Measure Theory

[Handwritten Study Material for MSc & Competition]

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Your Note/Remarks

Preliminaries

Δ - Symmetric difference

$A \Delta B = (A-B) \cup (B-A)$

= the symmetric difference of the sets A, B

$A-B$ = the set of elements of A not in B

$\wp(A)$ = the power set of A i.e the set of subsets of A

χ_A = Characteristic f^n of the set A

$$= \begin{cases} 1 & : x \in A \\ 0 & : x \notin A \end{cases}$$

F_σ Set $\coloneqq$ Countable union of closed sets.

G_δ Set = Countable intersection of open sets.

① **Result** : Let E & F, ~~are two~~ G & H are sets, then —

(1). $E \Delta F = F \Delta E$

(2). $(E \Delta F) \Delta G = E \Delta (F \Delta G)$

(3). $(E \Delta F) \Delta (G \Delta H) = (E \Delta G) \Delta (F \Delta H)$

(4). $E \Delta F = \emptyset$ iff $E = F$

(5) $E \Delta F \subseteq (E \Delta G) \cup (G \Delta F)$

(6). $\bigcup_{i=1}^{n} E_i \,\Delta\, \bigcup_{i=1}^{n} F_i = \bigcup_{i=1}^{n} (E_i \Delta F_i)$

②. Let $E_1 \supseteq E_2 \supseteq E_3 \dots \supseteq E_n \dots$ then

$$\bigcup_{i=1}^{\infty} (E_1 - E_i) = E_1 - \bigcap_{i=1}^{\infty} E_i$$

(Hint: For Proof, you may use De Morgan's laws)

③. Equivalence Relation ($\sim$)

An equivalence relation R on a set X is a subset of $X \times X$ with the following properties:

(i). $(x,x) \in R$ for each $x \in X$ i.e $x \sim x$

(ii). $(x,y) \in R \Rightarrow (y,x) \in R$ i.e if $x \sim y \Rightarrow y \sim x$

(iii) If $(x,y) \in R$, & $(y,z) \in R$ then $(x,z) \in R$

i.e if $x \sim y$ & $y \sim z \Rightarrow x \sim z$

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Measure Theory & Integration

Syllabus

UNIT-I 129-15 25 Countable & Uncountable sets, cardinality & Cardinal arithmetic, ~~schröder-Bernstein thm,~~ $a < 2^a$, $2^{\aleph_0} = c$, ~~the Canter's ternary set, semi-algebras~~, Algebras, monotone class, σ-Algebras, Measure and outer measures, ~~Caratheödory ext^n process of extending a measure on a semi-algebra to generated σ-algebras~~, Borel sets. (10L)

UNIT-II (155) lebesgue outer measure & lebesgue measure on R, translation invariance of lebesgue measure, ~~existence of a non-measurable set~~, characterizations of lebesgue measurable sets, the Cantor-lebesgue function, measurable f^n on a measure space and their properties, Borel and lebesgue measurable f^ns, simple f^ns and their integrals, Littlewood's three principle (statement only). (10L)

UNIT-III lebesgue integral on R and its properties, bounded convergence theorem, Fatou's lemma, lebesgue monotone convergence thm, lebesgue dominated convergence theorem, L-p spaces, Holder-Minkowski inequalities, parsenal's identity, Riesz Fisher's theorem. (10L)

Ref:

(1). H.L. Royden & P.M. Fitzpatrick, Real Analysis
(2). P.R. Halmos, Measure theory, Springer (1994)
(3). E. Hewit & K. stromberg, Real & Abstract Analy.
(4). K.R. Parthasarathy, Intro. to Prob. & Measure.
(5). I.K. Rana, An Intro. to Measure & Intro. (2Ed).

Riemann Integ.

f(x) b.dd in [a,b].

partition $P=\{x_i \mid 0\le i\le n\}$, $a=x_0,x_1,\dots x_n=b$

$m_i = \inf\{f(x);\ x_{i-1}<x<x_i\}$

$M_i = \sup\{f(x);\ x_{i-1}<x<x_i\}$

$U(f,P) = \sum M_i(x_i - x_{i-1}) \Rightarrow \overline{\int_a^b} f = \inf\{U(f,P)\}$

$L(f,P) = \sum m_i(x_i - x_{i-1}) \Rightarrow \underline{\int_a^b} f = \sup\{L(f,P)\}$

If U=L, then f(x) is Riemann Integrable.

Eg. $f(x) = \begin{cases} 1, & x\in[0,1], x \text{ is rational number} \\ 0 & x\in[0,1], x \text{ is an irrational no.} \end{cases}$

Here, $m_i=0$

$M_i = 1$.

$U(f,P) = \sum 1.(x_i - x_{i-1}) = 1$

$L(f,P) = \sum 0.(x_i - x_{i-1}) = 0$

$\Rightarrow V \neq L$

$\Rightarrow$ f(x) is NOT Riemann integrable.

Eg. Z is countable (To show it, we need to set-up 1-1 corresponds)

$f: N \to Z$, $f(x) = \begin{cases} n/2 & ; n \text{ even} \\ -\frac{n-1}{2} & : n \text{ odd} \end{cases}$ —(*)

one-one correspondence

(A is finite if A is 1-1 correspondence with $\{1,2,\dots n\}$, $n\in N$

Note: ~~Theorem~~: as Z: 0, 1, -1, 2, -2, ...

N: 1, 2, 3, 4, 5, ...

a/c this, we have to create a fn f(x) as (*)
which is 1-1 correspondence ~~with~~ N→Z.

Theorem: ① Let E is infinite subset of a countable set A, then E is countable.

pf: let elt-s of A are arranged in sequence $\{x_n\}$ of distinct elements. (Bcz A is countable)

Define a sequence n_k s.t

$x_{n_1} \in E$, where n_1 is the smallest integer

Choose $n_1, n_2 - - n_k$ ($k = 1, 2, --$) s.t n_k is the greatest integer s.t — $x_{n_k} \in E$

then $\{x_{n_k}\}$ is a sequence. and x_{n_k} is 1-1 correspondence with N. Hence E is countable

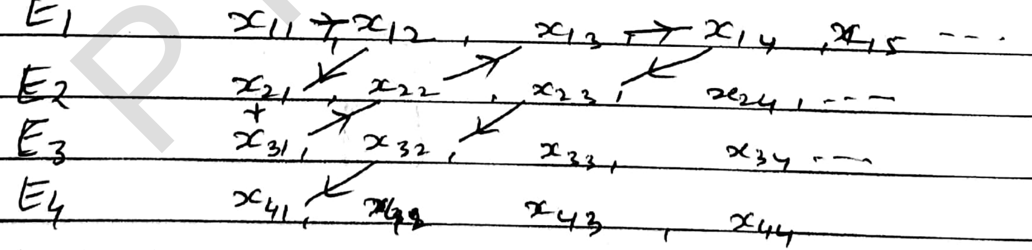
i.e we got a seq. in E which 1-1 correspond with N. so E ∫

$$\begin{array}{ccc} n_1 & n_2 & -- \; n_k \\ \downarrow & \downarrow & \downarrow \\ x_{n_1} & x_{n_2} & x_{n_k} \end{array} \leftarrow \text{one-one}$$

Theorem: ② let $\{E_i\}$ is sequence of countable sets. $S = \bigcup_{i=1}^{\infty} E_i$. Then S is countable. ~~union~~ for Union

or The union of countable collection of countable sets is countable.

pf:— let elements x_{nk} ($k = 1, 2, --$) of set E_n are arrange in the form —



$$\begin{array}{lccccc} E_1 & x_{11} \to x_{12}, & x_{13} \to x_{14}, & x_{15} & --- \\ E_2 & x_{21}, \; x_{22}, & x_{23}, & x_{24}, & --- \\ E_3 & x_{31}, \; x_{32}, & x_{33}, & x_{34} & -- \\ E_4 & x_{41}, \; x_{42} & x_{43}, & x_{44} & \end{array}$$

Constructing in such manner —

$$\{ \underbrace{x_{11}}_{\updownarrow \; n_1} ; \underbrace{x_{12} ; x_{21}}_{\updownarrow \; n_2} ; \underbrace{x_{31}, x_{22}, x_{13}}_{\updownarrow \; n_3} ; \underbrace{x_{14}, x_{23}, x_{32}, x_{41}}_{4} ; --- \}$$

1-1 correspon — → n_1 n_2 - - n_3 - - ② ④ ← sum of subscript

$$= \bigcup_{i=1}^{\infty} E_i = S$$

∵ $E_1, E_2 \cdots$ are countable, hence infinite set

A set is countable $\Leftrightarrow$ its elt.s can be written in the form of a sequence.

$\Rightarrow$ S is also infinite set.

Here, we got a seq $\{x_n\}$ corresponds to n_k which 1-1 corr to N.

Hence, S is infinite and have 1-1 Correspondence with N.

$\Rightarrow$ S is Countable.

Theorem: Set of Rational numbers Q is countable

Pf: ③ Let $A = \{0\}$

$$A_1 = \{\underbrace{\tfrac{1}{1}, \tfrac{-1}{1}}, \tfrac{2}{1}, \tfrac{-2}{1}, \tfrac{3}{1}, \tfrac{-3}{1} \cdots\}$$

$$A_2 = \{\tfrac{1}{2}, \tfrac{-1}{2}, \tfrac{2}{2}, \tfrac{-2}{2}, \cdots\}$$

$$\vdots$$

$$A_n = \{\underbrace{\tfrac{1}{n}, \tfrac{-1}{n}}, \underbrace{\tfrac{2}{n}, \tfrac{-2}{n}}, \cdots\}$$

$$\vdots$$

Here, $A_1, A_2, \ldots A_n$ all are Countable and A_0 is finite. (B/c all are infinite & have 1-1 Reln with N)

Hence,

$$Q = \bigcup_{i=0}^{\infty} A_i = \text{Countable union of countable sets.}$$

Hence, by previous theorem(2), Q is countable.

* $A = \{x;\ x \in Q,\ x \geq 1\}$ (By thm ①)

$\because A \subseteq Q$

$\Rightarrow$ A is countable ($\because$ Q is countable)

* **Theorem:**

④ $A = \{x \in R,\ x \in [0,1]\}$. Prove that A is NOT countable.

Pf. – $\because x_i \in [0,1]$ suppose

$\Rightarrow x_i = 0.a_1 a_2 a_3 \cdots$; $a_i = 0, 1, 2, \ldots 9$

Let one-one correspondence of x_i's are:

$n_1 \rightarrow x_1 \longleftrightarrow 0.\boxed{a_1} a_2 a_3 a_4 \cdots = 0.a_{11} a_{12} a_{13} a_{14} \cdots$

$n_2 \rightarrow x_2 \longleftrightarrow 0.b_1 \boxed{b_2} b_3 b_4 \cdots = 0.a_{21} a_{22} a_{23} a_{24} \cdots$

$n_3 \rightarrow x_3 \longleftrightarrow 0.c_1 c_2 \boxed{c_3} \cdots = 0.a_{31} a_{32} a_{33} a_{34} \cdots$

$\vdots$

y^2

Now let us consider a no. $y \in [0,1]$

let $y = 0 \cdot y_1 y_2 y_3 \cdots$, $y \in A$

$$y_1 = \begin{cases} 2 & : a_1 \geq 5 \\ 7 & a_1 \leq 4 \end{cases}$$

$$y_2 = \begin{cases} 2 & b_2 \geq 5 \\ 7 & b_2 \leq 4 \end{cases}$$

$$y_3 = \begin{cases} 2 & c_3 \geq 5 \\ 7 & c_3 \leq 4 \end{cases}$$

s.t $y_1 \neq a_{11}$

$y_2 \neq a_{22} = b_2$

$y_3 \neq a_{33} = c_3$

$y_m \neq a_{mm} = ()_m$

obviously $y \in [0,1]$ but $y \notin \{x_1, x_2, \cdots\}$

y doesn't have one-one correspondence of above. because $y \notin \{x_i\}$ (as $y \neq x_1, y \neq x_2, \cdots$ $y \neq x_n \cdots$

$\Rightarrow$ A is not countable.

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Theorem: ③ R is uncountable, Prove it (We prove it by contradiction)

Pf: $\because [0,1] \subset R$

Let R is countable so

$\Rightarrow$ [0,1] should not be uncountable (Bez $[0,1] \subset R$) but [0,1] is uncountable. (By thm ①)

Hence, we got contradiction.

$\Rightarrow$ R is uncountable.

Example: $X = \{x \in R, x \in [a,b]\}$

$\because [0,1]$ is uncountable.

let $y = a + (b-a)x \Rightarrow \boxed{x = \frac{y-a}{b-a}}$

Let $f: [0,1] \to [a,b]$ s.t $f(x) = a + (b-a)x$, $x \in [0,1]$

$x_1 \neq x_2$ if $a + (b-a)x_1 \neq a + (b-a)x_2$ $f_1 \neq f_2$

$\because f(x)$ is one-one and onto function

$\Rightarrow$ [0,1] and [a,b] are equivalent sets.

$\Rightarrow$ [a,b] is uncountable ($\because [0,1]$ is uncountable).

(thus every closed interval is uncountable)

Theorem: ⑥ A set A of all sequences, whose elements are digits 0 and 1. this set A is uncountable.

(pf like ④) Pf on (p-140)

#

Theorem ⑦ $\overset{N^2}{N \times N}$ is countable.

Theorem ⑧ Set I of irrational no.s is uncountable $(I = Q^c)$

② Proof: (I is uncountable)

$\because$ $R = I \cup Q$ (As we know that R is uncountable)

and Q is countable.

Let I is countable.

then $R = I \cup Q$ will be countable, but R is uncountable.

Hence I is uncountable. Proved

Pf: ⑦ Note that $N \times N = \{(u, v) \mid u, v \in N\}$, then clearly $N \times N$ set can be arranged as shown bellow—

$$N \times N = \left\{ \begin{array}{ll} (1,1), (1,2)\ (1,3), (1,4) \cdots & = A_1 \leftarrow \text{countable} \\ (2,1)\ (2,2), (2,3), (2,4) \cdots & = A_2 \\ (3,1)\ (3,2), (3,3), (3,4) \cdots & = A_3 \\ \vdots & \vdots \end{array} \right\} = \cup A_i$$

where $A_i = \{(i, n), n \in N\}$ which all are countable

Hence $\cup A_i$ is countable collection of countable sets. Hence by thm ②, $N \times N$ is countable.

Note:— The result can be extened to any no. of set. i.e $N \times N \times N \times \cdots \times N$ is also countable.

Algebra :-

(8) Let X be a non-empty set. Ω is collection of subsets of X, s.t —

(i) $\phi, X \in \Omega$

(ii) If $A \in \Omega$ then $A^c \in \Omega$

(iii) If $A, B \in \Omega$, then $A \cup B \in \Omega$ & $A \cap B \in \Omega$
(finite union) (finite intersection)

Then Ω is called Algebra.

Eg. (i) Power set $P(X)$ is Algebra.
(ii) N, $\Omega = \{\phi, N, \text{even}, \text{odd}\}$ then Ω is an algebra on N.

σ-Algebra :-

Let X be a n-e set & Ω be collection of subsets of X, which is algebra; If countable union of elements of Ω is also in Ω, then Ω is known as σ-Algebra, defined in X.

(9) Example: In R, Let $\Omega = \{A \subset R \,;\, A \text{ is finite or } A^c \text{ is finite}\}$

[PT Ω is not Algebra]

(1) $A = \{n\} = 1, 2, 3 \ldots$

singleton as finite set

$\cup \{n\} = N$

$\because N$ is infinite and N^c is also infinite.

$\Rightarrow N$ is NOT ~~as~~ Algebra.

$\Rightarrow \Omega$ is NOT σ-Algebra.

(2) In R, $\Omega = \{(a, b] \,,\, a, b \in R \,,\, -\infty \le a < b < \infty \,;$ also suppose (a, ∞) is right closed$\}$

TP Ω is algebra. (Not) (NOT Algebra)

$\because (0, 1 - 1/n] \in \Omega$ and

$\cup (0, 1 - 1/n] = (0, 1) \notin \Omega$ (Bcz $(0,1)$ is not of type $(a,b]$)

$\Rightarrow$ countable union of elements of Ω is NOT in Ω.

$\Rightarrow \Omega$ is NOT σ-algebra.

- countable union of closed sets may NOT be closed.

Theorem: (10) ① If F is any collection of subsets of X, then there exist a smallest σ-algebra M^* in X s.t $F \subset M^*$.

② If F is any collection of subsets of X, then ∃ a smallest algebra M^* in X s.t $F \subset M^*$.

Eg. In N, "$F = \{\phi, X, \boxed{\text{even}}\}$" is NOT algebra, (as $\text{even}^c \notin F$)

$\Omega_1 = \{\phi, X, \text{even}, \text{odd}\}$

$\Rightarrow F \subset M = \Omega_1$

$\Rightarrow \Omega \subset \Omega_1$

Proof ① Let Ω is collection of all σ-algebra M in X.

The Ω is non-empty, b/c $P(X) \in \Omega$, $\{P(X)$ is σ-algebra$\}$.

Let

$$M^* = \bigcap_{M \in \Omega} M.$$

We would like to show that M^* itself a σ-algebra. Let $A_i \in M^*$, $i = 1, 2, \cdots n$..

$\Rightarrow A_i \in M$, $(i = 1, 2, \cdots)$ $\forall M \in \Omega$.

$\bigcup A_i \in M$. $\forall M \in \Omega$, $\{\because M$ is σ-algebra$\}$

$\Rightarrow \bigcup A_i \in \bigcap_{\Omega} M = M^*$

$\Rightarrow \boxed{\bigcup A_i \in M^*} \Rightarrow$ Countable union of elt. of $M^* \in M^*$

if $A \in M^*$, then $A \in M$ $\forall M \in \Omega$.

$\Rightarrow A^c \in M$ $\forall M \in \Omega$ $\{M$ is σ-algebra$\}$.

$\Rightarrow A^c \in \bigcap_{\Omega} M$ $\forall M \in \Omega$

$\boxed{A^c \in M^*}$

Again, $\because X \in M$ $\forall M \in \Omega$

$\Rightarrow X \in \bigcap_{\Omega} M$. $\forall M \in \Omega$

(If $A_i \in A \Rightarrow A_i^c \in A \Rightarrow A$ is alg.)

(ii) $\Rightarrow \underline{X \in m^*}$

Also, $\phi \in m^*$ ✓   ($\because P(X)$ is the smallest σ-algebra and have $\{\phi\}, X\}$)

(ii) Again, let $A, B \in m^*$ $(m^* = \cap m)$, $A, B \in \cap m$

$\Rightarrow A, B \in m \quad \forall m \in \Omega$

$\Rightarrow A \cup B \in m \quad \forall m \in \Omega$ [$\because m$ is σ-algebra]

$\Rightarrow A \cup B \in \underline{\cap m} = m^*$

$\Rightarrow A \cup B \in m^*$ ✓

Similarly $\boxed{A \cap B \in m^*}$, Hence X.

m^* is the smallest σ-alge in

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Result:

let X be a topological space, we can find a smallest σ-Algebra containing X.
Its Elements are called "Borel set"

- * Closed sets are Borel sets.
- * open sets are Borel sets
- * Countable ~~sets~~ Union of closed sets is Borel set.
- * Finite intersection of open set is Borel set.

