Real Analysis

(Handwritten Classroom Study Material)

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Your Note/Remarks

Sequence

Defⁿ :- A seqⁿ is a function from the set of $\mathbb{N}$ to any set S.

$f: \mathbb{N} \to S$ where f is a function.

⇒ If $S = \{0, 1\}$ then f is called a binary seqⁿ.

⇒ If $S = \mathbb{N}$ then f is called a natural seqⁿ.

⇒ If $S = \mathbb{Q}$ then f is called a rational seqⁿ.

⇒ If $S = \mathbb{R}$ then f is called a real seqⁿ.

⇒ If $S = \mathbb{C}$ then f is called a complex seqⁿ.

Infact :- A function from a countably infinite set to any set is a seqⁿ.

Ex $f: A \to B$ where $f(x) = \sin x$.

$$A \times B := \{(a,b) : a \in A,\ b \in B\}$$

$$f = \{(a_1, b_1), (a_2, b_2), \dots\}$$

$$f = \{(x, f(x)) ; x \in A\}$$

$\therefore\ f \subseteq A \times B$.

☆ Real seqⁿ :-

A real seqⁿ is a function from $\mathbb{N}$ to $\mathbb{R}$.

i.e. $f : \mathbb{N} \to \mathbb{R}$

Infact,

it is a function of Countably infinite set of $\mathbb{R}$.

Eg. $f : \mathbb{N} \to \mathbb{R}$ s.t. $f(n) = \frac{1}{n}$

$f(n) = (\sqrt{2})^n$

$f(n) = \sin(n)$

$f(n) = \log(n)$.

$\Rightarrow$ $f : \mathbb{R} \to \mathbb{R}$ s.t. $f(x) = x^2$

$f = \{(x, x^2) : x \in \mathbb{R}\}$

$f = \{(x, f(x)) : x \in D\}$

$f = \{(x, f(x)) : x \in \mathbb{N}\}$

$f = \{(1, f(1)), (2, f(2)), (3, f(3)), \text{------}\}$ where $f_n = f(n)$

we express f as an ordered set.

$f = \{f_1, f_2, f_3, f_4, f_5, \text{---}\}$

☆ Let $f : \mathbb{N} \rightarrow \mathbb{R}$ be a function s.t. $f(n) = a_n$.

$\therefore f = \{(1, a_1), (2, a_2), (3, a_3), ---, (n, a_n), ----\}$ ← Tebular form

Now, $f = \{a_1, a_2, a_3, a_4, a_5, ----\}$ ← ordered set.
↳ Don't interchange order.

→ we drop pre-images from the ordered pair which makes f as an ordered set.

⇒ If f is a seq^n such that
$$f = \{\alpha, \beta, \alpha, \beta, ----\}$$
then $f \neq \{\alpha, \beta\}$.

Because, $f = \{(1, \alpha), (2, \beta), (3, \alpha), (4, \beta), ---\}$.

☆ A seq^n is denoted by $\langle a_n \rangle$, $\{a_n\}$ or (a_n) where a_n is a function of n.

and expressed as,
$$\langle a_n \rangle = \{a_1, a_2, a_3, a_4, ----\}.$$

☆ Examples of Sequences :-

① $a_n = 1$, $\forall n \in \mathbb{N}$.

$a_1 = 1$, $a_2 = 1$, $a_3 = 1$, $a_4 = 1$, --

$\langle a_n \rangle = \{1, 1, 1, 1, 1, ----\}$.

$R = \{1\}$
↳ Range.

② $a_n = (-1)^{n+1}$, $\forall n \in \mathbb{N}$

$a_1 = 1, a_2 = -1, a_3 = 1, a_4 = -1, ---$

$\langle a_n \rangle = \{1,-1,1,-1,1,-1,----\}$, $R = \{1,-1\}$

③ $a_n = \frac{(-1)^{n+1}}{n}$, $\forall n \in \mathbb{N}$

$a_1 = 1, a_2 = -\frac{1}{2}, a_3 = \frac{1}{3}, a_4 = -\frac{1}{4}, ---$

$\langle a_n \rangle = \{1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, ----\}$. , $R = \{a_n\}$

④ $a_n = 1 + \frac{(-1)^n}{n}$, $\forall n \in \mathbb{N}$

$a_1 = 0, a_2 = \frac{3}{2}, a_3 = \frac{2}{3}, a_4 = \frac{5}{4}, ---$

$\langle a_n \rangle = \{0, \frac{3}{2}, \frac{2}{3}, \frac{5}{4}, ---\}$, $R = \{a_n\}$

⑤ $a_n = (-1)^{n+1}\left(1 + \frac{1}{n}\right)$, $\forall n \in \mathbb{N}$

$a_1 = 2, a_2 = -\frac{3}{2}, a_3 = \frac{4}{3}, a_4 = -\frac{5}{4}, --$

$\langle a_n \rangle = \{2, -\frac{3}{2}, \frac{4}{3}, -\frac{5}{4}, ---\}$. , $R = \{a_n\}$.

☆ Range of a seqⁿ :- The range of a seqⁿ $\{a_n\}$ is the set of elements of $\{a_n\} = \{a_1, a_2, a_3, ---\}$ without repetition expressed in any order.

i.e. → The set of distinct element of the seqⁿ $\{a_n\}$.

Eg. ④ $a_n = \sin(\frac{n\pi}{3})$

$\Rightarrow \{a_n\} = \{ \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, 0, -\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}, 0, - - - \}$

$\therefore R = \{ -\frac{\sqrt{3}}{2}, 0, \frac{\sqrt{3}}{2} \}$.

★ Types of Sequences :-

(1) **Bounded above sequence :-**

If $\exists\, M \in \mathbb{R}$ s.t. $\boxed{a_n \leq M}$, $\forall n \in \mathbb{N}$.

(2) **Bounded below sequence :-**

If $\exists\, m \in \mathbb{R}$ s.t. $\boxed{a_n \geq m}$, $\forall n \in \mathbb{N}$.

(3) **Bounded seqⁿ :-**

If $\exists\, M \in \mathbb{R}$ s.t. $\boxed{|a_n| \leq M}$, $\forall n \in \mathbb{N}$.

Or // A seqⁿ $\{a_n\}$ is said to be bounded. if it's range set is bounded.

So, $\{a_n\}$ is said to be bounded.

If $\exists\, m$ and $M \in \mathbb{R}$ s.t.

$$m \leq a_n \leq M \quad \forall n \in \mathbb{N}.$$

⇒ m is called a lower bound of seqⁿ $\{a_n\}$.

⇒ and M is — — upper ——— — .

⇒ seqⁿ and range set same behaviour.

Hence, a seqⁿ $\{a_n\}$ is

(i) Bounded above iff it's range set is bounded above.

(ii) Bounded below iff it's range set is Bounded below.

(iii) M is the Supremum of seqⁿ $\{a_n\}$ iff M is the Supremum of it's range set.

(iv) m is the infimum of seqⁿ $\{a_n\}$ iff m is the infimum of it's range set.

☆ The range of a seqⁿ may be finite or Countably infinite But, Can never be empty.

<u>However</u>, A seqⁿ is always a Countably infinite set.

⇒ ④ Unbounded seqⁿ :- If $\{a_n\}$ is not a bounded seqⁿ.

(i) Unbounded above seqⁿ :- If $\forall M \in \mathbb{R}$, $\exists n \in \mathbb{N}$ s.t. $\boxed{a_n > M}$.

(iii) Unbounded below :-

If $\forall m \in \mathbb{R}$, $\exists n \in \mathbb{N}$ s.t. $\boxed{a_n < m}$

Eg:- $\{n \sin(n\pi)\}$ ← Bounded. $\Rightarrow \{a_n\} = \{0, 0, 0, ---\}$

$\{n \cos(n\pi)\}$ ← unbounded.

$\Rightarrow \{a_n\} = \{-1, 2, -3, 4, -5, 6, -7, ---\}$

$= \{-\infty, ---, \infty\}$

this is unbounded above and below Both:

☆ $\boxed{\text{A seq. is Bounded} \Leftrightarrow \text{it's range set is bounded.}}$

☆ Monotonic seq. :-

(i) Monotonic increasing :- (monotonic non-decreasing)

A seq. $\{a_n\}$ is said to be monotonic increasing.

If $\forall n \in \mathbb{N}$, $\boxed{a_n \le a_{n+1}}$

(ii) Strictly monotonic increasing :-

If $\forall n \in \mathbb{N}$, $\boxed{a_n < a_{n+1}}$

(iii) monotonic Decreasing :-

If $\forall n \in \mathbb{N}$, $\boxed{a_n \ge a_{n+1}}$

(iv) Strictly monotonic Decreasing :-

If $\forall n \in \mathbb{N}$, ~~s.t.~~ : $\boxed{a_n > a_{n+1}}$

⇒ If a seq^n $\{a_n\}$ is not monotonic then it is called a non-monotonic seq^n.

⇒ A monotonic Decreasing seq^n is always bounded above and has the Supremum a_1.

IIIy.
A monotonic increasing seq^n is always bounded below and has the infimum a_1.

Eg. ① $\{n\sin(n\pi)\} = \{0\}$.

this is monotonic increasing & decreasing both.

② $\{n\cos(n\pi)\}$ ← not a monotonic.

③ $\{a_n\} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \text{---}\}$

⇒ Strictly monotonic Decreasing.

④ $\{a_n\} = \{2, \frac{9}{4}, \frac{64}{27}, \text{---}\}$

⇒ Monotonic increasing.

⑤ $\{a_n\} = \{1, 4, 9, 16, 25, --\}$

$\Rightarrow$ Monotonic strictly increasing.

⑥ $\{a_n\} = \{e, e^2, e^3, e^4, ---\}$

$\Rightarrow$ Strictly monotonic increasing.

⑦ $\{a_n\} = \{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, ---\}$

$\Rightarrow$ Strictly monotonic decreasing.

⑧ $\{a_n\} = \{1, \sqrt{3}, \sqrt{3\sqrt{3}}, \sqrt{3\sqrt{3\sqrt{3}}}, ---\}$

$\Rightarrow$ monotonic increasing.

⑨ $\{1+\frac{1}{n}\} \rightarrow$ S.M.D. $= \{2, \frac{3}{2}, \frac{4}{3}, --\}$

⑩ $\{1-\frac{1}{n}\} \rightarrow$ S.M.I. $= \{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, ---\}$

⑪ $\{1+\frac{(-1)^n}{n}\} \rightarrow$ not a monotonic.

⑫ $a_n = \frac{100^n}{n!}$

$a_1 = \frac{100}{1}$, $a_2 = \frac{100}{1}\cdot\frac{100}{2}$, $a_3 = \frac{100}{1}\cdot\frac{100}{2}\cdot\frac{100}{3}$,

$$\begin{aligned} \cdots\cdots a_{99} &= \frac{100}{1}\cdot\frac{100}{2}, \cdots\cdots \frac{100}{99} \\ a_{100} &= \frac{100}{1}\cdot\frac{100}{2}\cdots\cdots\frac{100}{100} \\ a_{101} &= \frac{100}{1}\cdot\frac{100}{2}\cdots\cdots\frac{100}{101} \end{aligned}$$

So, $\{a_n\}$ = Non-monotonic seq^n.

Q. Find the largest term in $a_n = \frac{20^n}{n!}$.

[✓ (a)] a_{19} [✓ (b)] a_{20} (c) a_{21} (d) D.E.N.

Sol^n:- $a_1 = \frac{20}{1}$, $a_2 = \frac{20}{1} \cdot \frac{20}{2}$, $a_3 = \frac{20}{1} \cdot \frac{20}{2} \cdot \frac{20}{3}$, ----

-- , $a_{19} = \frac{20}{1} \cdot \frac{20}{2} \cdot \frac{25}{3} \cdot --- \boxed{\frac{20}{19}}$ -

~~$a_{20} = \frac{20}{1} \cdot \frac{20}{2} \cdot (20)^2$~~

$a_{20} = \frac{20}{1} \cdot \frac{20}{2} \cdot \frac{20}{3} \cdot \frac{20}{4} \cdot --- \boxed{\frac{20}{20}}$ -

$\Rightarrow a_n \rightarrow$ bounded. If $\sup \{a_n\} = a_{20}$

$\inf \{a_n\} = 0$

☆ Limit point of a seq^n :-

A number $a \in \mathbb{R}$ is said to be a limit point of a seq^n.

if $\forall \epsilon > 0$, $a_n \in (a-\epsilon, a+\epsilon)$ for infinitely many values of n.

Eg: ① $\langle a_n \rangle = \{1, 1, 1, 1, 1, ---\}$

So, limit point of $\{a_n\}$ is 1.

② $\langle b_n \rangle = \{1, -1, 1, -1, 1, -1, ---\}$

→ limit point of $\langle b_n \rangle$ is 1 & -1.

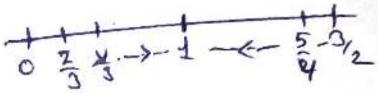
③ $\langle C_n \rangle = \{1, 2, 3, 1, 2, 3, ----\}$

→ limit point of $\langle C_n \rangle$ are $1, 2, 3$.

Q. Find the limit point of the following seqⁿ ?

① $\{a_n\} = \{1 + \frac{(-1)^n}{n}\}$

Solⁿ – $\{a_n\} = \{0, \frac{3}{2}, \frac{2}{3}, \frac{5}{4}, \frac{4}{5}, ---\}$



→ limit point $= 1$.

② $\{a_n\rangle = \{1 + (-1)^n\}$.

$\langle a_n \rangle = \{0, 2, 0, 2, 0, 2, ---\}$

→ limit point $= 0, 2$.

③ $\{a_n\} = \{(-1)^n (1 + \frac{1}{n})\}$

$\{a_n\} = \{-2, \frac{3}{2}, \frac{-4}{3}, \frac{5}{4}, \frac{-6}{5}, \frac{7}{6}, ---\}$

→ limit point $= 1, -1$.

④ $\{a_n\} = \{n^2\}$

$\{a_n\} = \{1, 4, 9, 16, 25, 36, --\}$

→ Does not limit point.

★ Limit of a seq^n :-

A number $a \in \mathbb{R}$ is said to be the limit of a seq^n $\{a_n\}$.

If $\forall \epsilon > 0, \exists n_0 \in \mathbb{N}$ s.t. $\boxed{|a_n - a| < \epsilon}$, $\forall n > n_0$.

$\Rightarrow \forall \epsilon > 0, \exists n_0 \in \mathbb{N}$ s.t. $-\epsilon < a_n - a < \epsilon$, $\forall n > n_0$

$\Rightarrow \forall \epsilon > 0, \exists n_0 \in \mathbb{N}$ s.t. $a - \epsilon < a_n < a + \epsilon$, $\forall n > n_0$

$\Rightarrow \forall \epsilon > 0, \exists n_0 \in \mathbb{N}$ s.t. $a_n \in (a-\epsilon, a+\epsilon)$, $\forall n > n_0$

So, 'a' is the limit of a seq^n $\{a_n\}$.

If $\forall \epsilon > 0$, $a_n \in (a-\epsilon, a+\epsilon)$ for all but finitely many values of n.

* $\Rightarrow$ If limit of a seq^n exist then it must be unique.

$\Rightarrow$ Every limit is a limit point but a limit point need not be the limit.

$\Rightarrow$ A unique limit point of a seq^n need not be the limit.

Eg:- $a_n = \{1, \frac{1}{2}, 3, \frac{1}{4}, 5, \frac{1}{6}, 7, \frac{1}{8}, ---\}$

$a_n = n$, if n is odd.

$\frac{1}{n}$, if n is even.

Now, 0 is the only limit point of the seq^n.

* ⇒ A unique limit point of a bounded seqⁿ must be the limit of the seqⁿ.

* ⇒ A monotonic seqⁿ has atmost one limit point.

* ⇒ A monotonic seqⁿ is bounded iff it has a unique limit point. (Limit).

★ Bolzano weierstras theorem :- Every infinite bounded seqⁿ has a limit point.

⇒ If a seqⁿ has no limit point then the seqⁿ must be unbounded seqⁿ.

→ Converse of Bolzano weierstras theorem is not true. an unbounded seqⁿ may or may not have a limit point.

Eg. $\{a_n\} = n$, if n is odd.

$\frac{1}{n}$, if n is even.

$\{a_n\} = \{1, \frac{1}{2}, 3, \frac{1}{4}, 5, \frac{1}{6}, ----\}$

→ limit point = 0.

unbounded seqⁿ.

✯ Subseqⁿ :-

Let $\{a_n\}$ be a seqⁿ and $\{n_k\}$ be a strictly monotonic increasing seqⁿ of natural numbers then $\{a_{n_k}\}$ is said to be a subseqⁿ of $\{a_n\}$.

Egs.- $n_k = \{6, 7, 9, 13, 69, 78, \ldots\}$

then

$\{a_{n_k}\} = \{a_6, a_7, a_9, a_{13}, a_{69}, a_{78}, \ldots\}$

is a subseqⁿ of $\{a_n\}$.

⇒ $\{a_{2n}\}$ is a subseqⁿ of $\{a_n\}$.

⇒ $\{a_{2n-1}\}$ is a subseqⁿ of $\{a_n\}$.

✯ Complementary Subseqⁿ :-

Two subseqⁿ $\{a_{n_k}\}$ and $\{a_{m_k}\}$ are said to be complementary subseqⁿ of $\{a_n\}$.

If $\{n_k\} \cap \{m_k\} = \phi$

and $\{n_k\} \cup \{m_k\} = \mathbb{N}$

Egs → $\{a_{2n}\}$ and $\{a_{2n-1}\}$ are complementary subseqⁿ.

Ex ① $a_n = \frac{(-1)^{n+1}}{n} = \{1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, -\frac{1}{6}, ---\}$

$\{a_{2n}\} = \{-\frac{1}{2}, -\frac{1}{4}, -\frac{1}{6}, -\frac{1}{8}, ---\}$

$\{a_{2n-1}\} = \{1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, ---\}$.

② $a_n = (-1)^{n+1}\left(1+\frac{1}{n}\right) = \{2, -\frac{3}{2}, \frac{4}{3}, -\frac{5}{4}, \frac{6}{5}, -\frac{7}{6}, --\}$

$\{a_{2n}\} = \{-\frac{3}{2}, -\frac{5}{4}, -\frac{7}{6}, ---\} \Rightarrow$ limit $= -1$

$\{a_{2n-1}\} = \{2, \frac{4}{3}, \frac{6}{5}, ---\} \Rightarrow$ limit $= 1$.

* ⇒ Every seqⁿ has a monotonic subseqⁿ.

* ⇒ A seqⁿ $\{a_n\}$ has a limit point 'a' iff there is a subseqⁿ $\{a_{n_k}\}$ of $\{a_n\}$ such that 'a' is the limit of the subseqⁿ $\{a_{n_k}\}$.

* ⇒ If 'a' is the limit of a seqⁿ $\{a_n\}$ then 'a' is the limit of every subseqⁿ of $\{a_n\}$.

** ⇒ Every Convergent seqⁿ ⇒ Bounded. [Converse not true.]

* ⇒ A monotonic seqⁿ never oscillates.

* ⇒ Every subseqⁿ of a $\frac{\text{M.I.}}{\text{M.D.}}$ seqⁿ is $\frac{\text{M.I.}}{\text{M.D.}}$

* ⇒ Every subseqⁿ of bounded seqⁿ is bounded.

** ⇒ Every subseqⁿ of unbounded seqⁿ is need not be unbounded. {∵ (can be bounded. Ex $\sin x$}.

If l is a limit point of a set S then $\exists$ a seq^n $\{S_n\}$ in S s.t. l is the limit of $\{S_n\}$. ([✓ T] / F)

Proof:-

If l is a limit point of a set S

then

$\forall \epsilon > 0$, $(l-\epsilon, l+\epsilon)$ has an infinite elements of S.

Then we can choose countably infinite subset of S which is contained in $(l-\epsilon, l+\epsilon)$.

The countably infinite subset of S is a seq^n in S which is contained in $(l-\epsilon, l+\epsilon)$

$\Rightarrow$ l is the limit of the seq^n.

Eg. $S = [1,2]$

$\rightarrow$ 2 is the limit point of S.

So, $(2-\epsilon, 2+\epsilon)$

$(2-\epsilon, 2]$

Subset of $S = \{1+\frac{1}{n} : n \in \mathbb{N}\}$.

★ If l is an isolated point of a set S. then $\exists$ a seq^n $\{S_n\}$ in S such that l is the limit of $\{S_n\}$. ([✓ T]/F)

Sol^n:- $l \in S$. $a_n = \{l, l, l, ---\}$.

★ Limit Superior and limit inferior:

Let $\{a_n\}$ be a seq^n then limit Superior of $\{a_n\}$ denoted by limsup.a_n or $\overline{\lim}\, a_n$ is the greatest limit point of $\{a_n\}$. if it exists.

and limit inferior, denoted by lim inf. a_n or $\underline{\lim}\, a_n$. is the least limit point of $\{a_n\}$, if it exists.

Eg: ① $a_n = (-1)^{n+1}\left(1+\frac{1}{n}\right)$

$\overline{\lim}\, a_n = 1$

$\underline{\lim}\, a_n = -1$.

② $a_n = \left(1 + \frac{(-1)^{n+1}}{n}\right)$

$\overline{\lim}\, a_n = 1$

$\underline{\lim}\, a_n = 1$

③ $a_n = \frac{1}{2^n}$, $n = 3K$

1, $n = 3K+1$

2^n, $n = 3K+2$.

$K \in \mathbb{N}$

$\underline{\lim}\, a_n = 0$

$\overline{\lim}\, a_n =$ D.N.E.

④ $a_n = n + \frac{(-1)^n}{n} = \left\{0, \frac{5}{2}, \frac{8}{3}, \frac{17}{4}, ---\right\}$

$\underline{\lim}\, a_n = \overline{\lim}\, a_n =$ D.N.E. (∞)

⑤ $a_n = \{1, 2, 3, 1, 2, 3, ---\}$

$\underline{\lim}\, a_n = 1$, $\overline{\lim}\, a_n = 3$.

⇒ If $\{a_n\}$ Converge to a then.

$$\underline{\lim}\, a_n = a = \overline{\lim}\, a_n$$

⇒ If $\{a_n\}$ diverges to ∞ or $-\infty$ then.

$$\overline{\lim}\, a_n = \infty \text{ or } -\infty = \underline{\lim}\, a_n$$

⇒ If $\{a_n\}$ oscillates finitely then.

$$-\infty < \underline{\lim}\, a_n < \overline{\lim}\, a_n < \infty$$

⇒ If $\{a_n\}$ oscillates infinite then

$$-\infty \leq \underline{\lim}\, a_n < \overline{\lim}\, a_n \leq \infty$$

⇒ For every bounded seq^n $\{a_n\}$.

$$\inf\{a_n\} \leq \underline{\lim}\{a_n\} \leq \overline{\lim}(a_n) \leq \sup.\{a_n\}$$

⇒ For every monotonic seq^n $\{a_n\}$.

$$\underline{\lim}\{a_n\} = \overline{\lim}\{a_n\} \leq \sup\{a_n\}$$

☆ properties of $\overline{\lim}$ & $\underline{\lim}$:-

If $\{a_n\}$ & $\{b_n\}$ are two bounded seq^n then.

① limit inferior may not be equal to $\inf.\{a_n\}$.
and limit superior of $\{a_n\}$ may not be equal to $\sup.\{a_n\}$.

E.g. $a_n = (-1)^{n+1}\left(1+\frac{1}{n}\right)$

$\sup\{a_n\} = 2$

$\inf\{a_n\} = -\frac{3}{2}$

$\overline{\lim}\, a_n = 1$

$\underline{\lim}\, a_n = -1$

② $\inf\{a_n\} \le \underline{\lim}\{a_n\} \le \overline{\lim}\{a_n\} \le \sup\{a_n\}$

③ If $\inf\{a_n\} \notin \{a_n\}$ then $\inf\{a_n\} = \underline{\lim}\{a_n\}$.
and $\sup\{a_n\} \notin \{a_n\}$ then $\sup\{a_n\} = \overline{\lim}\{a_n\}$.

④ $\overline{\lim}\,(\alpha n) = \alpha\, \overline{\lim}\, n$, if $\alpha > 0$.
$= \alpha\, \underline{\lim}\, n$, if $\alpha < 0$.

and $\underline{\lim}\,(\alpha n) = \alpha\, \underline{\lim}\, n$, if $\alpha > 0$.
$= \alpha\, \overline{\lim}\, n$, if $\alpha < 0$.

⑤ $\overline{\lim}\,(-a_n) = -\underline{\lim}\,(a_n)$

and $\underline{\lim}\,(-a_n) = -\overline{\lim}\,(a_n)$.

⑥ $\overline{\lim}\,(a_n) > 0$ then $\boxed{\underline{\lim}\left(\frac{1}{a_n}\right) = \frac{1}{\overline{\lim}\,(a_n)}}$.

and $\underline{\lim}\,(a_n) > 0$ then $\boxed{\overline{\lim}\left(\frac{1}{a_n}\right) = \frac{1}{\underline{\lim}(a_n)}}$

⑦ $\underline{\lim}\,(a_n) + \underline{\lim}\,(b_n) \le \underline{\lim}\,(a_n+b_n) \le \underline{\lim}(a_n) + \overline{\lim}\,(b_n)$
$\le \overline{\lim}\,(a_n+b_n) \le \overline{\lim}\,(a_n) + \overline{\lim}\,(b_n)$.

Eg. $a_n = \{0, 1, -1, 0, 1, -1, ---\}$

$b_n = \{1, -1, 0, 1, -1, 0, ---\}$

$\overline{\lim}\, a_n = 1$, $\underline{\lim}\, a_n = -1$.

$\overline{\lim}\, b_n = 1$, $\underline{\lim}\, a_n = -1$

$a_n + b_n = \{1, 0, -1, 1, 0, -1, ---\}$

$\overline{\lim}\,(a_n + b_n) = 1$, $\underline{\lim}\,(a_n + b_n) = -1$.

$\underline{\lim}\, a_n + \underline{\lim}\, b_n = -2$

$\underline{\lim}\,(a_n + b_n) = -1$

$\underline{\lim}\, a_n + \overline{\lim}\, b_n = 0$

$\overline{\lim}\,(a_n + b_n) = 1$

$\overline{\lim}\, a_n + \overline{\lim}\, b_n = 2$.

⑧ If $\underline{\lim}\,(a_n)$ and $\underline{\lim}\,(b_n) > 0$ then

$$(\underline{\lim}\, a_n)(\underline{\lim}\, b_n) \le \underline{\lim}\,(a_n b_n) \le (\underline{\lim}\, a_n)(\overline{\lim}\, b_n)$$

$$\le \overline{\lim}\,(a_n b_n) \le (\overline{\lim}\, a_n)(\overline{\lim}\, b_n).$$

Eg. $a_n = \{1, 2, 3, 1, 2, 3, ---\}$

$b_n = \{3, 1, 2, 3, 1, 2, ---\}$

$a_n b_n = \{3, 2, 6, 3, 2, 6, ---\}$

$\underline{\lim}\, a_n = 1$, $\overline{\lim}\, a_n = 3$

$\underline{\lim}\, b_n = 1$, $\overline{\lim}\, b_n = 3$

$\underline{\lim}\, a_n b_n = 2$, $\overline{\lim}\, a_n b_n = 6$

$\underline{\lim}\, a_n + \underline{\lim}\, b_n = 2$

$\overline{\lim}\, a_n + \overline{\lim}\, b_n = 6$

$\underline{\lim}\, a_n + \overline{\lim}\, b_n = 4$

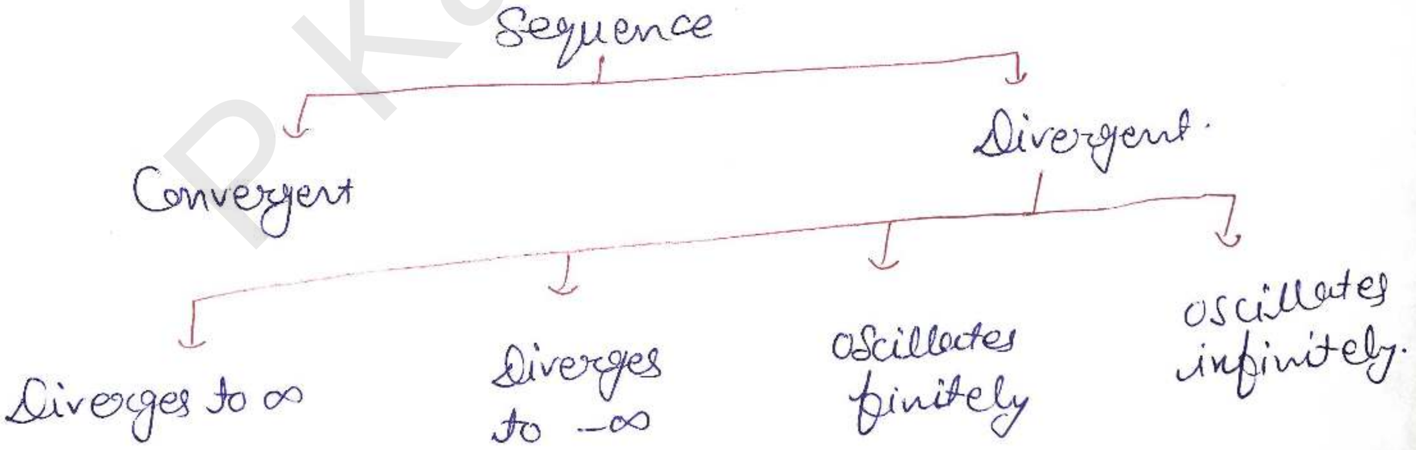
⑨ $\underline{\lim}\, a_n - \overline{\lim}\, b_n \le \underline{\lim}(a_n - b_n) \le \underline{\lim}\, a_n - \underline{\lim}\, b_n$

$\le \overline{\lim}\,(a_n - b_n) \le \overline{\lim}\, a_n - \underline{\lim}\, b_n$

⑩ $\dfrac{\underline{\lim}\, a_n}{\overline{\lim}\, b_n} \le \underline{\lim}\left(\dfrac{a_n}{b_n}\right) \le \dfrac{\underline{\lim}\, a_n}{\underline{\lim}\, b_n} \le \overline{\lim}\left(\dfrac{a_n}{b_n}\right) \le \dfrac{\overline{\lim}\, a_n}{\underline{\lim}\, b_n}$.

$\because \underline{\lim}\, b_n \neq 0 \neq \overline{\lim}\, b_n$

☆ Nature of a seqⁿ :-



① Convergent Seq^{n} :-

A seq^{n} $\{a_n\}$ is said to be Convergent.

If $\exists\, a \in \mathbb{R}$ s.t. $\forall\, \epsilon > 0$, $\exists\, n_0 \in \mathbb{N}$ s.t.

$|a_n - a| < \epsilon$, $\forall\, n > n_0$

OR ✓ $\{a_n\}$ is said to be Convergent. If the limit of $\{a_n\}$ exists.

→ In this Case, we say that $\{a_n\}$ Converges to a and denote it by $a_n \to a$ as $n \to \infty$

OR $\boxed{\lim_{n\to\infty} a_n = a}$.

② Divergent Seq^{n} :-

A seq^{n} $\{a_n\}$ is said to be divergent. if it not Convergent.

(i) Diverges to ∞ :-

A seq^{n} $\{a_n\}$ diverges to ∞.

If $\{a_n\}$ is bounded below, unbounded above and has no limit point.

OR A seq^{n} $\{a_n\}$ diverges to ∞.

