Classical Mechanics

(Handwritten Study Material for MSc, GATE, NET...etc)

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Your Notes/Remarks

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Classical Mechanics.

Our universe consists of material bodies which are in constant interaction and motion. The mechanical form of motion of body is studied in mechanics.

There are many fields of study in mechanics such as —

Size ↑ / Speed →	far less than 3×10^8 m/s	comparable to 3×10^8 m/s
for larger than 10^{-9} m	Classical Mechanics.	Relativistic Mechanics.
near or less than 10^{-9} m	Quantum Mechanics.	Quantum field theory.

Also, continuum Mechanics, space mechanics and so on.

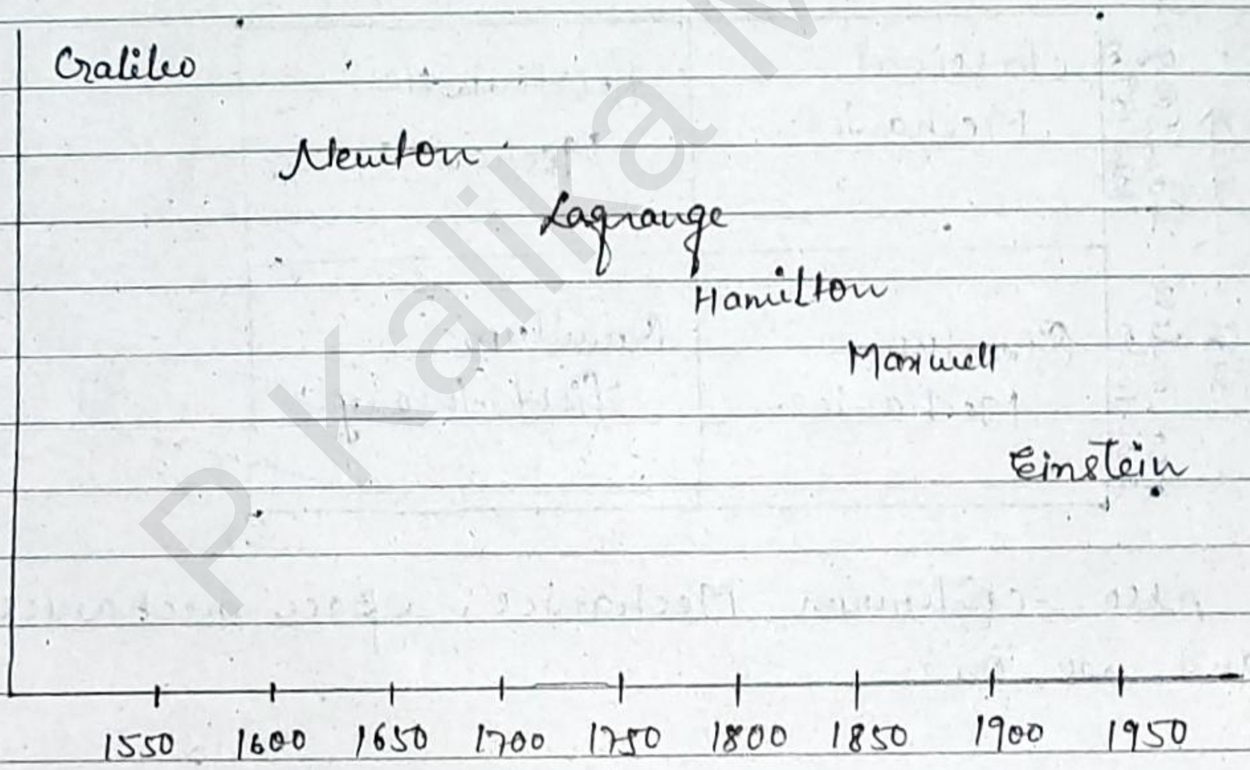
→ classical mechanics (also known as Newtonian mechanics) is usually sub-divided into three parts —

(1) Kinematics
(2) Statics
(3) Dynamics.

(1) Kinematics : Considering motion of bodies without considering the factors causing their motion.

(2) Statics : Deals with the laws of equilibrium of bodies.

(3) Dynamics : Study of laws of motion of bodies and the causes producing their motion



Development of concept of Mechanics.

31st October 2020.

Degree of freedom:

The degree of freedom (DoF) of a mechanical system is the number of independent parameters that define its configuration or state.

Eg: The position of a single railcar (engine) moving along a track has one degree of freedom because the position of the car is defined by the distance along the track.

(OR)

Degree of freedom is ~~depend~~ defined as the minimum number of independent variables required to define the position of a rigid body in space. In other words DoF defines the number of directions a body can move. Degree of freedom concept is used in kinematics to calculate the dynamics of a body.

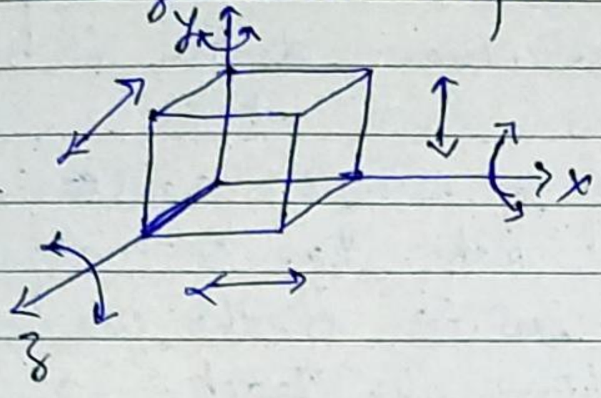
If DoF > 0 It's a mechanism.

If DoF ≤ = 0 It's a structure

If DoF < 0 It's a pre-loaded structure

Degree of freedom of a rigid body in space:

An ~~ideal~~ rigid body in space has six degrees of freedom: three translating motions along the x, y, ~~z~~ and z axes and three rotary motion around the x, y and z axes respectively.



formula:

for the N numbers of particles moving freely in d dimensional space degrees of freedom is represented by the following equation

$$dof = Nd$$

If there are constraints then

$$\boxed{dof = Nd - k}$$

where k is number of constraints

(OR) Degree of freedom is the number of independent ways in which a mechanical system can move ~~without~~ without violating the prescribed constraints on the system. ~~is~~ ~~kno~~

Eg:- If a particle moving freely in 3-D space then

$$dof = 3n - k \quad , \quad d = 3\text{-D space}$$
$$k = \text{constraints}.$$

Generalized Coordinates

- To describe the configuration of a system, we naturally select the smallest possible number of variables, which we call generalized coordinates of the system.