\# Theorem: (II) Let A be an algebra of subsets of a set X and $\{A_i\}$ be a seq. of sets in A. Then There is a sequence of sets $\{B_i\}$ in A s.t $B_m \cap B_n = \phi$ if $m \neq n$. and

$$\bigcup_{i=1}^{n} B_i = \bigcup_{i=1}^{n} A_i$$

Pf:- Let $\{A_i\}$ is an infinite seq. of subsets of X.

define $B_1 = A_1$

$B_2 = A_2 \setminus A_1 = A_2 \cap A_1^c$

$B_3 = A_3 \setminus [A_1 \cup A_2] = A_3 \cap A_1^c \cap A_2^c$

$\vdots$

$B_n = A_n \setminus [A_1 \cup A_2 \cdots \cup A_{n-1}]$

$= A_n \cap A_1^c \cap A_2^c \cap \cdots \cap A_{n-1}^c$

Then All $B_i \in A$.

and $B_n \subset A_n \ \forall n$ —— ①

let $m < n$, then $B_m \subset A_m$

take

$\underline{B_m \cap B_n} \subset A_m \cap B_n = A_m \cap B_n$ $\quad A_m^c$

$= A_m \cap [A_n \cap A_1^c \cap A_2^c \cap \cdots \cap A_{n-1}^c]$

$= (A_m \cap A_m^c) \cap \cdots$ other terms $(\because m < n)$

$= \phi$ satisfied for $m \neq n$

Again $\because B_n \subset A_n \ \forall n$ (from ①)

$\Rightarrow \bigcup_{i=1}^{n} B_i \subseteq \bigcup_{i=1}^{n} A_i$ —— ②

Again let $x \in \bigcup_{i=1}^{n} A_i$

$i = 1, 2, \ldots n$

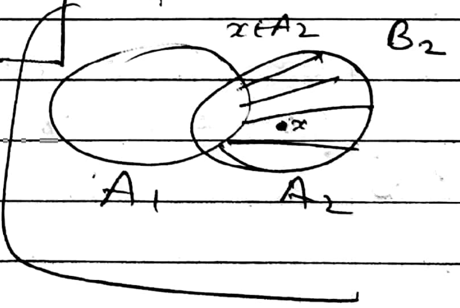
$\Rightarrow x$ is member of at least one of A_i's

let m be the smallest i, s.t $x \in A_m$

$\because B_m = A_m - \{A_1 \cup A_2 \cup \cdots \cup A_{m-1}\} \ni x \Rightarrow x \in B_m$ for some $i = m$.

$\Rightarrow x \in \bigcup_{i=1}^{n} B_i$

$\Rightarrow \boxed{\bigcup_{i=1}^{n} A_i \subseteq \bigcup_{i=1}^{n} B_i}$ —— ③



from ② & ③, we have

$$\bigcup_{i=1}^{n} B_i = \bigcup_{i=1}^{n} A_i$$

\# ⑫ **Monotone class:** let M be collection of subsets of X s.t ——

(i) If $A_i \in M$, $i = 1, 2, \ldots$ and $A_n \subset A_{n+1}$, then $\bigcup A_i \in M$

and (ii) If $A_i \in M$, $i = 1, 2, \ldots$ and $A_n \supset A_{n+1}$, then $\bigcap A_i \in M$

Theorem: But Converse need not be true:

⑫ (N) Every σ-Algebra is monotone class.

Pf:— let M be a σ-algebra, $A_i \in M$.
and $A_n \subset A_{n+1}$ $\forall n$.

then $\bigcup_{i=1}^{\infty} A_i \in M$ [defⁿ of σ-Alg.]

Again let $A_n \supset A_{n+1}$

$\because A_i \in M \Rightarrow A_i^c \in M$

$\Rightarrow \bigcup_{i=1}^{\infty} A_i^c \in M$. [∵ M is σ-Alg.]

$\Rightarrow (\cap A_i)^c \in M$ $(A \in M \Rightarrow A^c \in M)$

$\Rightarrow \cap A_i \in M$.

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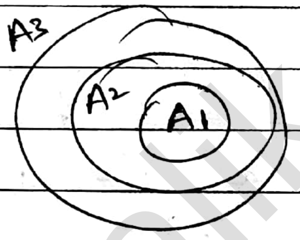
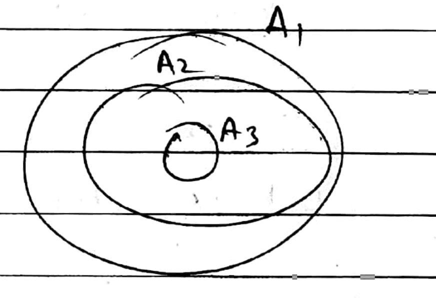


$A_n \subseteq A_{n+1}$



$A_n \supseteq A_{n+1}$

Every σ-algebra is monotone class, But Converse is NOT true.

EX ① Let X be uncountable set and
$M = \{A \subseteq X \mid A \text{ is countable}\}$
Then M is monotone class but NOT σ-algebra.

Pf: solⁿ. Let $A_n \in M$ $(n=1,2\dots)$ & $A_n \subseteq A_{n+1}$
then $\bigcup_{n=1}^{\infty} A_n \in M$ (count. union of count set)

Also suppose that, $A_n \in M$ $(n=1,\dots n)$ and $A_n \supseteq A_{n+1}$ then

$\because \bigcap_{n=1}^{\infty} A_n \subset \boxed{A_n} \Rightarrow \bigcap_{n=1}^{\infty} A_n \in M$

Since it is countable

so it is countable so $\in M$

X uncountable and
$A \in M$, $A \subseteq X$ then
either A or A^c is countable
Now, if A is countable $\Rightarrow$ $A \in M$
$\Rightarrow$ A^c is not-countable $\Rightarrow$ $A^c \notin M$
$\Rightarrow$ M is Not a σ- Algebra (By def^n of σ-Alg)

statement: Let C be (collection) class of subsets of X. If C is Alg.
⑬ , Also monotone class, then C is σ- Algebra.

Pf: ~~Given that~~ let C is algebra and monotone class
(i) $X, \phi \in C$
(ii) $A \in C$ then $A^c \in C$
(iii) $\bigcup_{i=1}^{n} A_i \in C$ if $A_i \in C$ $(i=1, \dots n)$, also $\bigcap_{i=1}^{n} A_i \in C$

let $B_n = \bigcup_{i=1}^{n} A_i$, if $B_i \subseteq B_{i+1}$ $\Rightarrow$ $\bigcup_{n=1}^{\infty} B_n \in C$

if ~~B_n~~ $\supseteq$ ~~B_{n+1}~~, so by monotone

$\Rightarrow \bigcap_{n=1}^{\infty} B_n \in C$

$\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} \left(\bigcup_{i=1}^{n} A_i\right) \in C \Rightarrow \bigcup_{n=1}^{\infty} A_i \in \underset{\sigma\text{-Alg}}{C}$

$\Rightarrow \left(\bigcap_{n}^{\infty} B_n\right)^c \in C$

— x —

Why measure theory is interesting

Some interesting facts

① Let $S = \{1, 2, 3\}$ = finite set
smallest closed interval contains S, $[1,3] = \Delta$
$\therefore \Delta - S = (1,2) \cup (2,3)$
$\therefore l(\Delta - S) = l[(1,2) \cup (2,3)]$
$l[\Delta] - l[S] = 2$
$\Rightarrow 2 - l[S) = 2 \Rightarrow \boxed{l(S] = 0}$
So length of finite set is '0'.

(ii) $S = \{0, 1, 1/2, 1/3, 1/4, \dots \frac{1}{n} \dots 0\} = \{1/n, n \in N\}$
smallest closed interval contains S, $[0,1] = \Delta$
$l(\Delta - S) = l\left[\bigcup_{i=1}^{\infty}\left(\frac{1}{n+1}, \frac{1}{n}\right)\right]$
$= \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right)$ $\left(\frac{1}{n} - \frac{1}{n+1} + \frac{1}{\ }\right.$

$= 1$

$\Rightarrow l(\Delta) - l(S) = 1 \Rightarrow 1 - l[S] = 1 \Rightarrow \boxed{l(S) = 0}$

I.K. Rana → Real Analysis (Indian)

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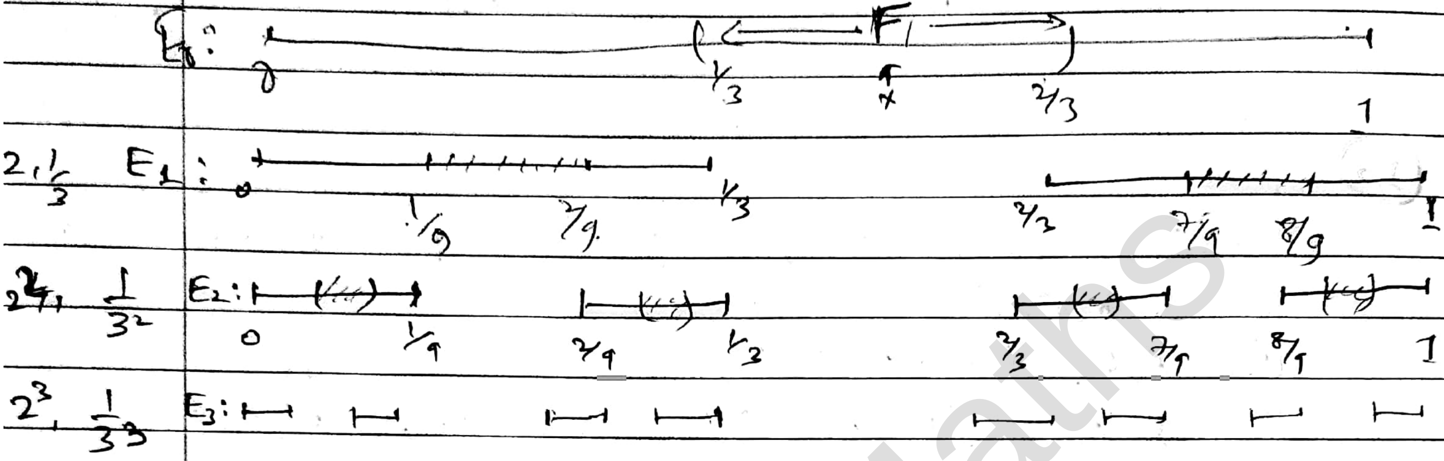
Eg. ④ Length is a set f^n defined over set of intervals.

$$l(I) = \begin{cases} \infty , & \text{If } I \text{ is unbounded} \\ \text{Difference of end pt.s} & \text{if is b.dd} \end{cases}$$

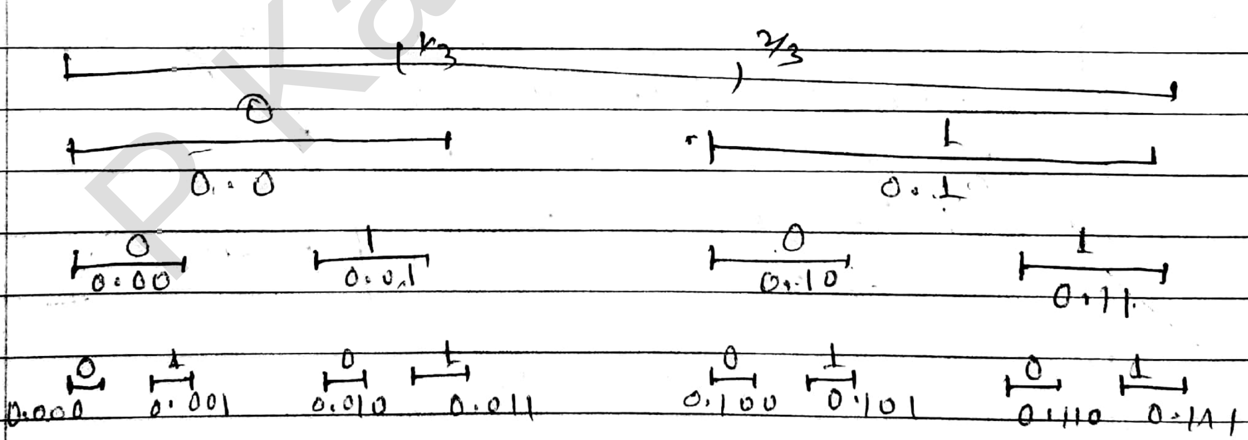


$$P = \bigcap_{i=0}^{\infty} E_i = \left\{0, 1, \frac{1}{3}, \frac{2}{3}, \frac{1}{9}, \frac{2}{9}, \frac{7}{9}, \frac{8}{9}, \dots \right\}$$

It is known as '<u>Cantor set</u>'.

(since [0,1] is uncountable set, so its subset may be countable/uncountable)

Now checking for uncountable



$P \in [0,1]$

For each element of P, We have a representation of the form $0.a_1a_2a_3a_4a_5a_6\dots$ where a_i's are 0 or 1.

let P is countable, then there is 1-1 Corresponding with $\mathbb{N}$.

$n_1 \longleftrightarrow 0.a_{11}a_{12}a_{13}a_{14}a_{15}\ldots$

$n_2 \longleftrightarrow 0.a_{21}a_{22}a_{23}a_{24}\ldots$ } — Ⓐ

$n_3 \longleftrightarrow 0.a_{31}a_{32}a_{33}a_{34}a_{35}\ldots$

$- - - -$

We ~~Construct~~ would like to Construct another number belonging to P, not the member of (A).

Let $b \in P$ s.t

$$b = 0.b_1b_2b_3b_4\ldots$$

Where $b_i = \begin{cases} 0 & \text{if } a_{ii} = 1 \\ 1 & \text{if } a_{ii} = 0 \end{cases} \Rightarrow b \notin P$

then P is not countable.

(Now goto A)

$E_1 = [0, 1/3] \cup [2/3, 1]$, $F = (1/2, 2/3)$ $\quad \mu(E_1) = 2/3$

$E_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$ $\quad \mu(E_2) = (2/3)^2$

$\vdots$

$\mu(E_n) = (2/3)^n$

$$l[[0,1]-P] = l\left[\bigcup_{n=1}^{\infty} E_n\right]$$

$$1 - l(P) = \sum_{n\geq1}^{\infty} (2/3)^n \qquad (\text{Replace } n \text{ by } n+1)$$

$\Rightarrow n-1 = 0 \qquad (n-1 = n)$

$$= \sum_{n=0}^{\infty} (2/3)^{n+1} = \frac{2}{3}\sum_{n=0}^{\infty} (2/3)^n$$

$$= \frac{2}{3} \times \frac{1}{1-2/3} = \frac{2}{3} \times 3 = 2$$

$$[0,1] - F = \frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \cdots + \left(\frac{2}{3}\right)^{n-1}\cdot\frac{1}{3} = \bigcup F_i$$

When $F_1 = E_1^c = (1/3, 2/3) \quad \Rightarrow l(F_1) = 1/3 = \frac{1}{3}\cdot 1$

$F_2 = E_2^c = (1/9, 2/9) \cup (7/9, 8/9) \Rightarrow l(F_2) = 2/9 = \frac{1}{3}\cdot\frac{2}{3}$

— — — —

$= \frac{1}{3}\cdot\left(\frac{2}{3}\right)^{n-1}$

$\frac{1}{3}\left(1 + \frac{2}{3} + \frac{2^2}{3^2}\right)$

$$\frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n = \frac{1}{3} \times \frac{1}{1-2/3} = \frac{1}{3} \times \frac{3}{1} = 1$$

Measure. $A \subseteq R$, ① $m(A) \geq 0$ (iv) $m(\cup E_k) = \Sigma m(E_k)$

$y \in R$ (ii) $m(A+y) = m(A)$

(iii) $m(I) = l(I)$

$\Rightarrow l([0,1] - F) = \frac{1}{3} + \frac{2}{9} + \frac{4}{27}$ ——— $= 1$

* * *

20/1/17

Measure: It should be a set function (14) which assign an extended real numbers to each set A of Real No's with the following properties.

(i) ~~∀~~ $\boxed{m(A) \geq 0}$

(ii) If y is any real no., then $A+y = \{x+y \mid x \in A\}$

Hence

$$\boxed{m(A+y) = m(A)}$$

interval shifted by α

$l[I+\alpha] = l(I)$

Eg

$l[(1,2)+5] = l(6,7)$
$= 1$
$= l(1,2)$

This is known as 'Translation Invariant'.

The measure should be translation invariant.

(iii) The measure of an interval is its length i.e. If I is an interval, then $m(I) = l(I)$.

Let $\{I_k\}_{k=1}^{\infty}$ is countable collection of disjoint intervals

$$l\left(\bigcup_{k=1}^{\infty} I_k\right) = \sum_{k=1}^{\infty} l(I_k)$$

this property is known as additive

(iv) The measure should be countably additive i.e if $\{E_k\}_{k=1}^{\infty}$ is countable collection of disjoint-sets then

$$m\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} m(E_k).$$

(14) * Outer Measure

For each set A of real no.s, consider the countable collection $\{I_k\}_{k=1}^{\infty}$ of open intervals that covers A, that is ~~that is~~ $A \subseteq \bigcup_{k=1}^{\infty} I_k$. and for each collection, consider the sum of length of intervals, then outer measure denoted by m^* is defined ~~by~~ as

$$m^*(A) = \inf\left\{ \sum_{k=1}^{\infty} l(I_k) \;\middle|\; A \subset \bigcup_{k=1}^{\infty} I_k \right\}$$

and satisfies

$$\boxed{m^*\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} m^*(E_k)}$$

Theorem: If A is countable set, then $m^*(A) = 0$.

Pf:— ⑮ Let A is a countable set given by

$$A = \{a_1, a_2, \dots a_k \dots\}$$

Each a_i's is covered by $I_k = \left(a_k - \frac{\varepsilon}{2^{k+1}}, a_k + \frac{\varepsilon}{2^{k+1}}\right)$

$$A \subseteq \bigcup_{k=1}^{\infty} I_k$$

$$\Rightarrow m^*(A) \le \underline{m^*\left(\bigcup_{k=1}^{\infty} I_k\right)} \overset{\le \sum_{k=1}^{\infty} m^*(I_k)}{\cancel{=}} \sum_{k=1}^{\infty} l(I_k) = \sum_{k=1}^{\infty} \varepsilon/2^k$$

↓ series

$$= \varepsilon \longrightarrow 0$$

$$\left.\begin{aligned} &\Rightarrow m^*(A) \le 0 \\ &\text{but } \underline{m^*(A) \ge 0} \end{aligned}\right\} \Rightarrow \boxed{m^*(A) = 0}$$

Using above theorem, prove that $[0,1]$ is uncountable. (Proving by contradiction)

Let $[0,1]$ is countable

$$\Rightarrow m^*([0,1]) = 0 \quad \text{— ⓐ}$$

But we know that m^* of an interval is equal to lengths of that interval.

$$\text{so } m^*([0,1]) = l([0,1]) = 1, \quad \text{— ⓑ}$$

But ⓐ, ⓑ are NOT equal. $(0 \ne 1)$

So, our assumption is wrong.