If $\boxed{\lim_{n\to\infty} a_n = \infty}$

Ex → $a_n = n^2$, $a_n = n + \frac{1}{n}$

(ii) Diverges to $-\infty$:

If $\{a_n\}$ is bounded above, unbounded below and has no limit point.

OR If $\overline{\lim}\, a_n = -\infty$

(iii) Oscillates finitely :

A bounded seq^n is said to ~~be~~ oscillates finitely.

If it has more than one limit point.

OR If $-\infty < \underline{\lim}\, a_n < \overline{\lim}\, a_n < \infty$.

Eg. $a_n = (-1)^n \left(1 + \frac{1}{n}\right)$.

(iv) Oscillates infinitely :

An unbounded seq^n. $\{a_n\}$ is said to oscillates infinitely.

If it is either unbounded and has at least one limit point OR Both unbounded above and unbounded below.

OR If $-\infty \le \underline{\lim}\, a_n < \overline{\lim}\, a_n \le \infty$

and $\{a_n\}$ is unbounded.

Q. Discuss the nature of the seq^n. ?

① $a_n = \frac{(-1)^{n+1}}{n}$

Sol^n- $\{a_n\} = \{1, \frac{-1}{2}, \frac{1}{3}, \frac{-1}{4}, \frac{1}{5}, \frac{-1}{6}, ---\}$

→ Convergent (limit = 0). $\left[\because \lim_{n\to\infty} \frac{(-1)^{n+1}}{n} = 0\right]$.

② $a_n = (-1)^{n+1}\left(1+\frac{1}{n}\right)$

$\{a_n\} = \{2, -\frac{3}{2}, \frac{4}{3}, -\frac{5}{4}, \frac{6}{5}, -\frac{7}{6}, ---\}$

→ oscillates finitely between 1 & -1.

③ $a_n = (-1)^{n+1}\left(n+\frac{1}{n}\right)$

$\{a_n\} = \{2, -\frac{5}{2}, \frac{10}{3}, -\frac{17}{4}, ---\}$

→ oscillates infinitely.

④ $a_n = \sin\left(\frac{n\pi}{4}\right)$

$\{a_n\} = \{\frac{1}{\sqrt{2}}, 1, \frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}, -1, -\frac{1}{\sqrt{2}}, ---\}$

→ oscillates finitely Between $\{1, \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}, -1\}$.

⑤ $a_n = n+\frac{1}{n}$

$\{a_n\} = \{2, \frac{5}{2}, \frac{10}{3}, \frac{17}{4}, \frac{26}{5}, \frac{37}{6}, ---\}$

→ Diverges to ∞.

⑥ $a_n = \frac{20^n}{n!}$

→ Convergent to 0.

⑦ $a_n = \frac{\sin n}{n}$

→ Convergent to 0.

⑧ $a_n = (-1)^{n+1}(n^2 - n)$

$\{a_n\} = \{0, -2, 6, -14, 20, -30, \cdots\}$

→ oscillates infinitely.

⑨ $a_n = -n + \frac{1}{n}$

$\{a_n\} = \{0, -\frac{3}{2}, -\frac{8}{3}, \cdots\}$

→ Diverges to $-\infty$.

⑩ $a_n = \frac{n^2 - n + 1}{n^2 + n + 1}$

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{n^2 - n + 1}{n^2 + n + 1} = \lim_{n \to \infty} \frac{1 - \frac{1}{n} + \frac{1}{n^2}}{1 + \frac{1}{n} + \frac{1}{n^2}} = ①$$

→ $\{a_n\}$ is Convergent.

☆ Properties of nature of seqⁿ :

① Every Convergent seqⁿ $\Rightarrow$ Bounded.
$\not\Leftarrow$

→ A Bounded seqⁿ is either Convergent or oscillates finitely.

② An unbounded seqⁿ Can never be Convergent and unbounded seqⁿ either diverges to ∞ or $-\infty$ or oscillates infinitely.

③ If $\{a_n\}$ is bounded below and unbounded above then either $\{a_n\}$ diverges to ∞ or it oscillates infinitely.

④ If $\{a_n\}$ is bounded above and unbounded below then either $\{a_n\}$ diverges to $-\infty$ or oscillates infinitely.
↳ has atleast one limit point.

⑤ A monotonic seqⁿ never oscillates. So, it is either Convergent or diverges to ∞ or diverges to $-\infty$.

⑥ A monotonic increasing seq^n is either Convergent or diverges to ∞.

⑦ A monotonic Decreasing seq^n is either Convergent or diverges to $-\infty$.

⑧ A monotonic bounded seq^n is always Convergent.

So, A monotonic seq^n is Convergent iff it is bounded.

$\Rightarrow$ A monotonic increasing seq^n which is bounded above, is Convergent and Converges to it's Supremum.

i.e. $\lim_{n\to\infty} a_n = \text{Sup}(a_n)$

$\Rightarrow$ A monotonic Decreasing seq^n which is bounded below, is Convergent and it Converges to it's infimum.

☆ Relation between a seq^n & a Sub-seq^n:

① If a seq^n has a limit point then $\exists$ a Sub-seq^n which Converges to that limit point.

② Every bounded seq^n has a Convergent Sub-seq^n.

③ If a seq^n is convergent then every sub-seq^n of the seq^n is convergent and converges to the same limit.

④ If proper sub-seq^n of a seq^n is convergent then the seq^n may or may not converge.

⑤ Every seq^n has a monotonic sub-seq^n.

Q. Let $\{a_n\}$ be a seq^n s.t. $\{a_{2n}\}$ and $\{a_{2n+1}\}$ are convergent then.

① If $\{a_n\}$ is convergent then $\{a_{2n}\}$ and $\{a_{2n+1}\}$ both converge to same limit.

② If $\{a_{2n}\}$ and $\{a_{2n+1}\}$ both converge to same limit then $\{a_n\}$ is convergent.

(∵ Complementary sub-seq^n).

③ $\{a_n\}$ is convergent.

④ $\{a_n\}$ may not be convergent.

Q. Let $\{a_n\}$ be a seq^n s.t. $\{a_{2n}\}$ and $\{a_{2n+1}\}$ or $\{a_{3n}\}$ are convergent then

(i) If $\{a_n\}$ is Convergent then $\{a_{2n}\}$ and $\{a_{2n+1}\}$ Both Converge to Same limit.

(ii) If $\{a_{2n}\}$ & $\{a_{3n}\}$ Converge to the Same limit then $\{a_n\}$ is Convergent.

(iii) $\{a_n\}$ is Convergent.

(iv) $\{a_n\}$ may not be Convergent.

Sol:- $\{a_{2n}\} \rightarrow L_1$, $\{a_{2n+1}\} \rightarrow L_2$, $\{a_{3n}\} \rightarrow L_3$

Now, $\{a_3, a_9, a_{15}, a_{21}, a_{27}, ---\}$ is a subseqⁿ of

$\{a_{3n}\} \rightarrow L_3$

———— is a subseqⁿ of $\{a_{2n+1}\} \rightarrow L_2$

$\Rightarrow L_2 = L_3$

Illy

$\{a_6, a_{12}, a_{18}, a_{24}, ---\}$ is a Sub-seqⁿ of

$\{a_{3n}\} \rightarrow L_3$

———— is a Subseqⁿ of $\{a_{2n}\} \rightarrow L_1$

$\Rightarrow L_1 = L_2$

So, Two Complementary Sub-seqⁿ $\{a_{2n}\}$ and $\{a_{2n+1}\}$ Converge to the Same limit.

So, $\{a_n\}$ Converges to the Same limit.

Q. Let $\{a_n\}$ be a seq^n s.t. $\overset{\swarrow L_1}{\{a_{3n}\}}$, $\overset{\swarrow L_2}{\{a_{3n+1}\}}$, $\overset{\swarrow L_3}{\{a_{3n+2}\}}$ and $\overset{\swarrow L_4}{\{a_{5n}\}}$ Convergent then is $\{a_n\}$ is Convergent.?

Sol^n:- $\{a_n\}$ is Convergent.

Sub seq^n of $\{a_{3n}\} := \{a_{15}, a_{30}, a_{45}, a_{60}, ---\} \to L_4$

——— $\{a_{3n+1}\} = \{a_{10}, a_{25}, a_{40}, a_{55}, ---\} \to L_2$

——— $\{a_{3n+2}\} = \{a_5, a_{20}, a_{35}, a_{50}, a_{65}, ---\} \to L_3$

$\Rightarrow L_1 = L_2 = L_3 = L_4$

So, $\{a_n\}$ is Convergent.

☆ **Cauchy seq^n :-** If seq^n $\{a_n\}$ is said to be Cauchy seq^n.

If $\forall \epsilon > 0, \exists n_0 \in N$ s.t. $|a_n - a_m| < \epsilon, \forall n, m > n_0$.

Or If $\forall \epsilon > 0, \exists n_0 \in N$ s.t. $|a_{n+p} - a_n| < \epsilon, \forall n \geq n_0, p \geq 1$.

Eg. ① $\{a_n\} = \frac{1}{n}$ is a cauchy seq^n.

Let $\epsilon > 0$ be given

Now, Consider, $|a_{n+p} - a_n|$

$$= \left|\frac{1}{n+p} - \frac{1}{n}\right|$$

$$= \frac{p}{n(n+p)}$$

$$= \frac{1}{n} \cdot \frac{p}{n+p} < \frac{1}{n} \qquad \left(\because \frac{p}{n+p} < 1 \ \forall p\right)$$

If $\frac{1}{n} < \epsilon \Rightarrow n > \frac{1}{\epsilon}$

So, $\forall n > \frac{1}{\epsilon}$, $\frac{1}{n} < \epsilon$

and ~~&~~ $\forall p \geq 1$, $|a_{n+p} - a_n| < \frac{1}{n} < \epsilon$

So, Let $n_0 = \left[\frac{1}{\epsilon}\right]$ then

$\forall \epsilon > 0$, $\exists n_0 \in \mathbb{N}$ s.t. $|a_{n+p} - a_n| < \epsilon$, $\forall n > n_0, p \geq 1$.

② $a_n = 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$ is a Cauchy seq^n.

Let $\epsilon > 0$ be given.

Consider, $$|a_{n+p} - a_n| = \left|\left(\cancel{1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}} + \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots + \frac{1}{(n+p)!}\right) - \left(\cancel{1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}}\right)\right|$$

$$= \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots + \frac{1}{(n+p)!}$$

$\because n! > 2^{n-1} \ \forall n \geq 3$

$\Rightarrow \frac{1}{n!} < \frac{1}{2^{n-1}}$

So, $$\frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots + \frac{1}{(n+p)!} < \frac{1}{2^n} + \frac{1}{2^{n+1}} + \cdots + \frac{1}{2^{n+p-1}}$$

$$< \frac{1}{2^n} + \frac{1}{2^{n+1}} + \cdots$$

$$= \frac{\frac{1}{2^n}}{1 - \frac{1}{2}} = \frac{1}{2^{n-1}}$$

If $\frac{1}{2^{n-1}} < \epsilon$

$\Rightarrow 2^{n-1} > \frac{1}{\epsilon} \Rightarrow n-1 > \lg\left(\frac{1}{\epsilon}\right)$. $\left(\because \lg x = \log_2 x\right)$

$\Rightarrow n > 1 + \log\left(\frac{1}{\epsilon}\right)$

Let $n_0 = \left[1 + \log\left(\frac{1}{\epsilon}\right)\right]$

then $\forall \epsilon > 0$, $\exists n_0 = \left[1 + \log\left(\frac{1}{\epsilon}\right)\right]$ s.t.

$|a_{n+p} - a_n| < \epsilon$, $\forall n > n_0$, $p \geq 1$.

③ $a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ is not a Cauchy seqⁿ:-

Let $\epsilon = \frac{1}{2}$ and let $p = n$ then

$$|a_{2n} - a_n| = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$$
$$\geq \frac{1}{2n} + \frac{1}{2n} + \cdots + \frac{1}{2n}$$
$$= \frac{n}{2n} = \frac{1}{2}$$

$\forall n \in \mathbb{N}$, $|a_{2n} - a_n| \geq \frac{1}{2}$

So, $\nexists n_0 \in \mathbb{N}$ s.t. $|a_{2n} - a_n| < \frac{1}{2}$, $\forall n > n_0$

So, $\nexists n_0 \in \mathbb{N}$ s.t. $|a_{n+p} - a_n| < \frac{1}{2}$, $\forall n > n_0$, $p \geq 1$.

$\therefore \{a_n\}$ is not a Cauchy seqⁿ.

☆ P: $\forall \epsilon > 0$, $\exists n_0 \in \mathbb{N}$ s.t. $|a_{n+p} - a_n| < \epsilon$, $\forall n > n_0$, $p \geq 1$.

$\sim P$: $\exists \epsilon > 0$, $\forall n_0 \in \mathbb{N}$, $\exists n > n_0$ and $p \geq 1$ s.t. $|a_{n+p} - a_n| \geq \epsilon$.

④ $a_n = 1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+ - - - + \frac{1}{2n-1}$. is not a Cauchy.

$$|a_{n+p}-a_n| = \left|1+\frac{1}{3}+\frac{1}{5}+ - - + \frac{1}{2n-1} + \frac{1}{2(n+1)-1} + \frac{1}{2(n+2)-1} + - - + \frac{1}{2(n+p)-1} - \left(1+\frac{1}{3}+\frac{1}{5}+ - - + \frac{1}{2n-1}\right)\right|$$

$$> \left|\frac{1}{2n+1} + \frac{1}{2n+3} + —— + \frac{1}{2n+(2p-1)}\right|$$

$$> \frac{2p-1}{2n+(2p-1)} > \frac{1}{2n}$$

∴ a_n is not a Cauchy seq^n.

☆ ⇒ A seq^n in $\mathbb{R}$ is Cauchy iff if it is Convergent in $\mathbb{R}$.

⇒ A Convergent seq^n is always a Cauchy seq^n. But a cauchy seq^n need not be Convergent.

Eg. ① If $a_n = \frac{1}{n}$ then $\{a_n\}$ is Cauchy seq^n in $\mathbb{R}^+$. But $\{a_n\}$ is not Convergent in $\mathbb{R}^+$.

② $a_n = \left(1+\frac{1}{n}\right)^n$ is a Cauchy seq^n in $\mathbb{Q}$. But it is not Convergent in $\mathbb{Q}$. ($\because e \notin \mathbb{Q}$)

☆ ⇒ a Cauchy seq^n is always bounded.

⇒ Every Cauchy seq^n has atmost one limit point. (Exactly one limit point in $\mathbb{R}$).

★ If a seq.^{n} is Cauchy then it has atmost one limit point in the given space. If it has a limit point in the space than it is Convergent in that space and if it does not have a limit point in that space then it is not Convergent in that space.

Q. Let $\{a_n\}$ be a seq.^{n} s.t. $\forall \epsilon > 0, \exists n_0 \in N$ s.t. $|a_{n+1} - a_n| < \epsilon, \forall n > n_0$ then $\{a_n\}$ is Cauchy.

[T / [✓ F]]

Ex $a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ is not a Cauchy.

But $|a_{n+1} - a_n| = \frac{1}{n+1} \rightarrow 0$ as $n \rightarrow \infty$.

★ Let $\{a_n\}$ be a seq.^{n} such that $\forall \epsilon > 0, \exists n_0 \in N$

s.t. $|a_{n+1} - a_n| < \epsilon \cdot b_n$

where b_n is a function of n.

If $\sum_{n=1}^{\infty} b_n$ is Convergent then $\{a_n\}$ is a Cauchy seq.^{n}.

$\sum_{n=1}^{\infty} \frac{1}{n^p} \Rightarrow$ Convergent, if $p > 1$

Divergent, if $p \leq 1$

Q. Which of the following Conditions ensure that the seqⁿ $\{a_n\}$ is Cauchy?

① $\frac{1}{n}|a_{n+1}-a_n|<\epsilon$ ② $n|a_{n+1}-a_n|<\epsilon$

③ $n^2|a_{n+1}-a_n|<\epsilon$ ④ $|a_{n+1}-a_n|<\epsilon$.

$\because \sum \frac{1}{n^2} \rightarrow$ Convergent, $\boxed{P>1}$ $P=2 \geq 1$

$\Rightarrow$ $\boxed{|a+b| \leq |a|+|b| \leftarrow \text{Traingle inequality}}$

IFR

Q. किसी function के zeros $(f(x)=0)$ का set alway Closed set.

Proof:- $Z(f) := \{x \in R : f(x)=0\}$

Let, if possible, $Z(f)$ is not Closed.

So, $\exists\, x \in \mathbb{R}$ s.t. x is a limit point of $Z(f)$ and $x \notin Z(f)$.

Since, x is a limit point of $Z(f)$.

So, $\exists$ a seqⁿ $\{x_n\}$ in $Z(f)$.

which Converges to x.

So, $\forall\, n \in \mathbb{N}$, $f(x_n)=0$ and $f(x) \neq 0$

Since f is Continuous function, So, if x_n Converges to x then $f(x_n)$ Converges to $f(x)$.

i.e. $\lim_{n\to\infty} f(x) = f(x) \Rightarrow \lim_{n\to\infty} 0 = f(x) \neq 0$

which is not possible.

Q. Which of the following Conditions ~~m~~ imply that $\{a_n\}$ is a Cauchy seq^n. ?

① $|a_{n+1} - a_n| \to 0$ ③ $(n+1)\,|a_{n+1} - a_n| \to 0$

② $\frac{n}{n+1}\,|a_{n+1} - a_n| \to 0$ ✓④ $(n+1)^2\,|a_{n+1} - a_n| \to 0$.

✯ Algebra of seq^n :-

(1.) If $\{a_n\}$ Converges to a and $\{b_n\}$ Converges to b. i.e. $a_n \to a$, $b_n \to b$ then.

(i) $\{a_n + b_n\}$ Converges to $(a+b)$.

(ii) $\{a_n - b_n\}$ " " $(a-b)$.

(iii) $\{a_n b_n\}$ " " (ab).

(iv) If $b \neq 0$ then $\left\{\frac{a_n}{b_n}\right\} \to \frac{a}{b}$

(2.) If $\boxed{\{a_n\} \to a}$ and $\boxed{f}$ is a Continuous fuction s.t. $\boxed{a \in D_f}$ and $\boxed{a_n \in D_f}$ $\forall\, n \in \mathbb{N}$.

then the seq^n $\boxed{\{f(a_n)\} \to f(a)}$!

Eg. ① $f : (0,1] \to \mathbb{R}$ s.t. $f(x) = \frac{1}{x}$ is a Continuous function. Let $a_n = \frac{1}{n} \in (0,1]$ then

$\lim_{n \to \infty} a_n = 0 \notin (0,1]$ //

Now, $\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} f\left(\frac{1}{n}\right) = \lim_{n\to\infty} n$ is not Convergent.

☆ Ⓐ If $a_n \to a$ then

(i) $a_n^2 \to a^2$

(ii) $a_n^K \to a^K$, $K \in \mathbb{N}$

(iii) $\sin(a_n) \to \sin(a)$

(iv) $e^{a_n} \to e^a$

(v) $|a_n| \to |a|$

(vi) If $a_n > 0$, $a > 0$, $\forall n \in \mathbb{N}$, $\ln(a_n) \to \ln(a)$.

③ (i) If $a_n > 0$, $\forall n \in \mathbb{N}$ and $a_n \to a$ then $a \geq 0$.

(ii) If $a_n \geq 0$, $\forall n \in \mathbb{N}$,, ,, ,, ,,

(iii) If $a_n < 0$, $\forall n \in \mathbb{N}$, , , , $a \leq 0$.

(iv) If $a_n \leq b_n$, $\forall n \in \mathbb{N}$ and $\{a_n\} \to a$, $\{b_n\} \to b$

then $a \leq b$.

$\forall n \in \mathbb{N}$
$a_n < b_n \Rightarrow a \leq b$
$a_n \leq b_n \Rightarrow a \leq b$
$a_n > b_n \Rightarrow a \geq b$
$a_n \geq b_n \Rightarrow a \geq b$.

Ex Let $C_n = a_n - b_n$, If $C_n \to C$

then $C = a - b$

$\because a_n \leq b_n \ \forall n \in \mathbb{N}$

$\Rightarrow a_n - b_n \leq 0$, $\forall n \in \mathbb{N}$

$\Rightarrow C_n \leq 0$, $\forall n \in \mathbb{N}$

$\Rightarrow C \leq 0$

$\Rightarrow a - b \leq 0$

$\Rightarrow a \leq b$.

④ If $\{a_n\}$ is Convergent and $\{b_n\}$ is divergent then.

(i) $\{a_n+b_n\}$ is divergent.

(ii) $\{a_n-b_n\}$ is divergent.

(iii) $\{a_nb_n\}$ is may or may not be divergent.

⑤ (i) If $\{a_n\}$ & $\{b_n\}$ are divergent seqⁿ then.

$$\left.\begin{array}{l}\{a_n+b_n\}\\ \{a_n-b_n\}\\ \{a_nb_n\}\\ \left\{\dfrac{a_n}{b_n}\right\}\end{array}\right\rangle \text{May Converge or Diverge.}$$

(ii) If $\{a_n\}$ & $\{b_n\}$ Both diverge to ∞ (or $-\infty$) then

$\Rightarrow$ $\{a_n+b_n\}$ will diverge to ∞ (or $-\infty$).

$\Rightarrow$ $\{a_nb_n\}$ will diverges to ∞.

$\Rightarrow$ Now, $$\left.\begin{array}{l}\{a_n-b_n\}\\ \left\{\dfrac{a_n}{b_n}\right\}\end{array}\right\rangle \text{may Converge or diverge.}$$

Ex ① $a_n = n^2+n-1$
$b_n = n^2+n+5$
$\Rightarrow \{a_n-b_n\} = -6 \rightarrow$ Converge.

② $a_n = n^2$
$b_n = n^3$
$\Rightarrow \dfrac{a_n}{b_n} = \dfrac{1}{n} \rightarrow 0 \rightarrow$ Converge.

③ $a_n = n^2$

$b_n = 2n^2+5 \Rightarrow \frac{a_n}{b_n} = \frac{n^2}{2n^2+5} \to \frac{1}{2} \to$ Converge.

☆ (iii) If $\{a_n\} \to$ divergent seq.ⁿ and ~~b_n~~ $\{b_n\}$ Converges to $b \neq 0$ then $\{a_n b_n\}$ is a divergent seq.ⁿ

IIIy $\left\{\frac{a_n}{b_n}\right\}$ is divergent seq.ⁿ

(iv) If $\{a_n\}$ is divergent seq.ⁿ and $\{b_n\}$ Converges to '0' then

$\{x_n y_n\} \to$ may Converge or diverge.

Eg:
~~①~~ ① $a_n = n^2$, $b_n = \frac{1}{n^2} \Rightarrow a_n b_n \to 1$.

② $a_n = n^3$, $b_n = \frac{1}{n^2} \Rightarrow a_n b_n \to \infty$

③ $a_n = n$, $b_n = \frac{1}{n^2} \Rightarrow a_n b_n \to 0$.

(v) If $\{a_n\}$ is a bounded seq.ⁿ and $\{b_n\}$ Converges to '0'. then $\{a_n b_n\}$. Converges to '0'.

⇒ product of two unbounded seq.ⁿ is divergent.

⇒ If $a_n \to a$ and $K \in \mathbb{R}$ then $Ka_n \to Ka$.

⇒ The nature of a seq.ⁿ is unchanged by inserting, deleting, replacing finitely many terms of the seq.ⁿ.

⑥ Sandwich theorem or Sqveeze theorem :-

If $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are three Seqⁿ Such that $\boxed{a_n \le b_n \le c_n}$, $\forall n \in \mathbb{N}$ and If $\{a_n\}$ and $\{c_n\}$ Converges to Same limit b then $\{b_n\}$ Converges to b.

$$\because a_n \le b_n \Rightarrow \lim_{n\to\infty} a_n \le \lim_{n\to\infty} b_n$$

$$\Rightarrow b \le \lim_{n\to\infty} b_n$$

$$c_n \ge b_n$$

$$\Rightarrow \lim_{n\to\infty} c_n \ge \lim_{n\to\infty} b_n$$

$$\Rightarrow b \ge \lim_{n\to\infty} b_n$$

$$\therefore \lim_{n\to\infty} b_n = b.$$

Q.① If $a_n = \frac{1}{n^2+1} + \frac{1}{n^2+2} + \cdots + \frac{1}{n^2+n}$ then find the limit of $\{a_n\}$.?

Solⁿ:- By Sandwich theorem:-

$$\frac{n}{n^2+n} \le a_n \le \frac{n}{n^2+1}$$

$$\rightarrow 0 \le a_n \le 0 \qquad \Rightarrow \boxed{a_n \to 0}$$

as $n\to\infty$

② $a_n = \frac{n}{n^2+1} + \frac{n}{n^2+2} + \text{———} + \frac{n}{n^2+n}$

then find the limit of $\{a_n\}$?

Sol^n:-

$$\frac{n^2}{n^2+n} \le a_n \le \frac{n^2}{n^2+1}$$

$$1 \le a_n \le 1$$

$$\Rightarrow a_n \to 1$$

☆ ⑦

$$\boxed{\lim_{n\to\infty} \frac{1}{n}\sum_{r=\alpha}^{\beta} f\left(\frac{r}{n}\right) = \int_a^b f(x)\,dx}$$

where $\frac{r}{n} \to x$, $\frac{1}{n} \to dx$

$$a = \lim_{n\to\infty} \frac{\alpha}{n}, \quad b = \lim_{n\to\infty} \frac{\beta}{n}$$

Q ① $a_n = \frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \text{———} + \frac{n}{n^2+n^2}$

then find the limit of $\{a_n\}$?

By Sandwith theorem:-

$$\frac{n^2}{n^2+n^2} \le a_n \le \frac{n^2}{n^2+1^2} \Rightarrow \frac{1}{2} \le a_n \le 1$$

Now,

$$a_n = \sum_{r=1}^{n} \frac{n}{n^2+r^2} = \sum_{r=1}^{n} \frac{1}{n} \cdot \frac{1}{1+\frac{r^2}{n^2}} = \int_0^1 \frac{1}{1+x^2}$$

$$= \left[\tan^{-1}(x)\right]_0^1 = \tan^{-1}(1) - \tan^{-1}(0)$$

$$= \boxed{\frac{\pi}{4}}$$

Q $a_n = \frac{1}{n^2+1} + \frac{2}{n^2+2^2} + \cdots + \frac{n}{n^2+n^2}$

$$a_n = \sum_{r=1}^{n} \frac{r}{n^2+r^2} = \sum_{r=1}^{n} \frac{1}{n} \cdot \frac{\frac{r}{n}}{1+\frac{r^2}{n^2}}$$

$$= \int_0^1 \frac{x}{1+x^2}\,dx = \boxed{\frac{1}{2}\ln(2)}$$

Q $a_n = \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{3n}$

$$a_n = \sum_{r=0}^{2n} \frac{1}{n+r} = \int_0^2 \frac{1}{1+x}\,dx = \left[\ln(1+x)\right]_0^2$$

$$= \ln(3)$$

☆ (8.) If $\{a_n\}$ is a seqⁿ of positive terms such that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = l$ then

(a) If $l < 1$ then $\lim_{n\to\infty} a_n = 0$

(b) If $l > 1$ then $\lim_{n\to\infty} a_n = \infty$

$\frac{a_{n+1}}{a_n} \approx L$

$\Rightarrow a_{n+1} \approx L a_n$

$\Rightarrow a_{n+2} \approx L^2 a_n$

$\Rightarrow a_{n+r} \approx L^r a_n$

(c) If $\{a_n\}$ converges to $a \neq 0$ or oscillates finitely or oscillates infinitely then either $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ does not exist or

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1.$$

☆ Some Standard seq^n:

① Geometric seq^n:

$$a_n = r^n \text{ , where } r \in \mathbb{R}$$

$$\{a_n\} = \{1, r, r^2, r^3, ---\}.$$

$a_n = r^n \rightarrow$ Converges to '0' , if $|r| < 1$;
Converges to '1' , if $r = 1$.
Diverges to ∞ , if $r > 1$.
oscillates finitely , if $r = -1$;
oscillates infinitely , if $r < -1$;

② If $a_n = \frac{P(n)}{Q(n)}$, where $P(n)$ & $Q(n)$ are polynomials in n;

then

$a_n \rightarrow 0$, if $\deg.(P(n)) < \deg(Q(n))$

$a_n \rightarrow \infty$, if $\deg.(P(n)) > \deg.(Q(n))$

$a_n \rightarrow \frac{P_0}{Q_0}$, if $\deg(P(n)) = \deg(Q(n))$

where P_0 & Q_0 are leading Coefficients of $P(x)$ & $Q(x)$ respectively.

E.g. ① $a_n = \dfrac{n^2-5n+6}{n^3+7n-8} \to 0$, is Convergent

② $a_n = \dfrac{n^{7/5}+2n^{8/9}+3n^{12/5}}{3n^{8/7}+2n^{9/4}+3n^{12/5}} = \left(\dfrac{3}{3} = 1\right)$ is Convergent

(3) Irrational Seqⁿs:-

(i) $\forall a>0 , \quad a^{\frac{1}{n}} \to 1$

(ii) $n^{1/n} \to 1$

(iii) $(n!)^{\frac{1}{n}} \to \infty$

(iv) $(n!)^{1/n^2} \to 1$

Solⁿ:- (iii) $a_n = n! \quad , a_{n+1} = (n+1)!$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{n!} = (n+1) = \infty \quad , \text{ as } n \to \infty$$

(iv) $\lim\limits_{n\to\infty} (n!)^{\frac{1}{n^2}} = \left[(n!)^{1/n}\right]^{1/n}$

$\lim\limits_{n\to\infty} a_n = \cancel{\lim\limits_{n\to\infty} (n!)^{\frac{1}{n}}}$

$= \cancel{\lim\limits_{n\to\infty} \dfrac{(n+1)!}{n!}}$

$= .$

$\underline{\lim}(n+1) \le \underline{\lim}(n!)^{\frac{1}{n}} \le \overline{\lim}(n!)^{\frac{1}{n}} \le \overline{\lim}(n+1)$

$\Rightarrow \underline{\lim}(n+1)^{\frac{1}{n}} \le \underline{\lim}(n!)^{\frac{1}{n^2}} \le \overline{\lim}(n!)^{\frac{1}{n^2}} \le \overline{\lim}(n+1)^{\frac{1}{n}}$

$\Rightarrow 1 \le \underline{\lim}(n!)^{\frac{1}{n^2}} \le \overline{\lim}(n!)^{\frac{1}{n^2}} \le 1$

$\Rightarrow \lim\limits_{n\to\infty} (n!)^{\frac{1}{n^2}} = 1$

Some exponential seqⁿ :-

$$\Rightarrow \left(1+\frac{a}{n}\right)^n \to e^a$$

$$\Rightarrow \left(1+\frac{a}{n}\right)^{bn} \to e^{ab} \quad \Bigg\} \text{ as } n\to\infty$$

(5.) Comparision Between different type of seqⁿ :-

$$\boxed{\log n \ll p(n) \ll a^n \ (a>1) \ll n! \ll n^n} \quad \text{as } n\to\infty$$

Eg :-

① $\lim\limits_{n\to\infty} \dfrac{2^n}{n!} = 0$

② $\lim\limits_{n\to\infty} \dfrac{(3n)^n}{(3n)!} = \infty$

③ $\lim\limits_{n\to\infty} \dfrac{n^2+2n+5}{2^n} = 0$

④ $\lim\limits_{n\to\infty} \dfrac{\log n^2}{n^3-3n+5} = 0$

(5.) $\lim\limits_{n\to\infty} \dfrac{\overset{\leftarrow p(n)}{n^{100}}}{\underset{\searrow a^n}{100^n}} = 0$

⑥ $\lim\limits_{n\to\infty} \dfrac{\overset{\nearrow a^n}{x^n}}{n!} = 0$

⑦ $\lim\limits_{n\to\infty} \dfrac{n!}{n^n} = 0$

$A_n = \max\{a_n, b_n\} \to \max\{a, b\}$

$B_n = \min\{a_n, b_n\} \to \min\{a, b\}$

⇒ If seqⁿ don't oscillate then $\underline{\lim} = \overline{\lim}$.