- The number of degrees of freedom coincides with the minimum number of independent ~~cor~~ coordinates necessary to describe a system uniquely.

- While selecting generalized coordinates, we should follow these three important point—

(i) They should determine the system completely.

(ii) They may be varied arbitrarily and independently of each other without violating the constraints on the system.

(iii) The generalized coordinates are conventionally denoted by the symbols $q_1, q_2, \cdots q_n$.

(They may not have physical meaning, nor are they unique. They should give us a reasonable mathematical simplification).

<u>Example-1</u> when a particle is moving in a plane, we may describe its motion in cartesian coordinates or, more conveniently, in polar coordinates (r, θ), we may then

$$q_1 = x, \quad q_2 = y.$$

or,

$$q_1 = r = (x^2+y^2)^{1/2}, \quad q_2 = \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Alternatively, we may also write as

$$x = q_1 \cos q_2, \quad y = q_1 \sin q_2.$$

or

$$x_i = x_i(q_1, q_2).$$

If the problem involves spherical symmetry, we may conveniently use spherical polar coordinates (r, θ, ϕ) as (q_1, q_2, q_3) s.t

$$x = r \sin\theta \cdot \cos\phi$$
$$y = r \sin\theta \cdot \sin\phi$$
$$z = r \cos\theta.$$

or,
$$x = q_1 \sin q_2 \cdot \cos q_3$$
$$y = q_1 \sin q_2 \sin q_3$$
$$z = q_1 \cos q_2.$$

or,
$$x_i = x_i(q_1, q_2, q_3).$$

In general, the relation b/w the old cartesian coordinates and the generalized coordinates can be written in the form $x_i = x_i(q_1, q_2, \ldots q_n)$.

Example – 2 / A rigid body moving freely in space has six degrees of freedom.

Example – 3 / A rod of mass m and length l which is free to move in any direction on a frictionless inclined plane has only 1 dof, as it can slide down vertically.

6.2 / Classification of a dynamical system.

There is an eight fold classification of all known dynamical system, they are —

(1,2) Scleronomic and Rheonomic System :

A configuration of the scleronomic system is given, when the values of the generalized coordinates $q_1, q_2, \dots q_n$ are given by the equations of the form

$$x_i = x_i(q_1, q_2, \dots q_n)$$

which do not depend explicitly on time.

In rheonomic system, it is necessary to specify the time t as well. Then the equations will be of the form

$$x_i = x_i(q_1\, q_2, \dots q_n, t).$$

Thus, in a scleronomic system, there will be only fixed constraints, whereas in a a rheonomic system there will be moving constraints.

(3,4): Conservative and non-Conservative System

In a conservative system, the generalized forces are derivable from a potential energy ~~V~~ ($V = V(q_1, q_2, \dots q_n)$) i.e if $Q_1, Q_2, \dots Q_n$ represents generalized forces then

$$Q_1 = -\frac{\partial V}{\partial q_1}, \quad Q_2 = -\frac{\partial V}{\partial q_2}, \dots \dots Q_n = -\frac{\partial V}{\partial q_n}$$

or,

$$Q = -\nabla \vec{V}$$

otherwise, the system is non-conservative.

(5,6): Holonomic and non-holonomic system.

Let $q_1, q_2, \dots, q_n$ denote the generalized coordinates, describing a system and let t denotes the time and if all the constraints of the system can be expressed in ~~at~~ equation of the form

$$f_i(q_1, q_2, \dots, q_n, t) = 0$$

which is independent of velocities $\dot{q}_1, \dot{q}_2, \dots \dot{q}_n$, where i is the ith component, then the system is called holonomic.

However, if the constraints cannot be expressed as relations among the generalized coordinates, then the system is ~~refree~~ called as non-holonomic.

(7, 8): Bilateral and Unilateral Systems

If at any point on the constraint surface, both the forward and backward motions are possible, such a system is called bilateral.

↳ In this system, the constraint relations are in the form of equations but not in the form of inequalities.

However, if at some point on the constraint surface, the forward motion is not possible, while the constraint relation are expressed in the form of inequalities, then such a system is called unilateral.

Simple System: Systems which are scleronomic, conservative and holonomic are called simple system.
It is also known as general system.

Note: Scleronomic → Time independent
Rheonomic → Time dependent
Holonomic → Velocity independent
Non-holonomic → velocity dependent
Conservative → generalized force derived from potential energy.
Non-conservative → otherwise, it is non-conservative.

Bilateral → If forward & backward motion is possible
↳ constraint relation are in the form of equation

Unilateral → If forward motion is not possible, only backward motion is possible.
↳ Constraint relation are in the form of inequalities.

* Lagrange's Equation for Simple System

Consider a simple dynamical system consisting of n-particles. Let m_i be the mass of a typical particle and $\vec{r_i}$ be its position vector. Suppose our simple system has n degrees of freedom and n generalized coordinates q_l $(l = 1, 2, \dots n)$. Then, the position vector $\vec{r_i}$ is a function of q_l, which can be expressed as (scleronomic system)

$$\vec{r_i} = \vec{r_i}(q_1, q_2, \dots, q_n) \qquad \text{--- (1)}$$

where q_l explicitly depends on t. As the system moves in a specified manner, we have by using the chain rule of partial differentiation that

$$\frac{d\vec{r_i}}{dt} = \sum_{l=1}^{n} \frac{\partial \vec{r_i}}{\partial q_l} \frac{dq_l}{dt}$$

$$\Rightarrow \dot{\vec{r_i}} = \sum_{l=1}^{n} \frac{\partial \vec{r_i}}{\partial q_l} \dot{q_l} \qquad \text{--- (2)}$$

Here the quantities $\dot{q}_\ell$ are called generalized velocities and, therefore we may write

$$\vec{\dot{r}_i} = \vec{v_i}(q_1, q_2, \ldots q_n, \dot{q}_1, \dot{q}_2, \ldots \dot{q}_n) \quad \text{---(3)}$$

which means that we have a function of $2n$ quantities $q_\ell, \dot{q}_\ell$.

