $f, g\colon [0,1] \to \mathbb{R}$ be non-negative continuous functions such that

$$\sup_{0\le x\le 1} f(x) = \sup_{0\le x\le 1} g(x).$$

Show that $f(t) = g(t)$ for some $t \in [0,1]$.

(2) Let $f\colon [0,1] \to [0,1]$ be a function. Assume that, for every sequence $\{x_n\}$ in $[0,1]$, whenever both the sequences $\{x_n\}$ and $\{f(x_n)\}$ converge, we have

$$\lim_{n\to\infty} f(x_n) = f\left(\lim_{n\to\infty} x_n\right).$$

Show that f is continuous.

(3) Let $f\colon [0,\infty) \to \mathbb{R}$ be a continuously differentiable function such that

$$f'(x) = \frac{1}{x^2 + \sin^2(x) + f(x)}, \ \forall x \ge 1,$$

and

$$f(x) \ge 0, \ \forall x \ge 1.$$

Show that $\lim_{x\to\infty} f'(x) = 0$. Deduce that $\lim_{x\to\infty} f(x)$ exists.

(4) Let $f(x)$ be a continuous function on $[0,1]$ satisfying

$$\int_0^1 f(x)dx = \int_0^1 xf(x)dx = 0.$$

Show that there exist $a, b \in [0,1]$, $a < b$, such that $f(a) = f(b) = 0$.

(5) Let V and W be vector spaces over $\mathbb{R}$ and $A\colon V \to W$ a linear transformation. Suppose there exists a unique $B\colon W \to V$ with $BA = I$, show that $AB = I$.

(6) Let $f\colon \mathbb{Z} \to \mathbb{Z}$ be a surjective homomorphism from the additive group of integers to itself. Show that f must be injective.

(7) Let $f\colon \mathbb{Q} \to \mathbb{Q}$ be an injective homomorphism from the additive group of rationals to itself. Show that f must be surjective.

(8) Let $p(z)$ and $q(z)$ be relatively prime polynomials with complex coefficients so that $\deg(q(z)) \ge \deg(p(z)) + 2$ and let $f(z) = p(z)/q(z)$. Show that the sum of the residues of $f(z)$ over all poles is 0.

Some Useful Links:

1. Free Maths Study Materials (https://pkalika.in/2020/04/06/free-maths-study-materials/**)**

2. BSc/MSc Free Study Materials (https://pkalika.in/2019/10/14/study-material/**)**

3. MSc Entrance Exam Que. Paper: (https://pkalika.in/2020/04/03/msc-entrance-exam-paper/**)**
[JAM(MA), JAM(MS), BHU, CUCET, ...etc]

4. PhD Entrance Exam Que. Paper: (https://pkalika.in/que-papers-collection/**)**
[CSIR-NET, GATE(MA), BHU, CUCET,IIT, NBHM, ...etc]

5. CSIR-NET Maths Que. Paper: (https://pkalika.in/2020/03/30/csir-net-previous-yr-papers/**)**
[Upto 2019 Dec]

6. Practice Que. Paper: (https://pkalika.in/2019/02/10/practice-set-for-net-gate-set-jam/**)**
[Topic-wise/Subject-wise]