✡ Some theorems related to Convergence :-

① Cauchy 1st Theorem on limit :-

If $\{a_n\}$ Converges to 'a' then the seqn of arithmetic mean of $\{a_n\}$ also Converge to 'a'.

i.e. If $\lim_{n\to\infty} a_n = a$ then

$$\boxed{\lim_{n\to\infty} \frac{a_1+a_2+\cdots+a_n}{n} = a}$$

Ex>

$\{a_n\}$	$\left\{\frac{a_1+a_2+\cdots+a_n}{n}\right\}$
$\downarrow$	$\downarrow$
a_1	a_1
a_2	$\frac{a_1+a_2}{2}$
a_3	$\frac{a_1+a_2+a_3}{3}$
|	|
|	|
a_n	$\frac{a_1+a_2+\cdots+a_n}{n}$
|	|
|	|
$\downarrow$	$\downarrow$
a	a

⇒ But Converse of Cauchy 1st theorem on limit is not true.

$\Rightarrow$ If $a_n = (-1)^n$ then $\{a_n\}$ is not Convergent

But $\left\{\frac{a_1+a_2+\cdots+a_n}{n}\right\} = 0$, if n is even.

$-\frac{1}{n}$, if n is odd.

is Convergent.

☆ Corollary:-

If $\{a_n\}$ is a seq^n of positive terms converges to 'a' then the seq^n of geometric mean of $\{a_n\}$ is also Converges to 'a'.

i.e. If $\lim\limits_{n\to\infty} a_n = a$ then

$$\boxed{\lim_{n\to\infty} (a_1 \cdot a_2 \cdot a_3 \cdots\cdots a_n)^{1/n} = a}$$

Ex> $\lim\limits_{n\to\infty} a_n = a$

$$\Rightarrow \lim_{n\to\infty} \log a_n = \log a$$

$$\Rightarrow \lim_{n\to\infty} \frac{\log a_1 + \log a_2 + \cdots + \log a_n}{n} = \log a$$

$$\Rightarrow \lim_{n\to\infty} \frac{1}{n}\log(a_1 \cdot a_2 \cdot a_3 \cdots a_n) = \log a$$

$$\Rightarrow \lim_{n\to\infty} e^{\log(a_1 a_2 a_3 \cdots a_n)^{1/n}} = e^{\log a}$$

$$\Rightarrow \lim_{n\to\infty} (a_1 a_2 a_3 \cdots a_n)^{1/n} = a$$

② CeSaro Theorem :-

If $\{a_n\}$ and $\{b_n\}$ be two seq^n which Converge to a & b respectively.

i.e. $\lim\limits_{n\to\infty} a_n = a$ & $\lim\limits_{n\to\infty} b_n = b$

then $$\boxed{\lim_{n\to\infty} \frac{a_1b_n + a_2b_{n-1} + a_3b_{n-2} + \cdots + a_nb_1}{n} = ab}$$

=)

$\{a_n\}$	$\{b_n\}$	$\left\{\frac{a_1b_n+a_2b_{n-1}+\cdots+a_nb_1}{n}\right\}$
↓	↓	↓
a_1	b_1	a_1b_1
a_2	b_2	$\frac{a_1b_2+a_2b_1}{2}$
a_3	b_3	$\frac{a_1b_3+a_2b_2+a_3b_1}{3}$
⋮	⋮	⋮
a_n	b_n	$\frac{a_1b_n+a_2b_{n-1}+\cdots+a_nb_1}{n}$
⋮	⋮	⋮
↓	↓	↓
a	b	ab

2nd theorem on limit :-

If $\{a_n\}$ is a Seqⁿ of (+)ive terms then

$$\boxed{\underline{\lim} \frac{a_{n+1}}{a_n} \leq \underline{\lim} (a_n)^{1/n} \leq \overline{\lim} (a_n)^{1/n} \leq \overline{\lim} \frac{a_{n+1}}{a_n}}$$

So, If $\left\{\frac{a_{n+1}}{a_n}\right\}$ is Convergent then $\{(a_n)^{1/n}\}$ is Convergent.
(∵ $\underline{\lim} = \overline{\lim}$)

and

$$\boxed{\lim_{n\to\infty} (a_n)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}}$$

⇒ If $\left\{\frac{a_{n+1}}{a_n}\right\}$ diverges to ∞ then $\{(a_n)^{1/n}\}$ diverges to ∞.

However:

$$\boxed{\text{If } \lim_{n\to\infty} (a_n)^{1/n} \text{ exist } \not\Rightarrow \lim_{n\to\infty} \left\{\frac{a_{n+1}}{a_n}\right\}}$$

Q. ① $\lim_{n\to\infty} \frac{[(n+1)(n+2) \cdots (n+n)]^{1/n}}{n} = ?$

$$\Rightarrow \lim_{n\to\infty} \left[\frac{(n+1)(n+2) \cdots (n+n)}{n^n}\right]^{1/n}$$

$$\Rightarrow \lim_{n\to\infty} \left[\left(\frac{n+1}{n}\right)\left(\frac{n+2}{n}\right) \cdots \left(\frac{n+n}{n}\right)\right]^{1/n}$$

→ Don't apply Cauchy 1st theorem on limit. [∵ first term in term of n Don't a term of Constant].

→ So, Apply Cauchy 2nd thᵒ. on limit :-

Let $a_n = \frac{(n+1)(n+2) - - - (n+n)}{n^n}$

$a_{n+1} = \frac{(n+2)(n+3) - - - - - (2n+2)}{(n+1)^{n+1}}$

$\Rightarrow \frac{a_{n+1}}{a_n} = \frac{\cancel{(n+2)(n+3)} - - \cancel{(2n)} (2n+1) \overset{2}{\cancel{(2n+2)}}}{(n+1)^n (n+1) \cancel{(n+1)} \cancel{(n+2) - - - (2n)}} \times n^n$

$= \frac{2(2n+1)}{(n+1)\left(1+\frac{1}{n}\right)^n} = \frac{4+\frac{2}{n}}{\left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{n}\right)}$

$= \boxed{\frac{4}{e}}$ as $n \to \infty$

② $\lim_{n\to\infty} \frac{1+\frac{1}{2}+\frac{1}{3}+ - - +\frac{1}{n}}{n} = ?$

By Cauchy 1st tho. on limit:

Let $a_n = \frac{1}{n} \Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$

So, $\lim_{n\to\infty} \frac{1+\frac{1}{2}+\frac{1}{3}+ - - - +\frac{1}{n}}{n} = 0$

③ $\lim_{n\to\infty} \frac{1+2^{1/2}+3^{1/3}+ - - - +n^{1/n}}{n} = ?$

By Cauchy 1st tho. on limit.

Let $a_n = n^{1/n} \Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} n^{1/n} = \boxed{1}$

④ $\lim\limits_{n\to\infty}\left(1\cdot 2^{1/2}\cdot 3^{1/3}\cdot --- n^{1/n}\right)^{1/n} = ?$

By Corollary of Cauchy 1st Tho. on limit -

Let $a_n = n^{1/n} \Rightarrow \lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} n^{1/n} = \boxed{1}$

⑤ $\lim\limits_{n\to\infty}\dfrac{(n!)^{1/n}}{n} = ?$

By Cauchy 2nd tho. on limit :-

Let $a_n = \dfrac{n!}{n^n}$, $a_{n+1} = \dfrac{(n+1)!}{(n+1)^{n+1}}$

$$\frac{a_{n+1}}{a_n} = \frac{\cancel{(n+1)!}}{(n+1)^n\cdot\cancel{(n+1)}}\times\frac{n^n}{\cancel{n!}} = \frac{1}{\left(1+\frac{1}{n}\right)^n} = \boxed{\frac{1}{e}} \text{ as } n\to\infty$$

Q ⑥ $\lim\limits_{n\to\infty}\left\{\left(\frac{2}{1}\right)^1\left(\frac{3}{2}\right)^2\left(\frac{4}{3}\right)^3 --- \left(\frac{n+1}{n}\right)^n\right\}^{1/n}$

By Corrolary. of Cauchy 1st tho. on limit -:-

Let $a_n = \left(1+\frac{1}{n}\right)^n$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \boxed{e}$$

⑦ $\lim\limits_{n\to\infty}\left[\dfrac{(3n)!}{(n!)^3}\right]^{1/n}$

By Cauchy 2nd tho. on limite :-

Let

$a_n = \dfrac{(3n)!}{(n!)^3}$, $a_{n+1} = \dfrac{(3n+3)!}{[(n+1)!]^3}$

$$\frac{a_{n+1}}{a_n} = \frac{(3n+3)!}{((n+1)!)^3}\times\frac{(n!)^3}{(3n)!} = \frac{(3n+3)(3n+2)(3n+1)}{(n+1)^3} = \boxed{27}$$

⑧ $$a_n = \frac{[x]+[2x]+[3x]+ - - +[nx]}{n^2}$$

Solⁿ:-

$$x-1 < [x] \le x$$
$$2x-1 < [2x] \le 2x$$
$$3x-1 < [3x] \le 3x$$
$$\vdots$$
$$nx-1 < [nx] \le nx$$

$$+ \quad \frac{n(n+1)}{2}x - n < a_n \le \frac{n(n+1)}{2}x.$$

$$\Rightarrow \frac{1}{2}\left(1+\frac{1}{n}\right)x - \frac{1}{n} < \frac{a_n}{n^2} \le \frac{1}{2}\left(1+\frac{1}{n}\right)x$$

as $n\to\infty$

$$\boxed{\frac{x}{2} < \frac{a_n}{n^2} \le \frac{x}{2}}$$

⑨ $$a_n = \frac{[(m+1)(m+2) - - - (m+n)]^{1/n}}{n}$$

Let $$a_n = \left[\frac{(m+1)(m+2) - - - (m+n)}{n^n}\right]^{1/n}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n^{1/n} = \lim_{n\to\infty} \frac{b_{n+1}}{b_n}$$

$$= \lim_{n\to\infty} \frac{\cancel{(m+1)(m+2) - - - (m+n)}\,(m+n+1)\cdot n^n}{(n+1)^{n+1}\,\cancel{(m+1)(m+2) - - - (m+n)}}$$

$$= \lim_{n\to\infty} \frac{m+n+1}{(n+1)} \cdot \frac{1}{(1+\frac{1}{n})^n} = \boxed{\frac{1}{e}}$$

(10.) $a_n = \left[\frac{(4n)!}{(n!)^4}\right]^{1/n}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n^{1/n} = \lim_{n\to\infty} \frac{b_{n+1}}{b_n}$$

Let $b_n = \frac{(4n)!}{(n!)^4}$

$$b_{n+1} = \frac{(4n+4)!}{[(n+1)!]^4}$$

$$\frac{b_{n+1}}{b_n} = \frac{(4n+4)!}{[(n+1)!]^4} \times \frac{(n!)^4}{(4n)!}$$

$$= \frac{(4n+4)(4n+3)(4n+2)(4n+1)\times\cancel{(4n)!} \times \cancel{(n!)^4}}{(n+1)^4 \times \cancel{(n!)^4} \times \cancel{(4n)!}}$$

$$= \frac{(4n+4)(4n+3)(4n+2)(4n+1)}{(n+1)^4}$$

$$= \frac{256\cancel{n^4}\left(1+\frac{1}{n}\right)\left(1+\frac{3}{4n}\right)\left(1+\frac{1}{2n}\right)\left(1+\frac{1}{4n}\right)}{\cancel{n^4}\left(1+\frac{1}{n}\right)^4}$$

$$= \boxed{256} \quad \text{as } n\to\infty$$

☆ Recursive definition of a seqⁿ :-

The definition of a seqⁿ is said to be recursive.

If n^{th} term of the seqⁿ is a function of it's previous terms.

Eg. ① $a_n = a_{n-1} + a_{n-2}$, $\forall n \geq 2$.

② $a_n = a_{n-1}^2$, $\forall n \geq 1$

③ $a_{n+1} = \frac{a_n + a_{n-1}}{2}$, $\forall n \geq 1$

☆ The recursive definition of a seqⁿ contains two parts :-

(i) Initial values.

(ii) Recursive relation.

Eg.:- ① $\{a_n\}$ is a seqⁿ given by.

$a_{n+1} = \sqrt{3a_n}$, $a_1 = 1$, $\forall n \geq 1$

$a_1 = 1$, $a_2 = \sqrt{3}$, $a_3 = \sqrt{3\sqrt{3}}$, $a_4 = \sqrt{3\sqrt{3\sqrt{3}}}$, - - - -

② $a_1 = 1$, $a_{n+1} = 2^{a_n}$, $\forall n \geq 1$

$a_1 = 1$, $a_2 = 2$, $a_3 = 4$, $a_4 = 16$, - - -

Q. If $a_n = 2^m$ then $m = 2^{2^{2^{\cdot^{\cdot}}}}$ (n-1) time.

If $a_n = 2^{n+a_{n-1}}$, $a_1 = 1$

If $a_n = 2^m$ then $m = ?$

$a_1 = 1$, $a_2 = 2^2$, $a_3 = 2^7$, $a_4 = 2^{132}$, $a_5 = 2^{261}$, - - -

Q. $a_n = n + a_{n-1}$, $a_1 = 1$

$a_1 = 1$
$a_2 = 2+1$
$a_3 = 3+2+1$
$a_4 = 4+3+2+1$
$\vdots$

$a_n = n+(n-1)+(n-2)+ --- +2+1$

$= \boxed{\frac{n(n+1)}{2}}$

monotonic + bd → convergent

⁂ Monotone Convergence theorem :-

⇒ If a monotonic increasing seq^n is bounded above then it converges to it's supremum.

⇒ If a monotonic decreasing seq^n is bounded below then it converges to it's infimum.

Q. If $\{a_n\}$ is a seq^n s.t. $a_1 = 1$, $a_{n+1} = \sqrt{7a_n}$, $\forall n \geq 1$ then find the limit of a_n if it converges.

Sol^n:- $a_1 = 1$, $a_{n+1} = \sqrt{7a_n}$

$a_1 = 1$, $a_2 = \sqrt{7}$, $a_3 = \sqrt{7\sqrt{7}}$, $a_4 = \sqrt{7\sqrt{7\sqrt{7}}}$, - -

$a_1 < a_2 < a_3 < a_4$ - - -

Let $a_{K+1} > a_K$

$\Rightarrow 7a_{K+1} > 7a_K$

$\Rightarrow \sqrt{7a_{K+1}} > \sqrt{7a_K}$

$\Rightarrow \quad a_{K+2} > a_{K+1}$

Now, From P.M.I. $a_{n+1} > a_n \quad \forall n \in \mathbb{N}$

So, $\{a_n\}$ is monotonic increasing.

$a_1 = 1 < 7$

$a_2 = \sqrt{7} < 7$

Let $a_K < 7$

$7a_K < 49$

$\sqrt{7a_K} < 7$

So, By P.M.I., $a_n < 7, \forall n \in \mathbb{N}$

So, $\{a_n\}$ is bounded above by 7.

So, By Monotone Convergence theorem $\{a_n\}$ is Convergent.

Now, Let $\lim\limits_{n\to\infty} a_n = l$

$\because a_{n+1} = \sqrt{7a_n}$

$\Rightarrow \lim\limits_{n\to\infty} a_{n+1} = \lim\limits_{n\to\infty} \sqrt{7a_n} \Rightarrow l = \sqrt{7l} \Rightarrow l^2 = 7l$

So, $l = 0, 7$, But $L \neq 0$

So, $\boxed{l = 7}$

= $a_1 = 0$, $a_{n+1} = \sqrt{7+a_n}$, $\forall n \geq 1$

$a_1 = 0$, $a_2 = \sqrt{7}$, $a_3 = \sqrt{7+\sqrt{7}}$, - - -

$a_1 < a_2$

Let $a_k < a_{k+1}$

$7 + a_k < 7 + a_{k+1}$

$\sqrt{7+a_k} < \sqrt{7+a_{k+1}}$

$a_{k+1} < a_{k+2}$

So, $\{a_n\}$ is monotonic increasing.

Now,

Let $a_k < \frac{1+\sqrt{29}}{2}$

$7 + a_k < \cancel{7+} \frac{15+\sqrt{29}}{2}$

$7 + a_k < \frac{30+2\sqrt{29}}{4}$

$7 + a_k < \left(\frac{1+\sqrt{29}}{2}\right)^2$

$\sqrt{7+a_k} < \frac{1+\sqrt{29}}{2}$

So, $\{a_n\}$ is bounded above by $\frac{1+\sqrt{29}}{2}$.

Now, $\lim_{n\to\infty} a_n = l$

$a_{n+1} = \sqrt{7+a_n}$

$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{7+a_n}$

$l = \sqrt{7+l}$

$l^2 - l - 7 = 0$

$l = \frac{1 \pm \sqrt{1+28}}{2}$

$l = \frac{1 \pm \sqrt{29}}{2}$

$\boxed{l = \frac{1+\sqrt{29}}{2}}$

So, $\{a_n\}$ is convergent.

Q. $a_{n+1} = 2 - \frac{1}{a_n}$, $a_1 = 3/2$, $\forall n \geq 1$

Solⁿ:- $a_1 = \frac{3}{2}$, $a_2 = \frac{4}{3}$, $a_3 = \frac{5}{4}$, $a_4 = \frac{6}{5}$, - -

$a_2 < a_1$

Let $a_{k+1} < a_k$

$$\frac{1}{a_k} < \frac{1}{a_{k+1}}$$

$$2-\frac{1}{a_k} > 2-\frac{1}{a_{k+1}}$$

$$a_{k+1} > a_{k+2}$$

So, $\{a_n\}$ is monotonic decreasing.

Now, $a_k > 1$

$$\frac{1}{a_k} < 1$$

$$2-\frac{1}{a_k} > 1$$

$$a_{k+1} > 1$$

So, $\{a_n\}$ is bounded below by 1.

$\because \lim_{n\to\infty} a_n = l$

$$a_{n+1} = 2-\frac{1}{a_n}$$

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty}\left(2-\frac{1}{a_n}\right) \Rightarrow l = 2-\frac{1}{l} \Rightarrow l^2-2l+1=0$$

$$\Rightarrow \boxed{l=1}$$

Q. $a_1 = 1$, $a_{n+1} = \frac{4+3a_n}{3+2a_n}$, $\forall n \geq 1$.

$a_1 = 1$, $a_2 = \frac{7}{5}$, $a_3 = \frac{41}{9}$, — —

$a_2 > a_1$

Let $a_{k+1} > a_k$

$$\frac{4+3a_n}{3+2a_n} = \frac{3}{2} - \frac{1}{2(3+2a_n)}$$

→ $\{a_n\}$ is monotonic increasing.

$$a_{n+1} = \frac{4+3a_n}{3+2a_n} = \frac{3}{2} - \frac{1}{2(3+2a_n)}$$

$$\Rightarrow \frac{1}{2(3+2a_n)} = \frac{3}{2} - a_{n+1}$$

$\because \frac{1}{2(3+2a_n)} > 0$

$\therefore a_{n+1} < \frac{3}{2}$ $\Rightarrow$ $\{a_n\}$ is bounded above by $\frac{3}{2}$.

$\Rightarrow \{a_n\} \to$ Convergent

Let $\lim_{n\to\infty} a_n = l$

$$a_{n+1} = \frac{4+3a_n}{3+2a_n}$$

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{4+3l}{3+2l}$$

$$\Rightarrow l = \frac{4+3l}{3+2l}$$

$$\Rightarrow 2l^2 + \cancel{3l} = 4 + \cancel{3l}$$

$$l = \pm\sqrt{2}$$

$$\therefore \boxed{l = \sqrt{2}}$$

Q. Let $\{x_n\}$ and $\{y_n\}$ be two seq^n s.t.

$$x_{n+1} = \frac{x_n + y_n}{2} \quad \text{and} \quad y_{n+1} = \sqrt{x_n y_n} \quad \text{for } n \geq 1, \; x_1 > 0, \; y_1 > 0$$

then.

① $\{x_n\}$ & $\{y_n\}$ both are divergent.

② $\{x_n\}$ is Convergent and $\{y_n\}$ is Divergent.

③ $\{x_n\}$ & $\{y_n\}$ both are Convergent But Converge to different limits.

✓④ $\{x_n\}$ & $\{y_n\}$ Both are Convergent But Converge to Same limit.

$$\boxed{A.M. \geq G.M. \geq H.M.}$$

Sol^n:- If $x_1 = y_1$

$$x_2 = \frac{x_1 + y_1}{2} = x_1 \quad , \quad y_2 = \sqrt{y_1 \cdot x_1} = y_1$$

then $\{x_n\}$ & $\{y_n\}$ are Constant seq^n and Converge to Same number x_1.

If $x_1 > y_1$ then

$$x_2 = \frac{x_1+y_1}{2} > \sqrt{x_1y_1} = y_2$$

Let $x_k > y_k$ then $x_{k+1} = \frac{x_k+y_k}{2} \geq \sqrt{x_ky_k} = y_{k+1}$

So, By P.M.I. :-

$$x_n > y_n, \forall n \in \mathbb{N}$$

Now,

$$x_2 = \frac{x_1+y_1}{2} < \frac{x_1+x_1}{2} = x_1$$

IIIy

$$x_3 = \frac{x_2+y_2}{2} < \frac{x_2+x_2}{2} = x_2$$

So, $x_{n+1} = \frac{x_n+y_n}{2} < \frac{x_n+x_n}{2} = x_n, \forall n \in \mathbb{N}$

$\Rightarrow \{x_n\}$ is a monotonic decreasing seq^n.

Now,

$$y_2 = \sqrt{x_1y_1} > \sqrt{y_1y_1} = y_1$$

$$y_3 = \sqrt{x_2y_2} > \sqrt{y_2y_2} = y_2$$

$$\vdots$$

$$y_{n+1} = \sqrt{x_ny_n} > \sqrt{y_ny_n} = y_n$$

$\{y_n\}$ is monotonic increasing seq^n.

then the seq^n will be like,

$$x_1 > x_2 > x_3 > \text{---} \dots y_4 > y_3 > y_2 > y_1$$

So, $\{x_n\}$ is a monotonic decreasing seq^n which is bounded below by y_1, So, $\{x_n\}$ is Convergent

IIIy $\{y_n\}$ is a monotonic increasing seq^n which is bounded above by x_1, So, $\{y_n\}$ is Convergent seq^n.

$x_K > y_K$ then $x_{K+1} = \frac{x_K + y_K}{2} > \sqrt{x_K y_K} = y_{K+1}$

Let $\lim_{n\to\infty} x_n = l$ & $\lim_{n\to\infty} y_n = m$

Now, $\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \frac{x_n + y_n}{2}$

$$l = \frac{l+m}{2}$$

$$\Rightarrow \boxed{l = m}$$

Q. Let $\{a_n\}$ be a seqⁿ s.t. $a_1 > 0, a_2 > 0, a_1 < a_2$,

$$a_{n+1} = \frac{a_n + a_{n-1}}{2}, \quad \forall n \geq 2.$$

① $\{a_n\}$ is divergent

② $\{a_n\}$ converges to $\frac{a_1 + a_2}{2}$

③ $\{a_n\}$ Converges to $\frac{2a_1 + a_2}{3}$

④ $\{a_n\}$ Converges to $\frac{a_1 + 2a_2}{3}$.

Solⁿ:- $a_1 < a_2$

$$\frac{a_1 + a_1}{2} < \frac{a_1 + a_2}{2} < \frac{a_2 + a_2}{2}$$

$$\Rightarrow a_1 < \frac{a_1 + a_2}{2} < a_2$$

$$\Rightarrow a_1 < a_3 < a_2$$

IIIy $a_3 < a_2$

$$\Rightarrow a_3 < \frac{a_2 + a_3}{2} < a_2$$

$$\Rightarrow a_3 < a_4 < a_2$$

Now, $a_3 < a_4$

$$\Rightarrow a_3 < \frac{a_3+a_4}{2} < a_4$$

$$\Rightarrow a_3 < a_5 < a_4$$

So, the seqⁿ $\{a_n\}$ can be expressed as

$$a_1 < a_3 < a_5 < a_7 < \text{- - - - - -} < a_6 < a_4 < a_2.$$

and hence $\{a_{2n+1}\}$ is monotonic increasing seqⁿ which is bounded above a_2.

So, $\{a_{2n+1}\}$ is convergent.

IIIy $\{a_{2n}\}$ is a monotonic decreasing seqⁿ which is bounded below by a_1, so, $\{a_{2n}\}$ is convergent.

Let $\{a_{2n}\}$ converges to l_1 and

$\{a_{2n+1}\}$ converges to l_2

then,

$$a_{2n+2} = \frac{a_{2n+1} + a_{2n}}{2}$$

$$\Rightarrow \lim_{n\to\infty} a_{2n+2} = \lim_{n\to\infty} \frac{a_{2n+1} + a_{2n}}{2}$$

$$\Rightarrow l_1 = \frac{l_2 + l_1}{2} \Rightarrow \boxed{l_1 = l_2}$$

$\therefore$ $\{a_n\}$ is convergent.

$$\cancel{2a_3} = a_1 + a_2$$
$$\cancel{2a_4} = a_2 + \cancel{a_3}$$
$$\cancel{2a_5} = \cancel{a_3 + a_4}$$
$$\vdots$$
$$\cancel{2a_{n-2}} = \cancel{a_{n-3}} + a_{n-4}$$
$$\cancel{2a_{n-1}} = a_{n-2} + \cancel{a_{n-3}}$$
$$2\,a_n = \cancel{a_{n-1}} + a_{n-2}$$

$$2a_n + a_{n-1} = a_1 + 2a_2$$

$$\Rightarrow \lim_{n\to\infty} (2a_n + a_{n-1}) = a_1 + 2a_2$$

$$\Rightarrow 3l = a_1 + 2a_2$$

$$\boxed{l = \frac{a_1 + 2a_2}{3}} \text{ Ans.}$$

Q. $0 < a_1 < a_2$ and $a_{n+2} = \sqrt{a_{n+1} a_n}$

Solⁿ:- Let $u_n = \log a_n$

then $u_1 < u_2$ ($\because a_1 < a_2 \Rightarrow \log a_1 < \log a_2$)

and $u_{n+2} = \frac{u_n + u_{n+1}}{2}$

(as, $a_{n+2} = \sqrt{a_{n+1} \cdot a_n}$)

$$\Rightarrow \log a_{n+2} = \log (a_n a_{n+1})^{1/2} = \frac{1}{2}(\log a_n + \log a_{n+1})$$

Since, $\{u_n\}$ Converges to $\frac{u_1 + 2u_2}{3}$

So, $\{\log a_n\}$ Converges to $\frac{\log a_1 + 2\log a_2}{3}$

$$= \frac{1}{3} \log a_1 a_2^2 \Rightarrow \log (a_1 a_2^2)^{1/3}$$

So, $\{a_n\}$ Converges to $(a_1 a_2^2)^{1/3}$

Q $0 < a_1 < a_2$ and $a_{n+2} = \frac{2a_{n+1} + a_n}{3}$, $n \geq 1$ then $\{a_n\}$ Converges to ..... ?

Solⁿ:-

$$3a_3 = 2a_2 + a_1$$
$$3a_4 = 2a_3 + a_2$$
$$3a_5 = 2a_4 + a_3$$
$$\vdots$$
$$3a_{n-2} = 2a_{n-3} + a_{n-4}$$
$$3a_{n-1} = 2a_{n-2} + a_{n-3}$$
$$3a_n = 2a_{n-1} + a_{n-2}$$

$$3a_n + a_{n-1} = a_1 + 3a_2$$

Let $\lim_{n\to\infty} a_n = l$

$\Rightarrow$ $3l + l = 3a_2 + a_1$

$\Rightarrow$ $\boxed{l = \frac{a_1 + 3a_2}{4}}$ Ans

☆ $\Rightarrow$ If $\{x_n\}$ & $\{y_n\}$ be two seqⁿ s.t.

$\boxed{x_{n+1} = \sqrt{x_n y_n}}$, $\boxed{\frac{2}{y_{n+1}} = \frac{1}{x_n} + \frac{1}{y_n}}$

$x_1 > 0$, $y_1 > 0$ then

Both $\{x_n\}$ & $\{y_n\}$ Converge to the Same limit.

$\Rightarrow$ If $\{x_n\}$ & $\{y_n\}$ be two seqⁿ s.t.

$\boxed{x_{n+1} = \frac{x_n + y_n}{2}}$ & $\boxed{\frac{2}{y_{n+1}} = \frac{1}{x_n} + \frac{1}{y_n}}$, $x_1 > 0$, $y_1 > 0$ then

Both $\{x_n\}$ & $\{y_n\}$ Converge to Same limit $\sqrt{x_1 y_1}$

$$\Rightarrow \quad a_{n+2} = \frac{a_n + a_{n-1}}{2} \longrightarrow \underline{\underline{\frac{2a_2 + a_1}{3}}}$$

$$\Rightarrow \quad a_{n+2} = \sqrt{a_n \cdot a_{n+1}} \longrightarrow \underline{\underline{(a_1 a_2^2)^{1/3}}}$$

$$\Rightarrow \quad \frac{2}{a_{n+2}} = \frac{1}{a_{n+1}} + \frac{1}{a_n} \longrightarrow \underline{\underline{\frac{3}{\left(\frac{1}{u_1} + \frac{2}{u_2}\right)}}}$$

proof Let $u_n = \frac{1}{a_n}$

$$2u_{n+2} = u_{n+1} + u_n$$

$$\Rightarrow u_{n+2} = \frac{u_{n+1} + u_n}{2}$$

$$u_n \longrightarrow \frac{u_1 + 2u_2}{3} \Rightarrow \frac{1}{u_n} \longrightarrow \frac{3}{u_1 + 2u_2}$$

$$\Rightarrow a_n \longrightarrow \underline{\underline{\frac{3}{\left(\frac{1}{u_1} + \frac{2}{u_2}\right)}}}$$

☆ ⇒ If $\{a_n\}$ is a seqⁿ s.t.

$$\boxed{\frac{2}{a_{n+2}} = \frac{1}{a_{n+1}} + \frac{1}{a_n}}, \quad n \geq 1. \quad , 0 < a_1 < a_2.$$

then

$\{a_n\}$ Converges to $\underline{\underline{\boxed{\frac{3}{\left(\frac{1}{a_1} + \frac{2}{a_2}\right)}}}}$

☆ **Newton repson method:** If α is a root of $f(x) = 0$ and x_1 is properly chosen, then a seqⁿ $\{a_n\}$ satisfying.

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad , n \geq 1}$$

Converges to α.

⇒ Recursive relation for p^{th} root of a positive number :-

Let $x^p = a$

$\Rightarrow x^p - a = 0$

Let $f(x) = x^p - a$

$f'(x) = px^{p-1}$

So, $x_{n+1} = x_n - \frac{x_n^p - a}{px_n^{p-1}}$

$$\boxed{x_{n+1} = \frac{(p-1)\,x_n^p + a}{p\,x_n^{p-1}}}$$

then $\{x_n\}$ Converges to $\boxed{a^{1/p}}$ for a Suitably chosen x_1.