Differentiating eq^n(2) partially with respect to $\dot{q}_\ell$, we get

$$\frac{\partial \vec{\dot{r}_i}}{\partial \dot{q}_\ell} = \sum_{\ell=1}^{n} \frac{\partial \vec{r_i}}{\partial q_\ell} \cdot \frac{\partial \dot{q}_\ell}{\partial \dot{q}_\ell}$$

$$\Rightarrow \quad \frac{\partial \vec{\dot{r}_i}}{\partial \dot{q}_\ell} = \sum_{\ell=1}^{n} \frac{\partial \vec{r_i}}{\partial q_\ell} \cdot 1$$

$$\Rightarrow \quad \boxed{\frac{\partial \vec{\dot{r}_i}}{\partial \dot{q}_\ell} = \frac{\partial \vec{r_i}}{\partial q_\ell}} \quad \text{---(4)}$$

(Since all other quantities are treated as constant)

This result is called cancellation of dots.

Now, let us consider a coordinate q_s $(1 \le s \le n)$ {i.e q_s must be equal to some q_ℓ $(\ell = 1, 2, \ldots n)$} then differentiating (2) with respect to q_s then

$$\frac{\partial \vec{\dot{r}_i}}{\partial q_s} = \sum_{\ell=1}^{n} \frac{\partial^2 \vec{r_i}}{\partial q_s \, \partial q_\ell} \cdot \dot{q}_\ell \quad \text{---(5)}$$

At the same time it may be denoted as that

from (1) $\Rightarrow$ $\frac{d}{dt}\left(\frac{\partial \vec{r_i}}{\partial q_s}\right) = \sum_{\ell=1}^{n} \frac{\partial}{\partial q_\ell}\left(\frac{\partial \vec{r_i}}{\partial q_s}\right) \frac{dq_\ell}{dt}$

$$= \sum_{\ell=1}^{n} \frac{\partial^2 \vec{r_i}}{\partial q_\ell \partial q_s} \dot{q}_\ell \quad \text{— (6)}$$

Now comparing eq^n (5) and 6, we get

$$\frac{d}{dt}\left(\frac{\partial \vec{r_i}}{\partial q_s}\right) = \frac{\partial \dot{\vec{r_i}}}{\partial q_s} \quad \text{— (7)}$$

— This result, which is equally important is called "Interchange of d and ∂".

— These two mathematical tricks, i.e "Cancellation of dots and interchange of d and ∂" are widely used in Lagrangian method.

The kinetic energy of the system considered can be written as

$$T = \sum_{i=1}^{n} \frac{1}{2} m_i (\dot{\vec{r_i}} \cdot \dot{\vec{r_i}}) = T(\vec{q}, \dot{\vec{q}}) \quad \text{— (8)}$$

Differentiating the expression, partially w.r.t q_ℓ, we get

$$\frac{\partial T}{\partial q_\ell} = \sum_{i=1}^{n} m_i \dot{\vec{r_i}} \frac{\partial \dot{\vec{r_i}}}{\partial q_\ell} \quad \text{——(9)}$$

Similarly, differentiating eq^n (8) with respect to $\dot{q}_\ell$, we obtain, on using cancellation of dot.

$$\frac{\partial T}{\partial \dot{q}_\ell} = \sum_{i=1}^{n} m_i \dot{\vec{r_i}} \cdot \frac{\partial \dot{\vec{r_i}}}{\partial \dot{q}_\ell} = \sum_{i=1}^{n} m_i \dot{\vec{r_i}} \frac{\partial \vec{r_i}}{\partial q_\ell} \quad \text{——(10)}$$

Now differentiating the above eq^n, w.r.t 't' we get

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\ell}\right) = \sum_{i=1}^{n} m_i \ddot{\vec{r_i}} \frac{\partial \vec{r_i}}{\partial q_\ell} + \sum_{i=1}^{n} m_i \dot{\vec{r_i}} \frac{d}{dt}\left(\frac{\partial \vec{r_i}}{\partial q_\ell}\right)$$

Interchanging d and ∂, we have,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\ell}\right) = \sum_{i=1}^{n} m_i \ddot{\vec{r_i}} \frac{\partial \vec{r_i}}{\partial q_\ell} + \sum_{i=1}^{n} m_i \dot{\vec{r_i}} \frac{\partial \dot{\vec{r_i}}}{\partial q_\ell} \quad \text{——(11)}$$

Subtracting eq^n (9) from (11) we get

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\ell}\right) - \frac{\partial T}{\partial q_\ell} = \sum_{i=1}^{n} m_i \ddot{\vec{r_i}} \frac{\partial \vec{r_i}}{\partial q_\ell} + \sum_{i=1}^{n} m_i \dot{\vec{r_i}} \frac{\partial \dot{\vec{r_i}}}{\partial q_\ell} - \sum_{i=1}^{n} m_i \dot{\vec{r_i}} \frac{\partial \dot{\vec{r_i}}}{\partial q_\ell}.$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\ell}\right) - \frac{\partial T}{\partial q_\ell} = \sum_{i=1}^{n} m_i \ddot{\vec{r_i}} \frac{\partial \vec{r_i}}{\partial q_\ell} \quad \text{——— (12)}$$

It may be observed that, this result is obtaining from, kinematics without using Newton's law of motion.

Suppose, $\vec{F_i}$ is the total force (applied force + internal force) acting on an i^{th} particle, then,

Newton's second law gives

$$m_i \ddot{\vec{r_i}} = m_i \vec{a_i} = \vec{F_i} \quad \text{——— (13)}$$

Now eq^n (12) can be written as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\ell}\right) - \frac{\partial T}{\partial q_\ell} = \sum_{i=1}^{n} \vec{F_i} \cdot \frac{\partial \vec{r_i}}{\partial q_\ell} = Q_\ell \ (\text{say}) \text{—(14)}$$

This is one form of Lagrange's equation, where Q_ℓ are ~~generated~~ generalized force.

