# Legendre & Bessel Differential Equations

[ODE Handwritten Study Material]

P. Kalika
PhD (NET(JRF), GATE, SET)
Email: maths.whisperer@gmail.com

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* **Analytic function** :- suppose $f(x)$ is any function, then ~~f(x)~~
$f(x)$ is analytic at point $x = x_0$ if $f(x)$ can be expressed as

$$\boxed{f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n}$$

# General form of Second order ordinary DEs :-

$$P(x)\, y'' + Q(x)\, y' + R(x)\, y = 0, \quad x \in I$$

where 'I' is a index set.

* **Ordinary Point** :

Consider the DE

$$P(x)\, y'' + Q(x)\, y' + R(x)\, y = 0$$

$$\Rightarrow y'' + \frac{Q(x)}{P(x)} y' + \frac{R(x)}{P(x)} y = 0$$

$x = x_0$ is called ordinary point for this differential eqn. if $\frac{Q(x)}{P(x)}$ & $\frac{R(x)}{P(x)}$ are analytic functions at $x = x_0$.

Example: $x^2 y'' + x^3 y' + x^2 y = 0$, $p(x) = x^2$, $Q(x) = x^3$, $R(x) = x^2$ then $x = 0, 1, 2, \ldots$ is ordinary point for DE.

Solⁿ: $\frac{Q(x)}{P(x)} = x$, $\frac{R(x)}{P(x)} = 1$. Use analytic at $x = 0$.

$$x^2 y'' + x^3 y' + x^2 y = 0$$

$x = 0$ is not an ordinary pt for this diff. eqn.
because $\frac{Q(x)}{P(x)} = x$ is not analytic at $x = 0$.

If $x = x_0$ is ordinary pt for any DEs then series sol$^n$ for that DE is —

$$\boxed{\sum_{k=0}^{\infty} a_k (x-x_0)^k}$$

**Que:** find series sol$^n$ of the DE

$$(1-x^2)y'' - 2xy' + 2y = 0 \quad \text{——— (1)}$$

Here $P(x) = 1-x^2$, $Q(x) = -2x$, $R(x) = 2$.

**Sol$^n$:** Here $\frac{Q(x)}{P(x)} = \frac{-2x}{1-x^2}$, $\frac{R(x)}{P(x)} = \frac{2}{1-x^2}$

these function are analytic at $x=0$.
so series sol$^n$ for this DE is —

$$Y(x) = \sum_{k=0}^{\infty} a_k (x-0)^k$$

$$\Rightarrow y(x) = \sum_{k=0}^{\infty} a_k x^k \quad \text{——— (2)}$$

$$\Rightarrow y'(x) = \sum_{k=0}^{\infty} k \cdot a_k \cdot x^{k-1} \quad \text{——— (3)}$$

$$y''(x) = \sum_{k=0}^{\infty} k(k-1) a_k x^{k-2} \quad \text{——— (4)}$$

Now use eqn (2), (3) & (4) in (1) —
we have

$$\Rightarrow (1-x^2) \sum_{k=0}^{\infty} k(k-1) a_k x^{k-2}$$

$$-2x \sum_{k=0}^{\infty} k a_k x^{k-1} + 2\sum_{k=0}^{\infty} a_k x^k = 0$$

$$\Rightarrow \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} - \sum_{k=0}^{\infty} k(k-1) a_k x^k -$$

$$-\sum_{k=0}^{\infty} 2k a_k x^k + \sum_{k=0}^{\infty} 2a_k x^k = 0$$

$$\Rightarrow \sum_{j=0}^{\infty}(j+2)(j+1)a_{j+2}x^j \quad \begin{array}{|l} j = k-2 \\ j = k-1 \end{array}$$

$$-\sum_{k=0}^{\infty}k(k-1)a_kx^k + \sum_{k=0}^{\infty}2a_kx^k = 0 - \sum_{k=0} 2ka_kx^k$$

$$\Rightarrow \sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}x^k - \sum_{k=0}^{\infty}k(k-1)a_kx^k$$

$$-\sum_{k=0}^{\infty}2ka_kx^k + \sum_{k=0}^{\infty}2a_kx^k = 0$$

$$\Rightarrow \sum_{k=0}^{\infty}\left[(k+2)(k+1)(a_{k+2} - k(k-1)a_k) - 2ka_k + 2a_k\right]x^k = 0$$

Now compare co-eff. of $x^k$. we have

$$(k+2)(k+1)a_{k+2} + [k(k-1) - 2k + 2]a_k = 0$$

$$\forall\ k = 0, 1, 2 \ldots$$

$$\Rightarrow a_{k+2} = \frac{k^2+k-2}{(k+2)(k+1)}a_k$$

$$= \frac{(k-1)\cancel{(k+2)}}{\cancel{(k+2)}(k+1)}a_k$$

$$= \frac{(k-1)}{(k+1)}a_k, \quad k = 0, 1, 2 \ldots$$

if $k = 0$.