So, $[0,1]$ is NOT countable.

*→ Converse of Above theorem is NOT true,

Now consider the example on p-141

$$E_1 = [0, 1/3] \cup [2/3, 1], \quad l(E_1) = 2/3$$

$$E_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1], \quad l(E_2) = (2/3)^2$$

Heine Borel theorem for covering

$P = \bigcap_{k=1}^{\infty} E_k$, $P \subset E_k$ for each k

$\therefore m^*(P) \le m^*(E_k) = \left(\frac{2}{3}\right)^k$.

$\Rightarrow \lim_{k\to\infty} \left[m^*(P) \le m^*(E_k) = \left(\frac{2}{3}\right)^k \right] \to 0$

means Cantor set is countable
But Cantor set is not countable
So converse of above theorem is NOT true.

* * *

Wed 25/01/17 Recap. $m^*(\emptyset) = 0$ &
if $A \subseteq B \Rightarrow m^*(A) \le m^*(B)$

Theorem: The outer measure of an interval is its length.

⑯ Pf:- for proving this, we take three cases.
Case-I, $m^*([a,b]) = l([a,b]) = b-a$
case-II, $m^*($ ~~any~~ for any bdd interval$)$ = length of that interval
case-III, $m^*($unbdd interval$) = \infty$

Case-I Let given interval is closed and bounded interval given by $[a,b]$, for $\varepsilon > 0$, there exists an interval $(a-\varepsilon, b+\varepsilon)$ that contain $[a,b]$

$$m^*([a,b]) \le m^*((a-\varepsilon, b+\varepsilon)) \le l((a-\varepsilon, b+\varepsilon)) = b-a+2\varepsilon$$

Since ε is arbitrary, thus
We conclude that $m^*([a,b]) \le b-a$

Next we would like to show that $m^*([a,b]) \ge b-a$

Let $\{I_k\}_{k=1}^{\infty}$ is a collection of open and bounded intervals such that

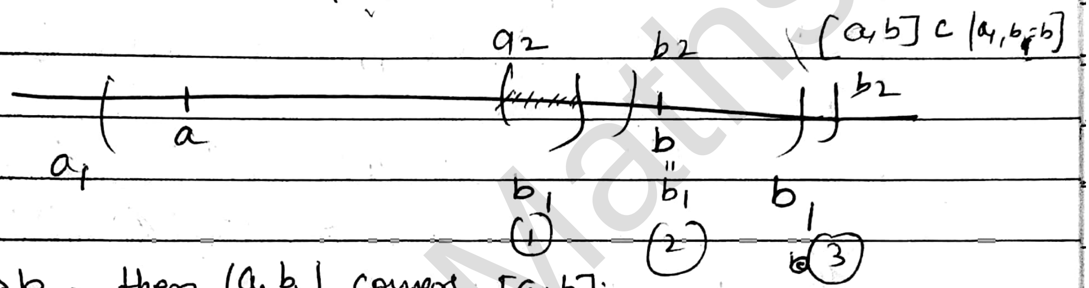
$$[a,b] \subseteq \bigcup_{k=1}^{\infty} I_k$$

finite
q ∃ a subcover of [a,b]

By Heine Borel theorem, there must be one natural number n s.t, $\bigcup_{k=1}^{n} I_k$ contains $[a,b]$ then it must be

$$m^*([a,b]) \geq \sum_{k=1}^{\infty} l(I_k) \geq \sum_{k=1}^{n} l(I_k) \geq b-a \longleftarrow \times 1)$$

Since $\{I_k\}_{k=1}^{n}$ is finite cover of $[a,b]$, then there must be one of I_k's that contain a; let it is (a_1, b_1) then $a_1 < a < b_1$



✓ If $b_1 > b$, then (a_1, b_1) covers $[a,b]$.

$$\sum_{k=1}^{\infty} l(I_k) \geq \sum_{k=1}^{n} l(I_k) \geq b_1 - a_1 > b-a$$

✓ If $b_1 < b$ $(b_1 \ngtr b)$, then $[a,b]$ is NOT fully covered by (a_1, b_1)

Hence there must be one of I_k's. let it be (a_2, b_2) then

$a_2 < b_1 < b_2$.

If $b_2 > b_1$,
$$\sum_{k=1}^{\infty} l(I_k) \geq (b_2 - a_2) + (b_1 - a_1)$$
$$= b_2 - (a_2 - b_1) - a_1 > b_2 - a_1$$
$$> b-a$$

further,

If $b_2 < b_1$, then there must be an interval say (a_3, b_3) ----

We continue this process/selection process until it terminates as it must as there is only n intervals in the collection $\{I_k\}_{k=1}^{n}$.

Thus we obtain a subcollection $\{(a_k, b_k)\}_{k=1}^{N}$ of $\{I_k\}_{k=1}^{n}$ for which $a_1 < a$ while

$a_{k+1} < b_k$ for $1 \leq k \leq N-1$

and since selection process terminates $b_N > b$.

Thus
$$\sum_{k=1}^{n} l(I_k) \geq \sum_{k=1}^{N} l((a_k, b_k)) = (b_N - a_N) + (b_{N-1} - a_{N-1})$$
$$+ \cdots + (b_1 - a_1)$$

$$= b_N - (a_N - a_{N-1}) - \cdots - a_1$$
$$> b_N - a_1 > b - a$$

Therefore, we conclude that $m^*[a,b] \geq b-a$

hence $\boxed{m^*([a,b]) = b-a}$

Case-II Let given interval I is bounded interval, then $\exists$ two open intervals, let J_1 and J_2 s.t

$$J_1 \subseteq I \subseteq J_2 \Rightarrow m^*(J_1) \leq m^*(I) \leq m^*(I_2) \quad (1)$$

with $l(I) - \varepsilon \geq l(J_1)$ and $l(J_2) \geq l(I) + \varepsilon$

Then $\Rightarrow l(I) - \varepsilon \leq m^*(I) \leq l(I) + \varepsilon$

~~$l(I) - \varepsilon < l(J_1) \leq m^*(J_1) \leq m^*(I) \leq m^*(J_2) \leq$~~

~~$\leq l(J_2) \leq l(I) + \varepsilon$~~ ~~$\leq l(J_2)$~~

Thus we conclude that

$$\boxed{m^*(I) = l(I)}$$

Case-III If I is unbounded $m^*(I) = \infty$ ← TP

then for every natural no. n s.t there is an interval J with $J \subseteq I$ and $m^*(J) = n$.

Thus

$$m^*(I) \geq m^*(J) = l(J) = n$$

hold for every natural n, Hence $m^*(I) = \infty$

* * *

1PM FRI
27/1/17
⑰

Theorem: The outer measure is translation invariant that is for any set A and any no. y

$$m^*(A+y) = m^*(A)$$

Pf:- let $\{I_k\}_{k=1}^{\infty}$ is countable collection of open, bounded intervals that covers A, then $\{I_k+y\}_{k=1}^{\infty}$ is also covers $A+y$.

$$m^*(A+y) = \inf\left\{\sum_{k=1}^{\infty} l(I_k+y) \;\middle|\; A+y \subseteq \bigcup_{k=1}^{\infty} [I_k+y]\right\}$$

$$= \inf\left\{\sum_{k=1}^{\infty} l(I_k) \;\middle|\; A \subseteq \bigcup_{k=1}^{\infty} I_k\right\} = m^*(A)$$

$$= m^*(A) \quad (\because l(I_k) = l(I_k+y))$$

also thorm (also finitely)

Theorem

statement: The outer measure is countably subadditive, that is

⑱ if $\{E_k\}_{k=1}^{\infty}$ is countable collection of sets, disjoint or NOT then $\boxed{m^*\left(\bigcup_{k=1}^{\infty} E_k\right\} \le \sum_{k=1}^{\infty} m^*(E_k)}$

Pf:- If any one of E_k's having infinite outer measure then result hold trivially.

If all E_k's have finite outer measure.

For fix k, let $\{I_{k,i}\}_{i=1}^{\infty}$ is countable collection of open, bounded intervals that cover E_k. s.t.—

$$E_k \subseteq \bigcup_{i=1}^{\infty} I_{k,i} \quad \text{and} \quad \sum_{i=1}^{\infty} l(I_{k,i}) \geq m^*(E_k) + \varepsilon/_{2^k}$$

Then $\{I_{k,i}\}_{1\le k,i<\infty}$ is cover $\bigcup_{k=1}^{\infty} E_k$

$$\bigcup_{k=1}^{\infty} E_k \subseteq \bigcup_{k=1}^{\infty}\bigcup_{i=1}^{\infty} I_{k,i}$$

$$m^*\left(\bigcup_{k=1}^{\infty} E_k\right) \le \sum_{k=1}^{\infty}\left(\sum_{i=1}^{\infty} l(I_{k,i})\right)$$

$$< \sum_{k=1}^{\infty}\left[m^*(E_k) + \varepsilon/_{2^k}\right]$$

$$= \sum_{k=1}^{\infty} m^*(E_k) + \varepsilon$$

$\varepsilon \to 0$ $\boxed{m^*\left(\bigcup_{k=1}^{\infty} E_k\right) \subseteq \sum_{k=1}^{\infty} m^*(I_k)}$

*

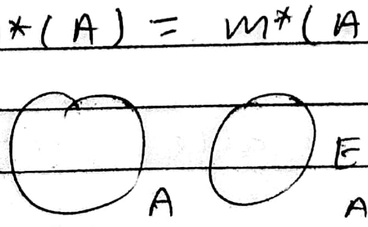
Result: The outer measure is finitely subadditive, that is $\{E_k\}_{k=1}^{n}$ is collection of sets, disjoint or NOT then $\boxed{m^*\left(\bigcup_{k=1}^{n} E_k\right) \le \sum_{k=1}^{n} m^*(E_k)}$ pf ($E_k = \phi$ for $k>n$)

M 1-8-II

* **Measurable set**

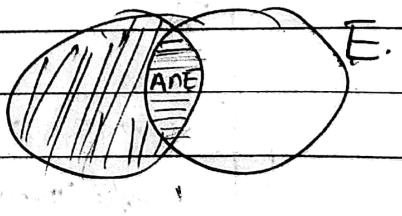
⑲ A set E is s.t.b measurable if for any set A

$m^*(A) = m^*(A\cap E) + m^*(A\cap E^c)$



$A\cap E = \phi$

OR



As $A = (A\cap E) \cup (A\cap E^c)$

$\Rightarrow \boxed{m^*(A) \le m^*(A\cap E) + m^*(A\cap E^c)}$ ← Always hold

So we have to show only that $m^*(A) \ge m^*(A\cap E) + m^*(A\cap E^c)$ → $m^*(A) =$ —

Result: If $\mathbb{R}$ is measurable ?

Ans: Yes,

let A be any set in $\mathbb{R}$ (extended $\mathbb{R}$)

then $A\cap \mathbb{R} = A$ & $A\cap \mathbb{R}^c = \phi$

$\Rightarrow m^*(A\cap \mathbb{R}) = m^*(A)$

$m^*(A\cap \mathbb{R}^c) = m^*(\phi) = 0$

$\Rightarrow m^*(A) = m(A\cap \mathbb{R}) + m^*(A\cap \mathbb{R}^c)$

* If null set $(\{\phi\})$ is measurable — YES

If E is measurable $\Rightarrow E^c$ is also measurable

Since $\mathbb{R}$ is measurable $\Rightarrow \mathbb{R}^c = \phi$ is measurable

Theorey: ⑳ Any set, which outer measure is zero, is measurable.

Pf:- Let E is a set s.t $m^*(E) = 0$

To show that E is measurable, we need to show that

$m^*(A) = m^*(A\cap E) + m^*(A\cap E^c)$ for any set A.

$A\cap E \subseteq E$ and $A\cap E^c \subseteq A$

$m^*(A\cap E) \le m^*(E) = 0$. $0 + m^*(A\cap E^c) \le m^*(A)$

$\Rightarrow m(A\cap E^c) + \underbrace{m^*(A\cap E)}_{=0} \le m^*(A)$ —(*)

Here (*) is sufficient as stated in (##).

So, E is measurable ⊠ ㉑

* * *

(Countable set is measurable set)

$m^*(\text{countable}) = 0 \Rightarrow$ mean

$\mathbb{R}$ Int

S = { set of measurable sets }

Theorem: The collection of measurable sets in $\mathbb{R}$ is algebra.

① ~~Sol~~ Pf: I The set $\mathbb{R}$ and ϕ is measurable, II also for any
(2) measurable set E in $\mathbb{R}$, its complement E^c is measurable. To prove this result,

We must show that union of finite collection of measurable sets i.e $\bigcup_{K=1}^{n} E_k$ is also measurable.

First we will show that, if E_1 & E_2 are two measurable sets, then $E_1 \cup E_2$ is also measurable

set E_1 is measurable space,

for any set A, by definition

$$m^*(A) = m^*(A\cap E_1) + \underline{m^*(A\cap E_1^c)}$$

$$\cap E_2^c]$$

① $\longrightarrow$ $$= m^*(A\cap E_1) + \underbrace{m^*([A\cap E_1^c]\cap E_2) + m^*([A\cap E_1^c]}$$

⇒ $[A\cap E_1^c]\cap E_2^c = A\cap[E_1\cup E_2]^c$ ——— ②

⇒ $$(A\cap E_1)\cup(A\cap E_1^c\cap E_2) = A\cap(E_1\cup(E_1^c\cap E_2))]$$

$$= A\cap[\underbrace{(E_1\cup E_1^c)}_{\mathbb{R}}\cap(E_1\cup E_2)]$$

$$= A\cap[R\cap(E_1\cup E_2)]$$

$$(A\cap E_1)\cup(A\cap E_1^c\cap E_2) = A\cap(E_1\cup E_2)$$ —(*)

By the finite subadditive of outer measure

∴ By (*) $\underline{m^*(A\cap(E_1\cup E_2))} \le m^*(A\cap E_1) + m^*(A\cap E_1^c\cap E_2)$

——— ③

using (1)–(3), We have

$$m^*(A) \ge m^*(A\cap[E_1\cup E_2]) + m^*(A\cap(E_1\cup E_2)^c]$$

⇒ holds ⇒ i.e $(E_1 \cup E_2)$ is measurable.

let $\bigcup_{K=1}^{n-1} E_k$ is measurable (i.e true for n=K-1)

then $\left[\bigcup_{K=1}^{n-1} E_k\right]\cup E_n = \bigcup_{K=1}^{n} E_k$ is measurable. $\in S$,

Proposition ② The intersection of two measurable set E_1 & E_2 is measurable (thus finite intersection belongs) self — As we know that ϕ & R are measurable set. The for any measurable set E ~~we have for~~ ~~any set~~ A ⇒ E^c is also measurable.

Let E_1 & E_2 be two measurable set.

TP: $E_1 \cap E_2$ is measurable.

∵ By de-Morgan's law we have $(E_1 \cap E_2)^c = E_1^c \cup E_2^c$

As we have E_1 & E_2 is measurable

⇒ E_1^c & E_2^c is also measurable

so by previous Result $E_1^c \cup E_2^c$ is also measurable.

so $E_1^c \cup E_2^c = (E_1 \cap E_2)^c$

⇒ $(E_1 \cap E_2)^c$ is measurable

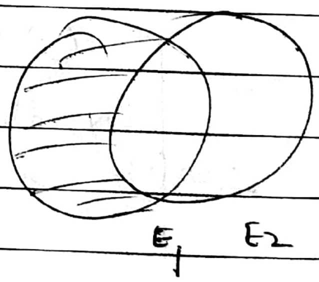
⇒ $E_1 \cap E_2$ is measurable.

Proposition ③ The difference of two measurable set is also measurable.

sol^n Let E_1 & E_2 be two measurable set

⇒ E_1^c & E_2^c be also measurable

TP: $E_1 - E_2$ or $E_2 - E_1$ is measurable.



∵ $E_1 - E_2 = E_1 \cap E_2^c$

Now using pr. result $E_1 \cap E_2^c$ is measurable.

Theorem ④ The collection of measurable sets in $\mathbb{R}$ is a σ-algebra. (Pf like-21)

Theorem ⑤ Let A be any set and $\{E_k\}_{k=1}^{n}$ is finite disjoint collection of measurable sets. then

$$m^*\left(A \cap \left(\bigcup_{k=1}^{n} E_k\right)\right) = \sum_{k=1}^{n} m^*(A \cap E_k)$$

{In particular

$$m^*\left(\bigcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} m^*(E_k)$$

Theorem ⑤ Let us, Prove that $\uparrow$ is countably additive. i.e

$$m^*\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} m^*(E_k)$$

* * *

03 THU
02/02/17

Proof ⑤ We prove it by induction method

The results holds for $n=1$ (assume)

Let it hold for $k=n-1$.

$$m^*\left(A\cap\left[\bigcup_{k=1}^{n-1} E_k\right]\right) = \sum_{k=1}^{n-1} m^*(A\cap E_k) \quad \text{—— (i)}$$

Consider. $A\cap\left[\underbrace{\bigcup_{k=1}^{n} E_k}\right]\cap E_n = A\cap E_n$

$$A\cap\left[\bigcup_{k=1}^{n} E_k\right]\cap E_n^c = A\cap\left[\left[\bigcup_{k=1}^{n-1} E_k\right]\cup E_n\right]\cap E_n^c$$

$$= A\cap\left(\left\{\bigcup_{k=1}^{n-1} E_k\right\}\cap E_n^c\right)\cup\left[\underbrace{E_n\cap E_n^c}_{=\phi}\right]$$

$$= A\cap\left[\bigcup_{k=1}^{n-1} E_k\right].$$

Since E_n is measurable, then by def^n —

$$m^*\left(A\cap\left[\bigcup_{k=1}^{n} E_k\right]\right) = m^*\left(A\cap\left[\bigcup_{k=1}^{n} E_k\right]\cap E_n\right) +$$

$$m^*\left(A\cap\left[\bigcup_{k=1}^{n} E_k\right]\cap E_n^c\right)$$

$$= m^*(A\cap E_n) + m^*\left(A\cap\left[\bigcup_{k=1}^{n-1} E_k\right]\right)$$

$$= m^*(A\cap E_n) + \sum_{k=1}^{n-1} m^*(A\cap E_k) \quad \text{(By (i))}$$

$$\boxed{m^*\left(A\cap\left(\bigcup_{k=1}^{n} E_k\right)\right) = \sum_{k=1}^{n} m^*\left(A\cap\left(\bigcup_{k=1}^{n} E_k\right]\right)}$$

✓

Suppose $A=\mathbb{R}$, then $\boxed{m^*\left(\bigcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} m^*(E_k)}$

S = {collection of measurable set}

Pf. (thm-4) Here we have to show that "the collection of measurable set is σ-algebra".