$\longrightarrow$ Now, consider a set of arbitrary infinitesimal increment δq_ℓ.

In view of eq^n 1 they give corresponding displacement to the particle a specified below :

$$\delta r_i = \sum_{\ell=1}^{n} \frac{\partial r_i}{\partial q_\ell} \cdot \delta q_\ell .$$

Then, the work done due to these displacement can be expressed as

$$\delta W = \sum_{i=1}^{n} \vec{F_i} \cdot \delta \vec{r_i} = \sum_{f=1}^{n} \left[\sum_{i=1}^{n} \vec{F_i} \cdot \frac{\partial \vec{r_i}}{\partial q_f} \right] \delta q_f \quad \text{——(15)(a)}$$

It may also be noted that the system has a potential energy V, which is a function of q's only. which may I write as

$$V = V(\vec{q})$$

Hence the work done in the above displacement can be expressed as

$$\delta W = -\delta V = \sum_{f=1}^{n} \frac{\partial V}{\partial q_f} \cdot \delta q_f \quad \text{——(15)(b)}$$

Comparing eq^n (15)(a) and 15(b) and noting that δq_f are arbitrary, we obtain.

$$\sum_{i=1}^{n} \vec{F_i} \cdot \frac{\cancel{\delta} \vec{r_i}}{\cancel{\partial}} \frac{\partial \vec{r_i}}{\partial q_f} = -\frac{\partial V}{\partial q_f} \quad \text{——(16)}$$

Consequently, eq^n (14) can be written as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_f}\right) - \frac{\partial T}{\partial q_f} = \frac{-\partial V}{\partial q_f} \quad \text{——(A)}$$

Now, defining the Lagrangian (L) as.

$$\boxed{L = T - V} = L(\vec{q}, \dot{\vec{q}}) \quad \text{———} (18)$$

Nothing, that V is a function of $\vec{q}$ only, we can write,

$$\frac{\partial L}{\partial \dot{q}_p} = \frac{\partial T}{\partial \dot{q}_p}$$

and $$\frac{\partial L}{\partial q_p} = \frac{\partial T}{\partial q_p} - \frac{\partial V}{\partial q_p}.$$

Substituting these expressions in eq^n (17), we arrive at the following final form

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_p}\right) - \frac{\partial L}{\partial q_p} = 0} \quad \text{———} (19)$$

which is known as the Lagrange's equation of motion for a conservative, Scleronomic (as there are no moving constraints) and holonomic. (As we can vary the generalized coordinates arbitrarily without violating the constraints) system.

In other word 19 represents the Lagrange's equation of motion for a simple system.

These are n – ordinary differential equation of second order and hence their solution contains 2n arbitrary constants, which can

~~by considering~~ be determined ~~by~~ by considering the initial conditions at $t=0$ on both the generalized coordinates q_p and generalized velocities $\dot{q}_p$.

In case of a non-conservative system, the Lagrange's equation of motion assumes the form

$$\boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_p}\right) - \frac{\partial T}{\partial q_p} = Q_p} \qquad \text{——— (20)}$$

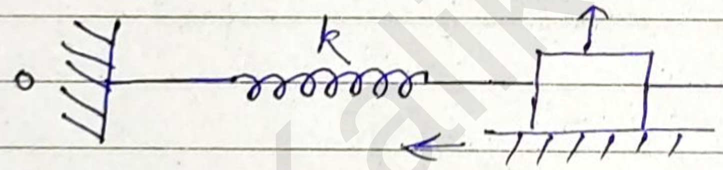
Where Q_p are generalized forces.

The Harmonic Oscillator

Many problem of oscillation can be discussed through a single mathematical model called Harmonic Oscillator.

A system undergoing periodic steddy-state motion under the action of a spring is called a Harmonic oscillator.

The Harmonic oscillator consists of a spring and a particle which can move on a straight line, which we shall take for convenience as x-axis.



(The Harmonic Oscillator).

The particle is attracted towards the origin by a contralling force $-kx\hat{i}$ (varying as the distance), where $\hat{i}$ is the unit vector in the positive direction of the x-axis and k is the spring constant.

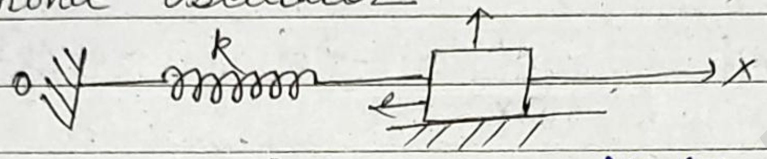
The eq^n of motion for this simple harmonic oscillator is given by

$$\boxed{m\frac{d^2x}{dt^2} = m\ddot{x} = -kx} \quad , k>0$$

It's general solution is found to be

$$x(t) = A \cos \omega t + B \sin \omega t .$$

Example 6.5 Use the Lagrangian method to obtain the equation of motion for one-dimensional harmonic oscillator.

Solⁿ:— 

(Harmonic Oscillator)

The harmonic oscillator consists of a spring and a particle, which can move on a straight line, which we shall take as x-axis.

The particle is attracted towards the origin by a controlling force, $-kx\hat{i}$, where k is spring constant.

The Harmonic oscillator is a simple system that is scleronomic, conservative and holonomic system. Hence, the kinetic energy is given by

$$T = \frac{1}{2} m\dot{x}^2 \quad \text{——— (1)}$$

The potential energy is

$$V = -\int f(x)\, dx = -\int -Kx\, dx = \frac{1}{2}Kx^2 + \text{Constant}$$

If we choose the horizontal plane passing through the position of equilibrium, then $V=0$, $x=0$, which gives constant of integration as zero.