$\Rightarrow a_2 = -a_0$

if $k = 2$

$a_3 = 0$

$$a_4 = \frac{a_2}{3} = \frac{1(-a_0)}{3} = -\frac{1}{3}a_0$$

$$a_5 = \frac{1}{2}a_3 = \frac{1}{2}(0) = 0$$

$$a_6 = \frac{3}{5}a_4 = \frac{3}{5}(-1/3)a_0 = -1/5\,a_0$$

$a_7 = 0$

$$a_8 = \frac{5}{7}a_6 = \frac{5}{7}(-1/5)a_0 = -\frac{1}{7}a_0$$

so $y(x) = \sum_{k=0}^{\infty} a_k x^k$

$= a_0 + a_1 x - a_0 x^2 - \frac{1}{3} a_0 x^4 - \frac{1}{5} a_0 x^5 - \frac{1}{7} a_0 x^8$

— — ———

$\Rightarrow y(x) = a_0 \left[ 1 - x^3 - \frac{1}{3} x^4 - \frac{1}{5} x^6 - \frac{1}{7} x^8 - \cdots + a_1 x \right.$

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2pr | Thursday
31/07/14

# Legendre Equation :—

Q $(1-x^2) y'' - 2xy' + n(n+1) y = 0, \ x \in I$

Sol$^n$ $P(x) = (1-x^2), \ Q(x) = -2x, \ R(x) = n(n+1)$

$\frac{Q(x)}{P(x)} = \frac{-2x}{1-x^2}, \quad \frac{R(x)}{P(x)} = \frac{n(n+1)}{1-x^2}$

these are analytic at $x=0$
so $x=0$ is an ordinary pt for this DE.

So Assume that sol$^n$ for eq$^n$ (1) is

$y(x) = \sum_{k=0}^{\infty} a_k (x-0)^k$

$y(x) = \sum_{k=0}^{\infty} a_k x^k$ ——— (2)

$y'(x) = \sum_{k=0}^{\infty} k \cdot a_k x^{k-1}$ ——— (3)

$y''(x) = \sum_{k=0}^{\infty} k(k-1) a_k \cdot x^{k-2}$ ——— (4)

Now, use eq$^n$ (2), (3) & (4) in (1) —
we get —

$(1-x^2) \sum_{k=0}^{\infty} k(k-1) a_k x^{k-2} - 2x \sum_{k=0}^{\infty} k \cdot a_k x^{k-1}$

$+ n(n+1) \sum_{k=0}^{\infty} a_k x^k = 0$

$$\Rightarrow \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} - \sum_{k=0}^{\infty} k(k-1) a_k x^k - \sum_{k=0}^{\infty} 2k a_k x^k + \sum_{k=0}^{\infty} n(n+1) a_k x^k = 0$$

$$\Rightarrow \sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k - \sum_{k=0}^{\infty} k(k-1) a_k x^k - \sum_{k=0}^{\infty} 2k a_k x^k + \sum_{k=0}^{\infty} n(n+1) a_k x^k = 0$$

$$\Rightarrow \sum_{k=0}^{\infty} \left[ (k+2)(k+1) a_{k-2} - k(k-1) a_k - 2k a_k + n(n+1) a_k \right] x^k = 0 \; ; \; k = 0, 1, 2, \ldots$$

this eqn is toue for all $k = 0, 1, 2 \ldots$

so, $(k+2)(k+1) a_{k+2} + \{-k(k-1) - 2k + n(n+1)\} a_k = 0$

$$\Rightarrow (k+2)(k+1) + \{n^2 - k^2 + n - k\} = 0$$

$$\Rightarrow \frac{\cancel{(k+2)}}{a_{k+2}} = \frac{-(n+k+1)(n-k)}{(k+2)(k+1)} a_k$$

for $k=0$, $a_2 = \dfrac{-(n+1)(n)}{2} a_0$

$k=1$, $a_3 = \dfrac{-(n+2)(n-1)}{6} a_1$

$k=2$, $a_4 = \dfrac{-(n+3)(n-2)}{12} a_2$

$$= \frac{-(n+3)(n-2)}{12} \left\{ \frac{-(n+1) n}{2} \right\} a_0$$

$$= \frac{(n+3)(n+1) n (n-2)}{24} a_0$$

$k=3$, $a_5 = \dfrac{-(n+6)(n-3)}{20} a_3$

$$= \frac{-(n+6)(n-3)}{20} \left\{ \frac{-(n+2)(n-1)}{6} \right\} a_1$$

$$= \frac{(n+6)(n-3)(n+2)(n-1)}{120} a_1$$

In general

$$a_{2m} = \frac{(-1)^m (n+2m-1)(n+2m-3)\cdots(n+1)n\cdots(n-2m+2)}{\lfloor 2m}$$

$$a_{2m+1} = \frac{(-1)^{m+1}(n+2m)(n+2m-2)\cdots(n+2)(n-1)\cdots(n-2m+1)}{\lfloor 2m+1}$$

So soln of eqn (1) is—

$$y(x) = a_0 \underbrace{\left[1 - \frac{(n+1)n}{\lfloor 2}x^2 + \frac{(n-2)n(n+1)(n+3)}{\lfloor 4}x^4 + \cdots\right]}_{y_1(x)}$$
$$+ a_1 \underbrace{\left[x - \frac{(n+2)(n-1)}{\lfloor 3}x^3 + \frac{(n+4)(n+2)(n-1)(n-3)}{\lfloor 5}x^5\right]}_{y_2(x)}$$

$\Rightarrow y(x) = a_0 y_1(x) + a_1 y_2(x)$.

where $y_1(x) = 1 - \frac{(n+1)n}{\lfloor 2}x^2 + \frac{(n-2)n(n+1)(n+3)}{\lfloor 4}x^4 + \cdots$

$y_2(x) = x - \frac{(n+2)(n-1)}{\lfloor 3}x^3 + \frac{(n+4)(n+2)(n-1)(n-3)}{\lfloor 5}x^5 + \cdots$

Here we assume that $n$ is a real no.. Now consider the case when $n$ is a non-neg. integer.