Since, collection of measurable sets is algebra then to prove result, we must show that union of countable collection of measurable sets $\{A_k\}_{k=1}^{\infty}$ is again measurable.

~~$\bigcup_{k=1}^{\infty} A_k =$~~ Let $\{E_k\}_{k=1}^{\infty}$ is collection of disjaint measurable sets defined by —

$$E_n = A_n \setminus \left\{ \bigcup_{k=1}^{n-1} A_k \right\}$$

$E_1 = A_1$
$E_2 = A_2 \setminus A_1$
$E_3 = A_3 \setminus (A_1 \cup A_2)$

Then $\bigcup_{k=1}^{\infty} A_k = \bigcup_{k=1}^{\infty} E_k = E$

Hence, it is sufficient to show that $\bigcup_{k=1}^{\infty} E_k$ is measurable. We need to prove that —

$$m^*(A) = m^*\left(A \cap \left(\bigcup_{k=1}^{\infty} E_k\right)\right) + m^*\left(A \cap \left[\bigcup_{k=1}^{\infty} E_k\right]^c\right)$$

Let $F = \bigcup_{k=1}^{n} E_k$, and $E = \bigcup_{k=1}^{\infty} E_k$ Since F is measurable, then

$$m^*(A) = m^*(A \cap F) + m^*(A \cap F^c) \quad \text{— ①}$$

And also, $F^c \supseteq E^c$ $(\because F \subseteq E)$
$\Rightarrow m^*(F^c) \geq m^*(E^c)$ (By monotonicity

$$\therefore m^*(A) \geq m^*(A \cap F) + m^*(A \cap E^c)$$

independent of n → $m^*(A) \geq \sum_{k=1}^{n} m^*(A \cap E_k) + m^*(A \cap E^c)$ (from thm 5)

Now let $n \to \infty$ $\quad m^*(A) \geq \sum_{k=1}^{\infty} m^*(A \cap E_k) + m^*(A \cap E^c)$ — (ii)

$$\geq m^*(A \cap E) + m^*(A \cap E^c)$$

∵ By the countable subadditive of outer measure —

$$m^*\Big(A \cap \underbrace{\Big[\bigcup_{k=1}^{\infty} E_k\Big]}_{=E}\Big) \leq \sum_{k=1}^{\infty} m^*(A \cap E_k)$$

* * *

Using this in (ii), then from (i)

AFTER 1st Internal

$m^*(A) \geq m^*\left(A \cap \left[\bigcup_{k=1}^{\infty} E_k\right]\right) + m^*(A \cap E^c)$

$\Rightarrow m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^c)$ — ③ $\left\{ \because \bigcup_{k=1}^{\infty} F_k = E \right\}$

Also, we know that for any set A :-

$m^*(A) \leq m^*(A \cap E) + m^*(A \cap E^c)$ — ④

from ③ and ④ —

$$\boxed{m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)}$$

$\Rightarrow$ E is measurable $\Rightarrow \bigcup_{k=1}^{\infty} E_k$ is measurable.

$\Rightarrow \bigcup_{k=1}^{\infty} A_k$ is measurable $\Rightarrow$ Hence σ-Algebra. Proved

* * *

THU 9/2/17 ②

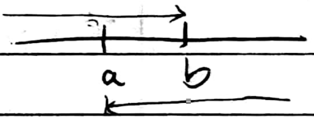
Theorem: The interval $\overset{B}{(a,\infty)}$ is measurable. and [Hence any interval is measurable]

Pf:- $\because (-\infty, a] = (a,\infty)^c = B^c$

$\because B$ $(-\infty, a) = \bigcup_{n=1}^{\infty} (-\infty, a - 1/n)$

and

$(a,b) = (-\infty, b) \cap (a, \infty)$



and $\because$ open set = union of countable collection of open intervals

Claim: $m^*(A) = m^*(A \cap B) + m^*(A \cap B^c)$

let A be any set, then suppose

$A_1 = A \cap \underbrace{(a,\infty)}_{B}$, $A_2 = A \cap (-\infty, a] \equiv A \cap (a,\infty)^c$

Then,

We need to show that $m^*(A) \geq m^*(A_1) + m^*(A_2)$

If $m^*(A) = \infty \Rightarrow$ Nothing to prove.

Now let

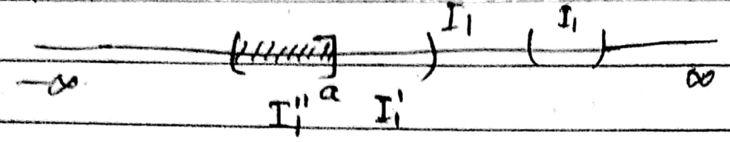
$m^*(A) < \infty$, then there is a countable collection of intervals $\{I_k\}_{k=1}^{\infty}$, that covers A. $\ni A \subseteq \bigcup_{k=1}^{\infty} I_k$

s.t $\sum_{k=1}^{\infty} l(I_k) \overset{\geq}{\leq} m^*(A) + \varepsilon$ — ①

$\rightarrow$ let $I_k' = I_k \cap (a, \infty)$ and

$I_k'' = I_k \cap (-\infty, a]$ $\forall k$

$\Rightarrow l(I_k) = l(I_k') + l(I_k'')$ $\forall k$



$= m^*(I_k') + m^*(I_k'')$ —— (*)

$\because I_k' \cup I_k'' = [I_k \cap (a,\infty)] \cup [I_k \cap (-\infty,a]]$

$= I_k \cap [(a,\infty) \cup (-\infty,a]]$

$= I_k \cap R = I_k$

$\Rightarrow l(I_k' \cup I_k'') = l(I_k)$

$\Rightarrow l(I_k') + l(I_k'') = l(I_k)$

$\Rightarrow A_1 \subseteq \bigcup_{k=1}^{\infty} I_k'$

$\Rightarrow m^*(A_1) \leq m^*\left(\bigcup_{k=1}^{\infty} I_k'\right) \leq \sum_{k=1}^{\infty} m^*(I_k')$ —— ②

Again, $A_2 \subseteq \bigcup_{k=1}^{\infty} I_k''$

$\Rightarrow m^*(A_2) \leq m^*\left(\bigcup_{k=1}^{\infty} I_k''\right) \leq \sum_{k=1}^{\infty} m^*(I_k'')$ —— ③

adding ② + ③ $\Rightarrow$

$m^*(A_1) + m^*(A_2) \leq \sum_{k=1}^{\infty} [m^*(I_k') + m^*(I_k'')]$

$= \sum_{k=1}^{\infty} l(I_k)$ (from (*))

$\leq m^*(A) + \varepsilon$ (from ①)

$\because \varepsilon \to 0$, then —

$$\boxed{m^*(A) \geq m^*(A_1) + m^*(A_2)}$$ ✓

Hence,

$m^*(A) \geq m^*[A \cap (a,\infty)] + m^*[A \cap (-\infty,a]]$

$\{\because A_1 = A \cap (a,\infty),\ A_2 = A \cap (-\infty,a]\}$

Hence,

(a,∞) is measurable.

Theorem:- ㉓ The translation of measurable set is measurable.

Pf:- Let E be a measurable. Let A be any set and, y be any real no.

$\because m^*(A) = m^*(A-y)$ ($\because m^*$ is translation invariant)

$= m^*([A-y] \cap E) + m^*([A-y] \cap E^c)$

($\because$ E is measurable)

$= m^*(A \cap [E+y]) + m^*(A \cap [E+y]^c)$

$=$ E+y is measurable

UNIT - II

∴ $m^*((A-y)\cap E) = m^*[A\cap(E+y)]$ because

bcz $A = \{3,4,5\}$, $y=5$, $E = \{0,1,2,3,4\}$

$A-y = \{-2,-1,0\}$

$A-y \cap E = \{0\}$ —— ①

again $E+y = \{5,6,7,8,9\}$

$A\cap(E+y) = \{5\}$

cardinality is same ⇒ outer measure is same.

[sets may not be equal]

* Definition :-

If we restrict set function outer measure to measurable set, then it is known as lebesgue measure (measure):

If E is measurable, then $m^*(E) = m(E)$

(24)* **Theorem:** Let lebesgue measure is countably additive i.e If $\{E_k\}_{k=1}^{\infty}$ is countable collection of disjoint measurable sets, then

$$m\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} m(E_k)$$

Pf:- By the countable subadditivity of outer measure

$$m\left(\bigcup_{k=1}^{\infty} E_k\right) = m^*\left(\bigcup_{k=1}^{\infty} E_k\right) \le \sum_{k=1}^{\infty} m^*(E_k)$$

$$= \sum_{k=1}^{\infty} m(E_k)$$

$$\Rightarrow \boxed{m\left(\bigcup_{k=1}^{\infty} E_k\right) \le \sum_{k=1}^{\infty} m(E)} \text{ —— ①}$$

??

Now ∵ We know that $m^*\left(\bigcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} m^*(E_k)$

∵ $\bigcup_{k=1}^{n} E_k \subseteq \bigcup_{k=1}^{\infty} E_k \Rightarrow m^*\left(\bigcup_{k=1}^{n} E_k\right) \le m^*\left(\bigcup_{k=1}^{\infty} E_k\right)$

$\Rightarrow m^*\left(\bigcup_{k=1}^{\infty} E_k\right) \ge m^*\left(\bigcup_{k=1}^{n} E_k\right)$ ↙

$\Rightarrow m\left(\bigcup_{k=1}^{\infty} E_k\right) \ge m\left(\bigcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} m(E_k)$ —— (II)

$\Rightarrow n\to\infty,\quad m\left(\bigcup_{k\geq 1}^{\infty}E_k\right)\geq\sum_{k\geq1}^{\infty}m(E_k)$ —②

∴ from ① & ②, $\boxed{m\left(\bigcup_{k\geq1}^{\infty}E_k\right)=\sum_{k=1}^{\infty}m(E_k)}$ Proved

* * *

FYS | M.C
10/02/17

㉕ **Theorem:** Let $\{E_i\}$ be an infinite decreasing sequence of measurable sets, that is $E_{i+1}\subset E_i$ $\forall i\in N$. Let $m(E_i)<\infty$ for at least one $i\in N$. Then,

$$m\left(\bigcap_{i=1}^{\infty}E_i\right)=\lim_{n\to\infty}m(E_n)$$

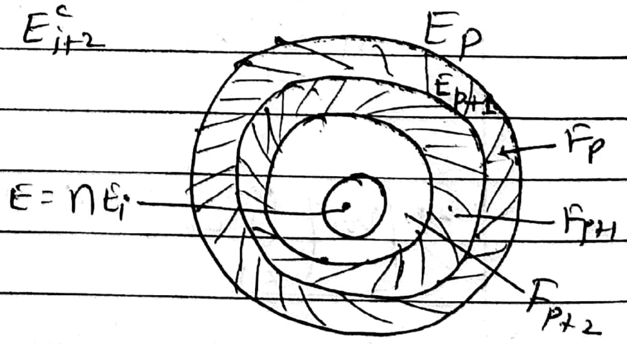
Pf:- Let p be the least number s.t $m(E_p)<\infty$, also suppose,

$$E=\bigcap_{i=1}^{\infty}E_i \quad \& \quad F_i=E_i-E_{i+1}$$

Here we will show that F_i's are pairwise disjoint
i.e $F_i\cap F_{i+1}=\phi \quad \forall \quad i\geq p$

$$\because F_i\cap F_{i+1}=(E_i-E_{i+1})\cap[E_{i+1}-E_{i+2}]$$
$$=(E_i\cap E_{i+1}^c)\cap(E_{i+1}\cap E_{i+2}^c)$$
$$=E_i\cap(E_{i+1}^c\cap E_{i+1})\cap E_{i+2}^c$$
$$=E_i\cap(\phi)\cap E_{i+2}^c$$
$$=\phi$$

$\Rightarrow F_i\cap F_{i+1}=\phi$



$$\because E_p-E=E_p\cap E^c$$
$$=\bigcup_{i\geq p}^{\infty}F_i$$

$$\Rightarrow m(E_p-E)=m\left(\bigcup_{i\geq p}^{\infty}F_i\right)$$

$m(E_p\cap E^c)=\sum_{i\geq p}^{\infty}m(E_i-E_{i+1})$ —① ($\because m$ is countably additive)

$\because E\subseteq E_p$ and $E_{i+1}\subseteq E_i$

~~$E = E_p \cap E$~~ $((E \cup E_p) \cap (E \cup E^c))$

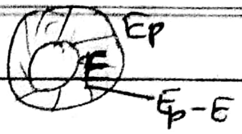
$i \geq p$, $m(E_p) < \infty$

$E \subseteq E_p$ $E \cup E^c$

$E_p = (E \cup E_p)$

$\Rightarrow E_p = E \cup (E_p - E)$ $(\because E \subseteq E_p)$

$= E \cup (E_p \cap E^c)$ — ②



and

$E_i = E_{i+1} \cup [E_i \cap E_{i+1}^c]$ — ③

from ② $\Rightarrow$ $m(E_p) = m(E) + m(E_p \cap E^c)$ (by ②)

from ③.

$m(E_i) = m(E_{i+1}) + m(E_i \cap E_{i+1}^c)$

④ — then $m(E_p) - m(E) = \sum_{i=p}^{\infty} [m(E_i) - m(E_{i+1})]$ (by ①)

$= \lim_{n \to \infty} \sum_{i=p}^{n} [m(E_i) - m(E_{i+1})]$

$\left(\because m(E_p) - m(E) = m(E_p \cap E^c) = m(E_p - E) \right.$

$= \sum_{i=p}^{\infty} m(E_i - E_{i+1}) = \sum_{i=p}^{\infty} m(E_i \cap E_{i+1}^c)$

$\left. = \sum_{i=p}^{\infty} (m(E_i) - m(E_{i+1}) \right)$

Now LHS ④ $\lim_{n \to \infty} [m(E_p) - m(E_n)]$

$= m(E_p) - \lim_{n \to \infty} m(E_n)$

$\Rightarrow m(E_p) - m(E) = m(E_p) - \lim_{n \to \infty} m(E_n)$

$\Rightarrow m(E) = \lim_{n \to \infty} m(E_n)$ $\left\{ \because E = \bigcap_{i=1}^{\infty} E_i \right\}$

$$\Rightarrow \boxed{m\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{n \to \infty} m(E_n)}$$ proved

Theorem (26): Let $\{E_i\}$ be an infinite increasing sequence of measurable sets, that is a sequence with $E_{i+1} \supseteq E_i$ for each $i \in \mathbb{N}$, Then—

$$m\left(\bigcup_{i=1}^{\infty} E_i\right) = \lim_{n \to \infty} m(E_n)$$

(Pf) (self)

Theorem (27):

If E_1 & E_2 are measurable sets, then

$$m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2)$$

Pf: (27) $\because \overbrace{(E_1\cup E_2)}^{A}\cup(E_1\cap E_2) = \left[\overbrace{(E_1\cup E_2)}^{A}\cup E_1\right]\cap\left[\overbrace{(E_1\cup E_2)}^{A}\cup E_2\right]$

$\Rightarrow m\left[(E_1\cup E_2)\cup(E_1\cap E_2)\right] = m\left[\left[(E_1\cup E_2)\cup E_1\right]\cap\left(E_1\cup E_2)\cup E_2\right]\right]$

$\because m(A\cup B) = m(A)+m(B)$ when A & B are ms ~~disjoint~~ —— (*)

$\because (E_1\cup E_2)\cup(E_1\cap E_2) = \left[(E_1\cup E_2)\cup E_1\right]\cap\left[(E_1\cup E_2)\cup E_2\right]$

$= \left[(E_1\cup E_2)\right]\cap\left[(E_1\cup E_2)\right]$

$= E_1\cup E_2$

So $m\left[(E_1\cup E_2)\cup(E_1\cap E_2)\right] = m\left[E_1\cup E_2\right)$

$\therefore$ by (*) $m(E_1\cup E_2)+m(E_1\cap E_2) = m(E_1)+m(E_2)$

—— Proof

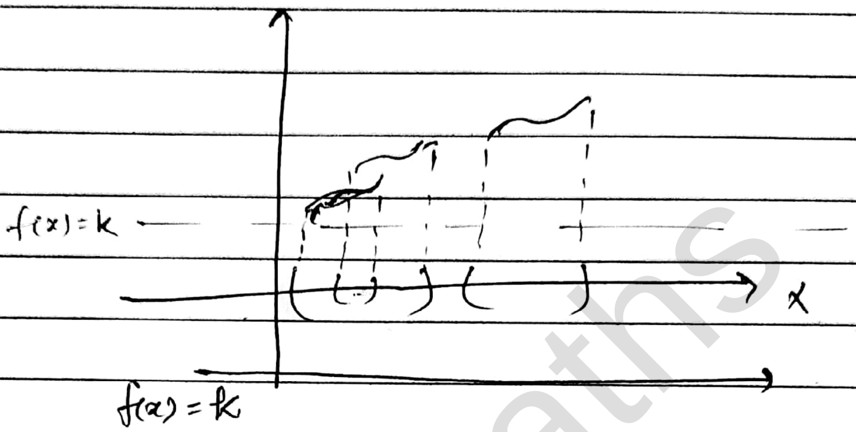
pf: (26)

16/02/17

* Measurable function:

Let E is a measurable set, f(x) is a real valued function defined on E, f is measurable if for any real k $\{x \in E \mid f(x) > k\}$ is measurable.
$k \in \mathbb{R}$

$f : E \to \mathbb{R}$



Eg. $f(x) = \begin{cases} 1 & ; \ 0 \le x \le 1 \\ 0 & ; \ \text{otherwise} \end{cases}$ is measurable f^n

$\because \{x \in E ; f(x) \le k\} = \underline{\{x \in E \mid f(x) > k\}^c}$

* A function f(x) defined on a measurable set E is measurable if and only if $\{x \in E \mid f(x) \le k\}$ is measurable.

Eg. ⓐ $\{x \in E , f(x) > k - 1/n\}$, $n = 1, 2, \ldots$ measurable for all n.

$\bigcap_{n=1}^{\infty} \{x \in E \mid f(x) > k - 1/n\} = \{x \in E \mid f(x) \ge k\}$ measurable

ⓜ ?? * $\boxed{f \text{ is measurable } \Leftrightarrow \{x \in E \mid f(x) \ge k\} \text{ is measurable}}$

Eg. ⓑ $\{x \in E \mid f(x) < k\} = \{x \in E \mid f(x) \ge k\}^c$ is measurable

Eg. ⓒ $\{x \in \mathbb{R} \mid f(x) > k\} = \begin{cases} \mathbb{R} & : k < 0 \\ [0,1] & : 0 \le k < 1 \\ \phi & , k \ge 1 \end{cases}$ measurable

(where f(x) is given in the prev. eg.)