⇒ If $p = 2$ then

$$x_{n+1} = \frac{x_n^2 + a}{2x_n} = \boxed{\frac{1}{2}\left(x_n + \frac{a}{x_n}\right)}$$ → form

then $\{x_n\}$ Converges to $\boxed{\sqrt{a}}$.

⇒ If $p = -1$ then

$$x_{n+1} = \frac{-2x_n^{-1} + a}{-x_n^{-2}} = x_n^2\left(\frac{2}{x_n} - a\right)$$

$$= \boxed{x_n(2 - ax_n)}$$

then $\{x_n\}$ Converges to $\boxed{\frac{1}{a}}$.

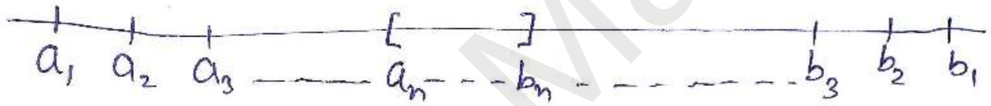
→ (एक के अन्दर दूसरा)

✫ Nested interval theorem :-

Let $[a_n, b_n]$ be seq^n of closed and bounded intervals s.t. $a_n < b_n$, $\forall n \in \mathbb{N}$ and

$$[a_1, b_1] \supseteq [a_2, b_2] \supseteq [a_3, b_3] \supseteq \cdots \text{ then}$$

$$\boxed{\bigcap_{n=1}^{\infty} [a_n, b_n] = \lim_{n\to\infty} [a_n, b_n]} \text{ is non-empty set.}$$



In this seq^n of nested intervals the seq^n of leftend points $\{a_n\}$ is monotonic increasing and bounded ~~bove~~ above by b_1.

So, $\{a_n\}$ is Convergent.

Ill^y $\{b_n\}$ is monotonic decreasing and bounded below by a_1.

So, $\{b_n\}$ is Convergent.

Let $\lim\limits_{n\to\infty} a_n = a$ and $\lim\limits_{n\to\infty} b_n = b$

$\because$ $a_n < b_n$ $\forall n \in \mathbb{N}$

So, $\lim\limits_{n\to\infty} a_n \leq \lim\limits_{n\to\infty} b_n \Rightarrow \boxed{a \leq b}$

So, $\bigcap_{n=1}^{\infty} [a_n, b_n] = \lim_{n \to \infty} [a_n, b_n] = [a, b]$.

→ If $a = b$ then $\bigcap_{n=1}^{\infty} [a_n, b_n]$ is a Singleton set.

→ If $a < b$ then $\bigcap_{n=1}^{\infty} [a_n, b_n]$ is a Closed interval $[a, b]$.

☆ If every Subseqⁿ of a seqⁿ $\{a_n\}$ has a Convergent Subseqⁿ then $\{a_n\}$ is bounded.

proof:- If possible, Let $\{a_n\}$ is unbounded.

$\exists$ a Subseqⁿ $\{a_{n_k}\}$ of $\{a_n\}$ which diverges to ∞ or $-\infty$.

$\Rightarrow \{a_{n_k}\}$ is a Subseqⁿ of $\{a_n\}$ which has no Convergent Subseqⁿ.

But every Subseqⁿ of $\{a_n\}$ has Convergent Subseqⁿ.

So, By Contradiction, $\{a_n\}$ is bounded.

~~Q Test the Convergence of the following~~ Series :-

Series

Series

Defn.:-

Let $\{a_n\}$ be a seqn of real no. then
$$A_n := a_1 + a_2 + --- + a_n = \sum_{k=1}^{n} a_k.$$
is called the seqn of partial sums of $\{a_n\}$. The ordered pair $(\{a_n\}, \{A_n\})$ is called an infinite series.

$\{a_n\}$ is called the seqn of terms of the series and $\{A_n\}$ is called the seqn of partial sum of the series.

Informly, $(\{a_n\}, \{A_n\})$ is denoted by $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} A_n$.

Eg.

$$\begin{array}{ll}
\{a_n\} & \{A_n\}. \\
\downarrow & \downarrow \\
a_1 & A_1 = a_1 \\
a_2 & A_2 = a_1 + a_2 \\
a_3 & A_3 = a_1 + a_2 + a_3 \\
\vdots & \vdots \\
a_n & A_n = a_1 + a_2 + --+ a_n \\
\vdots & \downarrow \\
 & \sum_{n=1}^{\infty} a_n.
\end{array}$$

The nature of a series is associated with the nature of seqn of partial sums, that is.

(i) **Bounded Series :-**

A Series $\sum_{n=1}^{\infty} a_n$ is Said to be bounded.

If the seqⁿ of partial Sum $\{A_n\}$ is bounded.

(ii) **Monotonic Series :-**

If it's seqⁿ of partial Sum $\{A_n\}$ is monotonic.

E.g. $\{a_n\} = \{1, 2, 1, 2, 1, 2, ---\}$

$\{A_n\} = \{1, 3, 4, 6, 7, 9, 10, ---\}$.

(iii) **Convergent Series :-**

If it's seqⁿ of partial Sum $\{A_n\}$ is Convergent.

* ⇒ The limit of $\{A_n\}$ is Called the Sum of the series $\sum_{n=1}^{\infty} a_n$.

E.g. $a_n = (-1)^n = \{-1, 1, -1, 1, -1, 1, ---\}$

$\sum a_n = ?$

$A_n = \{-1, 0, -1, 0, -1, ---\}$ ← oscillates finitely Between 0 & -1.

So, $\sum a_n$ is oscillates finitely between 0 & -1.

(iv) **Divergent Series :-**

If it's Seqⁿ of partial Sum $\{A_n\}$ is divergent.

⇒ A series of positive terms (i.e. $a_n > 0, \forall n \in N$) is always monotonic increasing. So, it is Convergent iff it is bounded.

as, $\sum_{n=1}^{\infty} a_n$ is monotonic iff $\{A_n\}$ is monotonic.

Now,
$$A_{n+1} - A_n = \sum_{r=1}^{n+1} a_r - \sum_{r=1}^{n} a_r = a_{n+1} > 0$$

$\therefore$ $\{A_n\}$ is monotonic increasing.

☆ Some basic properties of a series :-

① The nature of a series is unaltered (unchange) by inserting, deleting or replacing finitely many terms of the seqⁿ of terms of the series.

However, The sum of the series may be changed.

Eg:- $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_n - a_1$

$\sum_{n=1}^{\infty} a_n = a_0 + \sum_{n=1}^{\infty} a_n$.

② The Converges to a series remains unchanged if each of it's terms in multiplied by a non-zero Constant.

i.e. $\sum_{n=1}^{\infty} Ka_n = K \sum_{n=1}^{\infty} a_n$.

☆ ③ The Sum and difference of two Convergent Series is also Convergent.

Eg.

$\sum_{n=1}^{\infty} a_n = A$ & $\sum_{n=1}^{\infty} b_n = B$ then

$$\boxed{\sum_{n=1}^{\infty} (a_n \pm b_n) = A \pm B.}$$

④ Product of two Convergent Series need not be Convergent.

Eg.

$A_n = \sum \frac{(-1)^n}{n^{1/2}}$ ← Convergent & $B_n = \sum \frac{(-1)^n}{n^{1/3}}$ ← Convergent

$A_n B_n = \sum \frac{1}{n^{5/6}}$ ← divergent.

☆ A Necessary Condition for Convergence of a series :-

If a series $\sum_{n=1}^{\infty} a_n$ is Convergent then ⇒ $\boxed{\lim_{n\to\infty} a_n = 0.}$

So, → If $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n=1}^{\infty} a_n$ is not Convergent.

→ If $\lim_{n\to\infty} a_n = 0$ then $\sum_{n=1}^{\infty} a_n$ may Converge or Diverge.

⇒ p is necessary for q.

i.e. $Q \Rightarrow P$

Q	P	Q → P	~P	~Q	~P → ~Q
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Contraposition

$$\boxed{P \Rightarrow Q \Leftrightarrow \sim Q \Rightarrow \sim P}$$

Proof:

If $\sum_{n=1}^{\infty} a_n$ is Convergent.

$\Rightarrow \{A_n\}$ is Convergent.

$\Rightarrow \{A_n\}$ is a Cauchy seqⁿ.

$\Rightarrow \forall \epsilon > 0, \exists n_0 \in \mathbb{N}$ s.t. $|A_{n+1} - A_n| < \epsilon, \forall n > n_0$.

$\Rightarrow \forall \epsilon > 0, \exists n_0 \in \mathbb{N}$ s.t. $|a_{n+1}| < \epsilon, \forall n > n_0$.

$\Rightarrow \forall \epsilon > 0, \exists n_0 \in \mathbb{N}$ s.t. $|a_{n+1} - 0| < \epsilon, \forall n > n_0$

$\Rightarrow \lim_{n \to \infty} a_n = \underline{\underline{0}}$.

E.g. ① $\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^n$ is not Convergent. ~~as~~

as, $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e \neq \underline{0}$.

② $\sum_{n=1}^{\infty} \sin n$. is not Convergent.

as, $\lim_{n \to \infty} \sin(n) = \underline{\underline{\text{D.N.E.}}}$

$Q \nRightarrow P$, So, $\sim P \nRightarrow \sim Q$, But $P \nRightarrow Q$ T or F.

☆ Cauchy Criterion for Convergence of a Series :-

A series $\sum_{n=1}^{\infty} a_n$ is convergent iff.

$\forall\, \epsilon > 0$, $\exists\, n_0 \in \mathbb{N}$ s.t. $|a_{n+1} + a_{n+2} + \cdots + a_{n+p}| < \epsilon, \forall n > n_0, p \geq 1$

proof:- If $\sum_{n=1}^{\infty} a_n$ is convergent.

$\Rightarrow \{A_n\}$ is convergent.

$\Rightarrow \{A_n\}$ is Cauchy seq^n.

$\Rightarrow \forall\, \epsilon > 0, \exists\, n_0 \in \mathbb{N}$ s.t. $|A_{n+p} - A_n| < \epsilon, \forall n > n_0, p \geq 1.$

$\Rightarrow \forall\, \epsilon > 0, \exists\, n_0 \in \mathbb{N}$ s.t. $|a_{n+1} + a_{n+2} + \cdots + a_{n+p}| < \epsilon, \forall n > n_0, p \geq 1.$

☆ A series $\sum_{n=1}^{\infty} a_n$ is convergent iff $\lim_{m \to \infty} \sum_{r=m}^{\infty} a_r = 0.$

☆ Some standard series :-

① Geometric series :-

$$a_n = r^n$$

$$\therefore \sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} r^n = (1 + r + r^2 + r^3 + \cdots)$$

The seq^n of partial sum $A_n = \sum_{k=0}^{n} a_k$.

$$\Rightarrow A_n = \sum_{k=0}^{\infty} r^k = 1 + r + r^2 + \cdots + r^n$$

$$= \frac{1 \cdot (1 - r^{n+1})}{1 - r}, \text{ if } r \neq 1.$$

$$= (n+1) r \quad , \text{ if } r = 1.$$

$$\lim_{n\to\infty} A_n = \frac{1 - \lim_{n\to\infty} r^{n+1}}{1-r}, \quad \text{if } r \neq 1.$$

$$\infty, \quad \text{if } r = 1.$$

$$= \frac{1-0}{1-r}, \quad \text{if } |r| < 1.$$

$$\infty, \quad \text{if } r = 1$$

$$\infty, \quad \text{if } r > 1.$$

Oscillates finitely, if $r = -1$.

" infinite, if $r < -1$.

$\therefore \sum_{n=0}^{\infty} r^n$ is Convergent iff $|r| < 1$, and it's sum is $\frac{1}{1-r}$.

② Hyperharmonic series :-

$a_n = \frac{1}{n^p}$, $p \in \mathbb{R}$.

$$\therefore \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \text{---}$$

The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is Convergent, if $p > 1$.

Divergent, if $p \leq 1$

$\Rightarrow$ If $p = 0$ then $\sum \frac{1}{n^p} = \sum \frac{1}{n^0} = \sum \frac{1}{1} = \sum 1 \leftarrow$ divergent

③ Power Series :-

$\sum_{n=0}^{\infty} a_n x^n$, $a_n \in \mathbb{R}$ A series of the form is called a real power series.

$$\sum_{n=0}^{\infty} a_n . x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots$$

(i) $\Rightarrow$ $$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$ $\rightarrow a_n = \frac{1}{n!}$

↳ ~~e^{-x} → put $-x$ at e^{x}~~

put $x = 1$, to get

$$\boxed{1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots = e}$$

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n!} = e - 1}$$

$$\boxed{1 + \frac{1}{2!} + \frac{1}{3!} + \cdots = e - 1}$$

$$\boxed{\sum_{n=0}^{\infty} \frac{1}{n!} = e}$$

put $x = -1$, to get.

$$\boxed{\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \cdots = e^{-1} = \sum_{n=2}^{\infty} \frac{(-1)^n}{n!}}$$

put $x = a$, to get

$$\boxed{\sum_{n=0}^{\infty} \frac{a^n}{n!} = e^a}$$

$\Rightarrow$ $$\boxed{e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots}$$

$\Rightarrow$ $$\boxed{\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots}$$

$\Rightarrow$ $$\boxed{\frac{e^x - e^{-x}}{2} = \left[x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right]}$$

(ii) $$\boxed{\log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cdot x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ---}$$

for $\left(-1 < x \leq 1\right)$

$$\Rightarrow \log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - ......$$

$$\Rightarrow \frac{\log(1+x) + \log(1-x)}{2} = -\left[\frac{x^2}{2} + \frac{x^4}{4} + \frac{x^6}{6} + ---\right]$$

$$\Rightarrow \frac{\log(1+x) - \log(1-x)}{2} = x + \frac{x^3}{3} + \frac{x^5}{5} + ---$$

→ put $x = 1$ in $\ln(1+x)$, to get

$$= \boxed{1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + --- = \ln(1+1) = \ln 2 = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}}$$

(iii) $$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n \, x^{2n+1}}{(2n+1)!}, \ \forall x \in \mathbb{R} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ---$$

(iv) $$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ---$$

(v) $$\tan^{-1}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ---$$

(vi) $$(1-x)^{-n} = 1 + \sum_{r=1}^{\infty} \frac{n(n+1)(n+2) \cdots (n+r-1)}{r!} x^r.$$

$$= 1 + nx + \frac{n(n+1)}{2!} x^2 + \frac{n(n+1)(n+2)}{3!} x^3 + --- + \frac{n(n+1)(n+2) + -- + (n+r-1)}{r!} x^r$$

(vii) $$(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{2!} x^2 - \frac{n(n+1)(n+2)}{3!} x^3 + ---$$

(viii) $$(1-x)^n = 1 - nx + \frac{n(n-1)}{2!} x^2 - \frac{n(n-1)(n-2)}{3!} x^3 + ---$$

(ix) $$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 + ---$$

Q ① $1 + \frac{1+2}{2!} + \frac{1+2+3}{3!} + \frac{1+2+3+4}{4!} + — —$

$$a_n = \frac{1+2+3+ ---+n}{n!}$$

$\left[\because 1+2+3+ -+n = \frac{n(n+1)}{2}\right]$

$$= \frac{n(n+1)}{2 \cdot n!} = \frac{n+1}{2(n-1)!} = \frac{m+2}{2 \cdot m!} \quad \text{Let } n-1 = m$$

$$= \frac{1}{2}\sum \frac{m}{m!} + \sum \frac{1}{m!}$$

$$= \frac{1}{2}\sum \frac{1}{(m-1)!} + \sum \frac{1}{m!} \quad \text{Let } m-1 = p$$

$$= \frac{1}{2}\sum \frac{1}{p!} + \sum \frac{1}{m!}$$

$$= \frac{1}{2}e + e = \boxed{\frac{3e}{2}}$$

② $\sum_{n=2}^{\infty} \frac{(-1)^n}{n^2+n-2}$

$$a_n = \frac{(-1)^n}{n^2+n-2} = \frac{(-1)^n}{(n+2)(n-1)}$$

$$= \sum_{n=2}^{\infty} \frac{(-1)^n}{3}\left[\frac{1}{n-1} - \frac{1}{n+2}\right]$$

$$= \frac{1}{3}\left[\left(1-\frac{1}{4}\right) - \left(\frac{1}{2}-\frac{1}{5}\right) + \left(\frac{1}{3}-\frac{1}{6}\right) - \left(\frac{1}{4}-\frac{1}{7}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n+2}\right)\right.$$

$$= \frac{1}{3}\left[\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+-—\right) + \left(-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7} - \cdots\right)\right]$$

$$= \frac{1}{3}\left[2\ln 2 - \frac{5}{6}\right]$$

$$= \boxed{\frac{2}{3}\ln 2 - \frac{5}{18}}$$

Q. Find the sum of the series:

$$\frac{1}{2\pi}\left[\frac{\pi}{1!\cdot 3} - \frac{\pi^3}{3!\cdot 5} + \frac{\pi^5}{5!\cdot 7} - \cdots\right]$$

$$= \frac{1}{2\pi}\left[\frac{2\pi}{3!} - \frac{4\pi^3}{5!} + \frac{6\pi^5}{7!} - \cdots\right]$$

$$= \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\,2n\cdot\pi^{2n+1}}{(2n+1)!}$$

$$= \frac{1}{2\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(2n+1-1)\,\pi^{2n+1}}{(2n+1)!}$$

$$= \frac{1}{2\pi}\left[\frac{(-1)^{n-1}}{(2n)!}\pi^{2n+1} - \frac{(-1)^{n-1}}{(2n+1)!}\pi^{2n+1}\right]$$

$$= \frac{1}{2\pi}\left[\left(\frac{\pi^3}{2!} - \frac{\pi^5}{4!} + \frac{\pi^7}{6!} - \cdots\right) - \left(\frac{\pi^3}{3!} - \frac{\pi^5}{5!} + \frac{\pi^7}{7!} - \cdots\right)\right]$$

$$= \frac{1}{2}\left(\frac{\pi^2}{2!} - \frac{\pi^4}{4!} + \frac{\pi^6}{6!} - \cdots\right) - \frac{1}{2\pi}\left(\frac{\pi^3}{3!} - \frac{\pi^5}{5!} + \frac{\pi^7}{7!} - \cdots\right)$$

$$= -\frac{1}{2}\left(1 - \frac{\pi^2}{2!} + \frac{\pi^4}{4!} - \frac{\pi^6}{6!} + \cdots - 1\right) + \frac{1}{2\pi}\left(\pi - \frac{\pi^3}{3!} + \frac{\pi^5}{5!} - \frac{\pi^7}{7!} + \cdots - \pi\right)$$

$$= -\frac{1}{2}\cos(\pi) + \cancel{\frac{1}{2}} + \frac{1}{2\pi}\sin\pi - \cancel{\frac{1}{2}}$$

$$= -\frac{1}{2}(-1) + 0 = \boxed{\frac{1}{2}} \text{ Ans}$$

Q. $1 + \frac{1}{2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^4} + \frac{1}{2^5} + \frac{1}{3^6} + \cdots$

$$\Rightarrow \left(1 + \frac{1}{3^2} + \frac{1}{3^4} + \frac{1}{3^6} + \cdots\right) + \left(\frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^5} + \frac{1}{2^7} + \cdots\right)$$

$$\Rightarrow \left(1 + \frac{1}{3^2} + \frac{1}{3^4} + \frac{1}{3^6} + \cdots\right) + \frac{1}{2}\left(1 + \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \cdots\right)$$

$$\Rightarrow \frac{1}{1-\frac{1}{3^2}} + \frac{1}{2}\cdot\frac{1}{1-\frac{1}{2^2}}$$

$$= \frac{9}{8} + \frac{2}{3} = \boxed{\frac{43}{24}}$$

④ Telescoping Series :-

A series $\sum_{n=1}^{\infty} a_n$ is said to be telescoping series.

If $\exists K \in \mathbb{N}$ s.t. the seq^n of terms $\{a_n\}$ can be expressed as.

$$\boxed{a_n = b_{n+K} - b_n \quad \text{or} \quad b_n - b_{n+K}.}$$

for some seq^n $\{b_n\}$.

$\Rightarrow$ If $a_n = b_n - b_{n+K}$, $\forall n \in \mathbb{N}$ then

$$a_1 = b_1 - \cancel{b_{K+1}}$$

$$a_2 = b_2 - \cancel{b_{K+2}}$$

$$\vdots$$

$$a_K = b_K - \cancel{b_{2K}}$$

$$a_{K+1} = \cancel{b_{K+1}} - b_{2K+1}$$

$$\vdots$$

$$a_n = b_n - b_{n+K}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$$

$$\sum_{n=1}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8} - 1$$

$$A_n = \sum_{r=1}^{n} a_r = (b_1 + b_2 + \cdots + b_K) - (b_{n+1} + b_{n+2} + \cdots + b_{n+K})$$

So, $\{A_n\}$ is Convergent iff $\{b_n\}$ is Convergent.

and $\lim_{n\to\infty} A_n = \sum_{n=1}^{\infty} a_n = (b_1 + b_2 + \cdots + b_K) - Kb$

where $\lim_{n\to\infty} b_n = b.$

Eg. $\sum_{n=0}^{\infty} \frac{1}{(a+nd)(a+\overline{n+1}d)}$

$$= \frac{1}{(a+nd)(a+(\overline{n+1})d)} = \frac{1}{d}\left[\frac{1}{a+nd} - \frac{1}{(a+\overline{n+1}d)}\right]$$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{1}{(a+nd)(a+\overline{n+1}d)} = \frac{1}{d}\left[\frac{1}{a} - \lim_{n\to\infty}\frac{1}{a+\overline{n+1}d}\right]$$

$$= \boxed{\frac{1}{ad}}$$

Q. $\frac{1}{1\cdot3} + \frac{1}{3\cdot5} + \frac{1}{5\cdot7} + \text{—}$

$$= \frac{1}{2}\left(1-\frac{1}{3}\right) + \frac{1}{2}\left(\frac{1}{3}-\frac{1}{5}\right) + \frac{1}{2}\left(\frac{1}{5}-\frac{1}{7}\right) + \text{— — —} + \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)]$$

$$= \frac{1}{2}\left[1 - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{5}} + \cancel{\frac{1}{5}} - \cancel{\frac{1}{7}} + \text{———} + \cancel{\frac{1}{2}}\left(\cancel{\frac{1}{2n-1}} - \frac{1}{2n+1}\right)\right]$$

$$= \frac{1}{2}\left[1 - \frac{1}{2n+1}\right] = \boxed{\frac{1}{2}} \quad \boxed{\text{as } n\to\infty}$$

Q. $\frac{1}{3\cdot7} + \frac{1}{7\cdot11} + \frac{1}{11\cdot15} + \frac{1}{15\cdot19} + \text{———}$

$$\frac{1}{4}\left(\frac{1}{3}-\frac{1}{7}\right) + \frac{1}{4}\left(\frac{1}{7}-\frac{1}{11}\right) + \frac{1}{4}\left(\frac{1}{11}-\frac{1}{15}\right) + \frac{1}{4}\left(\frac{1}{15}-\frac{1}{19}\right) + \text{—}$$

$$\frac{1}{4}\left[\frac{1}{3} - \cancel{\frac{1}{7}} + \cancel{\frac{1}{7}} - \cancel{\frac{1}{11}} + \cancel{\frac{1}{11}} - \cancel{\frac{1}{15}} + \frac{1}{15} \text{ — - - - —}\right]$$

$$\boxed{\frac{1}{12}}$$ Ans

☆ $\sum_{n=1}^{\infty} \frac{1}{(a+(\overline{n-1})d)(a+nd)(a+\overline{n+1}d)}$

$$= \frac{1}{2d}\left[\frac{1}{(a+\overline{n-1}d)(a+nd)} - \frac{1}{(a+nd)(a+\overline{n+1}d)}\right]$$

firstly last term leave

first term leave

$\boxed{\$\ d = \text{Last} - \text{first}}$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{(a+\overline{n-1}\,d)(a+nd)(a+\overline{n+1}\,d)} = \frac{1}{2d}\left[\frac{1}{a(a+d)} - \lim_{n\to\infty}\frac{1}{(a+nd)(a+\overline{n+1}\,d)}\right]$$

$$= \boxed{\frac{1}{2ad(a+d)}}$$ Ans

Q. $\sum_{n=2}^{\infty} \frac{1}{n^3-n}$

$$a_n = \frac{1}{n^3-n} = \frac{1}{(n-1)(n)(n+1)} = \frac{1}{2}\left[\frac{1}{(n-1)n} - \frac{1}{n(n+1)}\right]$$

$$\sum_{n=2}^{\infty} a_n = \frac{1}{2}\left[\frac{1}{2} - \lim_{n\to\infty}\frac{1}{n(n+1)}\right] = \boxed{\frac{1}{4}}$$ Ans

☆ $\sum_{n=0}^{\infty} \frac{1}{(a+nd)(a+\overline{n+1}\,d)\cdots(a+\overline{n+k}\,d)}$

last - first k terms.

$$a = \frac{1}{Kd}\left[\frac{1}{(a+nd)(a+\overline{n+1}\,d)\cdots(a+\overline{n+k-1}\,d)} - \frac{1}{(a+\overline{n+1}\,d)\cdots(a+\overline{n+k}\,d)}\right]$$

$$= \frac{1}{Kd}\left[\frac{1}{a(a+d)\cdots(a+\overline{k-1}\,d)} - \frac{1}{(a+\overline{n+1}\,d)\cdots(a+\overline{n+k}\,d)}\right]$$

$$= \boxed{\frac{1}{Kd}\cdot\frac{1}{(a)(a+d)(a+2d)\cdots(a+\overline{k-1}\,d)}}$$

Q. $\frac{1}{2\cdot3\cdot4\cdot5} + \frac{1}{3\cdot4\cdot5\cdot6} + \frac{1}{4\cdot5\cdot6\cdot7} + \cdots$

$$a_n = \frac{1}{(n+1)(n+2)(n+3)(n+4)}$$

$$= \frac{1}{3}\left[\frac{1}{(n+1)(n+2)(n+3)} - \frac{1}{(n+2)(n+3)(n+4)}\right]$$

$$\sum_{n=1}^{\infty} a_n = \frac{1}{3}\left[\frac{1}{2\cdot3\cdot4} - \lim_{n\to\infty}\frac{1}{(n+2)(n+3)(n+4)}\right] = \boxed{\frac{1}{72}}$$ Ans

Q. $\sum_{n=1}^{\infty} \tan^{-1}\left(\frac{2}{n^2}\right)$

$$= \tan^{-1}\left(\frac{2+n-n}{n^2+1-1}\right)$$

$$= \tan^{-1}\left[\frac{(n+1)-(n-1)}{1+(n-1)(n+1)}\right]$$

$$= \tan^{-1}(n+1) - \tan^{-1}(n-1)$$

$\Rightarrow a_n = \tan^{-1}(n+1) - \tan^{-1}(n-1)$

$$a_1 = \cancel{\tan^{-1}(2)} - \tan^{-1}(0)$$

$$a_2 = \cancel{\tan^{-1}(3)} - \tan^{-1}(1)$$

$$a_3 = \cancel{\tan^{-1}(4)} - \cancel{\tan^{-1}(2)}$$

$$a_4 = \cancel{\tan^{-1}(5)} - \cancel{\tan^{-1}(3)}$$

$$|$$

$$a_{n-2} = \cancel{\tan^{-1}(n-1)} - \cancel{\tan^{-1}(n-3)}$$

$$a_{n-1} = \tan^{-1}(n) - \cancel{\tan^{-1}(n-2)}$$

$$+\ \ a_n = \tan^{-1}(n+1) - \cancel{\tan^{-1}(n-1)}$$

$$\sum a_n = \tan^{-1}(n) + \tan^{-1}(n+1) - \tan^{-1}(0) - \tan^{-1}(1)$$

as $n \to \infty$

$$\sum a_n = \frac{\pi}{2} + \frac{\pi}{2} - \frac{\pi}{4} = \boxed{\frac{3\pi}{4}}$$

Q. $\sum_{n=1}^{\infty} \tan^{-1}\left(\frac{1}{n^2-n+1}\right) \Rightarrow \tan^{-1}\left[\frac{n-(n-1)}{1+n(n-1)}\right]$

$a_n = \tan^{-1}(n) - \tan^{-1}(n-1)$ difference one है

as $n \to \infty$

$$\frac{\pi}{2} - 0 = \boxed{\frac{\pi}{2}}$$

Q. $\sum_{n=1}^{\infty} \tan^{-1}\left(\frac{1}{n^2+n+1}\right) = \tan^{-1}\left(\frac{(n+1)-n}{1+n(n+1)}\right)$

$a_n = \tan^{-1}(n+1) - \tan^{-1}(n)$, $\tan^{-1}(1)$

as $n \to \infty \Rightarrow \sum a_n = \frac{\pi}{2} - \frac{\pi}{4} = \boxed{\frac{\pi}{4}}$

Q $\sum_{n=0}^{\infty} \tan^{-1}\left(\frac{1}{n^2+n+1}\right)$

$$\Rightarrow \tan^{-1}\left(\frac{(n+1)-n}{1+n(n+1)}\right)$$

$a_n = \tan^{-1}(n+1) - \tan^{-1}(n)$

as $n\to\infty$

$$= \frac{\pi}{2} - 0 = \boxed{\frac{\pi}{2}}$$

Q $\lim_{n\to\infty} \frac{\pi}{n} \sum_{K=1}^{n} \sin\left(\frac{\pi}{2} + \frac{5\pi}{2}\cdot\frac{K}{n}\right)$.

$$= \pi \int_0^1 \sin\left(\frac{\pi}{2} + \frac{5\pi}{2}\cdot x\right) dx$$

$$= \pi \int_0^1 \cos\left(\frac{5\pi}{2}\cdot x\right) dx$$

$$= \frac{2}{5}\left[\sin\left(\frac{5\pi}{2}x\right)\right]_0^1 = \boxed{\frac{2}{5}}$$

Q. $\lim_{n\to\infty} \frac{1}{\sqrt{n}}\left(\frac{1}{\sqrt{3}+\sqrt{6}} + \frac{1}{\sqrt{6}+\sqrt{9}} + \cdots + \frac{1}{\sqrt{3n}+\sqrt{3n+3}}\right)$

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}}\left(\frac{\sqrt{6}-\sqrt{3}}{6-3} + \frac{\sqrt{9}-\sqrt{6}}{9-6} + \cdots + \frac{\sqrt{3n+3}-\sqrt{3n}}{3n+3-3n}\right)$$

$$= \lim_{n\to\infty} \frac{1}{3\sqrt{n}}\left(\cancel{\sqrt{6}} - \sqrt{3} + \sqrt{9} - \cancel{\sqrt{6}} + \cdots + \sqrt{3n+3} - \cancel{\sqrt{3n}}\right)$$

$$= \lim_{n\to\infty} \frac{1}{3\sqrt{n}}\left(\sqrt{3n+3} - \sqrt{3}\right)$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{3}}\left(\sqrt{1+\frac{1}{n}} - \frac{1}{\sqrt{n}}\right)$$

$$= \boxed{\frac{1}{\sqrt{3}}}$$

Q $\lim_{x\to 0^+} x\left(\left[\frac{1}{x}\right] + \left[\frac{2}{x}\right] + - - + \left[\frac{10}{x}\right]\right)$ $\boxed{\Rightarrow [x] = \lfloor x \rfloor}$

$$\frac{1}{x} - 1 < \left[\frac{1}{x}\right] \le \frac{1}{x}$$

$$\frac{2}{x} - 1 < \left[\frac{2}{x}\right] \le \frac{2}{x}$$

$$|$$

$$\frac{10}{x} - 1 < \left[\frac{10}{x}\right] \le \frac{10}{x}$$

So, $\boxed{55}$ $\underline{A}$

$$55 - 10x < f(x) \le 55$$

Q $\lim_{x\to 0^+} x\left(\left\lceil\frac{1}{x}\right\rceil + \left\lceil\frac{2}{x}\right\rceil + \left\lceil\frac{3}{x}\right\rceil + - - - + \left\lceil\frac{10}{x}\right\rceil\right)$

$$\frac{1}{x} \le \left[\frac{1}{x}\right] < \frac{1}{x} + 1$$

$$\frac{2}{x} \le \left[\frac{2}{x}\right] < \frac{2}{x} + 1$$

$$|$$

$$\frac{10}{x} < \left\lceil\frac{10}{x}\right\rceil < \frac{10}{x} + 1$$

So, $\boxed{55}$ $\underline{A}$

$$55 < f(x) < 55 + 10x$$

Q $\sum_{n=1}^{\infty} \frac{1}{(2n)^2}$

$$= \frac{1}{4}\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{4}\cdot\frac{\pi^2}{6} = \boxed{\frac{\pi^2}{24}} \quad \underline{A}$$

Q $\sum_{n=1}^{\infty} \frac{1}{(2n+1)^2}$

we know that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

$$\Rightarrow 1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+ - - — = \frac{\pi^2}{6}$$

$$\Rightarrow \left(1+\frac{1}{3^2}+\frac{1}{5^2}+ —\right)+\left(\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+ —\right) = \frac{\pi^2}{6}$$

$$\Rightarrow \left(1+\frac{1}{3^2}+\frac{1}{5^2}+ -\right)+\frac{1}{2^2}\left(1+\frac{1}{2^2}+\frac{1}{3^2}+ - -\right) = \frac{\pi^2}{6}$$

$$\Rightarrow \left(1+\frac{1}{3^2}+\frac{1}{5^2}+ —\right)+\frac{1}{4}\cdot\frac{\pi^2}{6} = \frac{\pi^2}{6}$$

$$= \frac{\pi^2}{\cancel{6}_2}\times\frac{\cancel{3}}{4} = \frac{\pi^2}{8}$$

$$\Rightarrow \frac{1}{3^2}+\frac{1}{5^2}+ — = \frac{\pi^2}{8}-1$$ Ans

☆ positive term series :- A series $\sum_{n=0}^{\infty} a_n$ is said to be positive term series.

if $\boxed{a_n > 0}$, $\forall n \in N$

→ If $\sum_{n=1}^{\infty} a_n$ is a positive term series then it's seqⁿ of partial sum $\{A_n\}$ is monotonic increasing seqⁿ. Ans.

as
$$A_1 = a_1$$
$$A_2 = a_1 + a_2 = A_1 + a_2 > A_1$$
$$A_3 = a_1 + a_2 + a_3 = A_2 + a_3 > A_2$$
$$A_4 = a_1 + a_2 + a_3 + a_4 = A_3 + a_4 > A_3$$
$$\vdots$$
$$A_n = a_1 + a_2 + a_3 + - - + a_{n-1} + a_n = A_{n-1} + a_n > A_{n-1}$$

monotonic increasing seq^n.
So, $\{A_n\}$ is Either Convergent or diverges to ∞.
and $\{A_n\}$ is Convergent iff $\{A_n\}$ is bounded.