This, $V = \frac{1}{2} kx^2$ ——— (2)

Now, we define the Lagrangian (1) as

$$L = T - V = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} kx^2$$

and get

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} \quad , \quad \frac{\partial L}{\partial x} = -kx \quad \text{——— (3)}$$

Hence, the Lagrangian equation of motion for a harmonic oscillator is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \quad \text{——— (4)}$$

using (3) & in (4) give

$$\frac{d}{dt}(m\dot{x}) + kx = 0$$

$$\Rightarrow \boxed{m\ddot{x} + kx = 0} \quad \text{——— (5)}$$

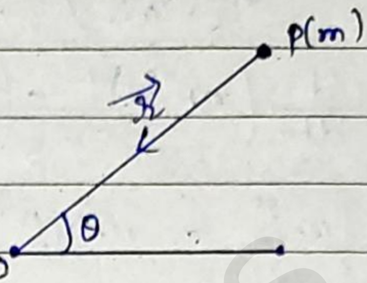
This is the Lagrangian eq^n of motion for a 1-D harmonic oscillator.

Example 6.6/ Use the Lagrangian method and obtain the equation of motion for planetary motion (Kepler's Problem).

Sol^n:— Here, we consider a particle of mass m, moving in a plane and attracted towards the origin O of coordinates with a force proportional to the inverse square of the distance from it

i.e

$$\vec{F} = \frac{-m\mu}{r^2}$$



If (r, θ) be the plane polar coordinates of the particle at a given instant, then its kinetic energy is given by

$$T = \frac{1}{2} m v^2 = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) \quad \text{———} (1)$$

while its potential energy is

$$V = -\int f(r)\,dr = \int \frac{m\mu}{r^2}\,dr = \frac{-m\mu}{r} \quad \text{———} (2)$$

Taking (r, θ) as generalized coordinates, we define the Lagrangian

$$L = T - V = \frac{1}{2}(m\dot{r}^2 + r^2\dot{\theta}^2) + \frac{m\mu}{r} \quad \text{———} (3)$$

which gives

$$\left.\begin{aligned} \frac{\partial L}{\partial \dot{r}} &= m\dot{r}, \quad \frac{\partial L}{\partial r} = mr\dot{\theta}^2 - \frac{m\mu}{r^2} \\ \frac{\partial L}{\partial \dot{\theta}} &= mr^2\dot{\theta}, \quad \frac{\partial L}{\partial \theta} = 0 \end{aligned}\right\} \quad \text{———} (4)$$

Thus, the Lagrangian equation for the planetary or central force motion can be written as follows:

(i) r – equation is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0$$

$$\Rightarrow \frac{d}{dt}(m\dot{r}) - \left(mr\dot{\theta}^2 - \frac{m\mu}{r^2}\right) = 0$$

$$\Rightarrow m\ddot{r} - mr\dot{\theta}^2 + \frac{m\mu}{r^2} = 0$$

$$\Rightarrow \boxed{m(\ddot{r} - r\dot{\theta}^2) + \frac{m\mu}{r^2} = 0} \quad \text{——— (5)}$$

(ii) θ – equation is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

i.e $\frac{d}{dt}(mr^2\dot{\theta}) - 0 = 0$

On integration $\Rightarrow mr^2\dot{\theta} = \text{Constant}$

$$\Rightarrow \boxed{r^2\dot{\theta} = \text{Constant}} \quad \text{——— (6)}$$

Equation (5) and (6) constitute the ~~a~~ required result eq^n (6) can be interpreted to state that the angular momentum of the particle about the centre of attraction is a constant.

6.4 Principle of virtual work – D'Alembert's Principle.

The adjective 'virtual' means possible or permissible. According to Goldstein, "A virtual displacement (infinetesimal) of a system refers to a change in the configuration of the system as a result of any arbitrary infinetesimal change of coordinate δr_i, consistent with the forces and constraint imposed on the system at a given ~~ent~~ instant t, and does not involve a change in time.

The displacement δr_i is called virtual to distinguish it ~~form a~~ from an actual displacement dr_i of the system occuring in a time interval dt, during which the forces and constraints may be changing".

Let us consider a system and any motion of the system during which it remains in equilibrium. Suppose, we apply the work energy principle $\Delta W = \Delta T$, to the system. Since each part of the system moves with at most a constant velocity, the change in kinetic energy, i.e $\Delta T = 0$. Thus for a system in equilibrium, the work done by all forces on the system is zero.

In otherwords, if the system is in equilibrium, then the total forces $\vec{F_i}$ acting on each particle of the system is zero.

Hence, the virtual work of the force $\vec{F_i}$ in the displacement $\delta\vec{r_i}$ is given by the dot product.

$$\vec{F_i} \cdot \delta\vec{r_i} = 0$$

∴ the sum over all particles is zero.

i.e $$\sum \vec{F_i} \cdot \delta\vec{r_i} = 0 \quad \text{——— (1)}$$

Suppose we decompose $\vec{F_i}$ into a sum of applied force $\vec{F_i}^{(a)}$ and the force of constraint f_i then

$$\vec{F_i} = \vec{F_i}^{(a)} + \vec{f_i}$$

∴ eqn (1) becomes $\Rightarrow \sum \left(\vec{F_i}^{(a)} + \vec{f_i}\right) \cdot \delta\vec{r_i} = 0$

$$\sum \vec{F_i}^{(a)} \cdot \delta\vec{r_i} + \sum \vec{f_i} \cdot \delta\vec{r_i} = 0 \quad \text{——— (2)}$$

Assume that the net virtual work of the forces of constraints is zero, which in fact is true. for, if a rigid body is constrained to move, on a surface, while the virtual displacement is tangential to it and hence, the virtual work vanishes.

For the condition of equilibrium, we therefore have

$$\boxed{\sum \vec{F_i}^{(a)} \cdot \delta\vec{r_i} = 0} \quad \text{——— (3)}$$

This is called the ~~virtual~~ Principle of virtual work.

This is the approach to statics and indeed to mechanics. But, we want to have a principle involving general motion of the system. This was developed by D' Alembert', which is explained as follows:

Newton's second law states that the rate of change of linear momentum is equal to the applied forces.

Therefore, $\vec{F_i} = \dot{\vec{p_i}}$

$\therefore$ Eqn (1) $\Rightarrow \sum (\vec{F_i} - \dot{\vec{p_i}}) \cdot \delta \vec{r_i} = 0$ ——— (4)

If $\vec{F_i}$ is the applied forces acting on the i^{th} particle of mass m_i, then the above equation becomes,

$$\boxed{\sum \left(\vec{F_i}^{(a)} - \dot{\vec{p_i}}\right) \cdot \delta \vec{r_i} = 0} \quad \text{——— (5)}$$

This is often called D' Alembert's principle

6.5| Lagrange's Equation for General Systems.