↳ $f(x) = \begin{cases} 1 & x \in [0,1] \\ 0 & \text{O.W} \end{cases}$

$$\because \{x \in R \mid f(x) > k\}^c = \{x \in R \mid f(x) \le k\} = \left\{\begin{array}{ll} R & ; k<0 \\ [0,1] & ; 0 \le k < 1 \\ \phi & ; k \ge 1 \end{array}\right\}^c$$

$$= \begin{cases} \phi & ; k<0 \\ (-\infty,0) \cup (1,\infty) & ; 0 \le k \le 1 \\ R & k \ge 1 \end{cases}$$ is also measurable.

(m) ✗ Let E is a measurable set, f(x) is a real valued f^n defined on E, f is measurable if for any real K, all statements are equivalent:—

(i) $\{x \in E \mid f(x) > k\}$ is measurable $= \overset{\infty}{\bigcup} \{x \in E \mid f(x) \ge k + 1/n\}$

(ii) $\{x \in E \mid f(x) \ge k\}$ is measurable $= \overset{\infty}{\bigcap} \{x \in E \mid f(x) > k - 1/n\}$

(iii) $\{x \in E \mid f(x) < k\}$ is measurable

(iv) $\{x \in E \mid f(x) \le k\}$ is measurable

complement

Theorem: (28) (m) If f(x) & g(x) are measurable on E, then for any real constant c, c, cf, f+c, f+g, f^2 and fg are measurable.

Pf:- $\because \{x \in E \mid f(x) = c\} = \underbrace{\{x \in E \mid f(x) \ge c\}}_{\text{measurable}} \cap \underbrace{\{x \in E \mid f(x) \le c\}}_{\text{measurable}}$

(i) f(x) = c is measurable

(ii) If c = 0, $\Rightarrow$ cf = 0 $\Rightarrow$ constant
$\Rightarrow$ cf is measurable.

If c ≠ 0 then

$$\{x \in E \mid c f(x) > k\} = \{x \in E \mid f(x) > \boxed{k/c}\} \quad \text{if } c>0$$
$$= \{x \in E \mid f(x) < k/c\} \quad \text{if } c<0$$
$$\{ k/c \text{ is constant} \}$$

$\Rightarrow$ cf is measurable.

(iii) If c = 0, $\Rightarrow$ f(x) + c = f(x) $\Rightarrow$ measurable.

$\Rightarrow f+c$ is measurable

If $c \neq 0$, then

$\{x \in E \mid f(x)+c > k\} = \{x \in E \mid f(x) > \underbrace{k-c}_{=k_1}\}$ $\{\because k-c = \text{const}\}$

(iv) $\{x \in E \mid (f+g)(x) > k\} = \{x \in E \mid k-g\}$

$\because f(x) > g(x)$ then we can find a rational no. r.t $f(x) > r > g(x)$. then

$= \bigcup_{r \in Q} \{x \in E \mid f > r > k-g\}$

$= \bigcup_{r \in Q} \left[\{x \in E \mid f(x) > r\} \cup \{x \in E \mid k-g < r\}\right]$

$= \bigcup_{r \in Q} \left[\{x \in E \mid f(x) > r\} \cup \{x \in E \mid g > r-k\}\right]$

$\searrow M \swarrow$

$=$ measurable $\{\because$ union of countable colln of measurable set is measurable$\}$

$\Rightarrow f+g$ is measurable.

(v) $\{x \in E \mid f > k\} = \{x \in E \mid f(x) > \sqrt{k}\} \cup \{x \in E \mid f(x) < -\sqrt{k}\}$

(vi) $\Rightarrow fg = \frac{1}{2}\left[(f+g)^2 - f^2 - g^2\right]$

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(29)

theorem (Internal) : Let f(x) is a cts f^n defined in a measurable set E, then f is measurable.

A = $\{x \in E \mid f(x) \geq k\}$ closed (TS) (To show closed simply we show for limit pt.)

Let α is a limit point of A, then there exist a sequence of points $\{x_n\}$ in A s.t

$$\lim_{n\to\infty} x_n = \alpha.$$

By the def^n of Continuity, we have

$$\lim_{n\to\infty} f(x_n) = f(\alpha) \quad ①$$

$$x_n \in A \ \forall n, \Rightarrow f(x_n) \geq k \quad \forall k \in \eta, \quad ②$$

∴ from ① & ②,

$$\underset{n\to\infty}{\text{lt}}\, f(x_n) = f(\alpha) \geq k \ ; \ \forall k \quad (\text{Bez } f(\alpha) \geq k)$$

"f(α)"

$\Rightarrow \alpha \in A \Rightarrow$ A is closed

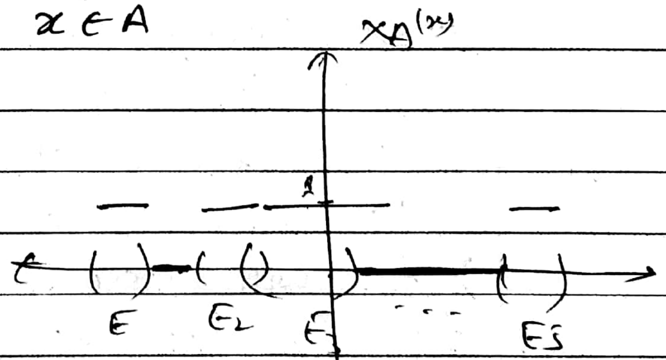
$\Rightarrow$ A is measurable

Characteristics function

(30)

Let E be measurable set. and $A \in E$, then characteristics f^n of A is defined by

$$\chi_A(x) = \begin{cases} 0, & x \notin A \\ 1, & x \in A \end{cases}$$



f(x) =

$$S = \{x \mid \chi_A(x) \geq k\} = \begin{cases} \phi, & k \geq 1 \\ A, & 0 \leq k < 1 \\ E, & k < 0 \end{cases}$$

∵ φ ∉ E and measurable so only depends on A

If A is measurable then S is measurable

$\Rightarrow \chi_A(x)$ is measurable.

㉑ Simple function

A f^n $f: E \to R$ is known to be SIMPLE if, $E = \bigcup_{i=1}^{n} E_i$ and each E_i ($i=1, \dots n$) are measurable and there exists a set of points $\{c_1, c_2 \dots c_n\}_n$ then —

$$\boxed{f(x) = \sum_{i=1}^{n} c_i \chi_{E_i}(x)}$$

Theorem ㉑-I A f^n f is a SIMPLE FUNCTION iff it is measurable and assume only a finite no. of values.

Pf:— Let f be a simple function defined on $E = \bigcup_{i=1}^{n} E_i$ where each E_i ($i = 1 \dots n$) is measurable then

$$f(x) = \sum_{i=1}^{n} c_i \chi_{E_i}(x).$$

where $E_i = \{x \mid f(x) = c_i\}$ & each E_i distinct & c_i are distinct if n-2.

As E_i's are measurable, then $\chi_{E_i}(x)$ are measurable, Hence $f(x)$ is measurable

let $x \notin E = \bigcup_{i=1}^{n} E_i$, Hence $x \notin E_i$ for any i.

$\chi_{E_i}(x) = 0$ for each E_i's.

Hence,

$\underline{f(x) = 0}$, Hence $f(x)$ assume only one value.

Let x belongs to only one E_i $\Rightarrow \chi_{E_i}(x) = 1$

$f(x) = \underline{c_i} \Rightarrow f(x)$ assume only one value. one set out of n set can be choosen by $^nc_1$ ways.

Let $x \in E_i \cap E_j$ $\Rightarrow f(x) = \underline{c_i + c_j}$

$\Rightarrow f(x)$ will assume $^nc_2$ values.

$f(x)$ will assume (in general)

$$^nc_1 + {^nc_2} + \dots + {^nc_n} \text{ values.} = 2^n - 1 \text{ Values}$$

$\Leftarrow$ Conversely $f(x)$ is measurable and assume only finite no. of values. Let it is assume $c_1, c_2 \dots c_n$.

let f(x) assumes $\{c_1, c_2, \dots c_n\}$

$$\underset{\text{mes (given)}}{f(x)} = \sum_{i=1}^{n} c_i \underset{\text{mes}}{\chi}_{\underset{\text{mes}}{E_i}}(x) \text{ ——— } (*)$$

where.

$$\chi_{E_i} = \begin{cases} 1 & x \in E_i \\ 0 & x \notin E_i \end{cases}$$

Since f(x) is L.C of c_i & some f^n so measurable

∵ f(x) is measurable ⇒ $f(x) = \underbrace{c_1 f_1(x) + c_2 f_2(x)}_{\to \text{ measurable}}$

⇒ f_1 & f_2 is also measurable

Suppose that $f_2(x)$ is not-mes. i.e $\underset{m}{f} = c_1 \underset{m}{f_1} + c_2 \underset{nm}{f_c}$

$\Rightarrow f_2 = \frac{1}{c_2}(f - c_1 f_1) \leftarrow$ meas.

so contradiction hence f_2 must be mes.

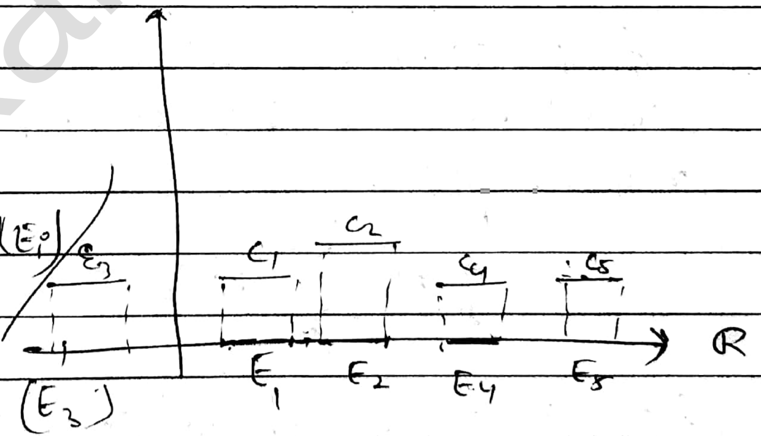
Thus If f(x) is defined as (*), then f(x) is SIMPLE function.

* * *

wed 1st
22/02/16

$$E = \bigcup_{i=1}^{5} E_i$$

$$\int_E f = \int \sum_{i=1}^{5} c_i (m(E_i))$$



Def^n (32): For a simple function ψ defined on a set of finite measurable E, represented by

$$\psi(x) = \sum_{i=1}^{n} c_i \chi_{E_i}, \text{ where } E = \bigcup_{i=1}^{n} E_i, \; c_i \in \mathbb{R} \; (i=1, \dots n)$$

its integration is defined by

$$\boxed{\int_E \psi = \sum_{i=1}^{n} c_i \, m(E_i)}$$

Theorem: Let $\{E_i\}_{i=1}^{n}$ be the finite disjoint collection of measurable subsets of a set of finite measure E. for $1 \le i \le n$, let a_i are real no-s.

Imp

32-I

(p-197)

If $$\phi = \sum_{i=1}^{n} a_i \chi_{E_i} \text{ on } E \text{ then}$$

$$\int_E \phi = \sum_{i=1}^{n} a_i m(E_i).$$

Prev. defn may also be written as —
for a simple fn ψ defined on a set of finite measure E, assumes distincts values $c_1, c_2, \ldots c_n$ on respectively $E_1, \ldots E_n$, then simple fn is then canonical representation is given by

$$\psi(x) = \sum_{i=1}^{n} c_i \chi_{E_i}$$

where $E = \bigcup_{i=1}^{n} E_i$, $c_i \in R$ $(i=1, \ldots n)$

Pf:- Imp — If ϕ is simple fn. Let it having $d_1, d_2, \ldots d_m$ distinct values, ~~then~~ on $A_j = \{x \in E \mid \phi(x) = d_j\}$ respectively.

$$\phi(x) = \sum_{j=1}^{m} d_j \chi_{A_j}$$

then integration of ϕ is given by

$$\int_E \phi = \sum_{j=1}^{m} d_j\, m(A_j) \quad \text{———} ①$$

To show above, we have — $I_j = \{i \mid i \in \{1, 2, \ldots n\}\}$

For $1 \le j \le m$. Let I_j be the set of indices i in $\{1, 2, \ldots n\}$ for which $a_i = d_j$, then

$\{1, 2, \ldots n\} = \bigcup_{j=1}^{m} I_j$. then

$\{1, \dots n\} = \bigcup_{j=1}^{m} I_j$ then

$$m(A_j) = \sum_{i \in I_j} m(E_i) \quad \text{for } 1 \le j \le m$$

Then,

$$\sum_{i=1}^{n} a_i \, m(E_i) = \sum_{j=1}^{m} \sum_{i \in I_j} a_i \, m(E_i)$$

$$= \sum_{j=1}^{m} d_j \Big[\sum_{i \in E_i} m(E_i) \Big] = \sum_{j=1}^{m} d_j \, m(A_j)$$

$$= \int \phi \qquad (\text{by } ①)$$

* * *

(Pr. week no any class of sir)

1Pr/wed
08/03/17

㉝ **Theorem:** Let ϕ and ψ are simple functions defined on a set E of finite measure, then for any real α & β

$$\int_E (\alpha\phi + \beta\psi) = \alpha \int_E \phi + \beta \int_E \psi .$$

If $\phi \le \psi$ on E, then $\int_E \phi \le \int_E \psi$

Pf:- ϕ and ψ are simple functions in E. Then $E = \bigcup_{i=1}^{n} E_i$, where $\{E_i\}_{i=1}^{n}$ is finite collection of disjoint measurable subsets of E. Let ϕ and ψ assumes values a_i and b_i respectively in E_i, $1 \le i \le n$, Then

~~$\phi(x) = \sum_{i=1}^{n} a_i \chi_{E_i}$, $\psi(x) = \sum_{i=1}^{n} b_i \chi_{E_i}$~~

~~$\int_E \phi(x) = \sum_{i=1}^{n} a_i \, m(E_i)$, [illegible]~~

~~$\int_E \psi(x) = \sum_{i=1}^{m} b_i \, m(F_i)$ $\quad \left(F = \sum_{i=1}^{m} F_i \right)$.~~

let ϕ & ψ assumes values a_i is being assumed in E_i & b_i is being assumed in F_i.

I

$$\phi(x) = \sum_{i=1}^{n} a_i \chi_{E_i}, \quad \psi(x) = \sum_{i=1}^{m} b_i \chi_{F_i}$$

$\alpha a_i + \beta b_j$ is being assumed by $\alpha\phi + \beta\psi$ in $E_i \cap F_j$, $1 \le i \le n, 1 \le j \le m$

also. $\int_E \phi = \sum_{i=1}^{n} a_i \cdot m(E_i)$ and $\int_E \psi = \sum_{j=1}^{m} b_j \cdot m(F_j)$

$$\alpha\phi + \beta\psi = \sum_{i=1}\sum_{j=1}(\alpha a_i + \beta b_j)\chi_{E_i\cap F_j}$$

$$\cancel{\int_E(\alpha\phi+\beta\psi)} = \sum_{i=1}^{n}\Big(\alpha a_i + \beta\sum_{j=1}^{m} b_j\Big)\chi_{E_i\cap\underbrace{(\bigcup_{j=1}^{m} F_j)}_{=E}}$$

$$= \sum_{i=1}^{n} \alpha a_i\chi_{E_i} + \sum_{j=1}^{m}\beta b_j\chi_{F_j}$$

$$\int_E(\alpha\phi+\beta\psi) = \cancel{\alpha\int\phi} + \alpha\sum_{i=1}^{n} a_i m(E_i) + \beta\sum_{j=1}^{m} b_j m(F_j)$$

$$= \alpha\int_E\phi + \beta\int_E\psi$$

as → $\sum_{i=1}^{n}\Big(\alpha a_i + \beta\sum_{j=1}^{m} b_j\Big)\chi_{E_i\cap E}$ $\quad$ [$\because E = \bigcup_{j=1}^{m} F_j$] [$\because E_i \subseteq E$]

$$= \sum_{i=1}^{n}\Big(\alpha a_i + \beta\sum_{j=1}^{m} b_j\Big)\chi_{E_i}$$

$$= \sum_{i=1}^{n} \alpha a_i\chi_{E_i} + \sum_{j=1}^{m}\beta b_j\chi_{E_i}$$

$$= \sum_{i=1}^{n} \alpha a_i\chi_{E_i} + \sum_{j=1}^{m}\beta b_j\chi_{E_i}$$ [if we sum on i]

Then

$$\int_E(\alpha\phi+\beta\psi) = \alpha\sum_{i=1}^{n} a_i m(E_i) + \beta\sum_{j=1}^{m} b_j m(F_j)$$

$$= \alpha\int_E\phi + \beta\int_E\psi$$ (Proof)

⇒ let ϕ and ψ assuming values a_i & b_i respectively in E_i, $1 \le i \le n$. Then

$\alpha\phi + \beta\psi$ will assume values $\alpha a_i + \beta b_i$ in E_i, $1 \le i \le n$.

$$\Rightarrow \alpha\phi + \beta\psi = \sum_{i=1}^{n} (\alpha a_i + \beta b_i)\chi_{E_i}$$

$$\Rightarrow \alpha\phi + \beta\psi = \sum_{i=1}^{n} \alpha a_i \chi_{E_i} + \sum_{i=1}^{n} \beta b_i \chi_{E_i}$$

then $\int_E \alpha\phi + \beta\psi = \int_E \sum_{i=1}^{n} \alpha a_i \chi_{E_i} + \int_E \sum_{i=1}^{n} \beta b_i \chi_{E_i}$

$\int_E \alpha \sum_{i=1}^{n} a_i \chi_{E_i} + \int_E \beta \sum^{n} b_i \chi_{E_i}$ Proof

$$\boxed{\int \alpha\phi + \beta\psi = \alpha \int_E \phi + \beta \int_E \psi}$$ ⓐ

Part - II Now let $\eta(x) = \psi(x) - \phi(x) \geq 0$ in E.

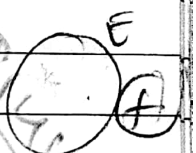
$\Rightarrow \eta$ is also simple. then

(by ⓐ) — $\int_E \psi - \int_E \phi = \int_E (\psi - \phi) = \int_E \eta(x) \geq 0$

(Integral of non-neg. simple f^n is simple)

($\because \eta$ is positive f^n)

$$\Rightarrow \boxed{\int_E \phi \leq \int_E \psi}$$ Proof



9/3/17 ③④ * f, bounded real valued f^n on a set E. $m(E) < \infty$, then for $\alpha, \beta \in \mathbb{R}$ s.t.

$\alpha = \inf\{f(x) \mid x \in E\}$ and $\beta = \sup\{f(x) \mid x \in E\}$ — ⓐ

$\alpha \leq f(x) \leq \beta$, then define

$L(f) = \{\phi \mid \phi$ is simple f^n on E, $\phi \leq f\}$ — ⓑ

$U(f) = \{\psi \mid \psi$ is simple f^n on E, $\psi \geq f\}$ — ⓒ

Pf:

$\because \alpha$ and β are constants

$\Rightarrow \alpha$ and β are simple function.