Since $\{A_n\}$ and $\sum_{n=1}^{\infty} a_n$ behave alike, so $\sum_{n=1}^{\infty} a_n$ is Either Convergent or diverges to ∞.
and. $\sum_{n=1}^{\infty} a_n$ is Convergent iff $\sum_{n=1}^{\infty} a_n < \infty$.
(i.e. is bounded).

☆ Convergence of a series of (+)ive terms:–

① **Abel's n^{th} term test:–** (Necessary Condition for Convergence of a series of (+)ive terms):–

If a positive term series $\sum_{n=1}^{\infty} a_n$ is Convergent then $\boxed{\lim_{n\to\infty} n a_n = 0.}$

So, If $\lim_{n\to\infty} n a_n \neq 0$ then the (+)ive term series $\sum_{n=1}^{\infty} a_n$ is not Convergent.

☆⇒ किसी series Σa_n को Convergent chake करने के लिए $\lim_{n\to\infty} a_n$ करते हैं यदि limit non-zero आती है, तो Series diverge होगी।

However,

If $\lim_{n\to\infty} n a_n ~~\neq~~ = 0$ then the positive term series $\sum_{n=1}^{\infty} a_n$ may converge or diverge.

Eg. ① $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent as $\lim_{n\to\infty} n a_n = \lim_{n\to\infty} n \cdot \frac{1}{n} = 1 \neq 0$

② $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is Convergent and $\lim_{n\to\infty} n a_n = \lim_{n\to\infty} n \cdot \frac{1}{n^2} = 0.$

③ $\sum_{n=1}^{\infty} \frac{1}{n \log n}$ is divergent as $\lim_{n\to\infty} n a_n = \lim_{n\to\infty} n \cdot \frac{1}{n \log n} = 0.$

* ② Comparition test :

Let $\sum_{n=1}^{\infty} a_n$ (Unknown) and $\sum_{n=1}^{\infty} b_n$ (Known) be two series of (+)ive terms.

(i) If $a_n \leq b_n \quad \forall n \in N$ and $\sum_{n=1}^{\infty} b_n$ is Convergent then $\sum_{n=1}^{\infty} a_n$ is Convergent.

(ii) If $a_n > b_n$ and $\sum_{n=1}^{\infty} b_n \leftarrow$ divergent then

$\sum_{n=1}^{\infty} a_n \leftarrow$ divergent.

(iii) If $a_n < b_n$ and $\sum_{n=1}^{\infty} b_n \leftarrow$ divergent then

$\sum_{n=1}^{\infty} a_n \leftarrow$ may converge or diverge.

(iv) If $a_n \geq b_n$ and $\sum_{n=1}^{\infty} b_n \leftarrow$ Convergent then

$\sum_{n=1}^{\infty} a_n \leftarrow$ may converge or diverge.

⇒ Limit form of Comparition test :-

If $\sum_{n=1}^{\infty} a_n$ (Unknown) and $\sum_{n=1}^{\infty} b_n$ (Known)

be two series of (+)ive terms s.t.

⇒ $\boxed{\lim_{n \to \infty} \frac{a_n}{b_n} = l}$ where l is a non-zero finite number (i.e. $0 < l < \infty$)

Then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ behave alike.

i.e. If $\sum_{n=1}^{\infty} b_n$ is Convergent then $\sum_{n=1}^{\infty} a_n \leftarrow$ Convergent.

and $\sum_{n=1}^{\infty} b_n \leftarrow$ divergent then $\sum_{n=1}^{\infty} a_n \leftarrow$ divergent.

$\Rightarrow$ If $l = 0$ then. $\left[\lim_{n\to\infty}\frac{a_n}{b_n} = 0 \Rightarrow a_n < b_n\right]$

(i) $\sum_{n=1}^{\infty} b_n \leftarrow$ Convergent then $\sum_{n=1}^{\infty} a_n \leftarrow$ ~~C~~ Convergent.

(ii) $\sum_{n=1}^{\infty} b_n \leftarrow$ Divergent then $\sum_{n=1}^{\infty} a_n \leftarrow$ May Converge or Diverge.

$\Rightarrow$ If $l = \infty$ then $\left[\lim_{n\to\infty}\frac{a_n}{b_n} = \infty \Rightarrow a_n > b_n\right]$.

(i) $\sum_{n=1}^{\infty} b_n \rightarrow$ divergent then $\sum_{n=1}^{\infty} a_n \rightarrow$ divergent.

(ii) $\sum_{n=1}^{\infty} b_n \rightarrow$ Convergent then $\sum_{n=1}^{\infty} a_n \rightarrow$ May Converge or Diverge.

Q Test the Convergence of the following Series :-

① $\sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}}$

$\sum_{n=1}^{\infty} \frac{1}{n \cdot n^{1/n}}$ Let $b_n = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\cancel{n}}{\cancel{n} \cdot n^{1/n}} = 1 \neq 0$$

So, divergent.

② $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n!}}$

$a_n = \frac{1}{\sqrt{n!}}$, Let $b_n = \frac{1}{n}$ then.

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{n}{\sqrt{n!}} = \lim_{n\to\infty}\sqrt{\frac{n^2}{n!}} = \sqrt{\lim_{n\to\infty}\frac{n^2}{n!}} = 0$$

$\sum b_n \leftarrow$ divergent then $\sum a_n \leftarrow$ may converge or diverge.

So, test fail.

Now, Let $b_n = \frac{1}{n^2}$

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{n^2}{\sqrt{n!}} = \lim_{n\to\infty}\sqrt{\frac{n^4}{n!}} = \sqrt{\lim_{n\to\infty}\frac{n^4}{n!}} = 0.$$

we know that,-

$\sum b_n \leftarrow$ Convergent then $\sum a_n \rightarrow$ Convergent.

So, $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n!}}$ is Convergent.

③ $\sum_{n=\phi}^{\infty} \sin\frac{1}{n}$

$a_n = \sin\frac{1}{n}$, $b_n = \frac{1}{n}$

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\sin\frac{1}{n}}{\frac{1}{n}} = 1 \neq 0$$

So, $\sum_{n=1}^{\infty}\sin\frac{1}{n} \rightarrow$ divergent.

④ $\sum_{n=1}^{\infty} \cos\frac{1}{n}$

$a_n = \cos\frac{1}{n}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \cos\frac{1}{n} = 1 \neq 0$

$\rightarrow \sum_{n=1}^{\infty} \cos\frac{1}{n} \rightarrow$ divergent.

⑤ $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \sin\frac{1}{n}$.

Let $b_n = \frac{1}{n^{3/2}}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} n \sin\frac{1}{n} = 1 \neq 0$

$\sum b_n \rightarrow$ Converge then $\sum a_n \rightarrow$ Converge

By Comparition test.

⑥ $\sum_{n=1}^{\infty} \frac{1}{n^2} \tan\frac{1}{n}$

$\lim_{n\to\infty} \frac{\tan\frac{1}{n}}{\frac{1}{n}} = 1$

Let $b_n = \frac{1}{n^3}$ $\left[\because \lim_{n\to\infty} \frac{a_n}{b_n} = 1 \neq 0\right)$

So, By Comparition test $\sum a_n \rightarrow$ Convergent.

⑦ $\sum_{n=1}^{\infty} \frac{1}{n^2} \cos\frac{1}{n}$

Let $b_n = \frac{1}{n^2}$ So, By Comparition test

$\sum_{n=1}^{\infty} \frac{1}{n^2} \cos\frac{1}{n} \rightarrow$ Convergent.

⑧ $\sum_{n=1}^{\infty} \sqrt{\frac{n+1}{n^3}}$

$\left(\because \lim_{n\to\infty} \frac{a_n}{b_n} = 1 \neq 0\right]$

Let $b_n = \frac{1}{n}$

By Comparition test $\sum_{n=1}^{\infty} \sqrt{\frac{n+1}{n^3}} \leftarrow$ Divergent.

⑨ $\sum_{n=1}^{\infty} \frac{n^3+7n^2-4n+8}{2n^5-3n^4+2n-7}$

$\frac{1+\frac{7}{n}-\frac{4}{n^2}+\frac{8}{n^3}}{n^2(2-3/n+\frac{2}{n^4}-7/n^5)}$ ~~$= \frac{1}{2}$~~ as

Let $b_n = \frac{1}{n^2}$

∴ this series is Convergent.

⑩ $\sum_{n=1}^{\infty} \frac{n^{15/4} - n^{14/5} + 3n^{7/2}}{n^{13/5} + 3n^{23/2} + 4n^{12/5}}$

$\frac{15}{4} - \frac{23}{2} = \frac{31}{4}$

Let $b_n = \frac{1}{n^{31/4}}$

↳ Convergent

⑪ $\frac{1\cdot2}{2\cdot3\cdot4\cdot5} + \frac{2\cdot3}{3\cdot4\cdot5\cdot6} + \frac{3\cdot4}{4\cdot5\cdot6\cdot7} + \text{———}$

$a_n = \frac{n(n+1)}{(n+1)(n+2)(n+3)(n+4)}$

Let $b_n = \frac{1}{n^2}$

↳ Convergent.

⑫ $\frac{1\cdot2\cdot3}{3\cdot5\cdot7\cdot9} + \frac{2\cdot3\cdot4}{5\cdot7\cdot9\cdot11} + \frac{3\cdot4\cdot5}{7\cdot9\cdot11\cdot13} + \text{———}$

~~$a_n = n(n+1)$~~

↳ Divergent

Let $b_n = \frac{1}{n}$

(13.) $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n-1}}{n}$

$$a_n = \frac{\sqrt{n+1}-\sqrt{n-1}}{n} \Rightarrow \frac{\sqrt{n+1}-\sqrt{n-1}}{n} \times \frac{\sqrt{n+1}+\sqrt{n-1}}{\sqrt{n+1}+\sqrt{n-1}}$$

$$= \frac{n+1-n+1}{n(\sqrt{n+1}+\sqrt{n-1})}$$

$$= \frac{2}{n^{3/2}\left(\sqrt{1+\frac{1}{n}}+\sqrt{1-\frac{1}{n}}\right)}$$

Let $b_n = \frac{1}{n^{3/2}}$

So, this is Convergent.

(14.) $\sum_{n=1}^{\infty} \sqrt{n^3+1}-\sqrt{n^3-1}$

$$a_n = \sqrt{n^3+1}-\sqrt{n^3-1} \Rightarrow \frac{\sqrt{n^3+1}-\sqrt{n^3-1}}{1} \times \frac{\sqrt{n^3+1}+\sqrt{n^3-1}}{\sqrt{n^3+1}+\sqrt{n^3-1}}$$

$$= \frac{n^3+1-\cancel{0}\,n^3+1}{\sqrt{n^3+1}+\sqrt{n^3-1}}$$

$$= \frac{2}{n^{3/2}\left(\sqrt{1+\frac{1}{n^3}}+\sqrt{1-\frac{1}{n^3}}\right)}$$

Let $b_n = \frac{1}{n^{3/2}}$

So, this series is Convergent.

$\sum_{n=1} \sqrt[3]{n^3+1} - n$

$a_n = \sqrt[3]{n^3+1} - n = n\left[\left(1+\frac{1}{n^3}\right)^{1/3} - 1\right]$

$= n\left[\left(\cancel{1}+\frac{1}{3n^3} + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\frac{1}{n^6} + \cdots\right) \cancel{-1}\right]$

$= \frac{n}{n^3}\left[\frac{1}{3} + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\frac{1}{n^3} + \cdots\right]$

$= \frac{1}{n^2}\left[\frac{1}{3} + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\cdot\frac{1}{n^3} + \cdots\right]$

Let $b_n = \frac{1}{n^2}$

So, $\sum_{n=1}^{\infty} \sqrt[3]{n^3+1} - n$ is Convergent.

OR $\quad \frac{(n^3+1)-n^3}{(n^3+1)^{2/3}+n(n^3+1)^{1/3}+n^2}$ $\quad \left[\because x^3-y^3 = (x-y)(x^2+xy+y^2)\right]$

$= \frac{1}{n^2\left[\left(1+\frac{1}{n^3}\right)^{2/3}+\left(1+\frac{1}{n^3}\right)^{1/3}+1\right]}$

Let $b_n = \frac{1}{n^2}$ So, this is Convergent.

(16) $\frac{e^{-1}}{1\cdot2\cdot3} + \frac{e^{-2}}{2\cdot3\cdot4} + \frac{e^{-3}}{3\cdot4\cdot5} + \text{———}$

$a_n = \frac{e^{-n}}{n(n+1)(n+2)}$, Let $b_n = \frac{1}{n^3}$

So ↳ Convergent.

⑰ $S_n = \frac{\sin\frac{\pi}{2}}{1\cdot 2} + \frac{\sin\frac{\pi}{2^2}}{2\cdot 3} + \frac{\sin\frac{\pi}{2^3}}{3\cdot 4} + \cdots + \frac{\sin\frac{\pi}{2^{n+1}}}{n(n+1)}$

$a_n = \frac{\sin\left(\frac{\pi}{2^{n+1}}\right)}{n(n+1)}$ Let $b_n = \frac{1}{n^2}$

Convergent.

Or

$$\frac{\sin\left(\frac{\pi}{2}\right)}{1\cdot 2} \leq \frac{1}{1\cdot 2}$$

$$\frac{\sin\frac{\pi}{2^2}}{2\cdot 3} \leq \frac{1}{2\cdot 3}$$

$$\vdots$$

$$\frac{\sin\frac{\pi}{2^n}}{n(n+1)} \leq \frac{1}{n(n+1)}$$

Since $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ → Convergent & Converges to 1.

$\Rightarrow \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)}$ is Cauchy.

$\Rightarrow S_n$ is Cauchy.

⑱ $\sum_{n=1}^{\infty} \frac{n+1}{n^p}$

Let $b_n = \frac{1}{n^{p-1}}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n+1}{n} = 1$

So, $\sum a_n$ → Convergent.

⑮ $\sum_{n=1}^{\infty} e^{-n^2}$

Let $b_n = \frac{1}{n^p}$, $p>1$

So, $\sum_{n=1}^{\infty} e^{-n^2} \rightarrow$ Convergent.

★ ③ Cauchy n^{th} root Test :-

Let $\sum_{n=1}^{\infty} a_n$ be a series of (+)ive terms. s.t.

$$\boxed{\lim_{n\to\infty} (a_n)^{\frac{1}{n}} = L}$$

⇒ If $L>1$ then $\sum_{n=1} a_n \leftarrow$ Divergent.

⇒ If $L<1$ then $\sum_{n=1}^{\infty} a_n \leftarrow$ Convergent.

⇒ If $L=1$ then ~~test~~ fail.

★ ④ D'Alembert Ratio Test :-

$$\boxed{\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = L}$$

⇒ If $L>1$ then $\sum a_n \rightarrow$ Convergent.

⇒ If $L<1$ then $\sum a_n \rightarrow$ Divergent.

⇒ If $L=1$ then test fail.

⑤ Raabe's Test :-

$$\boxed{\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}}-1\right)=L}$$

⇒ If $L>1$ then $\Sigma a_n \to$ Convergent.

⇒ If $L<1$ then $\Sigma a_n \to$ Divergent.

⇒ If $L=1$ then test fail.

*⑥ Logarithmic Test :-

$$\boxed{\lim_{n\to\infty} n\log\left(\frac{a_n}{a_{n+1}}\right)=L}$$

⇒ If $L>1$ then $\Sigma a_n \to$ Convergent.

⇒ If $L<1$ then $\Sigma a_n \to$ Divergent.

⇒ If $L=1$ then test fail.

⑦ De' Morgan & Bertrain Test :-

$$\boxed{\lim_{n\to\infty}\left[n\left(\frac{a_n}{a_{n+1}}-1\right)-1\right]\log n=L}$$

⇒ If $L>1$ then $\Sigma a_n \to$ Convergent.

⇒ If $L<1$ then $\Sigma a_n \to$ Divergent.

⇒ If $L=1$ then test fail.

⑧ Second Logarithmic Test :

$$\boxed{\lim_{n\to\infty}\left[n\log\left(\frac{a_n}{a_{n+1}}\right)-1\right]\log n = L}$$

⇒ If $L>1$ then $\sum a_n \to$ Convergent.

⇒ If $L<1$ then $\sum a_n \to$ Divergent.

⇒ If $L=1$ then ~~$\sum a_n$~~ test fail.

⑨ Gauss Test :

$$\frac{a_n}{a_{n+1}} = \alpha + \frac{\beta}{n} + \frac{\gamma_n}{n^p}, \quad p \geq 2.$$

⇒ If $\alpha>1$ then $\sum a_n \to$ Convergent

⇒ If $\alpha<1$ then $\sum a_n \to$ Divergent.

⇒ If $\alpha=1$ and $\beta>1$ then $\sum a_n \to$ Convergent.

⇒ If $\alpha=1$ and $\beta<1$ then $\sum a_n \to$ Divergent.

☆ ⑩ Cauchy Condensation Test :

Let $\sum_{n=1}^{\infty} u_n$ be a series of +ive terms and $a>1$ then $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} a^n u(a^n)$ Both Converge or Diverge. together:

(11.) Cauchy integral test :-

If $u(x)$ is a (+)ive and monotonic decreasing function of x defind on $[1, \infty)$ → Closed then

$\sum_{n=1}^{\infty} u_n$ and $\int_{1}^{\infty} u(x)\,dx$ Both Converge or Diverge together.

Q Test the Convergence of the following series :-

① $\sum_{n=2}^{\infty} \frac{1}{\log n}$

By Cauchy Condensation test.

$$\sum_{n=2}^{\infty} \frac{e^n}{\log e^n} = \sum_{n=2}^{\infty} \frac{e^n}{n \log e} = \sum_{n=2}^{\infty} \frac{e^n}{n} \leftarrow \text{Divergent}$$

✓ By Comparision test.
Let $b_n = \frac{1}{n}$.

② $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$

By Cauchy Condensation test.

$$\sum_{n=2}^{\infty} \frac{e^n}{e^n(\log e^n)^p} = \sum_{n=2}^{\infty} \frac{1}{(n \log e)^p}$$

$$\Rightarrow \sum_{n=2}^{\infty} \frac{1}{n^p} \leftarrow$$ Convergent, if $p>1$

Divergent, if $p\leq 1$

③ $\sum_{n=2}^{\infty} \frac{1}{(\log n)^p}$

By Cauchy Condensation test.

$$\sum_{n=2}^{\infty} \frac{e^n}{(\log e^n)^p} = \sum_{n=2}^{\infty} \frac{e^n}{n^p} \rightarrow$$ Convergent if $p>1$

Divergent, if $p\leq 1$

④ $\sum_{n=2}^{\infty} \frac{1}{n^p . (\log n)}$

By Cauchy Condensation test -

$$\sum_{n=2}^{\infty} \frac{e^n}{e^{np}(\log e^n)} = \sum_{n=2}^{\infty} \frac{1}{n e^{n(p-1)}}$$

$$a_n = \frac{1}{n e^{n(p-1)}}, \quad a_{n+1} = \frac{1}{(n+1)\, e^{(n+1)p-(n+1)}}$$

$$\frac{a_n}{a_{n+1}} = \frac{(n+1)\, e^{(n+1)p-(n+1)}}{n\, e^{np-n}} = \left(1+\frac{1}{n}\right) e^{p-1}$$

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = e^{p-1} \rightarrow$$ Convergent, if $p>1$

Divergent, if $p<1$

★ ⑤ $\frac{1^2}{2^2} + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2} + - -$

$$a_n = \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot --- (2n-1)^2}{2^2 \cdot 4^2 \cdot 6^2 \cdot --- (2n)^2}$$

$$a_{n+1} = \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot --- (2n+1)^2}{2^2 \cdot 4^2 \cdot 6^2 \cdot -- (2n+2)^2}$$

$$\therefore \frac{a_n}{a_{n+1}} = \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot --- (2n-1)^2}{2^2 \cdot 4^2 \cdot 6^2 \cdot -- (2n)^2} \times \frac{2^2 \cdot 4^2 \cdot 6^2 \cdot -- (2n+2)^2}{1^2 \cdot 3^2 \cdot 5^2 \cdot --- (2n+1)^2}$$

$$= \frac{(2n+2)^2}{(2n+1)^2} = 1 \quad \text{as } n \to \infty$$

then Ratio test fail.

$$\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) = \lim_{n\to\infty} n\left(\frac{4n^2+8n+4}{4n^2+4n+1} - 1\right)$$

$$= \lim_{n\to\infty} n\left(\frac{4n+3}{4n^2+4n+1}\right)$$

$$= \lim_{n\to\infty} \frac{4n^2+3n}{4n^2+4n+1} = 1$$ (Raabe's Test fail)

$$\lim_{n\to\infty} \left[n\left(\frac{a_n}{a_{n+1}} - 1\right) - 1\right] \log n$$

$$= \lim_{n\to\infty} \left[\frac{4n^2+3n}{4n^2+4n+1} - 1\right] \log n$$

$$= \lim_{n\to\infty} \left[\frac{-n-1}{4n^2+4n+1}\right] \log n$$

$$= -\lim_{n\to\infty} \frac{(n+1)\log n}{4n^2+4n+1}$$

$$\Rightarrow -\lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)+\log n}{8n+4}$$

$$\Rightarrow \lim_{n\to\infty} \frac{-\frac{1}{n^2}+\frac{1}{n}}{8} = 0 < 1$$ → Divergent. ~~Convergent~~.

By De'morgan or Bertrain test.

Q. $\left(\frac{2^2}{1^2}-\frac{2}{1}\right)^{-1}+\left(\frac{3^3}{2^3}-\frac{3}{2}\right)^{-1}+\left(\frac{4^4}{3^4}-\frac{4}{3}\right)^{-1}+$ ——

$$a_n = \left[\left(\frac{n+1}{n}\right)^{n+1}-\frac{n+1}{n}\right]^{-1} = \left(\frac{n+1}{n}\right)^{-1}\left[\left(\frac{n+1}{n}\right)^{n}-1\right]^{-1}$$

$$= \frac{n}{n+1}\left[\left(1+\frac{1}{n}\right)^{n}-1\right]^{-1}$$

as $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n+1}\left[\left(1+\frac{1}{n}\right)^{n}-1\right]^{-1}$

$$= (e-1)^{-1} = \frac{1}{e-1} \neq 0$$ ← Divergent

Q $\frac{1}{2}+\frac{1}{2\cdot 2^2}+\frac{1}{3\cdot 2^3}+\frac{1}{4\cdot 2^4}+$ - - - —— -

$$a_n = \sum_{n=1}^{\infty}\frac{1}{n\cdot 2^n} \Rightarrow a_n = \frac{1}{n\,2^n}.$$

By Cauchy n^{th} root test :-

$$\lim_{n\to\infty}(a_n)^{1/n} = \lim_{n\to\infty}\left(\frac{1}{n2^n}\right)^{1/n} = \frac{1}{2} < 1$$ → Convergent

Q. $\sum_{n=1}^{\infty} \frac{n^3+5}{3^n+2}$

$$a_n = \frac{n^3+5}{3^n+2}, \quad a_{n+1} = \frac{(n+1)^3+5}{3^{n+1}+2}$$

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} \frac{n^3+5}{3^n+2} \times \frac{3^{n+1}+2}{(n+1)^3+5}$$

$$= \lim_{n\to\infty} \left(\frac{3+\frac{2}{3^n}}{1+\frac{2}{3^n}}\right)\cdot\left(\frac{1+\frac{5}{n^3}}{\left(1+\frac{1}{n}\right)^3+\frac{5}{n^3}}\right)$$

$$= 3 > 1 \rightarrow \text{Convergent.}$$

Q. $\left(\frac{2^2}{1^2}-\frac{2}{1}\right)^{-1} + \left(\frac{3^3}{2^3}-\frac{3}{2}\right)^{-2} + \left(\frac{4^4}{3^4}-\frac{4}{3}\right)^{-3} + \text{——}$

$$a_n = \left[\left(\frac{n+1}{n}\right)^{n+1} - \frac{n+1}{n}\right]^{-n}$$

$$= \left(\frac{n+1}{n}\right)^{-n}\left[\left(\frac{n+1}{n}\right)^{n} - 1\right]^{-n}$$

By Cauchy n^{th} root test.

$$\lim_{n\to\infty} (a_n)^{\frac{1}{n}} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{-1}\cdot\left[\left(1+\frac{1}{n}\right)^n - 1\right]^{-1}$$

$$= \frac{1}{e-1} < 1 \rightarrow \text{Convergent.}$$

Q. $\frac{1}{4} + \left(\frac{1}{4}\right)^{1+\frac{1}{3}} + \left(\frac{1}{4}\right)^{1+\frac{1}{3}+\frac{1}{5}} + \left(\frac{1}{4}\right)^{1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}} + \text{———}$

$$a_n = \left(\frac{1}{4}\right)^{1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\cdots+\frac{1}{2n-1}}$$

$$a_{n+1} = \left(\frac{1}{4}\right)^{1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\cdots+\frac{1}{2n+1}}$$

$$\frac{a_n}{a_{n+1}} = \left(\frac{1}{4}\right)^{-\frac{1}{2n+1}} = (4)^{\frac{1}{2n+1}}$$

By Logarithmic test:

$$n\log\left(\frac{a_n}{a_{n+1}}\right) = \frac{n}{2n+1}\log(4)$$

$$A_n = \frac{1}{2+\frac{1}{n}}\log(4)$$

$$\lim_{n\to\infty} A_n = \frac{1}{2}\log(4) = \frac{\log 4}{\log e^2} < 1$$

So, this is Divergent

Q. $\tan\frac{\pi}{4} + \tan\frac{\pi}{8} + \tan\frac{\pi}{12} + ---$

$$a_n = \tan\frac{\pi}{4n}$$

Let $b_n = \frac{\pi}{4n}$

So, By Comparision test.

$\sum_{n=1}^{\infty} \tan\frac{\pi}{4n} \leftarrow$ ~~Con~~. Divergent

Q. $1 + \frac{1}{2} + \frac{1}{2\cdot 3} + \frac{1}{2^2\cdot 3} + \frac{1}{2^2\cdot 3^2} + \frac{1}{2^3\cdot 3^2} + ---$

Sol $1 + \frac{1}{2} + \frac{1}{2\cdot 3} + \frac{1}{2^2\cdot 3} + \frac{1}{2^2\cdot 3^2} + \frac{1}{2^3\cdot 3^2} + ---$

$\leq 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + ---$

Since $1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + ---$ is Convergent

Geometric series.

So, By comparision test, the given series is Convergent.

OR

$$= \left(1+\frac{1}{2\cdot 3}+\frac{1}{2^2\cdot 3^2}+\frac{1}{2^3\cdot 3^3}+ --\right)+\frac{1}{2}\left(1+\frac{1}{2\cdot 3}+\frac{1}{2^2\cdot 3^2}+ - - -\right)$$

$$= \frac{3}{2}\left(1+\frac{1}{2\cdot 3}+\frac{1}{2^2\cdot 3^2}+ ---\right)$$

⇒ Convergent.

By ~~Comparis~~ Cauchy nth root test $\left(a_n = \frac{1}{2^n \cdot 3^n}\right)$

Q If $\sum a_n$ is a series of (+)ive terms then,

(I) $\sum a_n$ is Convergent $\Rightarrow \sum a_n^2$ is Convergent.

(II) $\sum a_n^2$ is Convergent $\Rightarrow \sum a_n$ is Convergent.

ⓐ (I) & (II) Both are Correct.

[✓ ⓑ] (I) Correct But (II) is inCorrect.

ⓒ (I) is inCorrect But (II) is Correct.

ⓓ Both are inCorrect.

[✓ ①] $\sum a_n$ is Convergent $\Rightarrow \sum a_n^2$ is Convergent.

② $\sum a_n$ is divergent $\Rightarrow \sum a_n^2$ is divergent.

③ $\sum a_n^2$ is Convergent $\Rightarrow \sum a_n$ is Convergent.

[✓ ④] $\sum a_n^2$ is divergent $\Rightarrow \sum a_n$ is divergent.

Q If $\sum a_n$ is a series of (+)ive terms then.

[✓ ①] $\sum a_n$ is Convergent $\Rightarrow \sum \frac{a_n}{1+a_n}$ is Convergent.

[✓ ②] $\sum a_n$ is Convergent $\Rightarrow \sum \frac{a_n}{n}$ is Convergent.

③ $\sum a_n$ is Convergent $\Rightarrow \sum n a_n$ is Convergent.

④ $\sum a_n$ is Convergent $\Rightarrow \sum \frac{a_n^2}{1+a_n^2}$. is Convergent.