Case-I when $n$ is even

In this case $y_1(x)$ become poly. in $x$ of degn and $y_2(x)$ become an infinite series

for $n=0$, $y_1(x) = 1$

$n=2$, $y_1(x) = 1 - \frac{6}{2}x^2 = 1-3x^2$

$n=4$, $y_1(x) = 1 - 10x^2 + \frac{35}{3}x^4$

Case-II when $n$ is odd

2). $a_0$

## Legendre Polynomial

Kalika
01/08/14

Consider the diff eqn

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \quad \text{--- (1)}$$

$n \geq 0$ is an integ.

we have see that eqn (1) admits poly. soln. let's denoted by $P_n(x)$.

If $P_n(1) = 1$, then it is called legendre polynomial. of deg. $n$.

$$P_0(x) = 1$$
$$P_1(x) = x$$
$$P_2(x) = \frac{1}{2}(3x^2-1)$$
$$P_3(x) = \frac{1}{2}(5x^3-3x)$$

## Rodrigues Formulae (I8 in ex or test)

$n^{th}$ deg. legendre poly. is given by

$$P_n(x) = \frac{1}{2^n \lfloor n} \frac{d^n}{dx^n}(x^2-1)^n$$

Pf: let $v = (x^2-1)^n$

Consider the diff. eqn

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \quad \text{--- (1)}$$

Let $v = (x^2-1)^n$ --- (2)

then $\frac{dv}{dx} = n(x^2-1)^{n-1}\, 2x$

$$\Rightarrow \underbrace{(x^2-1)}_{u}\underbrace{\frac{dv}{dx}}_{v} - 2nxV = 0 \quad \text{--- (3)}$$

$$\boxed{\begin{aligned} &\text{Leibnitz's theorem} \\ &(uv)^{(n)} = \frac{d^n}{dx^n}(uv) \\ &= {}^nC_0 u^{(n)} v + {}^nC_1 u^{(n-1)} v' + {}^nC_2 u^{(n-2)} v'' \\ &\quad + \cdots + {}^nC_{n-1} u' v^{(n-1)} + {}^nC_n u v^{(n)} \end{aligned}}$$

Now diff. eqn (3), $(n+1)$ times we have

$$\Rightarrow (x^2-1)\frac{d^{n+2}}{dx^{n+2}}V + (n+1)2x\frac{d^{n+1}}{dx^{n+1}}V +$$

$$\frac{(n+1)n}{2}(2)\frac{d^n}{dx^n}V - 2nx\frac{d^{n+1}}{dx^{n+1}}V$$

$$-2n(n+1)\frac{d^n V}{dx^n} = 0$$

$$(x^2-1)\frac{d^{n+2}}{dx^{n+2}}V + 2x\frac{d^{n+1}}{dx^{n+1}}V - n(n+1)\frac{d^n V}{dx^n} = 0$$

$$\Rightarrow (1-x^2)\frac{d^2}{dx^2}(D^n V) - 2x\frac{d}{dx}(D^n V) + n(n+1)D^n V = 0$$

where $D = \frac{d}{dx}$

$\therefore D^n(V)$ satisfies eqn (1).

Now let us assume that nth Lengdre poly. is given by —

$$P_n(x) = AD^n V \quad \text{--- (4)}$$

for lengdre Polynomial

$$P_n(1) = 1 \quad \text{--- (5)}$$

from (4), ~~$P_n(x)$~~

$$P_n(x) = A\frac{d^n}{dx^n}(x^2-1)^n$$

$$= A\frac{d^n}{dx^n}\{(x-1)^n(x+1)^n\}$$

$$P_n(x) = A(x+1)^n \frac{d^n}{dx^n}(x-1)^n + \text{terms with } (x-1) \text{ as a factor}$$

$$= A\lfloor n (x+1)^n + \text{terms with } (x-1) \text{ as a factor}$$

Now using eqn ⑤ —

$P_n(1) = 1$ $\Rightarrow A = \frac{1}{2^n \lfloor n}$

$\Rightarrow A\lfloor n (1+1)^n + 0 = 1$ using it in eqn ④,

we have —

$$P_n(x) = \frac{1}{2^n \lfloor n} D^n V$$

$$\Rightarrow \boxed{P_n(x) = \frac{1}{2^n \lfloor n} \cdot \frac{d^n}{dx^n}(x^2-1)^n}$$

Tuesday | 10:00
05/08/14

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Q Find Series sol^n of the DE
$y'' - xy' + 2y = 0$ near $x = 1$ ①

Sol^n: Here $P(x) = 1$, $Q(x) = -x$, $R(x) = 2$

$\frac{Q(x)}{P(x)} = -x$, $\frac{R(x)}{P(x)} = 2$ are analytic at $x = 1$

So $x = 1$ is an ordinary pt. for this eqn.