Now

$A = \{\int_E \phi \mid \phi \in L(f)\}$ and $B = \{\int_E \psi \mid \psi \in U(f)\}$

$\because \phi \in L(f) \Rightarrow \boxed{\phi \leq f \leq \beta}$ by ⓐ & ⓑ

$\Rightarrow \int_E \phi \leq \int_{E} \beta = \beta\, m(E)$

$\Rightarrow$ A is b.dd above

$\because \alpha \leq f$ so $\alpha \in L(f)$ $\Rightarrow \int_E \alpha = \alpha\, m(E)$ $\Rightarrow A \neq \phi$

also (A) $\alpha \in L(f) \Rightarrow \alpha \cdot m(E) \in A \Rightarrow$ A is non-empty

hence A is non-empty and b.dd above

Simillarly $\beta \in U(f)$

$\Rightarrow \beta \cdot m(E) \in B \Rightarrow$ B is non-empty

and

$\psi \in U(f) \Rightarrow \alpha \leq f \leq \psi$

$\Rightarrow \int_E \psi \geq \int_E \alpha = \alpha\, m(E) \Rightarrow$ B is b.dd below

Hence B is non-empty and b.dd below

Now, sup A and inf B i.e

$\sup \{ \int_E \phi \mid \phi \in L(f) \}$ ($A = L(f)$, $B = U(f)$)

$\inf \{ \int_E \psi \mid \psi \in U(f) \}$ are well defined..

* ~~of~~ *

*3rd UNIT

Lebesgue Integral :-

~~24|3|17~~

A b.dd real valued f^n f, defined on a set E of finite measure is s.t.b Lebesgue Integral. if $\boxed{\sup A = \inf B}$ ie $\sup\{L(F)\} = \inf\{U$

$$\boxed{\begin{aligned} &\sup \{ \textstyle\int_E \phi \text{ ; where } \phi \text{ is simple } f^n \text{ on } E, \phi \leq f \} \\ &\qquad = \inf \{ \psi_E \text{ ; Where } \psi \text{ is simple } f^n \text{ on } E, \phi \geq f \} \end{aligned}}$$

Here, $\sup\{\ \} \Rightarrow$ Lower lebesgue integral (LLI)

$\inf\{\ \} \Rightarrow$ upper lebesgue integral (ULI)

* **Theorem:** Let f be b.dd real valued f^n defined on closed b.dd interval $[a, b]$, if f is Riemann integral over $[a, b]$, then it is lebesgue

③

integrable over $[a,b]$ and these two integrations are same. But the converse is NOT true. Consider the below eg. (eg-1)

Pf:- f is reimann integrable over $I=[a,b]$ if

$$\boxed{\begin{aligned}&\sup\left\{\int_I \phi \;\middle|\; \phi \text{ is } \overset{\text{step}}{\cancel{\text{simple}}} f^n \text{ on } I=[a,b],\ \phi \le f\right\}\\ &\qquad = \inf\left\{\int_I \psi \;\middle|\; \psi \text{ is } \overset{\text{step}}{\cancel{\text{simple}}} f^n \text{ on } I=[a,b],\ \psi \ge f\right\}\end{aligned}}$$

$\because$ ϕ and ψ are $\overset{\text{step}}{\cancel{\text{simple}}}$ f^n admitting finite no. of values hence ϕ and ψ are simple f^ns on I, f is measurable. hence -

$$\sup\left\{\int_I \phi : \phi \text{ is simple on } I ;\ \phi \le f\right\} = \inf\left\{\int_I \psi : \psi \text{ is simple on } I ;\ \psi \ge f\right\}$$

hence, f is also lebesgue integrable.
But

Theorem: ㊱ Let f be b.dd f^n on a set E of finite measure. Then f is integrable over E.
(means lebesgue integrable)

~~??, p. 173, ㊳~~

Eg. ① $f(x) = \begin{cases} 0 ; & x\in E_1,\ E_1 \text{ is set of irrational no.s on } [0,1] \\ 1 ; & x \in E_2,\ E_2 \text{ is set of rational no.s on } [0,1] \end{cases}$

$\because$ by the above theorem, f^n is lebesgue integrable (bcz f is b.dd on $[0,1]$ & $[0,1]$ is measurable)
but we know that →

Here we have to show $\sup(L(f)) \neq \inf(U(f))$

f is NOT Riemann integrable.

$\because f(x) = c_1 \chi_{E_1} + c_2 \chi_{E_2}$ $\Big[$ where $\left. \begin{array}{l} c_1 = 0 \\ c_2 = 1 \end{array} \right\}$ $\rightarrow L(f)$, $\rightarrow U(f)$

$$\Rightarrow \int_E f = c_1 m(E_1) + c_2 m(E_2)$$
$$= 0 \cdot m(E_1) + 1 \cdot 0$$
$$= 0$$

$$\Rightarrow \boxed{\int_E f = 0} \Rightarrow f \text{ is non. R.I}$$

Theorem.

2nd Internal

③

The simple approximation lemma:—

let f be measurable real valued f^n on a set E of finite measure. Assume f is bounded on E, i.e there is $M \geq 0$ for which $|f| \leq M$ on E.

For each $\varepsilon > 0$, there are simple f^ns ϕ_ε & ψ_ε on E, that have following approximation properties—

$$\phi_E \leq f \leq \psi_\varepsilon \;\&\; 0 \leq \psi_\varepsilon - \phi_\varepsilon < \varepsilon \quad \text{on } E$$

Pf:- $\because$ f(E) should contained on an open interval (c,d).

Assume, $c = y_0 < y_1 < \dots < y_n = d$ is partition of closed interval $[c,d]$. s.t $0 \leq y_k - y_{k-1} < \varepsilon$

let $I_k = [y_k, y_{k+1}]$ and $E_k = f^{-1}(I_k)$

then

each $E_k : 1 \leq k \leq n$ is measurable [$\because$ f. is measurable]

let ϕ_E and ψ_E are simple f^ns defined by

$$\phi_E(x) = \sum_{k=1}^{n} y_{k-1} \chi_{E_k}, \quad \psi_E(x) = \sum_{k=1}^{n} y_k \chi_{E_k}$$

$\Rightarrow$ For each $x \in f(E) \subset (c,d)$

i.e x belong to same I_k.

Then $\phi_E(x) = y_{k-1} \leq f(x) \leq y_k = \psi_\varepsilon(x)$.

$$\Rightarrow \phi_\varepsilon \leq f \leq \psi_\varepsilon \;\&\; 0 \leq \psi_\varepsilon - \phi_\varepsilon < \varepsilon \text{ on } E.$$

$\star$ & &

Theorem: ㊳ Let f be bdd measurable f^n on a set of finite measure E. Then f is integrable over E.

Pf:— Let n be the natural no. Then by simple approximation lemma for $\varepsilon = 1/n$ there exists simple f^n, ϕ_n and ψ_n s.t

$$\phi_n \leq f \leq \psi_n \quad \text{for } x \in E.$$

and $$0 \leq \psi_n - \phi_n < (\varepsilon) = 1/n$$

$$\int_E \psi_n - \int_E \phi_n = \int_E (\psi_n - \phi_n) < \frac{1}{n}\int_E = \frac{1}{n} m(E) \quad — ①$$

$$\inf\left\{\int_E \psi \mid \psi \text{ simple}, \psi \geq f\right\}$$
$$\leq \left\{\int_E \psi_n \mid \psi_n \text{ simple}, \psi_n \geq f\right\}$$
for each n

$$\sup\left\{\int_E \phi \mid \phi \text{ simple}, \phi \leq f\right\} \geq \left\{\int_E \phi_n \mid \phi_n \text{ simple}, \phi_n \leq f\right\}$$
for each n.

$$\underbrace{\inf\left\{\int_E \psi \mid \psi \text{ simple}, \psi \geq f\right\} - \sup\left\{\int_E \phi \mid \phi \text{ simple}, \phi \leq f\right\}}_{0} \leq \int_E \psi - \int_E \phi_n, \text{ for each } n$$
— ⑪

$$< \frac{1}{n} m(E) \quad (\text{from } ①)$$

As $n \to \infty$, ⑪ $\Rightarrow 0$ ie

lt $n \to \infty$

$$\inf\left\{\int_E \psi \mid \psi \text{ simple}, \psi \geq f\right\} - \sup\left\{\int_E \phi \mid \phi \text{ simple}, \phi \leq f\right\} \longrightarrow 0$$

Remarks: Hence by def^n of lebesgue integrable (p-170), f is integrable over E.

Theorem. (38) let f is bounded on a measurable set E of finite measure. ~~In order~~ that

L-Integrable

$$\inf\{\int_E \psi \mid \psi \text{ simple}, \psi \ge f\} = \sup\{\int_E \phi \mid \phi \text{ simple}, \phi \le f\}$$

for all simple functions ϕ and ψ, it is necessary and sufficient that f is measurable.

Pf: —

Let f is b.dd in a set E of finite measure. If f is measurable, then by prev. Result

$\inf\{- - - -\} = \sup\{- - -\}$

i.e f is integrable.

$\Leftarrow$ For converse part, let f is bdd and integrable, then we need to prove that f is measurable. then we can find simple fn ϕ_n & ψ_n ~~By simple approximation Lemma, there will be simple functions ϕ_n and ψ_n s.t~~ —

$$\phi_n \le f \le \psi_n \quad \text{for each } n$$

and

$$\int_E \psi_n - \int_E \phi_n < \frac{1}{n}^{m(E)} \quad n \text{ is natural number}$$

let $\phi^* = \sup \phi_n$, $\psi^* = \inf \psi_n$

let $\Delta = \{x : \phi^*(x) < \psi^*(x)\}$.

$\Delta_\nu = \{x : \phi^*(x) < \psi^*(x) - 1/\nu\}$

$\Delta_{\nu,n} = \{x : \phi_n(x) < \psi_n(x) - 1/\nu\}$

$\Delta = \bigcup_{\nu=1}^{\infty} \Delta_\nu$, $\Delta_\nu \subset \Delta_{\nu,n}$. Then

then $m(\Delta_{\nu,n}) < \nu/n$

(If not so, let $m^*(\Delta_{\nu,n}) \ge \nu/n$

[a.e → almost everywhere

Hence,

$$\int_{\Delta_{\gamma,n}} \psi_n - \int_{\Delta_{\gamma,n}} \phi_n = \int_{\Delta_{\gamma,n}} (\psi_n - \phi_n) > \frac{1}{\nu}\, m(\Delta_{\gamma,n})$$

$$\geq \frac{1}{\gamma} \cdot \frac{\gamma}{n} = 1/n$$

$\left(\because \exists \int \psi_n - \int \phi_n < 1/n \text{ on } E \right)$

this is not possible.

therefore $m(\Delta_{\gamma,n}) \to 0$

$m(\Delta) = m\{x : \phi^*(x) < \psi^*(x)\} = 0$ ⓐ

$\Rightarrow \boxed{\phi^*(x) \geq \psi^*(x) \quad a.e}$

Also $\phi^*(x) \leq \psi^*(x)$ for $x \in E$. $\Big\} \Rightarrow \phi^* = \psi^* = f_{a.e}$

[A condition hold a.e on a set E. s.t measure~~able~~ of set where condition not being hold is zero. i.e

$\{f_n\} \to g$ a.e on E.

$\{f_n\} \to g \ \forall n$ a.e on E.

$m\{x \in E \mid f_n \neq g\} = 0$

~~Also, $\because \phi^* \leq \psi$~~

* * *

Also $\because \phi^* \leq \psi^*$ for $x \in E$ Ⓑ

then from Ⓐ & Ⓑ $\Rightarrow$

$\phi^* = \psi^* = f$ a.e

$\because \phi$ and ψ are simple function

$\Rightarrow \phi$ and ψ are measurable $\Rightarrow f$.

Theorem: Let E be measurable set ~~in~~ with finite measure and $\{f_n\}$ be a seq. of measurable functions on E, f be a real valued f^n on E s.t for $x \in E$, $f_n \to f$. Then given $\varepsilon > 0$ and $\delta > 0$, there is $A \subset E$ with $m(A) < \delta$ and there is an integers N s.t —

$$|f_n(x) - f(x)| < \varepsilon, \text{ for all } x \in E-A \text{ and } n \geq N$$

Pf:- Let $G_n = \{x \in E \mid |f_n(x) - f(x)| \geq \varepsilon, \varepsilon > 0\}$.

Assume,

$$E_k = \bigcup_{n \geq k}^{\infty} G_n$$

$$= \{x \mid x \in G_n \text{ for some } n \geq k\}$$

$$= \{x \in E \mid |f_n(x) - f(x)| \geq \varepsilon \text{ for some } n \geq k\}$$

then $E_{k+1} \subseteq E_k$ and $\bigcap_{k \geq 1}^{\infty} E_k = \phi$ then

This gives $\lim_{k \to \infty} m(E_k) = 0 \Leftarrow \left\{ \begin{array}{l} m\left(\bigcap_{k \geq 1}^{\infty} E_k\right) = \lim_{n \to \infty} m(E_k) \\ \text{But } m\left(\bigcap_{k \geq 1}^{\infty} E_k\right) = m(\phi) = 0 \end{array} \right.$

Therefore there exist a $\delta > 0$ and an integer N s.t

~~with~~ $\boxed{m(E_k) < \delta \text{ for all } k \geq N}$

In particular

$$m(E_N) < \delta$$

i.e $m\left(\underbrace{\{x \in E \mid |f_n(x) - f(x)| \geq \varepsilon \text{ for some } n \geq N\}}_{\Downarrow}\right) < \delta$

If $E_N = A \Rightarrow m(A) < \delta$ then

$$|f_n(x) - f(x)| < \varepsilon \text{ for all } x \in E-A \text{ and } n \geq N$$

✓ **Theorem:** let E be a measurable set with finite measure & $\{f_n\}$ be a sequence of measurable functions. Converging a.e to f on E. Given $\varepsilon>0, \delta>0$, there is a set $A \subset E$. with $\underline{m(A)<\delta}$ and an integer N s.t —

$$|f_n(x)-f(x)| < \varepsilon \quad \text{for all} \quad x \in E \setminus A \quad \text{and } n \geq N$$

Pf: let $F = \{x \in E \mid f_n \nrightarrow f\}$, then $\underline{m(F)=0}$. Hence,

sequence $\{f_n\}$ converges to f, for $x \in E_1$, where $E_1 = E - F$

Therefore for $\varepsilon>0$. and $\delta>0$, there exist $A_1 \subset E_1$ with $m(A_1)<\delta$ and for an integer N s.t

$$|f_n(x) - f(x)| < \varepsilon \quad \text{for all} \quad x \in E_1 - A_1 \quad \forall\, n \geq N$$

let $A = F \cup A_1 \Rightarrow m(A) \leq m(F) + m(A_1)$.

$\because \underline{m(F)} = 0$ (By defn)

$\Rightarrow \boxed{m(A) < \delta}$

Theorem: Bounded convergence theorem

let $\{f_n\}$ be a sequence of measurable functions defined on a set E of finite measure. Suppose that $|f_n(x)| \leq M$ for all n for fixed $\underline{M \geq 0}$. ~~If~~

If $\lim\limits_{n\to\infty} f_n(x) = f(x)$ for ~~all~~ $x \in E$, then

$$\boxed{\int_E f = \lim_{n\to\infty} \int_E f_n}$$

$f_n \to f$

$\exists$ a set $A \subset E$ s.t $m(A) < \delta$, and there is a ~~prime~~ positive integer N s.t

$$|f_n(x) - f(x)| < \varepsilon \quad \forall\, x \in E \setminus A \quad \& \quad n \geq N$$

Also, $|f_n(x)| \leq M$ for all $n \Rightarrow |f(x)| \leq M$

$\int_E f = m(E)$

$|f_n(x) - f(x)| \le |f_n(x)| + |f(x)| \le 2M$

for all n and $x \in E$.

$\cancel{\text{Ref}_1}$ $\left| \int_E f_n - \int_E f \right| = \left| \int_E (f_n - F) \right| \le \int_{(E-A)\cup A} |f_n - f|$

let $\delta = \frac{\varepsilon}{4M}$

$\varepsilon = \frac{\varepsilon}{2m(E)}$

$= \int_{E-A} |f_n - f| + \int_A |f_n - f| \le \cancel{2M}$

$< \frac{\varepsilon}{2m(E)} m(E-A) + 2M.m(A)$

$\le$ ~~$2M\, m(E-A) + 2M.m(A)$~~

$< \varepsilon/_2 + \varepsilon/_2 = \varepsilon$

$\Rightarrow \int_E f_n = \int_E f \Rightarrow \lim_{n\to\infty} \int_E f_n = \int_E f$

$$\boxed{\int_E f = \lim_{n\to\infty} \int_E f_n}$$

Eg. Example based on last theorem.

Sol^n Let $f_n(x) = \frac{nx}{1+n^2x^2}$, $0 \le x \le 1$ ↳ is this set meas ↳ yes because it is interval.

$f_1(x) = \frac{x}{1+x^2}$

is continuous everywhere except $x = \pm i$

||'ly $f_2(x) = \frac{2x}{1+4x^2}$ " " " $x = \pm \frac{i}{2}$

ie $f_n(x)$

∴ $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x}{1/n + nx^2} = 0 = f(x)$

so hence $\{f_n\}$ is measurable function

Step-II. Now checking for boundedness

∵ $f_n(x) = \frac{nx}{1+n^2x^2}$, $0 \le x < 1$

$= \frac{1}{\frac{1}{nx} + nx} = \frac{1}{\left(\frac{1}{\sqrt{nx}} - \sqrt{nx}\right)^2 + 2} \le 1/_2$

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(As we know that all Riemann Integrable is Lebesgue Integral Also).

$$\lim_{n\to\infty} \int_E \frac{nx}{1+n^2x^2}\,dx = 0$$

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* * *

Theorem: Let $f(x)$ is measurable on E, let ~~then~~ f^+ & f^{-1} is defined by then $f^*(x) = \text{Max}(f(x), 0)$, $x \in E$
$f^-(x) = -\min(f(x), 0)$, $x \in E$

Que : $f(x) = \frac{1}{2} + \sin x$, $0 \le x \le 2\pi$

$$f^+(x) = \begin{cases} \frac{1}{2} + \sin x , & 0 \le x \le \frac{7\pi}{6} \\ 0 , & \frac{7\pi}{6} \le x \le \frac{11\pi}{6} \\ \frac{1}{2} + \sin x , & \frac{11\pi}{6} < x < 2\pi \end{cases}$$

How we chose interval

$\sin\left(\frac{7\pi}{6}\right) = \sin\left(\pi + \frac{\pi}{6}\right)$
$= -\sin \pi/6$
$= -1/2$
& $\sin\left(\frac{11\pi}{6}\right) = \sin\left(2\pi - \frac{\pi}{6}\right)$
$= -\sin \pi/6 = -1/2$

$$f^-(x) = \begin{cases} 0 & 0 \le x \le 7\pi/6 \\ -\frac{1}{2} - \sin x & 7\pi/6 \le x \le 11\pi/6 \\ 0 & \frac{11\pi}{6} < x \le 2\pi \end{cases}$$

$\because f = f^+ - f^-$
$\Rightarrow |f| = f^+ + f^-$

Defn: A bdd measurable f^n f in E, $m(E) < \infty$ is integrable over E iff f^+ and f^- both are integrable over E

Theorem: A f^n f is integrable over E iff $|f|$ is integrable over E.