Soln:- ① Let $b_n = \frac{a_n}{1+a_n}$ $\quad \because \lim_{n\to\infty} a_n = 0.$

$$\lim_{n\to\infty} \frac{b_n}{a_n} = \frac{\cancel{a_n}}{(1+a_n).\cancel{a_n}} = \frac{1}{1+0} = 1 \neq 0$$

($\because \sum a_n$ is Convergent $\Rightarrow \lim_{n\to\infty} a_n = 0$)

So, By Comparision test

$\sum \frac{a_n}{1+a_n} \to$ Convergent.

② Let $b_n = \frac{a_n}{n}$

$$\lim_{n\to\infty} \frac{b_n}{a_n} = \frac{\cancel{a_n}}{n.\cancel{a_n}} = \frac{1}{n} = 0$$

So, By Comparision test. $\sum \frac{a_n}{n} \to$ Convergent

③ Let $b_n = \frac{a_n^2}{1+a_n^2}$

$$\lim_{n\to\infty} \frac{b_n}{a_n} = \frac{a_n^{\cancel{2}}}{(1+a_n^2)\cancel{a_n}} = \frac{0}{1+0} = 0$$

So, By Comparision test, $b_n < a_n$

$\sum \frac{a_n^2}{1+a_n^2} \to$ Convergent

✯ Statement:-

If $\sum u_n$ is a series of (+)ive terms and $V_n = \frac{u_1+u_2+ \cdots +u_n}{n}$

then $\sum V_n$ is Divergent. [T ✓/F]

proof:

$$V_1 = u_1 = u_1$$

$$V_2 = \frac{u_1+u_2}{2} > \frac{u_1}{2}$$

$$V_3 = \frac{u_1+u_2+u_3}{3} > \frac{u_1}{3}$$

$$V_4 = \frac{u_1+u_2+u_3+u_4}{4} > \frac{u_1}{4}$$

$$|$$

$$V_n = \frac{u_1+u_2+ \text{---} +u_n}{n} > \frac{u_1}{n}$$

$$|$$

$$\therefore \quad V_1+V_2+V_3+ \text{---} +V_n > u_1\left(1+\frac{1}{2}+\frac{1}{3}+ \text{---} +\frac{1}{n}\right)$$

$$\sum V_n \geq u_1 \sum \frac{1}{n}$$

Since, $\sum \frac{1}{n}$ is Divergent.

So, By Comparision test $\sum V_n$ is Divergent.

✯ Series of arbitrary terms:-

① Absolutely Convergent Series:- A series $\sum_{n=1}^{\infty} a_n$ is said to be absolutely convergent.

If $\sum_{n=1}^{\infty} |a_n|$ is Convergent.

② Conditionally Convergent Series :-

A series $\sum_{n=1}^{\infty} a_n$ is said to be Conditionally Convergent.

If $\sum_{n=1}^{\infty} a_n \rightarrow$ Convergent.

But $\sum_{n=1}^{\infty} |a_n| \rightarrow$ Divergent.

☆

→ $\sum |a_n|$ is Convergent $\Rightarrow \sum a_n$ is absalutely Convergent.

→ $\sum a_n$ is divergent $\Rightarrow \sum |a_n|$ is divergent.

→ $\sum a_n$ is divergent and $\sum |a_n|$ is divergent $\Rightarrow \sum a_n$ is Conditionally Convergent.

→ If $\sum |a_n|$ is Convergent $\Rightarrow \sum a_n$ is Convergent.

☆ If $\sum |a_n|$ is Convergent then $\sum a_n$ is Convergent.

proof:- $\because \sum |a_n|$ is Convergent. So,

$\forall \epsilon > 0, \exists n_0 \in \mathbb{N}$ s.t.

$$\big| |a_{n+1}| + |a_{n+2}| + \cdots + |a_{n+p}| \big| < \epsilon, \ \forall n > n_0, p \geq 1$$

$$\Rightarrow |a_{n+1}| + |a_{n+2}| + \cdots + |a_{n+p}| < \epsilon, \ \forall n > n_0, p \geq 1$$

$$\because |a_{n+1} + a_{n+2} + \cdots + a_{n+p}| \leq |a_{n+1}| + |a_{n+2}| + \cdots + |a_{n+p}|$$

$\therefore \forall \epsilon > 0, \exists n_0 \in \mathbb{N}$ s.t.

$$|a_{n+1} + a_{n+2} + \cdots + a_{n+p}| < \epsilon, \ \forall n > n_0, p \geq 1.$$

$\Rightarrow \sum a_n$ is Convergent.

$\infty - \infty \rightarrow$ May be finite number.

Q $\sum_{n=1}^{\infty} \frac{\sin(nx)}{1+n^2}$

$\because |\sin nx| \leq 1$

$\therefore \frac{|\sin nx|}{1+n^2} \leq \frac{1}{1+n^2}$

Since $\sum_{n=1}^{\infty} \frac{1}{1+n^2}$ is Convergent.

So, By Comparision test

$\sum_{n=1}^{\infty} \frac{|\sin nx|}{1+n^2}$ is Convergent.

$\Rightarrow \sum_{n=1}^{\infty} \frac{\sin nx}{1+n^2}$ is absolutely Convergent.

$\Rightarrow \sum_{n=1}^{\infty} \frac{\sin nx}{1+n^2}$ is Convergent.

☆ $\Rightarrow$ Every absolutely Convergent series is Convergent.

☆ Let $\sum a_n$ be a series of real numbers. and $\sum P_n$ is the series of (+)ive terms of $\{a_n\}$. and $\sum N_n$ is the series of (−)ive terms of $\{a_n\}$.

→ If $\sum_{n=1}^{\infty} a_n$ is absolutely Convergent then $\sum P_n$ and $\sum N_n$ Both are Convergent.

(Converse true)

→ If $\sum a_n \leftarrow$ Conditionally Convergent then $\sum P_n$ & $\sum N_n$ Both are Divergent.

(Converse not true).

→ If one of $\sum P_n$ and $\sum N_n$ is Convergent and another is divergent then $\sum a_n \leftarrow$ Divergent.

(Converse not true)

☆ If a series $\sum_{n=1}^{\infty} a_n$ is absolutely Convergent and $\{b_n\}$ is a bounded seq^n then $\sum_{n=1}^{\infty} a_n b_n$ is absolutely Convergent.

proof:- If $\sum a_n$ is absolutely Convergent.

$\Rightarrow \sum |a_n|$ is Convergent.

$\Rightarrow \forall \epsilon > 0, \exists n_0 \in \mathbb{N}$ s.t.

$\big| |a_{n+1}| + |a_{n+2}| + - - - + |a_{n+p}| \big| < \epsilon, \ \forall n > n_0, p \geq 1$

Since, $\{b_n\}$ is a bounded seq^n. So,

$\exists M \in \mathbb{R}$ s.t. $|b_n| \leq M, \ \forall n \in \mathbb{N}$

$\therefore \ |b_{n+1} a_{n+1}| < M|a_{n+1}|$

$|b_{n+2} a_{n+2}| < M|a_{n+2}|$

$|b_{n+3} a_{n+3}| < M|a_{n+3}|$

|

$|b_{n+p} a_{n+p}| < M|a_{n+p}|$

So, $\forall \epsilon > 0, \exists n_0 \in N$ s.t.

$$|b_{n+1}a_{n+1}| + |b_{n+2}a_{n+2}| + \cdots + |b_{n+p}a_{n+p}| < M\epsilon, \forall n > n_0, p \geq 1$$

So, From Cauchy Criterion of Convergence

$\sum |a_n b_n|$ is Convergent.

$\Rightarrow \sum a_n b_n$ is absolutely Convergent.

Q If $\sum_{n=1}^{\infty} a_n$ is a Convergent ~~series~~ seq^n of (+)ive terms then which of the following seq^ns is/are Convergent.?

✓ ① $\sum \frac{a_n}{\sqrt{n}}$ $\downarrow$ $b_n = \frac{1}{\sqrt{n}}$

✓ ② $\sum \left(1+\frac{1}{n}\right) a_n$ $\downarrow$ $b_n = \left(1+\frac{1}{n}\right)$

✓ ③ $\sum \left(1+\frac{1}{n}\right)^n a_n$ $\downarrow$ $b_n = \left(1+\frac{1}{n}\right)^n$

④ $\sum e^{-n} a_n$ $\downarrow$ $b_n = e^{-n}$

$\because$ all b_n are bounded and $\sum a_n$ is Convergent $\Rightarrow \sum a_n b_n \rightarrow$ Convergent

So, all are Convergent.

✓ ⑤ $\sum \frac{a_n}{\log n}$ $\swarrow$ $b_n = \frac{1}{\log n}$

✓ ⑥ $\sum n^{\frac{1}{n}} a_n$ $\downarrow$ $b_n = n^{\frac{1}{n}}$

★ Alternating Series :-

A series $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$, $a_n > 0$, $\forall n \in \mathbb{N}$ is said to be an alternating series.

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n = a_1 - a_2 + a_3 - a_4 + - \text{———}$$

⇒ Leibnitz test for Convergence of alternating Series :-

An alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$, $a_n > 0$ $\forall n \in \mathbb{N}$ is Convergent if

✓ (i) $\lim_{n \to \infty} a_n = 0$.

✓ (ii) $\{a_n\}$ is monotonic decreasing. (i.e. $a_{n+1} < a_n$, $\forall n \in \mathbb{N}$)

Q $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p}$

Here, $a_n = \frac{1}{n^p}$

So, (i) $\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{1}{n^p} = 0$, if $p > 0$

(ii) $a_{n+1} = \frac{1}{(n+1)^p} < \frac{1}{n^p} = a_n$, if $p > 0$

So, from Leibnitz test

$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p}$ is Convergent, if $p > 0$

Divergent, if $p \leq 0$.

$\therefore \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p}$ is Divergent, if $p \leq 0$

Conditionally Convergent, if $0 \leq p < 1$

absolutely Convergent, if $p > 1$.

Q $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \log n}$ ← Conditionally Convergent.

$\because \left[\sum \frac{1}{n \log n} \right] \to$ Divergent.

☆ Test for Convergence of arbitrary term series :-

① Abel Test :-

If a seqn $\{b_n\}$ is a monotonic bounded seqn. and $\sum_{n=1}^{\infty} a_n$ is a Convergent series. then $\sum_{n=1}^{\infty} a_n b_n$ is Convergent series.

Q $\sum_{n=2}^{\infty} \frac{(-1)^{n+1} n^n}{(n+1)^{n+1}}$

$U_n = \frac{(-1)^{n+1} n^n}{(n+1)^{n+1}}$, $a_n = \frac{(-1)^{n+1}}{n+1}$, $b_n = \frac{1}{(1+\frac{1}{n})^n}$

$\{b_n\}$ is monotonic bounded.

$\sum_{n=1}^{\infty} a_n$ is Convergent (By Leibnitz test).

So, By Abel test

$\sum_{n=1}^{\infty} a_n b_n$ is Convergent.

$\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^n}{(n+1)^{n+1}}$ is Convergent.

② Dirichlet test :-

If $\{b_n\}$ is monotonic bounded seq^n Converges to '0'. and the seq^n of partial sum $\{A_n\}$ of series $\sum_{n=1}^{\infty} a_n$ is bounded. then

$\sum_{n=1}^{\infty} a_n b_n$ is a Convergent series.

Q. $\sum_{n=1}^{\infty} \frac{\sin nx}{n}$

Let $a_n = \sin nx$, $b_n = \frac{1}{n}$

Clearly, $\{b_n\} = \{\frac{1}{n}\}$ is monotonic Decreasing and Converging to '0'.

Now, $A_n = \sum_{k=1}^{n} a_k = \sin x + \sin 2x + \sin 3x + \cdots + \sin nx$

$$= \frac{\sin\left(\frac{nx}{2}\right)}{\sin\left(\frac{x}{2}\right)} \sin (n+1)\frac{x}{2} \leq \frac{1}{\sin\left(\frac{x}{2}\right)}$$

$\Rightarrow \{A_n\}$ is bounded.

By Dirichlet test. $\sum a_n b_n \leftarrow$ Convergent.

$\Rightarrow \sum_{n=1}^{\infty} \frac{\sin nx}{n}$ is Convergent.

☆ $S_n = \sin x + \sin 2x + \sin 3x + \text{---} + \sin nx$

$2\sin\frac{x}{2}\cdot S_n = 2\sin\frac{x}{2}\cdot\sin x + 2\sin\frac{x}{2}\cdot\sin 2x + 2\sin\frac{x}{2}\sin 3x + \text{---}$
$\text{---} + 2\sin\frac{x}{2}\cdot\sin nx.$

$= \cos\frac{x}{2} - \cancel{\cos\frac{3x}{2}} + \cancel{\cos\frac{3x}{2}} - \cos\frac{5x}{2} + \text{---}$
$\text{--} + \cancel{\cos\frac{2n-1}{2}x} - \cos\frac{2n+1}{2}x.$

$= \cos\frac{x}{2} - \cos\frac{(2n+1)}{2}x$

$= 2\sin\left(\frac{2n+1+1}{4}\right)\cdot\sin\left(\frac{2n+1-1}{4}\right)$

$2\sin\frac{x}{2}\cdot S_n = 2\sin\left(\frac{n+1}{2}\right)x\cdot\sin\left(\frac{nx}{2}\right)$

$$\boxed{S_n = \frac{\sin\left(\frac{nx}{2}\right)}{\sin\left(\frac{x}{2}\right)}\cdot\sin\left(\frac{n+1}{2}\right)x.}$$

☆

$$\boxed{\cos x + \cos 2x + \cos 3x + \text{---} + \cos nx = \frac{\sin\left(\frac{nx}{2}\right)}{\sin\left(\frac{x}{2}\right)}\cdot\cos(n+1)\frac{x}{2}}$$

Q $\frac{1}{1\cdot 2} + \frac{1}{3\cdot 4} + \frac{1}{5\cdot 6} + \text{---}$

$\Rightarrow \left(1-\frac{1}{2}\right) + \left(\frac{1}{3}-\frac{1}{4}\right) + \left(\frac{1}{5}-\frac{1}{6}\right) + \text{---}$

$\Rightarrow 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \text{---}$

$\Rightarrow \boxed{\ln(2)}$

Q $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+-$ ——

$\because \tan^{-1}(x) = x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+$ ——

$\therefore \tan^{-1}(1) = 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+$ ——

$\Rightarrow \boxed{\frac{\pi}{4}}$

Q $\frac{1}{1\cdot 3}+\frac{1}{5\cdot 7}+\frac{1}{9\cdot 11}+$ ——

$=\frac{1}{2}\left(1-\frac{1}{3}\right)+\frac{1}{2}\left(\frac{1}{5}-\frac{1}{7}\right)+\frac{1}{2}\left(\frac{1}{9}-\frac{1}{11}\right)+$ ——

$=\frac{1}{2}\left[1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+ \text{——}\right]$

$=\frac{1}{2}\times\frac{\pi}{4} = \boxed{\frac{\pi}{8}}$

Q $\sum_{n=1}^{\infty} \frac{\sqrt[3]{n^3+1}-n}{n(\log n)^p}$, $p>0$

$a_n = \frac{1}{n(\log n)^p}$, $b_n = \sqrt[3]{n^3+1}-n$

a_n is (+)ive, monotonic decreasing.

Now, $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \sqrt[3]{n^3+1}-n$

$= \cancel{n\left[\left(1+\frac{1}{n^3}\right)^{1/3}\right.} -$

$= \sum_{n=1}^{\infty} \frac{\left(\sqrt[3]{n^3+1}-n\right)\left((n^3+1)^{2/3}+n(n^3+1)^{1/3}+n^2\right)}{\left((n^3+1)^{2/3}+n(n^3+1)^{1/3}+n^2\right)}$

$= \sum_{n=1}^{\infty} \frac{n^3+1-n^3}{(n^3+1)^{2/3}+n(n^3+1)^{1/3}+n^2}$ By Comparision test.

$b_n = \frac{1}{n^2}$.

So, Convergent

★ $S_n = \sin x + \sin 2x + \sin 3x + \cdots\cdots + \sin nx$

$$2\sin\frac{x}{2}\cdot S_n = 2\sin\frac{x}{2}\cdot\sin x + 2\sin\frac{x}{2}\cdot\sin 2x + 2\sin\frac{x}{2}\cdot\sin 3x + \cdots\cdots + 2\sin\frac{x}{2}\cdot\sin x.$$

$$= \cos\frac{x}{2} - \cancel{\cos\frac{3x}{2}} + \cancel{\cos\frac{3x}{2}} - \cancel{\cos\frac{5x}{2}} + \cdots\cdots + \cancel{\cos\frac{2n-1}{2}x} - \cos\frac{2n+1}{2}x.$$

$$= \cos\frac{x}{2} - \cos\frac{(2n+1)}{2}x$$

$$= 2\sin\left(\frac{2n+1+1}{4}\right)\cdot\sin\left(\frac{2n+1-1}{4}\right)$$

$$2\sin\frac{x}{2}\cdot S_n = 2\sin\left(\frac{n+1}{2}\right)x\cdot\sin\left(\frac{nx}{2}\right)$$

$$\boxed{S_n = \frac{\sin\left(\frac{nx}{2}\right)}{\sin\left(\frac{x}{2}\right)}\cdot\sin\left(\frac{n+1}{2}\right)x.}$$

★ $$\boxed{\cos x + \cos 2x + \cos 3x + \cdots + \cos nx = \frac{\sin\left(\frac{nx}{2}\right)}{\sin\left(\frac{x}{2}\right)}\cdot\cos(n+1)\frac{x}{2}}$$

Q $\frac{1}{1\cdot 2} + \frac{1}{3\cdot 4} + \frac{1}{5\cdot 6} + \cdots\cdots$

$\Rightarrow \left(1-\frac{1}{2}\right) + \left(\frac{1}{3}-\frac{1}{4}\right) + \left(\frac{1}{5}-\frac{1}{6}\right) + \cdots$

$\Rightarrow 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots\cdots$

$\Rightarrow \ln(2)$

Q $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+-$ ———

$\because \tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} +$ ———

$\therefore \tan^{-1}(1) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} +$ ———

$\Rightarrow \boxed{\frac{\pi}{4}}$

Q $\frac{1}{1\cdot3} + \frac{1}{5\cdot7} + \frac{1}{9\cdot11} +$ ———

$= \frac{1}{2}\left(1-\frac{1}{3}\right) + \frac{1}{2}\left(\frac{1}{5}-\frac{1}{7}\right) + \frac{1}{2}\left(\frac{1}{9}-\frac{1}{11}\right) +$ ———

$= \frac{1}{2}\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \text{———}\right]$

$= \frac{1}{2} \times \frac{\pi}{4} = \boxed{\frac{\pi}{8}}$

Q $\sum_{n=1}^{\infty} \frac{\sqrt[3]{n^3+1} - n}{n(\log n)^p}, \; p>0$

$a_n = \frac{1}{n(\log n)^p}, \quad b_n = \sqrt[3]{n^3+1} - n$

a_n is (+)sive, monotonic decreasing.

Now, $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \sqrt[3]{n^3+1} - n$

$= \cancel{n\left[\left(1+\frac{1}{n^3}\right)^{1/3}\right.} -$

$= \sum_{n=1}^{\infty} \frac{(\sqrt[3]{n^3+1} - n)\left((n^3+1)^{2/3} + n(n^3+1)^{1/3} + n^2\right)}{\left((n^3+1)^{2/3} + n(n^3+1)^{1/3} + n^2\right)}$

$= \sum_{n=1}^{\infty} \frac{\cancel{n^3} + 1 - \cancel{n^3}}{(n^3+1)^{2/3} + n(n^3+1)^{1/3} + n^2}$

By Comparis test.

$b_n = \frac{1}{n^2}$

So, Convergent

☆ Rearrangement of terms of a series :-

① Dirichlet Theorem :-

Every rearrangement of terms of an absolutely convergent series is absolutely convergent series and the series obtained by rearrangement of terms converges to the same sum.

② Riemann Theorem :-

By appropriate rearrangement of terms, a conditionally convergent series can be made

(i) to converge to any number l.

(ii) to diverge to ∞.

(iii) to diverge to $-\infty$.

(iv) to oscillate finitely.

(v) to oscillate infinitely.

E.g. :- ① $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ ← Conditionally convergent series.

$= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \text{—} \text{—}$

(+)ive term $= 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots\cdots = \sum \frac{1}{2n-1}$

(−)ive term $= \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \cdots\cdots = \sum \frac{1}{2n}$

~~② $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$ is Convergent.~~

~~But $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.~~

E.g. $a_n^2 = \frac{1}{n^2}$, $a_n = \frac{1}{n}$

Q If $\sum a_n^2$ is Convergent then $\sum a_n$ is Convergent.
(False)

Q If $\sum a_n$ is Convergent then $\sum a_n^2$ is Convergent.
(False)

E.g. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}} \to$ Convergent

But $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \to$ Divergent

Q If $\sum a_n$ is Convergent then $\sum n^{\frac{1}{n}} a_n$ is Convergent
(True)

($\because$ By able test. and we know that
$n^{\frac{1}{n}} \to$ Don't monotonic
But after some term
$n^{\frac{1}{n}}$ is monotonic.

Q If $\sum a_n^2$ and $\sum b_n^2$ are Convergent then $\sum a_n b_n$ is absolutely Convergent. (True)

proof:- $(|a_n| - |b_n|)^2 \geq 0$

$$\Rightarrow |a_n|^2 + |b_n|^2 - 2|a_n b_n| \geq 0$$

$$\Rightarrow |a_n b_n| \leq \frac{|a_n|^2 + |b_n|^2}{2}$$

$$\Rightarrow |a_n b_n| \leq \frac{a_n^2 + b_n^2}{2}$$

$$\therefore \sum |a_n b_n| \leq \sum \frac{a_n^2 + b_n^2}{2}$$

Since, $\sum a_n^2$ and $\sum b_n^2$ are Convergent.

$\Rightarrow \sum \frac{a_n^2+b_n^2}{2}$ is Convergent.

So, by Comparision test.

$\therefore \sum |a_n b_n|$ is Convergent.

$\Rightarrow \sum a_n b_n$ is absolutely Convergent.

☆ Cauchy product of two series :-

$\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ term by term product

$$\sum_{n=1}^{\infty} a_n b_n = a_1b_1 + a_2b_2 + a_3b_3 + --$$

Cauchy product $\left(\sum_{n=1}^{\infty} a_n\right) \cdot \left(\sum_{n=1}^{\infty} b_n\right)$

$$= (a_1 + a_2 + a_3 + --) \cdot (b_1 + b_2 + b_3 + --)$$

$$= a_1b_1 + a_1b_2 + a_1b_3 + --$$
$$+ a_2b_1 + a_2b_2 + a_2b_3 + ---$$
$$+ a_3b_1 + a_3b_2 + a_3b_3 + --$$
$$+$$

$$= a_1b_1 + (a_1b_2 + a_2b_1) + (a_1b_3 + a_2b_2 + a_3b_1) + ---$$

$\Rightarrow$ If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are absolutely Convergent then their Cauchy product series $\left(\sum a_n\right)\left(\sum_{n=1}^{\infty} b_n\right)$ is absulutely Convergent.

more over, If $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$ then

$$\left(\sum_{n=1}^{\infty} a_n\right)\left(\sum_{n=1}^{\infty} b_n\right) = A \cdot B$$

⇒ If $\sum_{n=1}^{\infty} a_n$ is a convergent series and $\sum_{n=1}^{\infty} b_n$ is absolutely convergent series then their Cauchy product $\left(\sum_{n=1}^{\infty} a_n\right)\left(\sum_{n=1}^{\infty} b_n\right)$ is convergent.

But if may ~~or~~ not be absolutely convergent.

⇒ If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent series then But not absolutely convergent series then their Cauchy product $\left(\sum_{n=1}^{\infty} a_n\right)\left(\sum_{n=1}^{\infty} b_n\right)$ may not be convergent.

However, If $\left(\sum_{n=1}^{\infty} a_n\right)\left(\sum_{n=1}^{\infty} b_n\right)$ is convergent and $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$ then

$$\left(\sum_{n=1}^{\infty} a_n\right)\left(\sum_{n=1}^{\infty} b_n\right) = A \cdot B.$$

Power Series

|| Power Series ||

A series of the form

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \text{---} = \sum_{n=0}^{\infty} a_n x^n, \; a_n \in \mathbb{R}$$

is called a power series in x about $x=0$.

⇒ More general form of power series is

$$a_0 + a_1(x-\alpha) + a_2(x-\alpha)^2 + a_3(x-\alpha)^3 + \text{---} = \sum_{n=0}^{\infty} a_n (x-\alpha)^n$$

is a power series in x about $x=\alpha$.

Eg:

$$F(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \text{---}$$

Put $x=0$, $F(0) = 1+0+0+0+ \text{---}$

$$F\left(\tfrac{1}{2}\right) = 1 + \tfrac{1}{2} + \left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^3 + \text{---}$$

$$F(2) = 1 + 2 + 2^2 + 2^3 + \text{---}$$

$$F\left(-\tfrac{1}{2}\right) = 1 - \tfrac{1}{2} + \tfrac{1}{2^2} - \tfrac{1}{2^3} + \text{---}$$

$$F(-1) = 1 - 1 + 1 - 1 + 1 - 1 + \text{---}$$

$$F(-2) = 1 - 2 + 2^2 - 2^3 + \text{---}$$

So, The power series $\sum_{n=0}^{\infty} x^n$ is Convergent, if $|x|<1$

Divergent, if $|x| \geq 1$

⇒ A power series $\sum_{n=0}^{\infty} a_n x^n$ is always Convergent at $x=0$.

☆ Nowhere Convergent power series :-

A power series $\sum_{n=0}^{\infty} a_n x^n$ is said to be nowhere convergent. If it is convergent at $x=0$.

☆ Every where Convergent power Series :-

A power series $\sum_{n=0}^{\infty} a_n x^n$ is said to be everywhere convergent. If it is convergent at every real number.

☆ Region of Convergence :-

A set of real number S is called region of convergence of a power series $\sum_{n=0}^{\infty} a_n x^n$. If the series is convergent at every $x \in S$ and divergent at every $x \notin S$.

$\Rightarrow$ If power series is $\sum_{n=0}^{\infty} a_n (x-\alpha)^n$ then it can be reduced to $\sum_{n=0}^{\infty} a_n y^n$ by substituting $y = x - \alpha$.

Examples:-

① Geometric series:- $\sum_{n=0}^{\infty} K x^n$

where $a_n = K \quad \forall n \in \mathbb{N} \cup \{0\}$.

② Taylar series:- $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} \cdot (x-a)^n$, So, $a_n = \frac{f^{(n)}(a)}{n!}$

③ Maclaurin series:- $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$ is a power series

$a_n = \frac{f^{(n)}(0)}{n!}$

☆ If a power series $\sum_{n=0}^{\infty} a_n x^n$ is Convergent at $x = x_0$. then it is absolutely Convergent at every x_1. where $\boxed{|x_1| < |x_0|}$

Eg. $\sum a_n x^n$ is Convergent at $x = 3$, $\forall |x| < 3$.

proof:- Given that $\sum_{n=0}^{\infty} a_n x^n$ is Convergent at $x = x_0$

$\Rightarrow \sum_{n=0}^{\infty} a_n x_0^n$ is Convergent.

$\Rightarrow \lim_{n \to \infty} a_n x_0^n = 0$

$\Rightarrow \{a_n x_0^n\}$ is Convergent.

$\Rightarrow \{a_n x_0^n\}$ is bounded.

$\Rightarrow \exists K \in \mathbb{R}$ s.t. $|a_n x_0^n| \leq K$.

Now, Consider

$$\sum_{n=0}^{\infty} |a_n x_1^n| = \sum_{n=0}^{\infty} \left| a_n x_0^n \cdot \frac{x_1^n}{x_0^n} \right|$$

$$= \sum_{n=0}^{\infty} |a_n x_0^n| \left| \frac{x_1^n}{x_0^n} \right|$$

$$= \sum_{n=0}^{\infty} |a_n x_0^n| \left| \frac{x_1}{x_0} \right|^n$$

$$\leq K \sum_{n=0}^{\infty} \left| \frac{x_1}{x_0} \right|^n$$

So, if $|x_1| < |x_0|$ then $\left|\frac{x_1}{x_0}\right| < 1$

So, the geometric series

$\sum_{n=0}^{\infty} \left| \frac{x_1}{x_0} \right|^n$ is Convergent.

$\therefore$ By Comparision test.

$\sum_{n=0}^{\infty} |a_n x_1^n|$ is Convergent.

$\Rightarrow \sum_{n=0}^{\infty} a_n x_1^n$ is absolutely convergent.

★ If a power series $\sum_{n=0}^{\infty} a_n x^n$ is divergent at $x = x'$ then $\sum_{n=0}^{\infty} a_n x^n$ is divergent at every x'' with $|x''| > |x'|$.

Proof: Given that $\sum_{n=0}^{\infty} a_n x^n$ is divergent at x'.

If $\sum_{n=0}^{\infty} a_n x^n$ is Convergent at x'' with $|x''| > |x'|$ then it must be absolutely Convergent at every x with $|x| < |x''|$

since, $|x'| < |x''|$

So, $\sum_{n=0}^{\infty} a_n x^n$ Should be absolutely Convergent.

But $\sum_{n=0}^{\infty} a_n x^n$ is divergent.

So, By Contradiction,

$\sum_{n=0}^{\infty} a_n x^n$ Can never be Convergent at x'' with $|x''| > |x'|$

* ☆ If a power series $\sum_{n=0}^{\infty} a_n x^n$ is neither nowhere Convergent $(R=0)$ nor everywhere Convergent $(R=\infty)$ then their exists a (+)ive real no. R s.t. the series converges absolutely for every x with $\boxed{|x| < R}$ and diverges for every x with $|x| > R$.

R is Called the radius of Convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$.

⇒ Radius of Convergence Can never be negative.

☆ If the radius of Convergence of a power series $\sum_{n=0}^{\infty} a_n x^n$ is R then for $|x| = R$,

i.e. $\sum_{n=0}^{\infty} a_n R^n$ or $\sum_{n=0}^{\infty} (-1)^n a_n R^n$ may Converge or Diverge.

☆ Interval of Convergence :-

Let R be the radius of Convergence of a power series $\sum_{n=0}^{\infty} a_n x^n$.

then the interval of Convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$ is

(i) $(-R, R)$, If Both $\sum_{n=0}^{\infty} a_n R^n$ and $\sum_{n=0}^{\infty} (-1)^n a_n R^n$ is divergent.

(ii) $[-R, R)$, If $\sum_{n=0}^{\infty} a_n R^n$ is divergent and $\sum_{n=0}^{\infty} (-1)^n a_n R^n$ is Convergent.

(iii) $(-R, R]$, If $\sum_{n=0}^{\infty} a_n R^n$ is Convergent and $\sum_{n=0}^{\infty} (-1)^n a_n R^n$ is divergent.

(iv) $[-R, R]$, If Both $\sum_{n=0}^{\infty} a_n R^n$ and $\sum_{n=0}^{\infty} (-1)^n a_n R^n$ are Convergent.

★ Determination of radius of Convergence:-

① Cauchy Hadamard Theorem:-

Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series and $\boxed{\mu = \overline{\lim_{n\to\infty}} |a_n|^{1/n}}$ $R = \frac{1}{\mu}$

→ If $\mu = 0$, then the series is everywhere Convergent.

→ If $0 < \mu < \infty$, then the series is Convergent for every x with $|x| < R \left(= \frac{1}{\mu}\right)$ and Divergent for every x with $|x| > R \left(= \frac{1}{\mu}\right)$

→ If $\mu = \infty$, then the series is nowhere Convergent.