Consider a general system of 'n' particles in the sense that if it is rheonomic, non-conservative and non-holonomic.

Let m_i be the mass of a typical particle whose position vector is $\vec{r_i}$.

Also, let $\vec{F_i}$ be the total force acting on a typical particle (the reactions of the constraints being included). Suppose r_i is a function of $q_1, q_2, \ldots, q_n, t$.

i.e $\vec{r_i} = \vec{r_i}(q_1, q_2, \ldots, q_n, t)$ —— (1)

or,

$$\vec{r_i} = \vec{r_i}(\vec{q}, t)$$

Here, the time t is included to account for moving constraints (rheonomic system).

Using chain rule of partial differentiation, we have

$$\frac{d\vec{r_i}}{dt} = \dot{\vec{r_i}} = \sum_{f=1}^{n} \frac{\partial \vec{r_i}}{\partial q_f} \cdot \frac{dq_f}{dt} + \frac{\partial \vec{r_i}}{\partial t}$$

$$\text{or, } \frac{d\vec{r_i}}{dt} = \dot{\vec{r_i}} = \sum_{f=1}^{n} \frac{\partial \vec{r_i}}{\partial q_f} \cdot \dot{q}_f + \frac{\partial \vec{r_i}}{\partial t} \quad \text{—— (2)}$$

diff (2) partially w.r.t q_β, we get

$$\frac{\partial \dot{\vec{r_i}}}{\partial q_\beta} = \sum_{f=1}^{n} \frac{\partial^2 \vec{r_i}}{\partial q_\beta \partial q_f} \cdot \dot{q}_f + \frac{\partial}{\partial q_\beta}\left(\frac{\partial \vec{r_i}}{\partial t}\right)$$

changing the order of differentiation, we get-

$$\frac{\partial \dot{\vec{r_i}}}{\partial q_\beta} = \sum_{f=1} \frac{\partial}{\partial q_f}\left(\frac{\partial \vec{r_i}}{\partial q_\beta}\right) + \frac{\partial}{\partial t}\left(\frac{\partial \vec{r_i}}{\partial q_\beta}\right) = \frac{d}{dt}\left(\frac{\partial \vec{r_i}}{\partial q_\beta}\right)$$

$$\therefore \frac{\partial \dot{\vec{r_i}}}{\partial q_\beta} = \frac{d}{dt}\left(\frac{\partial \vec{r_i}}{\partial q_\beta}\right) \quad \text{---} (3)$$

Now, differentiating partially eq^n(2) w.r.t $\dot{q}_f$ we have,

$$\frac{\partial \dot{\vec{r_i}}}{\partial \dot{q}_f} = \frac{\partial \vec{r_i}}{\partial q_f} \quad \text{---} (4)$$

The kinetic energy of the system T is given by

$$T = \frac{1}{2} \sum m_i \left(\dot{\vec{r_i}}^2\right) \quad \text{---} (5)$$

differentiating this result partially w.r.t $\dot{q}_f$, we obtain after using cancellation of dot.

$$\frac{\partial T}{\partial \dot{q}_f} = \frac{1}{2} \sum m_i \, 2\dot{\vec{r_i}} \frac{\partial \dot{\vec{r_i}}}{\partial \dot{q}_f}$$

$$\Rightarrow \frac{\partial T}{\partial \dot{q}_f} = \sum m_i \dot{\vec{r_i}} \frac{\partial \dot{\vec{r_i}}}{\partial \dot{q}_f}$$

$$\Rightarrow \frac{\partial T}{\partial \dot{q}_f} = \sum m_i \dot{\vec{r_i}} \frac{\partial \vec{r_i}}{\partial q_f} \quad \text{(using cancellation of dot)}$$

$$\therefore \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_f}\right) = \sum m_i \left(\ddot{\vec{r_i}} \frac{\partial \vec{r_i}}{\partial q_f} + \dot{\vec{r_i}} \frac{\partial \dot{\vec{r_i}}}{\partial q_f}\right)$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_f}\right) = \sum m_i \ddot{\vec{r_i}} \frac{\partial \vec{r_i}}{\partial q_f} + \sum m_i \dot{\vec{r_i}} \frac{\partial \dot{\vec{r_i}}}{\partial q_f} \quad \text{---} (6)$$

Again, differentiating (5) partially w.r.t. q_f, we get

$$\frac{\partial T}{\partial q_f} = \frac{1}{2} \sum m_i \, 2 \dot{\vec{r_i}} \frac{\partial \dot{\vec{r_i}}}{\partial q_f}$$

$$\Rightarrow \frac{\partial T}{\partial q_f} = \sum m_i \dot{\vec{r_i}} \frac{\partial \dot{\vec{r_i}}}{\partial q_f} \quad \text{---} (7)$$

Now, we define S_f by writing

$$S_f = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_f}\right) - \frac{\partial T}{\partial q_f} \quad \text{---} (8)$$

Substituting (6) and (7) in (8) we get.

$$S_f = \sum m_i \ddot{\vec{r_i}} \frac{\partial \vec{r_i}}{\partial q_f} + \sum m_i \dot{\vec{r_i}} \frac{\partial \dot{\vec{r_i}}}{\partial q_f} - \sum m_i \dot{\vec{r_i}} \frac{\partial \dot{\vec{r_i}}}{\partial q_f}$$

$$S_f = \sum m_i \ddot{\vec{r_i}} \frac{\partial \vec{r_i}}{\partial q_f}$$

If $\vec{F_i}$ is the force acting on m_i, then

$$m_i \ddot{\vec{r_i}} = \vec{F_i}$$

$$\Rightarrow \sum_{i=1}^{n} \left(m_i \ddot{\vec{r_i}} - \vec{F_i} \right) = 0$$

$\because$ the system is in equilibrium, using the principle of virtual work is

$$\sum_{i=1}^{n} \left(m_i \ddot{\vec{r_i}} - \vec{F_i} \right) \cdot \delta \vec{r_i} = 0 \quad \text{——(9)}$$

where $\delta \vec{r_i}$ is the virtual displacement.

from (1) it follows that

$$\delta \vec{r_i} = \sum_{p=1}^{n} \frac{\partial \vec{r_i}}{\partial q_p} \cdot \delta q_p \quad \text{——(10)}$$

Here, a virtual displacement is interpreted as a first variation of the function $\vec{r_i}$.