Assume that sol^n is —

$$y(x) = \sum_{k=0}^{\infty} a_k (x-1)^k \quad \text{—— ②}$$

$$y'(x) = \sum_{k=0}^{\infty} k \cdot a_k (x-1)^{k-1} \quad \text{—— ③}$$

$$y''(x) = \sum_{k=0}^{\infty} k(k-1) a_k (x-1)^{k-2} \quad \text{—— ④}$$

using ②, ③ & ④ in ①, we get —

$$\sum_{k=0}^{\infty} k(k-1) a_k (x-1)^{k-2} - x \sum_{k=0}^{\infty} k \cdot a_k (x-1)^{k-1} + 2 \cdot \sum_{k=0}^{\infty} a_k (x-1)^k$$

$((x-1)+1)$ $= 0$

$$\Rightarrow \sum_{k=0}^{\infty} (k+2)\, a_{k+2}(x-1)^k (k+1) - \sum_{k=0}^{\infty} k\, a_k (x-1)^k$$

$$- \sum_{k=0}^{\infty} k\, a_k (x-1)^{k-1} + \sum_{k=0}^{\infty} 2 a_k (x-1)^k = 0$$

$$\Rightarrow \sum_{k=0}^{\infty} (k+2)(k+1)\, a_{k+2}(x-1)^k - \sum_{k=0}^{\infty} k\, a_k (x-1)^k - \sum_{k=0}^{\infty} (k+1)\, a_{k+1}(x-1)^k$$

$$+ \sum_{k=0}^{\infty} 2 a_k (x-1)^k = 0$$

$$\Rightarrow \sum_{k=0}^{\infty} \left[(k+2)(k+1)\, a_{k+2} - k a_k - (k+1)\, a_{k+1} + 2a_k\right](x-1)^k = 0$$

$$\forall\ k = 0, 1, 2, \ldots$$

$$\Rightarrow (k+2)(k+1)\, a_{k+2} - (k+1)\, a_{k+1} - (k-2)\, a_k = 0$$

$$\Rightarrow a_{k+2} = \frac{a_{k+1}}{(k+2)} + \frac{(k-2)}{(k+1)(k+2)} a_k \quad \forall\ k = 0, 1, \ldots$$

$$\Rightarrow a_2 = \cancel{\tfrac{1}{2} a_1 - 2a_0} = \tfrac{1}{2} a_1 - a_0$$

$$a_3 = \cancel{\tfrac{1}{3} a_2 - \tfrac{1}{2} a_1} = \tfrac{1}{3} a_2 - \tfrac{1}{6} a_1$$

$$= \frac{1}{3}\left(\frac{1}{2} a_1 - a_0\right) - \frac{1}{6} a_1$$

$$= \cancel{\frac{1}{3} a_1 - \frac{2}{3} a_0} = -\tfrac{1}{3} a_0$$

$$a_4 = \frac{a_3}{4} = \frac{1}{4}\left(-\tfrac{1}{3} a_0\right) = -\frac{1}{12} a_0$$

$$a_5 = \frac{a_4}{5} + \frac{1}{20} a_3$$

$$= \frac{1}{5}\left(\frac{1}{12} a_0\right) + \frac{1}{20}\left(-\tfrac{1}{3} a_2\right) = \frac{1}{30} a_0$$

$$y(x) = \sum_{k=0}^{\infty} a_k (x-1)^k$$

$$= a_0 + a_1(x-1) + a_2(x-1)^2 + a_3(x-1)^3 + a_4(x-1)^4 + a_5(x-1)^5 + \ldots$$

$$y(x) = a_0 + a_1(x-1) + \left(\frac{1}{2} a_1 - a_0\right)(x-1)^2 + \left(-\frac{1}{3} a_0\right)(x-1)^3$$

$$+ \left(-\frac{1}{12} a_0\right)(x-1)^4 + \left(-\tfrac{1}{30} a_0\right)(x-1)^5 + \ldots$$

$$\boxed{\Rightarrow y(x) = a_0\left[1 - (x-1)^2 - \frac{1}{3}(x-1)^3 - \frac{1}{12}(x-1)^4 - \frac{1}{30}(x-1)^5 + \ldots\right] + a_1\left[(x-1) + \tfrac{1}{2}(x-1)^2 \ldots\right]}$$

where $a_0$ & $a_1$ are arbitrary constant.



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## Ordinary and Singular points

Consider the DE

$$y'' + P(x)y' + Q(x)y = 0$$

If $P(x)$ & $Q(x)$ are analytic fn at $x = x_0$ then $x = x_0$ is called ordinary pt. of this DE.

If $x = x_0$ is not an ordinary pt. then it is called SINGULAR POINT.

(i) **Regular Singular Point:**

If $x_0$ is a singular pt & $(x-x_0)P(x)$ & $(x-x_0)^2 Q(x)$ both are analytic at $x = x_0$ then it is called Regular Singular Point

(ii) **Irregular Singular Pt.**

If $x_0$ is a singular pt. & $(x-x_0)P(x)$ & $(x-x_0)^2 Q(x)$ both are not analytic at $x = x_0$ then it is called Irregular Singular Point.

(a) $x^2 y'' + xy' + y = 0$, Check $x=0$.