So, $\boxed{\text{Radius of Convergence } R = \frac{1}{\mu}}$

② Ratio Test Theorem:-

Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series and $\boxed{\mu = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|}$ $R = \frac{1}{\mu}$

OR, $R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$

→ If $\mu = 0$ then the series is everywhere Convergent.

→ If $0 < \mu < \infty$, then the series is convergent for every x with $|x| < R$ and divergent for every x with $|x| > R$.

→ If $\mu = \infty$, then the series is nowhere convergent.

So, Radius of Convergence $R = \frac{1}{\mu}$.

$$\Rightarrow \boxed{\underline{\lim}\left|\frac{a_{n+1}}{a_n}\right| \le \underline{\lim}(a_n)^{\frac{1}{n}} \le \overline{\lim}(a_n)^{\frac{1}{n}} \le \overline{\lim}\left|\frac{a_{n+1}}{a_n}\right|}$$

we have find radius of convergence only Cauchy hadamard thororem.

Q Find the radius of Convergence :-

① $\sum_{n=1}^{\infty} \frac{(-1)^n n^n}{n!\, 2^n} x^n$

$$a_n = \frac{n^n}{n!\, 2^n}, \quad a_{n+1} = \frac{(n+1)^{n+1}}{(n+1)!\, 2^{n+1}}$$

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \frac{n^n}{n!\, 2^n} \times \frac{(n+1)!\, 2^{n+1}}{(n+1)^{n+1}} = \boxed{\frac{2}{e} = R}$$

② $\sum_{n=1}^{\infty} \frac{2^n}{n^2} \cdot x^n$

$$a_n = \frac{2^n}{n^2} \quad \Rightarrow \mu = \lim_{n\to\infty} (a_n)^{\frac{1}{n}} = \lim_{n\to\infty} \left(\frac{2^n}{n^2}\right)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{2}{(n^{\frac{1}{n}})^2}$$

$\boxed{\mu = 2}$ $\boxed{R = \frac{1}{2}}$

③ $\sum_{n=1}^{\infty}(2^n+3^n)\cdot x^n$

$$a_n = 2^n+3^n = 3^n\left[\left(\frac{2}{3}\right)^n+1\right]$$

$\left(\because \left(\frac{2}{3}\right)^n=0 \quad \because r^n=0 \text{ wher } |r|<1\right)$

$$\mu = \lim_{n\to\infty}(a_n)^{\frac{1}{n}} = \lim_{n\to\infty} 3\left[\left(\frac{2}{3}\right)^n+1\right]^{\frac{1}{n}} = 3$$

$$\boxed{R=\frac{1}{3}}$$

(4.) $\sum_{n=1}^{\infty}\frac{2^n}{3^n}x^{2n}$

$$a_{2n} = \left(\frac{2}{3}\right)^n$$

$$\mu = \lim_{n\to\infty}(a_{2n})^{\frac{1}{2n}} = \lim_{n\to\infty}\left[\left(\frac{2}{3}\right)^n\right]^{\frac{1}{2n}} = \sqrt{\frac{2}{3}}$$

$$\boxed{R=\frac{1}{\mu}=\sqrt{\frac{3}{2}}}$$

⑤ $\sum_{n=1}^{\infty}a_n x^n$, $a_1=1$ where a_n is the no. of prime divisors of n.

Sol^n:- $\forall n\in N$

$$1\le a_n\le n$$

$$1^{\frac{1}{n}}\le a_n^{\frac{1}{n}}\le n^{\frac{1}{n}}$$

$$\lim_{n\to\infty}1^{\frac{1}{n}}\le\lim_{n\to\infty}(a_n)^{\frac{1}{n}}\le\lim_{n\to\infty}n^{\frac{1}{n}}$$

$$\Rightarrow \mu \Rightarrow \lim_{n\to\infty}(a_n)^{\frac{1}{n}} = 1$$

$$\boxed{R=\frac{1}{\mu}=1}$$

⑥ $\sum_{n=1}^{\infty}(n!)\,x^{n^2}$, $a_{n^2}=n!$

$$\mu = \lim_{n\to\infty}(a_{n^2})^{\frac{1}{n^2}} = \lim_{n\to\infty}(n!)^{\frac{1}{n^2}} = 1 \qquad \boxed{R=1}$$

✓ ⑦ $\sum_{n=1}^{\infty} x^{n!}$

$a_{n!} = 1$

$\mu = \lim_{n\to\infty} (a_{n!})^{\frac{1}{n!}} = \lim_{n\to\infty} 1^{\frac{1}{n!}} = 1$

$\boxed{R = \frac{1}{\mu} = 1}$

✓ ⑧ $\sum_{n=1}^{\infty} n!\, x^{n!}$

$a_{n!} = n!$

$\mu = \lim_{n\to\infty} (a_{n!})^{\frac{1}{n!}} = \lim_{n\to\infty} (n!)^{\frac{1}{n!}} = 1$

$\boxed{R=1}$

☆ ⑧ $\sum_{n=1}^{\infty} a_n x^n$ where $a_n = \frac{1}{3^n}$, n is odd.

$\frac{1}{2^n}$, n is even.

$\mu = \lim_{n\to\infty} (a_n)^{\frac{1}{n}} = \lim_{n\to\infty} \left(\frac{1}{3^n}\right)^{\frac{1}{n}} = \frac{1}{3}$, n is odd.

$\lim_{n\to\infty} \left(\frac{1}{2^n}\right)^{\frac{1}{n}} = \frac{1}{2}$, n is even.

$\mu = \left(\frac{1}{2}, \frac{1}{3}\right) \to \max \Rightarrow \mu = \frac{1}{2}$

$\boxed{R = \frac{1}{\mu} = 2}$

Q Find the interval of convergence of the following power series.

① $\sum_{n=1}^{\infty} \frac{x^n}{2^n \cdot n^2}$

$a_n = \frac{1}{2^n \cdot n^2} \Rightarrow \mu = \lim_{n\to\infty} (a_n)^{\frac{1}{n}} = \lim_{n\to\infty} \left(\frac{1}{2^n \cdot n^2}\right)^{\frac{1}{n}} = \frac{1}{2}$

$\boxed{R = \frac{1}{\mu} = 2}$

$$\sum_{n=1}^{\infty} \frac{\cancel{2^n}}{\cancel{2^n}\cdot n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$ is Convergent.

$$\sum_{n=1}^{\infty} \frac{(-1)^n \cancel{2^n}}{\cancel{2^n}\cdot n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$ is Convergent.

$[-2, 2]$ &

② $$\sum_{n=1}^{\infty} \frac{x^n}{2^n \cdot n}$$

$a_n = \frac{1}{2^n \cdot n}$ so, $\boxed{R = 2}$

$$\sum_{n=1}^{\infty} \frac{\cancel{2^n}}{\cancel{2^n}\cdot n} = \sum_{n=1}^{\infty} \frac{1}{n}$$ is divergent.

$$\sum_{n=1}^{\infty} \frac{(-2)^n}{2^n \cdot n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$ is Convergent.

$[-2, 2)$.

③ $$\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{2^n \cdot n}$$

$a_n = \frac{1}{2^n \cdot n} \Rightarrow$ so, $\boxed{R = 2} \rightarrow$

$$\sum_{n=1}^{\infty} \frac{(-1)^n \cancel{2^n}}{\cancel{2^n}\cdot n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$ is Convergent.

$$\sum_{n=1}^{\infty} \frac{(-1)^n (-2)^n}{2^n \cdot n} = \sum_{n=1}^{\infty} \frac{(-1)^{2n} \cancel{2^n}}{\cancel{2^n}\cdot n} = \sum_{n=1}^{\infty} \frac{1}{n} \rightarrow$$ divergent.

$(-2, 2]$

④ $\sum_{n=1}^{\infty} \frac{x^{2n}}{2^n \cdot n}$

$$a_{2n} = \frac{1}{2^n \cdot n}$$

$$\mu = \lim_{n\to\infty} (a_{2n})^{\frac{1}{2n}} = \lim_{n\to\infty} \left(\frac{1}{2^n \cdot n}\right)^{\frac{1}{2n}} = \frac{1}{\sqrt{2}}$$

$$\boxed{R = \sqrt{2}}$$

$$\sum_{n=1}^{\infty} \frac{(\sqrt{2})^{2n}}{\cancel{2^n} \cdot n} = \sum_{n=1}^{\infty} \frac{1}{n} \to \text{divergent}$$

$$\sum_{n=1}^{\infty} \frac{(-\sqrt{2})^{2n}}{\cancel{2^n} \cdot n} = \sum_{n=1}^{\infty} \frac{1}{n} \to \text{Divergent}$$

$$(-\sqrt{2}, \sqrt{2})$$

✯ Properties of power series :-

① Let $\sum_{n=0}^{\infty} a_n x^n$ is a power series with radius of convergence $R>0$.

Let $f(x)$ be the sum of the power series in $(-R, R)$ then $f(x)$ is continuous in $(-R, R)$.

② Power series can be integrated (differentiated) term by term on any closed and bounded interval within the interval of convergence.

③ If R is the radius of Convergence of a power series $\sum_{n=0}^{\infty} a_n x^n$ then the series obtained by term by term integration.

$\sum_{n=1}^{\infty} \frac{a_n}{n+1} \underline{\underline{x^{n+1}}}$ has radius of Convergence R $\underline{\underline{\text{only}}}$

$$\left[\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = R \Rightarrow \lim_{n\to\infty}\left|\frac{a_n}{n+1}\cdot\frac{n+2}{a_{n+1}}\right| = R\right]$$

④ If R is the radius of Convergence of a power series $\sum_{n=0}^{\infty} a_n x^n$ then the series obtained by term by term differentiation

$\sum_{n=1}^{\infty} a_n \cdot n x^{n-1}$ has radius of Convergence R only.

⑤ Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of Convergence R.

Let the Sum of the series be f(x) on (-R, R). If $\underline{\sum_{n=0}^{\infty} a_n R^n}$ is Convergent then

$$\boxed{\sum_{n=0}^{\infty} a_n R^n = \lim_{x\to R^{-}} f(x)}$$

If $\underline{\sum_{n=0}^{\infty} (-1)^n a_n R^n}$ is Convergent then

$$\boxed{\sum_{n=0}^{\infty} (-1)^n a_n R^n = \lim_{x\to -R^{+}} f(x)}$$

<u>However</u>,:- If $\lim_{x \to R^-} f(x)$ or $\lim_{x \to -R^+} f(x)$ exists finitely then $\sum_{n=0}^{\infty} a_n R^n$ or $\sum_{n=0}^{\infty} (-1)^n a_n R^n$ may not be Convergent.

Eg. Let $\sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + \cdots = \frac{1}{1+x}$

$\forall \; -1 < x < 1.$

Now,

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \frac{1}{1+x} = \frac{1}{2}$$

But $\sum_{n=0}^{\infty} (-1)^n 1^n = \sum_{n=0}^{\infty} (-1)^n$ is not Convergent.

⑥ If $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ be two power series which have Same radius of Convergence and have the Same Sum function $f(x)$ than the two power series are identical.

i.e. $a_n = b_n \;, \forall n \in \mathbb{N} \cup \{0\}$

(Uniquence Convergence theorem)

⑦ If R_1 and R_2 are radii of Convergence of two power series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ respectively.

and If $\sum_{n=0}^{\infty} a_n x^n = f(x)$ on $(-R_1, R_1)$ and

$\sum_{n=0}^{\infty} b_n x^n = g(x)$ on $(-R_2, R_2)$ then radius of Convergence of $\sum_{n=0}^{\infty} (a_n+b_n)x^n$ is R,

where $\boxed{R = \min\{R_1, R_2\}}$

and $\sum_{n=0}^{\infty}(a_n+b_n)x^n = f(x)+g(x)$, $\forall x \in (-R, R)$

☆ If power series is $\sum_{n=0}^{\infty} a_n(x-\alpha)^n$ then it has Same radius of Convergence as that of $\sum_{n=0}^{\infty} a_n x^n$.

But if

(i) $\sum_{n=0}^{\infty} a_n R^n$ and $\sum_{n=0}^{\infty} (-1)^n a_n R^n$ Both are divergent then interval of Convergence is $(\alpha-R, \alpha+R)$.

(ii) $\sum_{n=0}^{\infty} a_n R^n$ and $\sum_{n=0}^{\infty} (-1)^n a_n R^n$ Both are Convergent then interval of Convergence is $[\alpha-R, \alpha+R]$.

(iii) If $\sum_{n=0}^{\infty} a_n R^n$ is Convergent and $\sum_{n=0}^{\infty} (-1)^n a_n R^n$ is divergent then interval of Convergence is $(\alpha-R, \alpha+R]$

(iv) $\sum_{n=0}^{\infty} a_n R^n$ is Divergent and $\sum_{n=0}^{\infty} (-1)^n a_n R^n$ is Convergent then interval of Convergence is $[\alpha-R, \alpha+R)$.

Q ① $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n+1}(x+1)^n$

$a_n = \frac{1}{n+1}$, $a_{n+1} = \frac{1}{n+2}$

$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\frac{n+2}{n+1} = 1 \Rightarrow \boxed{R=1}$

$|x+1| < 1$

$-1 < x+1 < 1$

$-2 < x < 0$

$\Rightarrow \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{n+1}(1)^n = \sum_{n=0}\frac{(-1)^{n+1}}{n+1} \to$ Convergen

$\Rightarrow \sum_{n=0}^{\infty}\frac{(-1)^{n+1}(-1)^n}{n+1} = -\sum_{n=0}^{\infty}\frac{(-1)^{2n}}{n+1} = -\sum_{n=0}^{\infty}\frac{1}{n+1} \leftarrow$ Divergen

$(-2, 0]$

② $\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(n+1)(n+2)}(x-2)^n$

$a_n = \frac{1}{(n+1)(n+2)}$, $a_{n+1} = \frac{1}{(n+2)(n+3)}$

$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{\cancel{(n+2)}(n+3)}{(n+1)\cancel{(n+2)}}\right| = 1 \Rightarrow \boxed{R=1}$

$|x-2| < 1$

$\Rightarrow -1 < x-2 < 1$

$\Rightarrow 1 < x < 3$

$\Rightarrow \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(n+1)(n+2)}(1)^n \to$ Convergent

$\Rightarrow \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(n+1)(n+2)}(-1)^n = -\sum_{n=0}^{\infty}\frac{1}{(n+1)(n+2)} \to$ Converge

$\Rightarrow [1, 3]$

(i) $\sum a_n x^n$

(ii) $\sum a_n (x-\alpha)^n$

Radius of Convergence (R) always Same.

But interval of Convergence

(ii) $|x-\alpha| < R \Rightarrow \alpha - R < x < \alpha + R$

Uniform Continuity

Uniform Continuity.

Continuity :-

A fuction is said to be Cont. in I.
Iff $\forall \epsilon > 0$ and $\underline{\forall x; y \in I \; , \exists \delta > 0}$ s.t.
$|x-y| < \delta \Rightarrow |f(x)-f(y)| < \epsilon$.

Uniform Continility :-

A fuction f is said to be Uniformly Continuous on I.
Iff $\forall \epsilon > 0 \; \underline{, \exists \delta > 0 \text{ s.t. } \forall x, y \in I}$, $|x-y| < \delta$
$\Rightarrow |f(x)-f(y)| < \epsilon$.

E.g. ① $f(x) = x^2$ on $[0,1]$ ← uniform Cont.

~~Let $\epsilon = \frac{1}{10}$ and $x=1$~~

Consider, $\forall x, y \in [0,1]$,
$$|f(x)-f(y)| = |x^2-y^2| = |x-y||x+y|$$
Now, Iff $|x-y| < \delta$ then $|x+y||x-y| < |x+y|\delta < 2\delta$ ($=\epsilon$ say)

So, ~~$|f(x)-f(y)| < 2\delta$~~ $\qquad \sup_{x \in [0,1]} |x+y| = 2$

So, $|x-y| < \delta \Rightarrow |f(x)-f(y)| < 2\delta$

Hence, $\forall \epsilon > 0$, $\exists \delta = \frac{\epsilon}{2} > 0$ s.t.
$|x-y| < \delta \Rightarrow |f(x)-f(y)| < \epsilon$, $\forall x, y \in [0,1]$

② $f(x) = x^2$ on $[0, \infty)$ is not U.C.

Consider, $|f(x)-f(y)| = |x^2-y^2| = |x-y|(x+y)$
then $|x-y| < \delta \Rightarrow |x^2-y^2| < |x+y|\delta$
which is not bd above. as in $[0,\infty)$.

$\sup_{x \in [0,\infty)} |x+y| = \infty$ So, $f(x) = x^2$ is not U.C. in $[0,\infty)$

③ $f(x) = \frac{1}{x}$ on $(0,1)$ is not U.C.

Consider, $|f(x) - f(y)| = \left|\frac{1}{x} - \frac{1}{y}\right| = \left|\frac{x-y}{xy}\right| = \frac{|x-y|}{|xy|}$

So, $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \frac{1}{|xy|}\delta$

But $\sup\limits_{x\in(0,1)} \frac{1}{|xy|} = \infty$

So, $f(x) = \frac{1}{x}$ is not U.C. on $(0,1)$.

④ $f(x) = \frac{1}{x}$ on (a,∞), $a>0$ is U.C.

$\because |f(x) - f(y)| = \frac{|x-y|}{|xy|}$

$\therefore |x-y| < \delta \Rightarrow |f(x) - f(y)| < \frac{1}{|xy|}\delta$

Now, $\sup\limits_{x\in(a,\infty)} \frac{1}{|xy|} = \frac{1}{a^2}$

$\therefore |f(x) - f(y)| < \frac{\delta}{a^2}$ ($= \epsilon$ say)

Hence, $\forall \epsilon > 0$, $\exists \delta = a^2\epsilon > 0$ s.t.

$|x-y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$, $\forall x, y \in (a,\infty)$.

⑤ $f(x) = \sqrt{x}$ on $[0,\infty)$ is U.C.

~~Consider $|f(x) - f(y)| = |\sqrt{x} - \sqrt{y}| = \left|\frac{x-y}{\sqrt{x}+\sqrt{y}}\right| = \frac{|x-y|}{|\sqrt{x}+\sqrt{y}|}$~~

~~$\therefore |x-y| < \delta \Rightarrow |f(x) - f(y)| < \frac{\delta}{|\sqrt{x}+\sqrt{y}|}$~~

Consider, $|f(x) - f(y)| = |\sqrt{x} - \sqrt{y}| \leq \sqrt{|x-y|}$

So, $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \sqrt{\delta}$ ($= \epsilon$ Say)

So, $\forall \epsilon > 0$, $\exists \delta = \epsilon^2 > 0$ s.t.

$|x-y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$, $\forall x, y \in [0,\infty)$

∴ $\sqrt{x}$ is U.C. on $[0,\infty)$

Infact, x^{α} is U.C. on $[0,\infty)$ where $0\le\alpha\le 1$

Every U.C. function on I is Cont. on I.

U.C. on I ⇒ Cont. on I.

Cont. on I ⇏ U.C. on I.

Not Cont. ⇒ Not. U.C.

If a function $f : I\to\mathbb{R}$ is diff. on I and $f'(x)$ is bd on I then $f(x)$ is U.C. on I.

(Converse not true e.g. $\sqrt{x}$)

proof:- From L.M.V.T.,

$$\left|\frac{f(x)-f(y)}{x-y}\right| = |f'(c)| \quad \text{for some } c\in(x,y)$$

$\Rightarrow |f(x)-f(y)| = |f'(c)||x-y|$

Since, $f'(x)$ is bd on I, So, $\exists M\in\mathbb{R}^+$ s.t.

$|f'(x)|\le M \ , \ \forall x\in I$

Hence, $|f(x)-f(y)|\le M|x-y|$

So, $|x-y|<\delta \Rightarrow |f(x)-f(y)|\le M\delta$ (say ϵ)

∴ $\forall\epsilon>0, \exists\ \delta=\frac{\epsilon}{M}>0$ s.t.

$|x-y|<\delta \Rightarrow |f(x)-f(y)|<\delta \ , \forall x,y\in I.$

Eg $f(x)=\tan^{-1}x$ on $\mathbb{R}$ is U.C.

∵ $|f'(x)| = \frac{1}{1+x^2}\le 1$

∴ $f(x)$ is U.C. on $\mathbb{R}$.

$\Rightarrow f(x) = \sin x$ is U.C. on $\mathbb{R}$.

The Converse is not true.

i.e. If $f : I \to \mathbb{R}$ is U.C on I then $f'(x)$ need not be bd on I.

E.g. $f(x) = \sqrt{x}$ is U.C. on $[0, \infty)$

But $f'(x) = \frac{1}{2\sqrt{x}}$ is not bd in $[0, \infty)$

A function $f : I \to \mathbb{R}$ is said to satisfy Lipschitz Condition of order α.

If $\exists L > 0$ s.t.

$$|f(x) - f(y)| \leq L|x-y|^{\alpha}, \quad \forall x, y \in I$$

L is called lipschitz Condition.

proof:- $|f(x) - f(y)| \leq L|x-y|^{\alpha}$

So, if $|x-y| < \delta \Rightarrow |f(x) - f(y)| \leq L|x-y|^{\alpha} < L\delta^{\alpha} (= \epsilon \text{ say})$

So, $\forall \epsilon > 0$, $\exists \delta = \left(\frac{\epsilon}{L}\right)^{1/\alpha} > 0$ s.t.

$|x-y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$.

$\therefore$ If a function satisfies the condition

$$|f(x) - f(y)| \leq L|x-y|^{\alpha}, \quad \forall x, y \in I$$

then $f(x)$ is U.C on I, $\forall \alpha > 0$.

U·C· on $[a,b]$:

$f:[a,b]\to\mathbb{R}$ is U·C· **iff**
f is Continuous on $[a,b]$.

U·C· on (a,b) :

$f:(a,b)\to\mathbb{R}$ is U·C· on (a,b)
iff (i) f is Cont. on (a,b)
(ii) $\lim_{x\to a^+} f(x)$ exists finitely.
(iii) $\lim_{x\to b^-} f(x)$ exists finitely.

Eg. $f(x)=\frac{\sin x}{x}$ on $(0,1)$ is U·C· on $(0,1)$

Q which of the following is not Correct?

① $\frac{\sin x}{x}$ is U.C. on $(0,1)$
② $\frac{\sin x}{x}$ is U.C. on $(0,2)$
③ $\frac{\tan x}{x}$ is U.C. on $(0,1)$
[✓ ④] $\frac{\tan x}{x}$ is U.C. on $(0,2)$

U·C· on $[a,\infty)$ or $(-\infty,a]$:

$f:[a,\infty)\to\mathbb{R}$ is U·C· on $[a,\infty)$. iff
(i) f is Cont. in $[a,\infty)$ and
(ii) $\lim_{x\to\infty} f(x)$ is finite.

(But if $\lim_{x\to\infty} f(x)$ is not finite then $f(x)$ may or may not be U·C·)

E.g. $f(x) = \sin x$ is U.C. on $[0, \infty)$. But $\lim_{x \to \infty} \sin x$ D.N.E.

★ **U.C. on (a, ∞):**

$f : (a, \infty) \to \mathbb{R}$ is U.C. on (a, ∞)

If (i) f is Cont. on (a, ∞) and

(ii) $\lim_{x \to a^+} f(x)$ exists finitely

(iii) $\lim_{x \to \infty} f(x)$ exists finitely.

★ If $f : (a, \infty) \to \mathbb{R}$ is not Cont. then f is not U.C.

★ If $\lim_{x \to a^+} f(x)$ D.N.E. finitely then f is not U.C.

★ If $\lim_{x \to \infty} f(x)$ D.N.E. finite then f may or may not be U.C.

★ **U.C. on $\mathbb{R}$:**

$f : \mathbb{R} \to \mathbb{R}$ is U.C. on $\mathbb{R}$.

If (i) f is Cont. on $\mathbb{R}$.

(ii) $\lim_{x \to \infty} f(x)$ exists finitely.

(iii) $\lim_{x \to -\infty} f(x)$ exists finitely.

⇒ If $\lim_{x \to \infty} f(x)$ D.N.E. finitely then f may or may not be U.C.

⇒ If $\lim_{x \to -\infty} f(x)$ ____________________

①

* properties of U.C :

① If f is Cont. on $\mathbb{R}$ then f is U.C. on every bd Sub-interval of $\mathbb{R}$.

② A Cont. bd function on bd open interval need not be U.C.

E.g. $\sin\frac{1}{x}$ on $(0,1)$

③ Every Cont. function defined on a Compact set is U.C.

④ If $f: D \to \mathbb{R}$ is Cont. then every seq. $\{x_n\}$ in D which Converges to x in D, $\{f(x_n)\}$ Converges to $f(x)$.

⑤ If $f: D \to \mathbb{R}$ is Cont. then for every seq^n $\{x_n\}$ in D, which is Cauchy, $\{f(x_n)\}$ need not be Cauchy.

E.g. $f: (0,1) \to \mathbb{R}$ s.t. $f(x) = \frac{1}{x}$

$\{x_n\} = \{\frac{1}{n}\}$ then $\{f(x_n)\} = n$

⑥ If $f: D \to \mathbb{R}$ is U.C. then for every seq^n $\{x_n\}$ in D, which is Cauchy, $\{f(x_n)\}$ is Cauchy seq^n.

⑦ If two functions f & g are U.C. on some interval I then for $\forall \alpha, \beta \in \mathbb{R}$,

$\alpha f + \beta g$ is ~~Cauchy~~ U.C on I.

⑧ If f $ g are U.C. on I then fg need not be U.C. on I.

Eg. $f(x) = x$ is U.C. on $\mathbb{R}$ But
$f^2(x) = x^2$ is not U.C. on $\mathbb{R}$

⑨ If f $ g are U.C. on Some bd interval I then fg is U.C. on I.

⑩ If f $ g are bd functions on I and are U.C. on I then fg is U.C. on I.

⑪ A Cont. periodic function on $\mathbb{R}$ is U.C. on $\mathbb{R}$.
→ Eg. sinx, cosx, tanx φ

⑫ Composition of two U.C. functions is a U.C. function.

⑬ If f(x) is U.C. on I and $|f(x)| \geq K > 0$ on I then $\frac{1}{f}(x)$ is U.C. on I.

⑭ If f(x) is U.C. on finitely many intervals then f is U.C. on their Union.

Eg → $\sin(x^2)$ ← bd $ Cont. on $\mathbb{R}$ But not U.C.

⑮ If f: $[a, \infty) \to \mathbb{R}$, $a > 0$ Satisfies $\lim_{x \to \infty} \frac{|f(x)|}{x} = \infty$ then f(x) is not U.C. on $[a, \infty)$

Eg: ① $f(x) = x\sin\frac{1}{x}$ on $(0,\infty)$ is U.C.

Sol^n (i) $f(x)$ is Cont. at $(0,\infty)$.

(ii) $\lim_{x\to\infty} x\sin\frac{1}{x} = \lim_{x\to\infty} \frac{\sin\frac{1}{x}}{\frac{1}{x}} = 1$

(iii) $\lim_{x\to 0} x\sin\frac{1}{x} = 0$

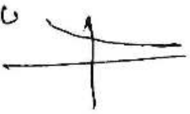
So, $f(x)$ is U.C. on $(0,\infty)$.

② $\tan^{-1}x$ on $\mathbb{R}$ is U.C. ∵ $f'(x) = \frac{1}{1+x^2} \le 1$ ← bd.

③ $\frac{x}{1+x^2}$ on $\mathbb{R}$ is U.C.

$f'(x) = \frac{1+x^2-2x^2}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2}$ ← bd.

④ e^{-x} on $[0,\infty)$ is U.C



$f'(x) = -e^{-x}$ ← bd

⑤ $|x|$ on $\mathbb{R}$ is U.C.

∵ $f'(x)$ is bd.

⑥ $|\sin x| + |\cos x|$ on $\mathbb{R}$ is U.C.

∵ $f'(x)$ is bd.

⑦ $\tan^{-1}(\sin^2 x)$ on $\mathbb{R}$

$f'(x) = \frac{2\sin x\cos x}{1+\sin^4 x} = \frac{|\sin 2x|}{|1+\sin^4 x|} \le \frac{1}{|1+\sin^4 x|}$ ← bd.

⑧ $\sqrt{|\sin x| + |\cos x|}$ on $\mathbb{R}$ is U.C.

∵ $\sqrt{x}$ is U.C.

⑨ $f(x) = x$ on $\mathbb{R}$ ← U.C.

⑩ $f(x) = x^{2/3}$ on $[0,\infty)$ ← U.C.

⑪ $f(x) = x^2$ on $[0,1]$ ← U.C.

⑫ $f(x) = x^2$ on $\mathbb{R}$. ← Not U.C.

⑬ $f(x) = \sin\frac{1}{x}$ on $(0,\infty)$ ← Not. U.C.

⑭ $f(x) = \frac{1}{1+x^2}$ on $\mathbb{R}$ $\cancel{\Leftrightarrow}$ $f'(x) = \frac{-2x}{1+x^2}$ ← U.C.

⑮ $f(x) = \sin x$ on $\mathbb{R}$ ← U.C.

⑯ $f(x) = \cos x$ on $\mathbb{R}$ ← U.C.

⑰ $f(x) = \tan x$ on $(0,\infty)$ ← Not. U.C.

⑱ $f(x) = \frac{1}{x}$ on $(0,\infty)$ ← ~~U.C.~~ Not. U.C.

⑲ $f(x) = \frac{1}{x}$ on $(1,\infty)$ ← U.C.

⑳ $f(x) = x^3$ on $[1,2]$ ← U.C.

㉑ $f(x) = \frac{x}{1+x^2}$ on $(0,\infty)$ $\cancel{\Leftrightarrow}$ $f'(x) = \frac{1+x^2-2x^2}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2} \leq$ ← U.C.

㉒ $f(x) = \cot^{-1}(x)$ on $\mathbb{R}$ → U.C.

㉓ $f(x) = \frac{\sin x}{x}$ on $(0,\infty)$ → U.C.

㉔ $f(x) = \begin{cases} \sin \pi x & , x \in (0,1] \\ x^2-1 & , x \in (1,2) \end{cases}$ → U.C

㉕. $f(x) = \sin(x^2)$ on $(0,\infty)$ → Not. U.C.

$|f(x) - f(y)| = |\sin(x^2) - \sin(y^2)| = |x^2 - y^2| = |x-y|\,(x+y) \leq |x+y|\,\delta$

$\underset{x\in(0,\infty)}{\sup}|x+y| = \infty$

(26) $f(x) = \begin{cases} x & , 0 \leq x < 1 \\ x^3 & , 1 \leq x \leq 3 \end{cases}$ ← U.C.

(27) $f(x) = \begin{cases} e^{\frac{-|x|}{2}} & , -1 \leq x < 0 \\ x^2 & , 0 \leq x < 2 \end{cases}$ ← Not U.C. (∵ not Cont.)

(28) $f(x) = \sqrt[3]{\log_3 \sqrt{x}} - 5\sqrt[4]{x} + \frac{x}{7}$ on $[1, \infty)$ ← U.C.

(29) $f(x) = \frac{\sin x}{x} + \sqrt{x}$ on $[1, \infty)$ ← U.C.

(31) $f(x) = \frac{x^2 + 2x + \sin x}{x+1}$ on $[0, 1]$ ← U.C.

(32) $f(x) = \frac{1}{2-x}$ on $(0, 1)$ ← U.C.

(33) $f(x) = \sqrt{|x|}$ on $\mathbb{R}$. ← U.C.

(34) $f(x) = \frac{\sin(x^2)}{\sin^2 x}$ on $(0, 1)$ ← U.C.