$\therefore$ Eq^n (9) becomes,

$$\sum_{i=1}^{n} \left(m_i \ddot{\vec{r_i}} - \vec{F_i} \right) \cdot \sum_{p=1}^{n} \left(\frac{\partial \vec{r_i}}{\partial q_p} \cdot \delta q_p \right) = 0$$

$$\Rightarrow \sum_{p=1}^{n} \left(\sum_{i=1}^{n} m_i \ddot{\vec{r_i}} \cdot \frac{\partial \vec{r_i}}{\partial q_p} \right) \delta q_p - \sum_{p=1}^{n} \left(\sum_{i=1}^{n} \vec{F_i} \cdot \frac{\partial \vec{r_i}}{\partial q_p} \right) \delta q_p = 0$$

$$\Rightarrow \sum_{p=1}^{n} \left(\sum_{i=1}^{n} m_i \ddot{\vec{r_i}} \cdot \frac{\partial \vec{r_i}}{\partial q_p} \right) \delta q_p = \sum_{p=1}^{n} \left(\sum_{i=1}^{n} \vec{F_i} \cdot \frac{\partial \vec{r_i}}{\partial q_p} \right) \delta q_p = 0 \quad \text{——(11)}$$

Now, if we define the generalized forces Q_p by

$$\sum_i \vec{F_i} \cdot \frac{\delta \vec{r_i}}{\partial q_p} = Q_p \quad \text{——— (12)}$$

then using (12) and S_p in (11)

$$\sum_{p=1}^{n} S_p \, \delta q_p = \sum_{p=1}^{n} Q_p \, \delta q_p$$

$$\Rightarrow \sum_{p=1}^{n} (S_p - Q_p) \cdot \delta q_p = 0 \quad \text{——— (13)}$$

If δq_p were arbitrary, we can conclude that

$$S_p = Q_p \quad \text{——— (14)}$$

which is same as Lagrange's equation.

Hamilton's Equation

Consider a dynamical system with n degrees of freedom whose behaviour obeys the Lagrange's equation of motion of the form.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_p}\right) - \frac{\partial L}{\partial q_p} = 0 \quad \text{——(1)}$$

where L is a Lagrangian and is a function of n quantities q_p.
Their derivatives $\dot{q}_p$ are w.r.t 't' and t itself.
Thus we write —

$$L = L(\vec{q}, \dot{\vec{q}}, t) \quad \text{——(2)}$$

In Hamiltonian formulation, we introduce a new independent variable called Generalized momenta p_p, defined as

$$p_p = \frac{\partial L(\vec{q}, \dot{\vec{q}}, t)}{\partial \dot{q}_p} = \frac{\partial L}{\partial \dot{q}_p} \quad \text{——(3)}$$

Like Lagrangian $L(\vec{q}, \dot{\vec{q}}, t)$, a new function of this formalism is Hamiltonian ~~$H(\vec{q}, \dot{\vec{q}}$~~ $H(\vec{q}, \vec{p}, t)$.

It means that there is a change of basis from $(\vec{q}, \dot{\vec{q}}, t)$ set to $(\vec{q}, \vec{p}, t)$ set.

In Hamiltonian Mechanics, q_p, p_p and t are independent variables and $\dot{q}_p$ is a dependent quantity.

i.e $\dot{q}_p = \dot{q}_p(\vec{q}, \vec{p}, H)$ ——(4)

The Hamiltonian function H is defined as

$$H = \sum_{p=1}^{n} \dot{q}_p \frac{\partial L}{\partial \dot{q}_p} - L$$

$$= \sum_{p=1}^{n} \dot{q}_p p_p - L \quad \text{——(5)}$$

To establish the physical significance of Hamiltonian. Let us consider a simple dynamical system, with $T = T(\vec{q}, \dot{\vec{q}})$, which does not contain time t explicitly

we can then show that—

$$T = \frac{1}{2} \sum_{i=1}^{n} m_i \left(\sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial \vec{r}_i}{\partial q_j} \cdot \frac{\partial \vec{r}_i}{\partial q_k} \dot{q}_j \dot{q}_k \right)$$

$$T = \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} a_{jk} \dot{q}_j \dot{q}_k \quad \text{——(6)}$$

Which is a Homogeneous Quadratic function in the generalized velocities.

But from eq^n (5) we can write

$$H = \sum_{p=1}^{n} \dot{q}_p \frac{\partial L}{\partial \dot{q}_p} - L$$

$\because$ $L = T - V = T(\vec{q}, \dot{\vec{q}}) - V(\vec{q})$

we have

$$\frac{\partial L}{\partial \dot{\vec{q}}} = \frac{\partial T}{\partial \dot{\vec{q}}} \quad (\because V \text{ is not a function of } \dot{\vec{q}})$$

$\therefore$ using Euler's theorem for Homogeneous function

$$H = \sum \dot{q}_p \frac{\partial T}{\partial \dot{\vec{q}}} - (T - V)$$

$$\Rightarrow H = 2T - (T - V)$$

$$\boxed{H = T + V} \quad \text{——(7)}$$

which is the sum of kinetic and potential energy.