(b) $(x-1)^2 y'' + \frac{1}{(x-1)} y' + (x-1)y = 0$, Check $x=1$

(a) $\Rightarrow$ $y'' + \frac{1}{x}y' + \frac{1}{x^2}y = 0$, $P(x) = \frac{1}{x}$, $Q(x) = \frac{1}{x^2}$

$P(x)$ & $Q(x)$ are not both analytic at $x=0$, so it is singular point.

$$\left.\begin{aligned} (x-x_0)P(x) &= x \cdot \frac{1}{x} = 1 \\ (x-0)^2 Q(x) &= x^2 \cdot 1/x^2 = 1 \end{aligned}\right\} \text{these both are analytic}$$

so $x=0$, is a regular singular pt.

(b) $\Rightarrow y'' + \frac{1}{(x-1)^3} y' + \frac{x}{(x-1)} y = 0$, $x = 1$

$P(x) = \frac{1}{(x-1)^3}$ & $Q(x) = \frac{1}{x-1}$

are not both analytic at $x=1$, so it is singular pt.

$\therefore (x-x_0) \cdot P(x) = (x-1) \cdot \frac{1}{(x-1)^3} = \frac{1}{(x-1)^2}$ } Irregular ~~singular point~~

& $(x-x_0)^2 Q(x) = (x-1)^2 \cdot \frac{1}{(x-1)} = (x-1) \rightarrow$ ~~Regular singular pt.~~

both are not analytic so

$\Rightarrow x = 1$ is irregular singular pt.

(c) $(x-2) y'' + x y' + y = 0$ check at $x=2$.

$\Rightarrow y'' + \frac{x}{x-2} y' + \frac{1}{x-2} y = 0$

$P(x) = \frac{x}{x-2}$ & $Q(x) = \frac{1}{x-2}$

are not both analytic at $x=2$

so it is singular pt.

$(x-2) \cdot \frac{x}{x-2} = x = 2$

$(x-2)^2 \cdot \frac{1}{x-2} = x-2 = 0$

} both analytic at $x=2$

(d) $x y'' + x^2 y' + x y = 0$ check at $x=0$

$\Rightarrow y'' + x y' + y = 0$

$P(x) = x$ & $Q(x) = 1$ are analytic at $x=0$.

$\therefore x=0$ is a Regular Singular pt.

(e) $(x-3)^2 y'' + x y' + x y = 0$ check at $x=3$.

$y'' + \frac{x}{(x-3)^2} y' + \frac{x}{(x-3)^2} y = 0$

$P(x) = \frac{x}{(x-3)^2}$ & $Q(x) = \frac{x}{(x-3)^2}$

are not both analytic at $x=3$
so it is singular pt.

$$(x-3) \cdot \frac{x}{(x-3)^2} = \frac{x}{(x-3)}$$

$$\& \ (x-3)^2 \cdot \frac{x}{(x-3)^2} = x$$

both are not analytic at $x=3$

so $x=3$ is irregular singular pt.

(f) $9x(1-x)y'' - 12y' + 4y = 0$ check at $x=0$

$$\Rightarrow y'' - \frac{12}{9x(1-x)} y' + \frac{4}{9x(1-x)} y = 0$$

$$P(x) = \frac{-12}{9x(1-x)} \quad \& \quad Q(x) = \frac{4}{9x(1-x)}$$

both are not analytic at $x=0$
so now

$$(x-0) \cdot \frac{-12}{9x(1-x)} = \frac{-12}{9(1-x)}$$

$$\& \ (x-0)^2 \cdot \frac{4}{9x(1-x)} = \frac{4}{9\left(\frac{1}{x}-1\right)}$$

both are analytic
so $x=0$ is regular singular pt

# Method Frobenius

This method is used to find Frobenius soln near regular singular point.

Step I

If $x=x_0$ is Regular Singular pt. then assumption for soln is —

$$y(x) = \sum_{k=0}^{\infty} a_k (x-x_0)^{k+m}$$

when $a_0 \neq 0$.

Find $y'$ & $y''$.

Step-II find $y'$ & $y''$
put the value of $y, y', y''$ in given DE & equating to zero. the co-efficient of smaller power of $x$ then we will get the quarditik eqn in $m$. This equation is called Initial Equation.
The find out roots of initial Equation

Step-III
Now equating to zero, the co-efficient of the general power of $x$
$(x^{k+m}, x^{k+m-1}, \text{etc})$

## Recurrence Relation

It is the Relation between the coeffients $(a_k, a_{k-1}, a_{k-2})$

Consider the Series solution of the following differential Eqn –

$$9x(1-x)y'' - 12y' + 4y = 0 \quad \text{--- (1)}$$

$x = 0$ is regular singular pt.

$$Y(x) = \sum_{k=0}^{\infty} a_k (x-0)^{k+m}, \quad a_0 \neq 0 \quad \text{--- (2)}$$

$$\Rightarrow y(x) = \sum_{k=0}^{\infty} a_k x^{k+m} \quad \text{--- (2)}$$

$$\Rightarrow Y'(x) = \sum_{k=0}^{\infty} (k+m) a_k x^{k+m-1} \quad \text{--- (3)}$$

$$\Rightarrow Y''(x) = \sum_{k=0}^{\infty} (k+m)(k+m-1) a_k x^{k+m-2} \quad \text{--- (4)}$$

$$\Rightarrow 9x(1-x) \cdot \sum_{k=0}^{\infty} (k+m)(k+m-1) a_k x^{k+m-2}$$
$$-12 \sum_{k=0}^{\infty} a_k x^{k+m} \cdot (k+m) + 4 \sum_{k=0}^{\infty} a_k x^{k+m} = 0$$