Pf. Let f is integrable
so by definition f^+ & f^- are integrable
Claim: $|f|$ is integrable
we know that $|f| \le f^+ + f^-$

∴ Integrable

$$\Rightarrow \int_E |f| \leq \int_E f^+ + \int f^-$$

∵ f^+ & f^- are inte
$\Rightarrow \int_E f^+ < \infty$ & $\int_E f^- < \infty$
$\Rightarrow \int f^+ + \int f^- < \infty$

Hence $|f|$ ~~is~~ integrable.

⇐ Conversely

Let $|f|$ is integrable.

Claim: f is integrable

$$∵ \int_E f^+ \leq \int_E |f| < \infty \quad (∵ f^+ \leq |f|)$$

as $|f|$ is integrable.

Similarly $\int_E f^- \leq \int_E |f| < \infty$

$$\Rightarrow \int_E f^+ + \int_E f^- < \infty$$

$$\Rightarrow \int_E (f^+ + f^-) < \infty \Rightarrow \int_E f < \infty$$

Hence f is integrable over E.

Remark: ① This statement doesn't hold for the case of Riemann Integrable (integration).

Let $f : [0,1] \to \mathbb{R}$ ~~set~~ is defined by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational in } [0,1] \\ -1 & \text{if } x \text{ is irrational in } [0,1] \end{cases}$$

$|f(x)| = 1$, $x \in [0,1]$ But $\int f^+ + \int f^- = 0$

so $\int |f| \neq \int f^+ + \int f^-$ so not Hold

Eg: $f(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ 0 & x = 0 \end{cases}$, $x \in [0, \infty)$

P.T $f(x)$ is not Lebesgue Integral.

Sol^n: If $\int_0^\infty f(x) = \infty$, i.e. $\int_0^\infty \left|\frac{\sin x}{x}\right| = \infty$

i.e. If $\left|\frac{\sin x}{x}\right|$ is not Lebesgue integral over $[0, \infty)$.

$$\int_0^{n\pi} \left|\frac{\sin x}{x}\right| dx = \sum_{k=1}^{n} \int_{(k-1)\pi}^{k\pi} \left|\frac{\sin x}{x}\right| dx$$

let $x = y + (k-1)\pi$

$$= \sum_{k=1}^{n} \int_0^{\pi} \left|\frac{\sin(y+(k-1)\pi)}{y+(k-1)\pi}\right| dy$$

$$\geq \sum_{k=1}^{n} \frac{1}{k\pi} \int_0^{\pi} |\sin y| dy .$$

$$\left[\begin{array}{l} 0 \leq y \leq \pi, \\ (k-1)\pi \leq y + (k-1)\pi \leq k\pi, \Rightarrow \dfrac{1}{k\pi} \leq \dfrac{1}{y+(k-1)\pi} \end{array}\right.$$

$$\geq \sum_{k=1}^{n} \frac{1}{k\pi} \left| \int_0^{\pi} \sin y \, dy \right| = \sum_{k=1}^{n} \frac{2}{k\pi}$$

Now taking limit $n \to \infty$, we have, R.H.S

$$= \sum_{k=1}^{\infty} \frac{2}{k\pi} = \infty \qquad \left(\lim_{n\to\infty} \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{1}{k} \right)$$

↑ it is p=1, p-series

Thus f is NOT lebesgue integral.

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def^n: A f^n with finite support: If f is measurable in a set E & if $E_0 \subset E$ s.t $m(E_0) < \infty$ & $f \equiv 0$ in $E \sim E_0$. Then f is known to be f^n with finite support

$$E_0 = \{ x \in E \mid f(x) \neq 0 \}$$

let f is a non-neg measurable f^n in E and $0 \leq h \leq f$ for all $x \in E$ s.t h is b.dd measurable f^n on E with finite support, then

$$\int_E f = \sup\left\{ \int h \right\}$$

let $f(x) = \begin{cases} 1 & \text{if } x \text{ is rational in } \mathbb{R} \\ 0 & \text{if } x \text{ is irrational in } \mathbb{R} . \end{cases}$

suppose we define a f^n

$$h_n = \begin{cases} 0, & \text{if } x \text{ is irrational in } \mathbb{R} \quad (\mathbb{Q}^c) \\ 1/n, & \text{if } \text{——— irration ———} \quad (\mathbb{Q}) \end{cases}$$

s.t.

$0 \le h_n \le f$ for all n.

$$\int_{\mathbb{R}} f = \sup\{\int_E h_n\} = \sup\{0 \cdot m(\overset{\mathbb{Q}^c}{\widetilde{\mathbb{R}-\mathbb{Q}}}) + \tfrac{1}{n} m(\mathbb{Q})\}$$

#. Chebychev's Inequality. (imp)

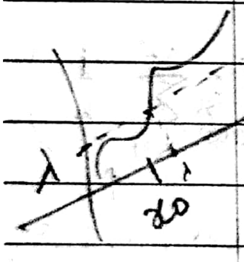
Let f be a non-neg. measurable f^n on E.
Then for any $\alpha > 0$.

$$m\{x \in E \mid f(x) \ge \alpha\} \le \frac{1}{\alpha}\int_E f$$

Pf:- let $E_\alpha = \{x \in E \mid f(x) \ge \alpha\}$.

Case-I first suppose that $m(E_\alpha) = \infty$ $\qquad \int \psi_n = \alpha m(E_{\alpha,n})$

Assume that $E_{\alpha,n} = E_\alpha \cap [-n, n]$ & $\underbrace{\psi_n = \alpha \chi_{E_{\alpha,n}}}_{\text{measurable (s.t. simple } f^n\text{) b.dd, having finite support}}$



$\lim_{n\to\infty} E_{\alpha,n} = E_\alpha$

also, $0 \le \psi_n \le f$ for all n.

$$\infty = \alpha m(E_\alpha) = \alpha \lim_{n\to\infty} m(E_{\alpha,n}) = \int_{E_\alpha} \psi_n \le \int_{E_\alpha} f = \infty$$

$\left(\because \int_{E_\alpha} \psi_n \le \int_{E_\alpha} f \qquad \int_E \psi_n = \alpha m(E_{\alpha,n}) \right)$

This gives the inequality.

Case-I suppose $m(E_\alpha) < \infty$ define $h = \alpha \chi_{E_\alpha}$.
$0 < h \le f$.

$$\alpha m(E_\alpha) = \int_{E_\alpha} h \le \int_{E_\alpha} f$$

$$\Rightarrow \alpha m(E_\alpha) = \int_{E_\alpha} h \le \int_E f \Rightarrow$$

(replace E_α by E)
(bcz E_α is 0 afterc)

$$\Rightarrow \boxed{m(E_\alpha) \le \tfrac{1}{\alpha}\int_E f}$$

Theorem: let f be a non-neg. measurable f^ns on E. Then $\int_E f = 0$ iff $f = 0$ a.e on E.

Pf: First suppose that $\int_E f = 0$, Let n is ~~first~~ a natural no., then using ~~by~~ Chebychev's ineq. for $\alpha = 1/n$, we have —

$$m\{x\in E \mid f(x) \ge 1/n\} \le n\int_E f = 0 \quad ①$$

$$\{x\in E \mid f(x) > 0\} = \bigcup_{n=1}^{\infty} \{x\in E \mid f(x) \ge 1/n\}$$

$$\Rightarrow m\{x\in E \mid f(x) > 0\} = \sum_{n\ge 1}^{\infty} \underbrace{m\{x\in E \mid f(x) \ge 1/n\}} = 0 \text{ by } ①$$

$\Rightarrow f = 0$ a.e (almost everywhere) on E.

$\Leftarrow$ Conversely, let $f=0$ a.e. on E

Claim: $\int_E f = 0$ a.e. on E

Let ψ are simple fn on E & h are b.dd measurable fn's on E with finite support s.t $0 \le \psi \le h \le f$ on E.

$\Rightarrow \psi = 0$ a.e on E $\forall \psi \Rightarrow \int_E \psi = 0$

$\Rightarrow \int_E h = 0$ for all h

$\Rightarrow \int_E f = 0 = \sup\{\int_E h\}$

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Fatou's lemma:

Let $\{f_n\}$ be a seq. of non-neg. measurable fn's on E. If $f_n \to f$ pointwise a.e on E, then

$$\boxed{\int_E f \le \liminf \int_E f_n}$$

Pf:- function f is also non-neg. and measurable. Since it is pointwise limit of seq. $\{f_n\}$. Let h are non-neg. measurable & bdd fn's with finite support on E with $0 \le h \le f$. Then it is necessary & sufficient to show that $\int_E h \le \lim \int_E f_n$

then $\Rightarrow$ $\boxed{\int_E h \le \liminf \int_E f_n}$

If h is bdd then, $0 \le h \le M$ and let

$E_0 = \{x \in E \mid h(x) \neq 0\}$, Define a seq.

$h_n(x) = \min.\{h(x), f_n(x)\}$

$$\lim_{n\to\infty} h_n(x) = \lim_{n\to\infty} \min.\{h(x), f_n(x)\}$$

$$= \min\{h(x), \lim_{n\to\infty} f_n(x)\}, \text{ for } x \in E$$

$$= \min.\{h(x), f(x)\} = h(x), \text{ for } x \in E \quad \text{—①}$$

$h_n(x)$ is bounded in E s.t $h_n(x) \le h(x)$ for $x \in E$.

~~$\lim_{n\to\infty} \int_E h_n(x) \le \int_E h(x)$.~~

$\lim_{n\to\infty} h_n(x) = h(x)$

Also $h_n(x) \le f_n(x)$ . . (from ①)

$$\int_E h(x) \le \lim_{n\to\infty} \int_E h_n(x) \le \lim_{n\to\infty} \int f_n(x).$$

Now on taking supremum — (by def^n of finite supp

$$\int_E f \le \lim\inf \int_E f_n$$

Remark: Fatou's lemma need not hold unless the f^{n}s are non-negative. (For this consider the below example).

Let

$$f_n(x) = \begin{cases} -1/n & \text{if } x \in [0, n] \\ 0 & \text{elsewhere.} \end{cases}$$

$\lim_{n\to\infty} f_n(x) = 0 = f(x)$ if $x \in [0, \infty)$ & $\int_E f = 0$

& $\lim\inf \int_0^n f_n(x) = \frac{-1}{n}[n-0] = -1$

so above cond^n not satisfied.

i.e $0 \not\le -1$

Monotone convergence theorem.

① Let $\{f_n\}$ be a incaing seq. of non-neg. measurable f^ns on E. If $\{f_n\} \to f$ pointwise a.e on E, then $\lim_{n\to\infty} \int_E f_n = \int_E f$ (for this consider the below eg.)

Eg. Suppose that $f_n(x) = \frac{1}{\sqrt{x+1/n}}$ for $x \in [0,1]$ & $n \geq 1$. If f is pointwise limit of $\{f_n\}$ then $\lim_{n\to\infty} \int f_n(x) = ?$

Now checking for increasing let $n_1 \leq n_2$ be the $\rightarrow \frac{1}{n_1} \geq \frac{1}{n_2}$

$\frac{1}{\sqrt{x_1+1/n}}$ $\quad x_1 + 1/n_2 \leq x_1 + 1/n_1 \Rightarrow \sqrt{x_1+1/n_2} \leq \sqrt{x_1+1/n_1}$

$$\Rightarrow \frac{1}{\sqrt{x_1+1/n_1}} \leq \frac{1}{\sqrt{x_1+1/n_2}} \Rightarrow f_2(x_2) \geq f(x_1)$$

$$\Rightarrow f_{n_1}(x) \leq f_{n_2}(x) \Rightarrow ~~f(x_1) \leq f(x_2)~~$$

$\{f_n\}$ is increasing seq. of non-neg. measurable f^ns

$$\lim_{n\to\infty} \int f_n(x) = \frac{1}{\sqrt{x}}$$

Now proof of above theorem

Proof: From fatou's lemma. $\int_E f \leq \liminf \int_E f_n$ — ①

for each index n,

$f_n(x) \leq f(x) \Rightarrow \int_E f_n(x) \leq \int f(x)$

$$\Rightarrow \limsup \int_E f_n(x) \leq \int f(x) \quad — ②$$

∴ from ① & ②

$$\boxed{\limsup \int_E f_n \leq \int_E f \leq \liminf \int_E f_n}$$

$$\Rightarrow \int_E f = \boxed{\limsup \int f_n = \liminf \int f_n}$$

* * *

Theorem# Bounded Convergence thm (P.177)

② If $\{f_n\}$ be a seq. of measurable f^ns on E of finite measure. suppose that $|f_n(x)| \leq M$ $\forall n$ for fixed M
If $\underset{n \to \infty}{\text{lt}} f_n(x) = f(x)$ for x $x \in E$ then

$$\boxed{\int_E f = \lim_{n \to \infty} \int_E f_n}$$

Theorem# Dominated Convergence theorem

③ If $\{f_n\}$ be a seq. of measurable f^ns on E. If g is a f^n integrable on E & dominate each f^n in the sense that $|f_n| \leq g$ on E.
* If $\{f_n\} \to f$ pointwise a.e on E, then $\boxed{\int_E f = \lim_{n \to \infty} \int_E f_n}$

$$\int_E f < \infty$$

(to prove above thm, we need two other theorem)

Theorem: (a) Let f be non-negative f^n (ie measurable f^n on E). & integrable over E. Then f is finite a.e on E.

Pf: Let n be natural no.

$$m\{x \in E \mid f(x) = \infty\} \leq m\{x \in E \mid f(x) \geq n\} \leq \frac{1}{n} \int_E f$$

By using Chebychev's ineq., we have ↺ finite

as $\int_E f < \infty$ then

$$\Rightarrow \boxed{m\{x \in E \mid f(x) = \infty\} = 0}$$

Theorem: (b) Let f be non-negative measurable f^ns on E. let g be a f^n integrable over E and dominent f s.t $|f| \leq g$, then f is integrable over E and

$$\boxed{\left|\int_E f\right| \leq \int_E |f|}$$ (integral comparision test)

Pf. – $\int_E |f| \leq \int_E g < \infty$ (By lebesgue integrable)

$\Rightarrow |f|$ is integrable.

(As we know that $|f|$ is integrable $\Leftrightarrow$ f is integrable)

(bcz $|f| = f^+ + f^-$)

⇒ f is integrable.

& $\left|\int_E f\right| = \left|\int_E (f^+ - f^-)\right| \le \left|\int_E f^+\right| + \left|\int f^-\right|$

$= \int_E f^+ + \int_E f^- = \int_E (f^+ + f^-)$

$= \int_E |f|$.

Now proof for main theorem

Pf:- Given, $|f_n| \le g$ on E ⇒ $|f| \le g$ on E. Here f and for each n, f_n are integrable over E, also f_n (for each n) and f are finite a.e on E.

g-f

∴ $g \ge \begin{cases} f, & f \ge 0 \\ -f, & f < 0 \end{cases}$

g-f and g - fn (for each n) are finite measurable and non-neg. on E.

$\{g-f_n\} \to g-f$ pointwise a.e on E.

Now, using Fatou's lemma, we have

$\int_E (g-f) \le \liminf \int_E (g-f_n)$ (∵ inf f ≤ sup f)

$\cancel{\int_E g} - \int_E f \le \cancel{\int_E g} - \limsup \int_E f_n$

⇒ $\boxed{\int_E f \ge \limsup \int_E f_n.}$ —①

Similarly for g+f

g+f & g+fn (for each n) are finite, measurable & non-neg. on E

⇒ $\{g+f_n\} \to g+f$ pt.wise a.e. on E using Fatou's lemma,

$\int_E (f+g) \le \liminf \int_E (g+f_n) \Rightarrow \int_E f \le \liminf \int_E f_n$ —②

∴ from ① & ②

$\boxed{\limsup \int_E f_n \le \int_E f \le \liminf \int f_n}$

⇒ f is integrable.

① # Norm- linear space

X is linear space. If $x, y \in X$ & $\alpha \in \mathbb{R}$

$\|\cdot\|$ (i) $\|x+y\| \le \|x\| + \|y\|$

(ii) $\|x\| = 0 \Leftrightarrow x = 0$

(iii) $\|\alpha x\| = |\alpha| \|x\|$

then $(X, \|\cdot\|)$ is n.l.s.

② **Def^n**: Let $L^p(E)$ $(1 \le p < \infty)$ be collections of measurable f^{n}s on E s.t $\int_E |f|^p < \infty$ $\le \|f\|_p^p$

If $p = 1$,

$L^1(E)$ is collection of all integrable f^n on E.

Let $\|\cdot\|_p$ is norm in $L^p(E)$ defined by-

$$\boxed{\|f\|_p = \left(\int_E |f|^p\right)^{1/p}}$$

* $(L^1(E), \|\cdot\|_1)$ is a normed linear space. Check ?

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$$\|f\|_p = \left(\int_E |f|^p\right)^{1/p}$$

(i) $\|f\|_p = 0$ iff $f = 0$

(ii) $\|\alpha f\|_p = |\alpha| \|f\|_p$

(p-192) (iii) $\|f+g\|_p \le \|f\|_p + \|g\|_p$ (Minkowski's inequality)

where $f, g \in L^p(E)$, $\alpha \in \mathbb{R}$

for KPL^∞

Pf.

(i) As

$$\|f\| = \left(\int_E |f|^p\right)^{1/p} = 0$$

$$\Rightarrow \int_E |f|^p = 0$$

then

$|f|^p = 0$ a.e on E

$f = 0$ a.e on E

(ii) $\|\alpha f\|_p = \left(\int_E |\alpha f|^p\right)^{1/p} = |\alpha| \left(\int_E |f|^p\right)^{1/p}$

$= |\alpha| \, \|f\|_p$

(3) # Essential upper bound (EUB) $M \geq 0$

The EUB of a f^n f is some/no. in $\hat{R} = R \cup (-\infty, \infty)$ s.t

$|f| < M$ for almost all $x \in E$.

$L^\infty(E)$ is collection of measurable f^ns in E. that are essentially bounded.

$\|f\|_\infty = \inf\{$ essential upper bounds of f in $E \}$

H.W, Prove that —

$\|\cdot\|_\infty$ is a norm in $L^\infty(E)$.