(35) $f(x) = \sin^2 x$ on $\mathbb{R}$. ← U.C.

(36) $f(x) = \sqrt{x} \sin \frac{1}{x^3}$ on $(0, 1)$ ← U.C.

(37) $f(x) = \frac{1}{1-x}$ on $(0, 1)$ ← Not. U.C. (∵ Limit D.N.E. at 1)

(38) $f(x) = \sin(x \sin x)$ on $(0, \infty)$ ← ~~Not U.C.~~ Not U.C.

(39) $f(x) = \ln x$ on $(0, \infty)$ → Not U.C.

(40) $f(x) = \ln x$ on $(0, 1)$ → Not U.C.

} (Limit D.N.E.) at 0

(41) $f(x) = \ln x$ on $(1, \infty)$ → U.C.

⑥

Sol^n (38). $x_n = 2n\pi + \frac{\pi}{n}$

$y_n = 2n\pi$

$x_n - y_n = \frac{\pi}{n} \to 0$ as $n \to \infty$

$\sin(x\sin x) - \sin(y\sin y)$

$x\sin x - y\sin y$

$$\lim_{n\to\infty} \left(2n\pi + \frac{\pi}{n}\right)\sin\left(2n\pi + \frac{\pi}{3}\right) - 2n\pi\sin(2n\pi)$$

$$= \lim_{n\to\infty} \left(2n\pi + \frac{\pi}{n}\right)\sin\frac{\pi}{n}$$

$$\lim_{n\to\infty} \sin\left(\left(2n\pi + \frac{\pi}{n}\right)\sin\left(2n\pi + \frac{\pi}{n}\right)\right) - \sin\left(2n\pi \sin 2n\pi\right)$$

$\lim_{n\to\infty}$

Uniform Convergence

✪ Let $\{f_n\}$ be a seq^n of functions defined on an interval I. Such that $f_n(x)$ exists for every $n \in \mathbb{N}$ and $x \in I$.

E.g. $f_n(x) = x^n$ on $[0,1]$

$f_n(x) = \tan^{-1}(nx)$ on $[0,\infty)$

$f_n(x) = \frac{nx}{1+n^2x^2}$ on $\mathbb{R}$

$f_n(x) = \frac{n^2x^2}{1+n^2x^2}$ on $\mathbb{R}$

✪ So, for a fixed $x = \alpha$, the seq^n of functions $\{f_n\}$ reduces to $\{f_n(\alpha)\}$, a seq^n of real numbers.

E.g. $f_n(x) = x^n$ on $[0,1]$ is seq^n of functions as

$\{f_n(x)\} = \{x, x^2, x^3, x^4, x^5, --\}$

and $x = \frac{1}{2}$ it reduces to a seq^n of No., as

$\{f_n(\frac{1}{2})\} = \{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, - -\}$

✪ Let $\sum f_n$ be a series of functions defined on an interval I s.t. $f_n(x)$ exists for every $n \in \mathbb{N}$ and $x \in I$.

E.g. $\sum f_n(x) = \sum x^n$ on $[0,1]$

$\sum f_n(x) = \sum \tan^{-1}(nx)$ on $[0,\infty)$

$\sum f_n(x) = \sum \frac{nx}{1+n^2x^2}$ on $\mathbb{R}$ / $\sum f_n(x) = \frac{n^2x^2}{1+n^2x^2}$ on $\mathbb{R}$

Ⓑ

So, for a fixed $x = \alpha$, the series of function $\sum f_n$ reduces to $\sum f_n(\alpha)$, a series of real No.

E.g. $\sum f_n(x) = \sum x^n$ on $[0,1]$ is series of functions, as

$$\sum f_n(x) = \sum_{n=1}^{\infty} x^n = \{x, x+x^2, x+x^2+x^3, ---\}$$

and $x = \frac{1}{2}$ it reduces to a series of a No. as

$$\sum f_n(\tfrac{1}{2}) = \sum_{n=1}^{\infty} \frac{1}{2^n} = \left\{\frac{1}{2}, \frac{1}{2}+\frac{1}{2^2}, \frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}, ---\right\}$$

☒ pointwise Convergent :-

Let $\{f_n\}$ $(\sum f_n(x))$ be a seqⁿ (series) of functions defined on some interval I. Seqⁿ $\{f_n\}$ (or series $\sum f_n$) is Said to be pointwise Convergent.

If $\forall \alpha \in I$, the seqⁿ of real numbers $\{f_n(\alpha)\}$ is Convergent.
(the series of real No. $\sum f_n(\alpha)$ is Convergent)

E.g. → $\sum x^n$ is pointwise convergent in $[0,1)$
But not ——————— $[0,1]$

→ $\{x^n\}$ is pointwise Convergent in $[0,1]$
But not ——————— $[0,2]$

ⓒ

If a seq^n of fuctions $\{f_n\}$ is paintwise Convergent on I then $\forall x \in I$, $\{f_n(x)\}$ Converges to Some number, Say l_x.

then Let $f: I \to \mathbb{R}$ s.t. $f(x) = l_x$ then $f(x)$ exists for Every $x \in I$. This fuction is Called limit fuction of seq^n $\{f_n\}$.

If $\{f_n\}$ is not paintwise Convergent then limit fuction does not exist on I.

$$\text{So,} \quad f(x) = \lim_{n\to\infty} f_n(x)$$

If a series of fuctions $\sum f_n$ is paintwise Convergent on I then $\forall x \in I$, $\sum f_n(x)$ Converges to Some number, Say S_x.

then Let $f: I \to \mathbb{R}$ s.t $f(x) = S_x$ then $f(x)$ exits for every $x \in I$. This fuction is Called Sum fuction of Series $\sum f_n$.

- If $\sum f_n$ is not paintwise convergent then Sum fuction does not exist on I.

$$\text{So,} \quad f(x) = \sum_{n=1}^{\infty} f_n(x)$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} f_r(x)$$

Q Find the limit or Sum function if the seqⁿ or series is point wise Convergent ?

① $\{f_n(x)\}$ where $f_n(x) = \frac{\sin nx}{\sqrt{n}}$ on $\mathbb{R}$.

② $\sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n}$ on $\mathbb{R}$

③ $\{nx(1-x^2)^n\}$ on $[0,1]$

④ $\{x^n\}$ on $[0,1]$

⑤ $\{\tan^{-1}(nx)\}$ on $[0,1]$

⑥ $\left\{\frac{nx}{1+n^2x^2}\right\}$ on $[0,1]$

⑦ $\left\{\frac{1}{x+n}\right\}$ on $[0,2]$

⑧ $\sum_{n=0}^{\infty} x^n(1-x^2)$ on $[0,1]$

⑨ $\left\{\frac{nx}{1+n^3x^2}\right\}$ on $[0,1]$

⑩ $\left\{\frac{x}{n+x}\right\}$ on $[0,1]$

⑪ $\{nx e^{-nx^2}\}$ on $[0,1]$

⑫ $\left\{\frac{x}{1+nx^2}\right\}$ on $\mathbb{R}$

⑬ $\{e^{-nx}\}$ on $[0,\infty)$

⑭ $\{n^2x(1-x^2)^n\}$ on $[0,1]$

⑮ $\left\{\frac{n^2x}{1+n^3x^2}\right\}$ on $[0,1]$. ⓔ

Sol^n – ① $f(x) = \lim_{n\to\infty} f_n(x)$

$= \lim_{n\to\infty} \frac{\sin nx}{\sqrt{n}}$ → U.C.

$= \underline{\underline{0}}$

$f(x) = 0, \ \forall x \in \mathbb{R}$

② $f(x) = \sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n}$

$= x^2 \sum_{n=0}^{\infty} \frac{1}{(1+x^2)^n}$ ←

$= x^2 \cdot \frac{1}{1-\frac{1}{1+x^2}}$

$= 1+x^2 \ , \ x \neq 0$

$\quad 0 \ , \ \underline{\underline{x=0}}$

③ $f(x) = \{nx(1-x^2)^n\}$

$\lim_{n\to\infty} nx(1-x^2)^n$ ← Not U.C.

$= 0 \quad \forall x \in [0,1]$

④ $f(x) = x^n$ on $[0,1]$

$= \lim_{n\to\infty} x^n$ → Not U.C.

$= 0 \ , \ x \in [0,1)$

$\quad 1 \ , \ \underline{\underline{x = 1}}$

ⓕ

⑤ $f(x) = \tan^{-1}(nx)$ on $[0,1]$

$$= \lim_{n\to\infty} \tan^{-1}(nx)$$

Not U.C.

$$= 0 \quad , x = 0$$

$$\frac{\pi}{2} \quad , x \in (0,1]$$

⑥ $\left\{ \frac{nx}{1+n^2x^2} \right\}$ on $[0,1]$

Not U.C.

$$f(x) = \lim_{n\to\infty} \frac{nx}{1+n^2x^2}$$

$$= 0 \quad , \forall x \in [0,1]$$

⑦ $\left\{ \frac{1}{x+n} \right\}$ on $[0,2]$

U.C.

$$f(x) = \lim_{n\to\infty} \frac{1}{x+n}$$

$$= 0 \quad \forall x \in [0,2]$$

⑧ $\sum_{n=0}^{\infty} x^n (1-x^2)$ · on $[0,1]$

$$f(x) = \lim_{n\to\infty} \sum_{n=0}^{\infty} x^n (1-x^2)$$

~~$= 0 \quad , \forall x \in [0,1]$~~

$$(1-x^2) \lim_{n\to\infty} \sum_{n=0}^{\infty} x^n$$

⑧

$$(1-x^2) \frac{1}{1-x} = 1+x$$

$$= 0 \quad , \quad x = 0, 1$$

$$1+x \quad , \quad x \in (0,1)$$

(g)

⑨ $\left\{ \frac{nx}{1+n^3x^2} \right\}$ on $[0,1]$ ← U.C.

$$f(x) = \lim_{n\to\infty} \frac{nx}{1+n^3x^2}$$

$$= x \lim_{n\to\infty} \frac{n}{1+n^3x^2}$$

$$= 0 \quad , \forall x \in [0,1]$$

⑩ $\left\{ \frac{x}{n+x} \right\}$ on $[0,1]$ ← U.C.

$$f(x) = \lim_{n\to\infty} \frac{x}{n+x}$$

$$= x \lim_{n\to\infty} \frac{1}{n+x}$$

$$= 0 \quad , \forall x \in [0,1]$$

$\frac{e^{-nx^2}}{1/n}$ $\frac{-x^2 e^{-nx^2}}{-1/n^2}$

⑪ $\{ nx e^{-nx^2} \}$ on $[0,\infty)$ ← Not U.C.

$$\cancel{\lim} f(x) = \lim_{n\to\infty} nx e^{-nx^2}$$

$$= \cancel{x} \lim_{n\to\infty} n e^{-nx^2}$$

$$= 0 \quad , \forall x \in [0,\infty)$$

⑫ $\left\{ \frac{x}{1+nx^2} \right\}$ on $\mathbb{R}$ → U.C.

$$f(x) = \lim_{n\to\infty} \frac{x}{1+nx^2}$$

$$= x \lim_{n\to\infty} \frac{1}{1+nx^2}$$

$$= 0$$

ⓗ

⑬ $\{e^{-nx}\}$ on $[0, \infty)$ ← Not U.C

$$f(x) = \lim_{n\to\infty} e^{-nx}$$

$$= 0 \;, x \neq 0$$
$$ 1 \;, x = 0$$

⑭ $\{n^2x(1-x^2)^n\}$ on $[0,1]$

$$f(x) = \lim_{n\to\infty} n^2x(1-x^2)^n \quad \leftarrow \text{Not U.C}$$

$$= x \lim_{n\to\infty} n^2(1-x^2)^n$$

$$= 0$$

⑮ $\left\{\frac{n^2x}{1+n^3x^2}\right\}$ on $[0,1]$ → Not. U.C

$$f(x) = \lim_{n\to\infty} \frac{n^2x}{1+n^3x^2}$$

$$= x \lim_{n\to\infty} \frac{n^2}{1+n^3x^2} = \underline{\underline{0}}$$

point wise Convergent :- A seq^n $\{f_n\}$ defined on I is said to be pointwise convergent to f(x) on I.

If $\forall \epsilon > 0$ and $\underline{\forall x \in I}$, $\exists n_0 \in \mathbb{N}$ s.t.

$$|f_n(x) - f(x)| < \epsilon \quad \forall n > n_0$$

Uniform Convergence :- A seq^n $\{f_n\}$ defined on I. is said to be Uniform Convergence to f(x) on I,

If $\forall \epsilon > 0$, $\underline{\exists n_0 \in \mathbb{N} \text{ s.t. } \forall x \in I}$,

$$|f_n(x) - f(x)| < \epsilon \quad \forall n > n_0$$

Sup. exists Every n_0 finitely.

⊛ Uniform Convergence of a seq^n :- (I)

M_n - test :-

A seq^n $\{f_n\}$ defined on I, which Converges pointwise to $f(x)$ on I. is uniformly Convergent on I.

iff $\lim_{n\to\infty} M_n = 0$

where $M_n = \sup_{x \in I} |f_n(x) - f(x)| \leftarrow$ (~~m~~ $n \leftarrow$ fixed. & $x \leftarrow$ varies in I)

Eg: $f_n(x) = n^2 x (1-x^2)^n$ on $[0,1]$

$f(x) = 0$

$M_n = \sup_{x \in [0,1]} |n^2 x(1-x^2)^n - 0|$

Let $g(x) = n^2 x (1-x^2)^n$

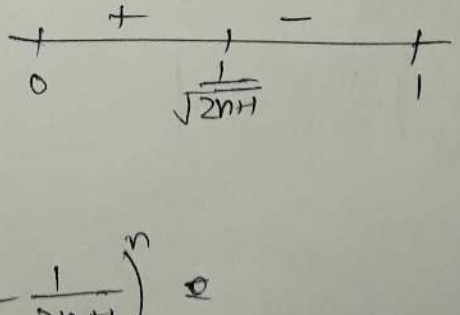
$g'(x) = n^2(1-x^2)^n \cancel{+} n^3 x (1-x^2)^{n-1}(-2x)$

$= n^2(1-x^2)^n - 2x^2 n^3 (1-x^2)^{n-1}$

$= n^2(1-x^2)^{n-1}[1 - x^2 - 2n x^2]$

$= n^2(1-x^2)^{n-1}[1 - (1+2n)x^2] = 0$

$x = 1, \frac{1}{\sqrt{2n+1}}$



~~$\therefore M_n = \frac{1}{\sqrt{2n+1}}$~~

$\therefore M_n = \frac{n^2}{\sqrt{2n+1}}\left(1 - \frac{1}{2n+1}\right)^n$

$\lim_{n\to\infty} M_n = \lim_{n\to\infty} \frac{n^2}{\sqrt{2n+1}}\left(1 - \frac{1}{2n+1}\right)^n = \infty \neq 0$

So $\{f_n(x)\}$ is not uniformly Convergent.

Ⓙ

Q $f_n(x) = \frac{nx}{1+n^2x^2}$ · on $[0,1]$

$f(x) = 0$

$$M_n = \sup_{x\in[0,1]} \left| \frac{nx}{1+n^2x^2} - 0 \right|$$

Let $g(x) = \frac{nx}{1+n^2x^2}$

$$g'(x) = \frac{n(1+n^2x^2) - 2n^3x^2}{\underbrace{(1+n^2x^2)^2}_{\neq 0}} = 0$$

~~$n + n^3x^2 - 2n^3x^2 = 0$~~

~~$n + n^3x^2(1 - 2x) = 0$~~

~~$x = 0, \frac{1}{2} \leftarrow \text{Sup}$~~

~~$\therefore M_n = \frac{n/2}{1+n^2/4}$~~

$n + n^3x^2 - 2n^3x^2 = 0$

$n - n^3x^2 = 0$

$x^2 = \frac{1}{n^2} \Rightarrow x = \frac{1}{n}$

$$\therefore M_n = \frac{n \times \frac{1}{n}}{1+n^2\left(\frac{1}{n}\right)^2} = \frac{1}{2} \neq 0$$

So, ~~not~~ $\{f_n(x)\}$ is not U.C. on $[0,1]$.

Q $f_n(x) = \frac{1}{x+n}$ on $[0,2]$

$f(x) = 0$

$$M_n = \sup_{x\in[0,2]} \left| \frac{1}{x+n} - 0 \right|$$

Let $g(x) = \frac{1}{x+n}$

$$g'(x) = \frac{-1}{(x+n)^2} = 0$$

$M_n = \frac{1}{n} \to 0$ as $n \to \infty$

U.C.

Ⓚ

Q $f_n(x) = x^n$ on $[0,1]$

Not U.C.

$\sup_{x\in[0,1)} |x^n - 0|$, 0

$1 \neq 0$

★ Uniform Convergence of a series :-

⊛ Weierstrass - M Test :-

A series $\sum_{n=1}^{\infty} f_n$ of fuctions defined on I . such that $|f_n(x)| \leq M_n$, $\forall x \in I$

If $\sum_{n=1}^{\infty} M_n$ is Convergent then $\sum_{n=1}^{\infty} f_n(x)$ is ~~the~~ uniform Convergent on I.

E.g. ① $\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$ on IR is U.C. as $\left|\frac{\sin nx}{n^2}\right| \leq \frac{1}{n^2}$ $\forall x \in IR$

② $\sum r^n \sin\theta$, ~~$\sum r^n \cos\theta$~~ $\sum r^n \cos n\theta$, $\sum r^n \cos(n^2\theta)$, $\sum r^n \cos(a^n x)$ are U.C. if $0 < r < 1$, $\forall x \in R$

Q ① If $\sum a_n$ is absolutely Convergent then

(i) $\sum \frac{a_n x^n}{1+x^{2n}}$ (ii) $\sum \frac{a_n x^{2n}}{1+x^{2n}}$ on IR

② $\sum \frac{\sin(x^2+n^2x)}{n(n+1)}$ on IR

③ $\sum n^{-x}$ on $[2, \infty)$

④ $\frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{8x^7}{1+x^8} + \text{———}$ on $[-\frac{1}{2}, \frac{1}{2}]$

⑤ $\sum \frac{x}{n^p + x^2 n^q}$ over any finite interval [a, b] if

(i) $p>1$, $q \geq 0$ (ii) $0 < p \leq 1$, $p+q > 2$

Solⁿ

② $\sum \frac{\sin(x^2+n^2x)}{n(n+1)}$ on $\mathbb{R}$ is U.C. as $\left|\frac{\sin(x^2+n^2x)}{n(n+1)}\right| \leq \frac{1}{n(n+1)}$, $\forall x \in \mathbb{R}$

① (i) $\sum \frac{a_n x^n}{1+x^{2n}}$ on $\mathbb{R}$ is U.C. as $\left|\frac{a_n x^n}{1+x^{2n}}\right| \leq a_n$, $\forall x \in \mathbb{R}$

(ii) $\sum \frac{a_n x^{2n}}{1+x^{2n}}$ on $\mathbb{R}$ is U.C. as $\left|\frac{a_n x^{2n}}{1+x^{2n}}\right| \leq a_n$, $\forall x \in \mathbb{R}$

③ $\sum n^{-x}$ on $[2,\infty)$ is U.C. as $\frac{1}{n^x} \leq \frac{1}{n^2}$, $\forall n \in \mathbb{N}$

④ $\frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{8x^7}{1+x^8} + \text{——}$ on $[-\frac{1}{2}, \frac{1}{2})$

$\sum \frac{2^n x^{2^n-1}}{1+x^{2^n}}$ ~~is not U.C.~~ as ~~$\left|\frac{2^n x^{2^n-1}}{1+x^{2^n}}\right| \leq 2^n$~~

⑤ ~~$\sum \frac{x}{n^p + x^2 n^q}$~~ on $[a,b]$

~~$\sum \frac{x}{n^p(1+x^2 n^{q-p})}$~~

$p+q > 2$
$p > 2-q$
~~$q-p > 1$~~
$q-2+q > 1$
$q-p > 0$
~~$q-2+q > 0$~~

$$f_n(x) = \frac{2^n x^{2^n-1}}{1+x^{2^n}}$$

$$g(x) = \frac{x^{2^n-1}}{1+x^{2^n}}$$

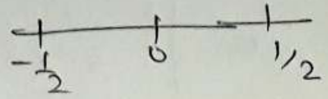
$$g'(x) = (1+x^{2^n})(2^n-1)x^{2^n-2} - x^{2^n-1} \cdot 2^n \cdot x^{2^n-1} = 0$$

$$\Rightarrow x^{2^n-2}\left[(1+x^{2^n})(2^n-1) - 2^n x^{2^n}\right] = 0$$

$$\Rightarrow x^{2^n-2}\left[2^n - 1 + \cancel{2^n x^{2^n}} - x^{2^n} - \cancel{2^n x^{2^n}}\right] = 0$$

$$\Rightarrow x^{2^n-2}\left[2^n - 1 - x^{2^n}\right] = 0$$

$x = 0$, $x^{2^n} = 2^n - 1 \Rightarrow x = (2^n - 1)^{\frac{1}{2^n}}$



$\therefore |f_n(x)| \leq \dfrac{2^{n+1}}{2^{2^n}+1}$

Now $\sum \dfrac{2^{n+1}}{2^{2^n}+1}$

$$\lim_{n\to\infty} \frac{2^{n+1}}{2^{2^n}+1} \cdot \frac{2^{2^{n+1}}+1}{2^{n+2}} = \lim_{n\to\infty} \frac{2^{2 \cdot 2^n}+1}{2^{2^n}+1} \cdot \frac{1}{2}$$

$$= \lim_{n\to\infty} \frac{(2^{2^n})^2+1}{2^{2^n}+1} \cdot \frac{1}{2} = \infty > 1$$

So, U.C.

⑤ $\sum \dfrac{x}{n^p + x^2 n^q}$ on $[a,b]$

(i) $p > 0$, $q \geq 0$

$$\left| \frac{x}{n^p + x^2 n^q} \right| \leq \frac{|x|}{n^p} \leq \frac{\text{Max}\{|a|, |b|\}}{n^p} = \frac{\alpha}{n^p}$$

Now $\sum \dfrac{\alpha}{n^p}$ is Convergent iff $p > 1$

So, if $p > 0$, $q \leq 0$ then $\sum \dfrac{x}{n^p + x^2 n^q}$ is U.C. over $[a,b]$

(ii) $0 \leq p \leq 1$, $p + q > 2$

$$|f_n(x)| = \left| \frac{x}{n^p + x^2 n^q} \right| = \frac{|x|}{n^p + x^2 n^q}$$

Let $g(x) = \frac{x}{n^p + x^2 n^q}$

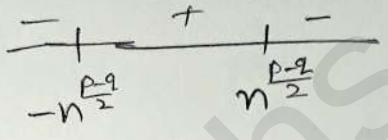
$$g'(x) = \frac{n^p + x^2 n^q - x(2x\, n^q)}{(n^p + x^2 n^q)^2} = 0$$

$$= n^p + x^2 n^q - 2x^2 n^q = 0$$

$$= n^p - x^2 n^q = 0$$

$$\Rightarrow x^2 = n^{p-q}$$

$$\Rightarrow x = \pm n^{\frac{p-q}{2}}$$



$$g\left(n^{\frac{p-q}{2}}\right) = \frac{n^{\frac{p-q}{2}}}{n^p + n^q\, n^{\frac{p-q}{\cancel{2}}}} = \cancel{\frac{n^{\frac{p-q}{2}}}{n^p + n^{\frac{p+q}{2}}}}$$

$$= \frac{n^{\frac{p-q}{2}}}{2n^p} = \frac{1}{2}\frac{1}{n^{\frac{p+q}{2}}}$$

Now, $\frac{1}{2}\sum \frac{1}{n^{\frac{p+q}{2}}}$ is Convergent if $p+q > 2$

So, $\sum \frac{x}{n^p + x^2 n^q}$ is U.C.

✪ Abel Test :-

If $\{b_n(x)\}$ is a positive monotonic seq^n for every $x \in [a,b]$ and x ~~etc~~. and $\{b_n(x)\}$ is bd for every n and x and if a series $\sum u_n(x)$ is uniformly convergent over $[a,b]$ then $\sum u_n(x) b_n(x)$ is U.C. over $[a,b]$

✪ Dirchlet Test :-

If $b_n(x)$ is a monotonic function of n for each value $x \in [a,b]$ and $\{b_n(x)\}$ converges uniformly to '0' in $[a,b]$

If a series $\sum U_n(x)$ is uniformly bd in $[a,b]$ then $\sum U_n(x) b_n(x)$ is uniformly convergent over $[a,b]$

Eg. $f_n(x) = n \sin(nx)$ ← bd But not uniformly bd.

Q $\sum_{n=1}^{\infty} (-1)^n \frac{x^n + n}{n^2}$ over $[0, 10]$

Sol^n :- $b_n(x) = \frac{x^n + n}{n^2} \le \frac{100+n}{n^2}$ $U_n(x) = (-1)^n$ → bd uniformly.

↓ Convergent

So, By Dirchlet Test

$\sum b_n(x) U_n(x) = \sum (-1)^n \frac{x^n+n}{n^2}$ is U. C.

Q If $\sum a_n$ is Convergent then over $[0,1]$

(i) $\sum a_n x^n$ (ii) $\sum a_n \frac{x^n}{1+x^n}$ (iii) $\sum \frac{a_n x^n}{1+x^{2n}}$

(iv) $\sum \frac{n a_n x^n (1-x)}{1+x^n}$ (v) $\sum \frac{2n\, a_n x^n (1-x)}{1+x^{2n}}$

(i) Let $b_n(x) = x^n$, $U_n(x) = a_n$

$\{b_n\}$ is monotonic, +ive, bd seq^n. on every n & x

& $\sum U_n(x)$ is uniformly convergent.

So, By Abel Test $\sum a_n x^n$ is Convergent.

(ii) $\sum a_n \frac{x^n}{1+x^n}$

Let $u_n(x) = a_n$, $b_n(x) = \frac{x^n}{1+x^n} \leq 1$

So, By Abel test

$\sum a_n \frac{x^n}{1+x^n}$ is U.C.

(iii) $\sum \frac{a_n x^n}{1+x^{2n}}$

Let $u_n(x) = a_n$, $b_n(x) = \frac{x^n}{1+x^{2n}} \leq \frac{1}{2}$

So, By Abel test $\sum \frac{a_n x^n}{1+x^{2n}}$ is U.C.

(iv) $\sum \frac{n a_n x^n(1-x)}{1+x^n}$

Let $g(x) = \frac{nx^n(1-x)}{1+x^n}$

~~$g'(x) = (1+x^n)[n^2x^{n-1}(1-x) - nx^n] -$~~

$$g'(x) = \frac{(1+x^n)\left[n^3x^{n-1}(1-x) - nx^n\right] - n^2x^n(1-x)\underset{x^{n-1}}{\cancel{n}}}{(1+x^n)^2}$$

$$\Rightarrow (1+x^n)\left[n^2x^{n-1} - n^2x^n - nx^n\right] - (n^2x^n + n^3x^{n+1})\,x^{n-1} = 0$$

$$\Rightarrow (1+x^n)\left[n^2x^{n-1} - n^2x^n - nx^n - n^2x^{2n-1}\right] = 0$$

Q $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^p}$, $\sum_{n=1}^{\infty} \frac{\cos nx}{n^p}$ over $[\alpha, 2\pi-\alpha]$, $\alpha>0$

$\sum_{n=1}^{m} \sin(nx) \leq \frac{1}{\sin\frac{x}{2}} \leq \operatorname{cosec}\frac{\alpha}{2} \rightarrow$ U. bd.

$\frac{1}{n^p} \leftarrow$ m. D. seqn converges to 0.

So, By Dirchlet Test

$\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^p}$ is U. C.

✪ properties of U.C. seq.^{n} & series :

① If a seq.^{n} $\{f_n\}$ Converges uniformly to f in $[a,b]$ and $x_0 \in [a,b]$ s.t. $\lim_{x\to x_0} f_n(x) = a_n$

then

(i) $\{a_n\}$ is Convergent and

(ii) $\lim_{n\to\infty} a_n =$ ~~$f(x_0)$~~ $\lim_{x\to x_0} f(x)$

i.e. $\lim_{n\to\infty}\lim_{x\to x_0} f_n(x) = \lim_{x\to x_0}\lim_{n\to\infty} f_n(x)$

② If a Series $\sum f_n$ Converges uniformly to f in $[a,b]$ and $x_0 \in [a,b]$ s.t. $\lim_{n\to\infty} f_n(x) = a_n$

then

(i) $\sum a_n$ is Convergent and

(ii) $\sum_{n=1}^{\infty} a_n = \lim_{x\to x_0} f(x)$

i.e. $\sum_{n=1}^{\infty}\left(\lim_{x\to x_0} f_n(x)\right) = \lim_{x\to x_0}\sum_{n=1}^{\infty} f_n(x)$

③ If $\{f_n\}$ be a seq.^{n} of Continuous fuctions in $[a,b]$ which Converges uniformly to f over $[a,b]$ then f is Continuous in $[a,b]$

④ If $\sum f_n$ be a series of Continuous fuctions in $[a,b]$ which Converges uniformaly to f over $[a,b]$ then f is Continuous in $[a,b]$

✪ Converse of this Statement is not true as if $\{f_n\}$ is Cont. in $[a,b]$ and limit fuction $f(x)$ is also Continuous in $[a,b]$ then the Convergence need not be uniform.

E.g. $\{f_n(x)\} = \left\{\frac{nx}{1+n^2x^2}\right\}$

✪ **Dini's Theorem:** If a seq.^{n} of Continuous funtion f_n defined on $[a,b]$ is monotonic and Converges pointwise to ~~Cont~~ a Continuous fuction f then the ~~Cont.~~ Convergence is Uniform in $[a,b]$.

✪ If a seq.^{n} $\{f_n\}$ Converges uniformly to f on $[a,b]$ and each term ~~function~~ $f_n(x)$ is integrable then 'f' is integrable on $[a,b]$ and the seq.^{n} $\left\{\int_a^x f_n(t)\,dt\right\}$ converges uniformly to $\int_a^x f(t)\,dt$.

i.e. $\lim_{n\to\infty} \int_a^x f_n(t)\,dt = \int_a^x \lim_{n\to\infty} f_n(t)\,dt.$

★ If a series $\sum f_n$ Converges uniformly to on $[a, b]$ and each term $f_n(x)$ is integr
then f is integrable on $[a, b]$
and $\sum \int_a^x f_n(t)\,dt = \int_a^x f(t)\,dt = \int_a^x \sum f_n(t)\,dt.$

⇒ This statement is not true in $[a, \infty)$.

→ $\boxed{\text{Differention is not U.C.}}$ ← of the above Statem

Some Useful Links:

1. Free Maths Study Materials (https://pkalika.in/2020/04/06/free-maths-study-materials/**)**

2. BSc/MSc Free Study Materials (https://pkalika.in/2019/10/14/study-material/**)**

3. MSc Entrance Exam Que. Paper: (https://pkalika.in/2020/04/03/msc-entrance-exam-paper/**)**
[JAM(MA), JAM(MS), BHU, CUCET, ...etc]

4. PhD Entrance Exam Que. Paper: (https://pkalika.in/que-papers-collection/**)**
[CSIR-NET, GATE(MA), BHU, CUCET,IIT, NBHM, ...etc]

5. CSIR-NET Maths Que. Paper: (https://pkalika.in/2020/03/30/csir-net-previous-yr-papers/**)**
[Upto 2019 Dec]

6. Practice Que. Paper: (https://pkalika.in/2019/02/10/practice-set-for-net-gate-set-jam/**)**
[Topic-wise/Subject-wise]