Now putting (2), (3) & (4) in (1) –

$$\Rightarrow 9x(1-x)\sum_{k=0}^{\infty}(k+m)(k+m-1)a_k x^{k+m-2}$$

$$\Rightarrow \sum_{k=0}^{\infty} 9(1-x)(k+m)(k+m-1)a_k x^{k+m-1}$$

$$-\sum_{k=0}^{\infty} 12(k+m)a_k x^{k+m-1} + \sum_{k=0}^{\infty} 4a_k x^{k+m} = 0 \quad (5)$$

~~taking~~ Now equating to zero the smallest power of 'x'. (namely $x^{m-1}$)

$$9m(m-1)a_0 - 12ma_0 = 0$$

$$\Rightarrow (3m^2 - 7m)a_0 = 0$$

$$\Rightarrow 3m^2 - 7m = 0, \quad a_0 \neq 0$$

Initial equation.

* * *

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Q $9x(1-x)y'' - 12y' + 4y = 0$

$$y = \sum_{k=0}^{\infty} a_k x^{k+m}, \quad a_0 \neq 0$$

$$m = 0, \; 7/3$$

$$\sum_{k=0}^{\infty}\left[9(k+m)(k+m-1) - 12(k+m)\right]a_k x^{k+m-1} + \sum_{k=0}^{\infty}\left[-9(k+m)\cdot(k+m-1)+4\right]a_k \cdot x^{k+m} = 0$$

Now, equating to zero the co-efficient of $x^{k+m-1}$

$$\left[9(k+m)(k+m-1) - 12(k+m)\right]a_k - \left[9(k+m-1)\cdot(k+m-2)+4\right]a_{k-1} = 0$$

$$\Rightarrow \cancel{3(k+m)}\left[3(k+m-1) - 4\right]a_k \cancel{=0}$$

$$\Rightarrow \cancel{3(k+m)=0}$$

$$\Rightarrow 3(k+m)\left[3(k+m) - 7\right]a_k - \left[9(k+m-1)(k+m-2)+4\right]a_{k-1} = 0$$

$$a_k = \frac{\left[9(k+m-1)(k+m-2)+4\right]}{3(k+m)\left[3(k+m)-7\right]}a_{k-1}$$

$\forall \; k = 0, 1, 2, 3$

$(k=1,$ ~~m~~ $)$ $a_k = \dfrac{9\{(k+m)^2 - 3(k+m)+2\}+4}{3(k+m)\{3k+3m-7\}} a_{k-1}$

~~$a_0 =$~~

$a_1 = \dfrac{9\{(1+m)^2 - 3(m+1)+2\}+4}{8(m+1)(3+3m-7)} a_0$

<u>$k=2$</u>

$a_2 =$

## Bessel Equation

$$x^2y'' + xy' + (x^2 - n^2)y = 0 \quad \text{—— (1)}$$

It is Bessel eqn of degree $n$.

$x = 0$ is a regular singular pt.

so (check at 0)

Assume that the soln is —

$$y(x) = \sum_{k=0}^{\infty} a_k x^{k+m}, \quad a_0 \neq 0 \quad \text{—— (2)}$$

$$y'(x) = \sum_{k=0}^{\infty} (k+m) a_k x^{k+m-1} \quad \text{—— (3)}$$

$$y''(x) = \sum_{k=0}^{\infty} (k+m)(k+m-1) a_k x^{k+m-2} \quad \text{—— (4)}$$

Now use eqn (2), (3) & (4) in (1) —

$$\sum_{k=0}^{\infty} (k+m)(k+m-1) a_k x^{k+m} + \sum_{k=0}^{\infty} (k+m) a_k x^{k+m} + (x^2 - n^2)\sum_{k=0}^{\infty} a_k x^{k+m} = 0$$

$$\Rightarrow \sum_{k=0}^{\infty} \{(k+m)(k+m-1) + (k+m) - n^2\} a_k x^{k+m} + \sum_{k=0}^{\infty} a_k x^{k+m+2} = 0 \quad \text{—— (5)}$$

Now, equating to zero the coefficient of minimum power of $x$. (namely, $x^m$)

$$\{m(m-1) + m - n^2\} a_0 = 0$$

$$\Rightarrow (m^2 - n^2) a_0 = 0$$

$$\Rightarrow m^2 - n^2 = 0 \longrightarrow \text{Initial eqn}$$

$$\Rightarrow m = \pm n \quad \text{—— (6)}$$

Now from eqn (5) —

$$\sum_{k=0}^{\infty} \{(k+m)(k+m-1) + (k+m) - n^2\} a_k x^{k+m} + \sum_{k=2}^{\infty} a_{k-2} x^{k+m} = 0$$

$$\Rightarrow \sum_{K=0}^{\infty}\left[\left\{(k+m)(k+m-1)+(k+m)-n^2\right\}a_k + a_{k-2}\right\} x^{k+m} = 0$$