Let p=1 (i) then $\|f\|_1' = \int_E |f|' = 0 \Leftrightarrow |f| = 0$ (2) $f=0$

(ii) $\because \|\alpha f\|_1' = \int_E |\alpha f|' = \int |\alpha||f| = |\alpha| \int_E |f| = |\alpha| \|f\|$

(iii) $\|f+g\|_1' = \int_E |f+g|' \leq \int_E |f|' + \int_E |g|' = \|f\|_1 + \|g\|_1$

Let p=∞ $\because |f| \leq \|f\|_\infty \Rightarrow \|f\|_\infty = 0 \Leftrightarrow |f| = 0 \Rightarrow f=0$

Lemma (a): for $0<\mu<1$ and any pair of non-negative real no.s a and b, we have—

$$a^{\mu}.b^{1-\mu} \leq \mu a+(1-\mu)b \qquad -①$$

pf:

Case-I if $a=0$ ~~&~~ or $b=0$

then ineq. (1) ~~ob~~ trivially true.

Case-II. let $a>0$ & $b>0$ and $\phi(t)=1-\mu+\mu t-t^{\mu}$, $t\in[0,\infty)$

$$\phi'(t)=\mu(1-t^{\mu-1})$$

$$\phi''(t)=-\mu(\mu-1)t^{\mu-2}$$

$t=1$. min. $\Rightarrow$ $\phi(t)\geq\phi(1)$ $\quad t\in[0,\infty)$

$(\because \phi(1)=0)$

$$\Rightarrow 1-\mu+\mu t-t^{\mu}\geq 0$$

put $t=a/b$

$$\Rightarrow 1-\mu+\frac{a\mu}{b}-\frac{a^{\mu}}{b^{\mu}}\geq 0$$

$$\Rightarrow (1-\mu)b+a\mu \geq \cancel{a\mu b}\ a^{\mu}b^{-\mu+1}$$

i.e $\boxed{a^{\mu}b^{-\mu+1} \leq \mu a+(1-\mu)b}$ Proved

Conjugate number

The conjugate no. of $p\in[1,\infty]$ is a unique no. $q\in[1,\infty]$ that satisfy

$$\frac{1}{p}+\frac{1}{q}=1 \Rightarrow \boxed{q=\frac{p}{p-1}}$$

* The conjugate of 1 is ∞ and for ∞ is 1.

Theorem: (Holder's inequality) Let E be measurable set, $1\leq p<\infty$ and q be conjugate no. of p. If $f\in L^{p}(E)$ and $g\in L^{q}(E)$, then $|f.g|$ is integrable over E and

$$\int_E |f.g| \leq \|f\|_p \|g\|_q$$

(Holder's Inequality)

∴ By lemma (4), we have also (by putting $\mu = 1/p$)

$$\left[a^{1/p} b^{1-1/p = 1/q} \le \frac{a}{p} + \frac{b}{q}\right]$$

Pf: If $p=1$, $q=\infty$, $g \in L^{\infty}(E)$

$$|g| \le |g|_{\infty}$$

then $\int_E |f g| = \int_E |f||g| \le \|g\|_{\infty} \int_E |f|$

$$= \|g\|_{\infty} \|f\|_1 \le \|g\|_{\infty} \|f\|_p$$

Next If $p \in (1,\infty)$ & ~~g~~ q is conjugate of p.

then taking $\mu = \frac{1}{p}$, $a = \frac{|f|^p}{\|f\|_p^p} = F$ and $b = \frac{|g|^q}{\|g\|_q^q} = G$

Now putting these value of a & b in prev. lemm we have

$$a^{\mu} b^{1-\mu} \le \mu a + (1-\mu) b \Rightarrow \boxed{a^{1/p} b^{1/q} \le \frac{a}{p} + \frac{b}{q}}$$

$$\Rightarrow \left(\frac{|f|^p}{\|f\|_p^p}\right)^{\mu} \left(\frac{|g|^q}{\|g\|_q^q}\right)^{1-\mu} \le \mu \left(\frac{|f|^p}{\|f\|_p^p}\right) + (1-\mu)\left(\frac{|g|^q}{\|g\|_q^q}\right)$$

$$\Rightarrow \left(F^{p}\right)^{\mu}\left(G^{q}\right)^{1-\mu} \le$$

$$\Rightarrow \left(\frac{|f|^p}{\|f\|_p^p}\right)^{1/p} \left(\frac{|g|^q}{\|g\|_q^q}\right)^{1/q} \le \frac{1}{p}\left(\frac{|f|^p}{\|f\|_p^p}\right) + \frac{1}{q}\left(\frac{|g|^q}{\|g\|_q^q}\right)$$

(*)

$$\left(\because \mu = \frac{1}{p} \Rightarrow 1-\mu = 1-\frac{1}{p} = \frac{1}{q}\right)$$

$$\Rightarrow \cancel{\frac{|f|}{\|f\|_p}\frac{|g|}{\|g\|_q}} \le \frac{1}{p} \quad \boxed{|F(x)G(x)| \le \frac{|F(x)|^p}{p} + \frac{|G(x)|^q}{q}}$$

Integrate both side on E

$$\int_E |F(x)G(x)| \le \int \frac{|F(x)|^p}{p} + \int \frac{|G(x)|^q}{q}$$

$$\le \frac{1}{p}\int \frac{|f|^p}{\|f\|_p^p}\,dx + \frac{1}{q}\int \frac{|g|^q}{\|g\|_q^q}\,dx$$

$$\le \frac{1}{p}\left[\frac{\int |f|^p}{\|f\|_p^p}\right] + \frac{1}{q}\left[\frac{\int |g|^q}{\|g\|_q^q}\right]$$

$$\le \frac{1}{p}\left[\frac{\|f\|_p^p}{\|f\|_p^p}\right] + \frac{1}{q}\left[\frac{\|g\|_q^q}{\|g\|_q^q}\right] = \frac{1}{p} + \frac{1}{q} = 1$$

now goto p(94)

MINKOWSKI INEQUALITY

theorem Let E be a measurable set & $1 \le p \le \infty$. If $f, g \in L^p(E)$, then, so does their sum $f+g$ ① and ②

$$\|f+g\|_p \le \|f\|_p + \|g\|_p$$

(Minkowski Inequality) $\because \|f\|_p^p = \int_E |f|^p$

case-I

Pf:— If $p = 1$ (i.e. $\|f+g\|_1 \le \|f\|_1 + \|g\|_1$)

$$\therefore \|f+g\|_1 = \int_E |f+g|^1 \le \underbrace{\int_E |f|_1 + \int_E |g|_1}_{<\infty} \le \|f\|_1 + \|g\|_1$$

$\Rightarrow f+g \in L^1(E)$

case-II $(1 \le p < \infty)$

proof for 1st part

Next let $p \in (1,\infty)$. If a & b are non-neg. numbers, then

bcz $a,b \ge 0$ $\rightarrow (a+b)^p \le 2^p\{|a|^p + |b|^p\}$

If $f,g \in L^p(E)$ then $\int_E |f|^p < \infty$, $\int_E |g|^p < \infty$

Now

$$\int_E |f+g|^p \le \int_E (|f|+|g|)^p \le 2^p\left\{\int_E |f|^p + \int_E |g|^p\right\} \quad — ⓐ$$

$< \infty$

$\Rightarrow f+g \in L^p(E)$ (first part proved)

second part Now

$$\int_E (f+g)^p \le \int_E |f+g|^{p-1}[|f| + |g|]$$

$$= \int_E |f+g|^{p-1}\cdot|f| + \int_E |f+g|^{p-1}|g| \quad — ①$$

$\frac{1}{p}+\frac{1}{q}=1$

or $\frac{1}{p} = \frac{q-1}{q}$

or $p = \frac{q}{q-1}$

or $q = p(q-1)$

Since, $\int_E ||f+g|^{p-1}|^q = \int_E |f+g|^{q(p-1)}$

$$= \int_E |f+g|^p < \infty \quad (\text{by } ⓐ)$$

$\Rightarrow |f+g|^{p-1} \in L^q(E)$, $f \in L^p(E)$

Then by using Holder's Inequality—

$$\int_E |f+g|^{p-1}|f| \le \||f+g|^{p-1}\|_q \|f\|_p \quad — ②$$

Similarly

$\int |f+g|^{p-1}|g| = \int_E |f+g|^{p-1}|g| \le \| |f+g|^{p-1} \|_q \|g\|_p$ — ③

using ① - ③.

$$\|f+g\|_p^p \le \underbrace{\| |f+g|^{p-1} \|_q}_{\downarrow} \left[\|f\|_p + \|g\|_p\right]$$

$$\because \| |f+g|^{p-1} \|_q = \left(\int_E |f+g|^{\overset{=p}{\boxed{(p-1)q}}}\right)^{1/q}$$

$$= \left[\underbrace{\left\{\int_E |f+g|^p\right\}^{1/p}}\right]^{p/q}$$

$$= \|f+g\|_p^{p/q}$$

$$\Rightarrow \|f+g\|_p^p \le \|f+g\|_p^{p/q} \left[\|f\|_p + \|g\|_p\right]$$

$\Rightarrow \|f+g\|_p$ [illegible] $\le \left[\|f\|_p + \|g\|_p\right]$

$p - p/q = 1$

$\Rightarrow 1 - \frac{1}{q} = \frac{1}{p}$

$\because \frac{1}{p} + \frac{1}{q} = 1$

$\Rightarrow -1/q = 1/p - 1$

$p - \frac{p}{q} = p(1 - 1/q)$

$= p(1/p) = 1$

Case-III if $p = \infty$

$\because |f| \le \|f\|_\infty$ a.e and $|g| \le \|g\|_\infty$ a.e

$\therefore |f+g| \le |f| + |g| \le \|f\|_\infty + \|g\|_\infty$ a.e

$\Rightarrow \boxed{\|f+g\|_\infty \le \|f\|_\infty + \|g\|_\infty}$

thus results holds for $p = \infty$

× Continue of Holder's Inequality —

* Here

$$\int_E \frac{|f|}{\|f\|_p} \cdot \frac{|g|}{\|g\|_p} \le 1$$

$$\Rightarrow \frac{\int_E |f||g|}{\|f\|_p \|g\|_p} \le 1 \Rightarrow \int_E |f||g| \le \|f\|_p \|g\|_p$$

$$\Rightarrow \boxed{\int_E |f \cdot g| \le \|f\|_p \|g\|_p}$$

Or $\Rightarrow \boxed{\|fg\| \le \|f\|_p \|g\|_p}$

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① $\Omega = \{A \subseteq R \mid A$ is finte or A^c is finite$\}$

(i) clearly $\phi \in \Omega$ as ϕ is finite
also $\Rightarrow \phi^c = R \subseteq R$ then $\phi, R \in \Omega$.

(ii) let $B \in \Omega$, and B is finite or B^c is finte
let $B \subseteq R$ then $R - B = B^c$

(1) **Set function:** A f^n f is s.t.b a set f^n, if its domain consists of a family of sets viz, the f^n $f : \{A_i : i \in N\} \to N$ defined by $f(A_r) = r$ is set f^n.

(2) **finitely Additive**

$$f\left(\bigcup_{r=1}^{n} A_r\right) = \sum_{r=1}^{n} f(A_r)$$

(3) **Countable additive:** $f\left(\bigcup_{r=1}^{\infty} A_r\right) = \sum_{r=1}^{\infty} f(A_r)$

(4) **Countable subadditive:** $f\left(\bigcup_{r=1}^{\infty} A_r\right) \leq \sum_{r=1}^{\infty} f(A_r)$

(5) **Characteristic f^n:** $\phi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$

(6) **Signum f^n:**

$$\text{sgn}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$$

(7) **Drichlet f^n:**

$$f(x) = \begin{cases} 1 & x \text{ is rational} \\ 0 & x \text{ is irrational} \end{cases}$$

(8) **Cantor f^n:**

If F be the cantor set, then the mapping ϕ $\phi : F \to R$ defined as

$$\phi(x) = \sum_{n=1}^{\infty} \frac{a_n}{2^{n+1}}$$

when $a_n = 0, 1$ or 2 $x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$ is s.t.b Cantor f^n.

(9) By a step f^n, we mean a f^n ϕ which has the form $\phi(x) = c_i$, $y_{i-1} < x < y_i$ for subdivision of $[a,b]$ & some set of constants c_i.

P-166 - F-32-I let $\phi = \sum_{i=1}^{n} a_i \chi_{E_i}$. with $E_i \cap E_{i+1} = \phi$ for $i \neq j$. suppose that each E_i is a measurable set. of finite measure. then —

$$\int \phi = \sum_{i=1}^{n} a_i m(E_i)$$

Pf: ∵ set $A_a = \{ x \mid \phi(x) = a \}$

- ∵ the set $A_a = \bigcup_{a_i = a} E_i$

Hence ~~$a A_a$~~ $a\, m(A_\alpha) = m\left(a \bigcup_{a_i = a} E_i\right)$

$= \sum_{a_i = a} a_i m(E_i)$ — ⓐ

(By additivity of m)

So,

$$\int \phi(x)\, dx = \sum a\, m(A_a)$$

$$= \sum a_i m(E_i) \quad (\text{by } ⓐ)$$

① RIESZ-FISCHER THEOREM

statement: The normed L^p-space are complete for $p \geq 1$

or L^p-space is B.S (or complete space)

or A cgt seq. $\{f_n\}$ in L^p-space has a limit in L^p.

or Every C.S ~~in~~ $\{f_n\}$ in L^p cgs to f in L^p.

pf: In order to prove the above theorem, we shall show that every cauchy seq. in $L^p(E)$ cgs to some elt. f^n f in $L^p(E)$.

Let $\{f_n\}$ be one of such (cauchy) seq. in $L^p(E)$ -
then for $\varepsilon > 0$, $\exists$ a $n_0 \in \mathbb{N}$ s.t $\forall$ $m, n \geq n_0$.
$\Rightarrow$ $\|f_m - f_n\|_p < \varepsilon$ (let $\varepsilon = 1/2$)

Let $\varepsilon = 1/2$ then we can find a $n_1 \in \mathbb{N}$ s.t
for $m, n \geq n_1$ $\Rightarrow$ $\|f_m - f_n\|_p < \varepsilon = 1/2$

Similarly for taking $\varepsilon = 1/2^k$, $k = 1, 2, \ldots$ $(k \in \mathbb{N})$ we can find a natural no. n_k s.t
$\forall$ $m, n \geq n_k$ $\Rightarrow$ $\|f_m - f_n\| < 1/2^k$

Now In particular for $n = n_k$, $m > n_k$
$$\|f_m - f_{n_k}\| < \frac{1}{2^k}$$

Obviously $n_1 < n_2 < \ldots < n_k < \ldots$ (Monotonic Inc. seq of natural no.)

let $g_k = f_{n_k}$, then from above -

$$\|g_2 - g_1\|_p = \|f_{n_2} - f_{n_1}\|_p < 1/2$$

$$\|g_3 - g_2\| = \|f_{n_3} - f_{n_2}\|_p < 1/2^2$$

- - - - - - - -

$$\|g_{k+1} - g_k\| = \|f_{n_{k+1}} - f_{n_k}\| \leq \frac{1}{2^k}$$

- - - - - - - -

on adding these inequalities,

we get — $\sum_{k=1}^{\infty} \|g_{k+1}-g_k\|_p < \sum_{k=1}^{\infty} \frac{1}{2^k} = 1$ —①

thus $\sum_{k=1}^{\infty} \|g_{k+1}-g_k\|_p$ is cgt. $\left(\frac{1/2}{1-1/2} = 1 \right)$

Now define g as

$$g(x) = \begin{cases} |g_1(x)| + \sum_{k=1}^{\infty} \|g_{k+1}-g_k\|_p & \text{if RHS is cgt} \\ \infty & \text{if RHS is dgt} \end{cases}$$

$\|g(x)\|_p =$ Now $\left(\int_E |g(x)|^p \right)^{1/p} = \lim_{n\to\infty} \left\{ \int_E \left| g_1(x) + \sum_{k=1}^{n} |g_{k+1}-g_k| \right|^p \right\}^{1/p}$

$\Rightarrow \|g(x)\|_p \le \lim_{n\to\infty} \left[\|g_1(x)\|_p + \sum_{k=1}^{\infty} \|g_{k+1}-g_k\| \right]$

$= \|g_1(x)\|_p + 1$ (By ①)

thus $\|g(x)\|_p < \infty \Rightarrow g \in L^p(E)$

Now let

$$E_1 = \{ x \in E \mid g(x) = \infty \}$$

Now define a map f as

$$f(x) = \begin{cases} 0 & \forall x \in E_1 \\ g_1(x) + \sum_{k=1}^{\infty} (g_{k+1}-g_k) & \text{for } x \in E \text{ but } x \notin E_1 \end{cases}$$

$$f(x) = \begin{cases} 0 & \\ \lim_{m\to\infty} \left(g_1 + \sum_{k=1}^{m-1} (g_{k+1}-g_k) \right) & \text{for } x \notin E_1 \end{cases}$$

$$= \lim_{m\to\infty} g_m(x) \qquad \text{for } x \in E_1^c$$

thus $f(x) = \begin{cases} 0 & \text{for } x \in E_1 \\ \lim_{m\to\infty} g_m(x) & \text{for } x \notin E_1 \end{cases}$

$\therefore f(x) = \lim_{m\to\infty} g_m(x)$ a.e in E.

$\therefore \lim_{m\to\infty} |g_m(x) - f| = 0$ a.e. in E. —(ii)

$\because g_m(x) = g_1 + \sum_{k=1}^{m-1} (g_{k+1} - g_k)$

$\Rightarrow |g_m(x)| \leq |g_1| + \sum_{k+1}^{m-1} |g_{k+1} - g_k| \leq |g_1| + \sum_{k=1}^{\infty} |g_{k+1} - g_k|$

$\leq g$

$\Rightarrow |g_m(x)| \leq g \quad \forall m \in N$

$\Rightarrow \therefore \lim_{m \to \infty} |g_m(x)| \leq g$

$\therefore$ by (iii) $\underline{|f| \leq g}$,

Some Useful Links:

1. Free Maths Study Materials (https://pkalika.in/2020/04/06/free-maths-study-materials/**)**

2. BSc/MSc Free Study Materials (https://pkalika.in/2019/10/14/study-material/**)**

3. MSc Entrance Exam Que. Paper: (https://pkalika.in/2020/04/03/msc-entrance-exam-paper/**)**
[JAM(MA), JAM(MS), BHU, CUCET, ...etc]

4. PhD Entrance Exam Que. Paper: (https://pkalika.in/que-papers-collection/**)**
[CSIR-NET, GATE(MA), BHU, CUCET,IIT, NBHM, ...etc]

5. CSIR-NET Maths Que. Paper: (https://pkalika.in/2020/03/30/csir-net-previous-yr-papers/**)**
[Upto 2019 Dec]

6. Practice Que. Paper: (https://pkalika.in/2019/02/10/practice-set-for-net-gate-set-jam/**)**
[Topic-wise/Subject-wise]

7. **List of Maths Suggested Books** (https://pkalika.in/suggested-books-for-mathematics/)

8. CSIR-NET Mathematics Details Syllabus (https://wp.me/p6gYUB-Fc)

9. Free Video Lectures for CSIR-NET, GATE, SET, Asst. Prof. ..etc
https://www.youtube.com/pkalika