Now, compare the coefficient of $x^{k+m}$, we have

$$\left[(k+m)(k+m-1)+(k+m)-n^2\right]a_k + a_{k-2} = 0$$

$$\Rightarrow \left[(k+m)\{k+m\cancel{-1}\cancel{+1}\} - n^2\right]a_k + a_{k-2} = 0$$

$$\Rightarrow \left((k+m)^2 - n^2\right)a_k + a_{k-2} = 0$$

$$\Rightarrow a_k = -\frac{a_{k-2}}{(k+m+n)(k+m-n)} \quad \forall\ k = 0, 1, 2, \ldots$$

Now, compare coefficient of $x^{m+1}$ in eqn (8)

$$\{(m+1)m + (m+1) - n^2\}a_1 = 0$$

$$\Rightarrow \{(m+1)^2 - n^2\}a_1 = 0$$

$$\Rightarrow \{(m+1-n)(m+1+n)\}a_1 = 0$$

for $m = n$

$$(2n+1)a_1 = 0 \Rightarrow a_1 = 0$$

for $m = -n$

$$(-2n+1)a_1 = 0 \Rightarrow a_1 = 0$$

from eqn (P)

$$a_1 = a_3 = a_5 = \cdots = 0$$

$$a_2 = \frac{1}{(m+n-2)(m-n+2)}a_0$$

$$a_4 = -\frac{1}{(m+n+4)(m-n+4)}a_2$$

$$= \frac{1}{(m+n+4)(m-n+4)(m+n+2)}$$

$$= \frac{1}{\cancel{(m+n+4)(m+n+2)}(m-n+2)}a_0$$

$$Y(x) = \sum_{K=0}^{\infty} a_k x^{k+m}$$

$\therefore \; y(x) = x^m \sum_{k=0}^{\infty} a_k x^k$

$$= x^m \left\{ a_0 - \frac{1 \; a_0 x^2}{(m+n+2)(m-n+2)} + \frac{x^4}{(m+n+4)(m-n+4)(m+n+2)(m-n+2)} + \ldots \right\}$$

$$y(x) = a_0 x^m \left[ 1 - \frac{x^2}{(m+n+2)(m-n+2)} + \frac{x^4}{(m+n+4)(m-n+4)(m+n+2)(m-n+2)} + \ldots \right\} \longleftarrow (8)$$

Put $m = n$ and replace $a_0$ by $a$ in (8) —

$$y(x) = a x^n \left[ 1 - \frac{x^2}{2^2(n+1)} + \frac{x^4}{2^4(n+2)(n+1)} + \ldots \right]$$

$= a \cdot u$ (say)

Put $m = -n$ & replace $a_0$ by $b$ in eqn (8) —

$$y(x) = b x^{-n} \left[ 1 + \frac{x^2}{2^2(n-1)} + \frac{x^4}{4 \times 8 (n-1)(n-2)} + \ldots \right]$$

$= b v$ (say)

So, general soln of (1) is given by

$y = au + bv$

If we replace $a$ by $\frac{1}{2^n \Gamma(n+1)}$ in eqn (9)

then we have

$$y(x) = \frac{x^n}{2^n \Gamma(n+1)} \left[ 1 - \frac{x^2}{2^2(n+1)} + \frac{x^4}{48(n+2)(n+1)} - \ldots \right]$$

$$= \sum_{r=0}^{\infty} (-1)^r \frac{1}{r!\,\Gamma(n+r+1)} \left( \frac{x}{2} \right)^{n+2r} \longrightarrow (10)$$

$$= J_n(x)$$

where $J_n(x)$ is Bessel $f^n$ of order $n$.

If we replace $b$ by $\frac{1}{2^{-n}\Gamma(-n+1)}$ in eq$^n$ (10)

then, we have

$$y(x) = \frac{x^{-n}}{2^{-n}\Gamma(-n+1)}\left[1 - \frac{x^2}{2^2(-n+1)} + \frac{x^4}{48(-n+2)(-n+1)} - \ldots\right]$$

$$= \sum_{r=0}^{\infty} (-1)^r \frac{1}{\lfloor r \, \Gamma(-n+r+1)} \left(\frac{x}{2}\right)^{-n+2r}$$

$$= J_{-n}(x)$$

Here we can see that $J_n x$ & $J_{-n}(x)$ are ~~two~~ dependent sol of Bessel eqn

~~So~~, general eqn is given by

$$\boxed{Y = A J_n(x) + B J_{-n}(x)}$$

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* * *

# $9x(1-x)y'' - 12y' + 4y = 0$ — (1)

$$Y(x) = \sum_{k=0}^{\infty} a_k x^{k+m}, \quad a_0 \neq m \quad \text{— (2)}$$

$$\boxed{m = 0, \ 7/3}$$

Not memo

$$Y'(x) = \sum_{k=0}^{\infty} (k+m) a_k x^{k+m-1} \quad \text{— (3)}$$

$$Y''(x) = \sum_{k=0}^{\infty} (k+m)(k+m-1) a_k x^{k+m-2} \quad \text{— (4)}$$

Using eq$^n$ (2), (3) & (4) in (1) ⟶

$$9x(1-x)\sum_{k=0}^{\infty} (k+m)(k+m-1) a_k x^{k+m-2} - 12\sum_{k=0}^{\infty} (k+m) a_k x^{k+m-1}$$

$$+ 4\sum_{k=0}^{\infty} a_k x^{k+m} = 0$$

$$\Rightarrow \sum_{k=0}^{\infty}\{-9(k+m)(k+m-1)a_k x^{k+m} + 4a_k x^{k+m}\} + \sum_{k=0}^{\infty}\{9(k+m)(k+m-1)a_k x^{k+m-1} - 12(k+m)a_k x^{k+m-1}\} = 0$$

Replacing $k$ by $k-1$ —

$$\Rightarrow \sum_{k=0}^{\infty}\{-9(k+m-1)(k+m-2)a_{k-1}x^{k+m-1} + 4a_{k-1}x^{k+m-1}\} + \sum_{k=0}^{\infty} 3(k+m)\{3k+3m-3-4\}a_k x^{k+m-1} = 0$$

Now equating to zero the coeff of $x^{k+m-1}$, we have —

$$\{-9(k+m-1)(k+m-2)+4\}a_{k-1} + \{3(k+m)(3k+3m-7)\}a_k = 0$$

$$\Rightarrow 3(k+m)(3k+3m-7)a_k - \{9(k+m-1)(k+m-2)-4\}a_{k-1} = 0$$

$$\Rightarrow \cancel{3(k+m)}\{\cancel{(3k+3m-7)}a_k -$$

$$\{9(k+m-1)(k+m-2)-4\}$$

$$= (3k+3m-3)(3k+3m-6)-4$$

$$= (3k+3m)^2+18-4-9(3k+3m)$$

$$= (3k+3m-2)(3k+3m-7)$$

$\cancel{2(3k+3m-2)(3k+3m}$

$$\Rightarrow 3(k+m)(3k+3m-7)a_k = (3k+3m-2)(3k+3m-7)a_{k-1}$$

$$\Rightarrow a_k = \frac{(3k+3m-2)\cancel{(3k+3m-7)}}{3(k+m)\cancel{(3k+3m-7)}}a_{k-1}$$

$$= \frac{3k+3m-2}{3(k+m)}a_{k-1}$$

$$a_1 = \frac{3m+1}{3(m+1)} a_0$$

$$a_2 = \frac{3m+4}{3(m+2)} a_1 = \frac{(3m+4)}{3(m+2)} \cdot \frac{(3m+1)}{3(m+1)} a_0$$

$$a_3 = \frac{3m+7}{3(m+3)} a_2 = \frac{(3m+7)(3m+4)(3m+1)}{3^3(m+3)(m+2)(m+1)} a_0$$

from eqn (1), soln is —

$$y(x) = \sum_{k=0}^{\infty} a_k x^{k+m} = x^m \sum_{k=0}^{\infty} a_k x^k$$

$$= x^m \left[ a_0 + \frac{3m+1}{3(m+1)} a_0 x + \frac{(3m+1)(3m+4)}{3^2(m+1)(m+2)} x^2 + \frac{(3m+1)(3m+4)(3m+7)}{3^3(m+1)(m+2)(m+3)} x^3 + \cdots \right]$$

$$y(x) = a_0 x^m \left[ 1 + \frac{3m+1}{3(m+1)} x + \frac{(3m+1)(3m+4)}{3^2(m+1)(m+2)} x^2 + \frac{(3m+1)(3m+4)(3m+7)}{3^3(m+1)(m+2)(m+3)} x^3 + \cdots \right] \quad (6)$$

put $m=0$, and replace $a_0$ by $a$ in eqn (6)

$$y(x) = a_0 \left[ 1 + \frac{x}{3} + \frac{2}{9} x^2 + \frac{21}{\cancel{243}} \cdots \right]$$

Put $m = 7/3$, and replace $a_0$ by $b$ in eqn (6), we have —

$$y(x) = b x^7 \left[ 1 + \frac{4}{5} x + \frac{88}{130} x^2 + \cdots \right]$$

So general soln is given by —

$$\boxed{y(x) = au + bv}$$

## Some Useful Links:

**1. Free Maths Study Materials (**https://pkalika.in/2020/04/06/free-maths-study-materials/**)**

**2. BSc/MSc Free Study Materials (**https://pkalika.in/2019/10/14/study-material/**)**

**3. MSc Entrance Exam Que. Paper: (**https://pkalika.in/2020/04/03/msc-entrance-exam-paper/**)**
[JAM(MA), JAM(MS), BHU, CUCET, ...etc]

**4. PhD Entrance Exam Que. Paper: (**https://pkalika.in/que-papers-collection/**)**
[CSIR-NET, GATE(MA), BHU, CUCET,IIT, NBHM, ...etc]

**5. CSIR-NET Maths Que. Paper: (**https://pkalika.in/2020/03/30/csir-net-previous-yr-papers/**)**
[Upto 2019 Dec]

**6. Practice Que. Paper: (**https://pkalika.in/2019/02/10/practice-set-for-net-gate-set-jam/**)**
[Topic-wise/Subject-wise]

7. **List of Maths Suggested Books** (https://pkalika.in/suggested-books-for-mathematics/)

**8. CSIR-NET Mathematics Details Syllabus (**https://wp.me/p6gYUB-Fc)

**9. Free Video Lectures for CSIR-NET, GATE, SET, Asst. Prof. ..etc**
https://www.youtube.com/pkalika