<u>Hamilton's Canonical Equation of Motion.</u>

Hamilton is a function of generalized coordinates, generalized momenta and time

i.e $H = H(\vec{q}, \vec{p}, t)$

The total differential of H. is given by

$$dH = \sum_{p=1}^{n} \frac{\partial H}{\partial q_p} dq_p + \sum_{p=1}^{n} \frac{\partial H}{\partial p_p} dp_p + \frac{\partial H}{\partial t} dt \quad \text{—(1)}$$

from the def^n of Hamiltonian function is given by

$$H = \sum_{p=1}^{n} \dot{q}_p p_p - L(\vec{q}, \dot{\vec{q}}, t).$$

then we can also write its total differential in the form

$$dH = \sum_{f=1}^{n} \dot{q}_f \, dp_f + \sum_{f=1}^{n} p_f \, d\dot{q}_f - \sum_{f=1}^{n} \frac{\partial L}{\partial q_f} dq_f - \sum_{f=1}^{n} \frac{\partial L}{\partial \dot{q}_f} d\dot{q}_f - \frac{\partial L}{\partial t} dt \quad \text{——(2)}$$

from the def^n of generalized momenta $p_f = \frac{\partial L}{\partial \dot{q}_f}$

Hence the 2^nd and 4^th Term of the R.H.S of (2) cancel and we are left with.

$$dH = \sum_{f=1}^{n} \dot{q}_f \, dp_f - \sum_{f=1}^{n} \frac{\partial L}{\partial q_f} dq_f - \frac{\partial L}{\partial t} dt \quad \text{——(3)}$$

Let us now define the generalized conjugate momenta as

$$\dot{p}_f = \frac{\partial L}{\partial q_f} \quad \text{——(4)}$$

Eq^n (3) becomes

$$dH = \sum \dot{q}_f \, dp_f - \sum \dot{p}_f \, dq_f - \frac{\partial L}{\partial t} dt \quad \text{——(5)}$$

Now comparing eq^n (1) and (5), we get

$$\left.\begin{aligned} \dot{p}_f &= \frac{\partial L}{\partial q_f} = -\frac{\partial H}{\partial q_f} \\ \dot{q}_f &= \frac{\partial H}{\partial p_f} \end{aligned}\right\} \quad \text{——(6)}$$

and $\frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t}$ ——— (7)

Eq^n (6) are called Hamilton's Canonical Equation of motion.

These are 2n differential equation of first order.

Eq^n (6) can also be rewritten as

$$\frac{dq_f}{\partial H/\partial p_f} = \frac{dp_f}{-\partial H/\partial q_f} = dt \quad \text{——— (8)}$$

Example 6.12/ Use Hamilton method and obtain the equation of motion for one dimensional harmonic oscillator.

Sol^n:- $\because T = \frac{1}{2} m\dot{x}^2$ ——— (1)

$$V = \frac{1}{2} kx^2 \quad \text{——— (2)}$$

$$L = T - V = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} kx^2 \quad \text{——— (3)}$$

Taking $q = x$, the generalized momenta p is found to be

$$p = \frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \quad \text{——— (4)}$$

The Hamiltonian

$$H = \sum \dot{q}_f p_f - L = p\dot{x} - \left(\frac{1}{2} m\dot{x}^2 - \frac{1}{2} kx^2\right)$$

using eq^n (4), we get

$$H = \frac{p^2}{m} - \frac{p^2}{2m} + \frac{1}{2}kx^2$$

$$= \frac{p^2}{2m} + \frac{1}{2}kx^2 \quad \text{———(5)}$$

Hamilton Canonical equations of motion

$$\frac{\partial H}{\partial q_f} = -\dot{p}_f \quad , \quad \frac{\partial H}{\partial p_f} = \dot{q}_f$$

Therefore, $\frac{\partial H}{\partial x} = -\dot{p}$, $\frac{\partial H}{\partial p} = \dot{x}$ ———(6)

using eq^n (5), we get

$$kx = -\dot{p} \; , \quad \frac{p}{m} = \dot{x} \quad \text{———(7)}$$

These are two first-order differential equation Eliminating p between them, we obtain

$$-m\ddot{x} = kx \quad \text{or,} \quad m\ddot{x} + kx = 0 \quad \text{———(8)}$$

This is the required equation of motion.

Example 6.13| A mass m is placed on a frictionless plane, which is tangential to the surface of the earth as shown. Determine the equation of motion using Hamilton's method, taking x as the independent generalized coordinates.

Sol^n:— The kinetic energy T of the system is

$$T = \frac{1}{2} m\dot{x}^2 \quad \text{——— (1)}$$

While the potential energy V is given by

$$V = lmg - Rmg = \left(\sqrt{x^2+R^2} - R\right)mg \quad \text{——— (2)}$$

Since $q = x$, the generalized momenta is

$$p_x = \frac{\partial L}{\partial \dot{x}} = \frac{\partial(T-V)}{\partial \dot{x}} = m\dot{x} \quad \text{——— (3)}$$

But

$$H = \sum \dot{q}_f p_f - L = m\dot{x}^2 - \frac{1}{2} m\dot{x}^2 + V$$

$$= \frac{m\dot{x}^2}{2} + V$$

$\therefore$ using (3), we find

$$H = \frac{p_x^2}{2m} + \left(\sqrt{x^2+R^2} - R\right) mg$$

Now, ~~apply~~ applying Hamilton's canonical equation,

$$\left.\begin{aligned} \dot{x} &= \frac{\partial H}{\partial p_x} = \frac{p_x}{m} \\ \dot{p}_x &= -\frac{\partial H}{\partial x} = \frac{-mgx}{\sqrt{x^2+R^2}} \end{aligned}\right\} \quad \text{——— (4)}$$

6.7 Ignorable Coordinates

There exist certain system in which a particular coordinates, say q_f is absent from the Lagrangian L, although its time derivative $\dot{q}_f$ is parent in L. If such a system is holonomic, then the corresponding Lagrange's equation takes the form.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_f}\right) = 0 \quad \text{——— (1)}$$

Now let us recall the definition of generalized momenta.

$$p_f = \frac{\partial L}{\partial \dot{q}_f} \quad \text{——— (2)}$$

from (1) & (2), we can write

$$\frac{d p_f}{dt} = 0 \quad \text{——— (3)}$$

Integrating w.r.t time t, we get

$$p_f = \text{constant} \quad \text{——— (4)}$$

Hence we conclude that the generalized momenta associated with an ignorable coordinate is conserved.

→ If there is an ignorable coordinate, the number of degree of freedom is reduced by unity.

→ Similarly, if there are m ignorable coordinates, then number of degrees of freedom of the system is at once reduced by m.

In case we start with Hamilton instead of Lagrangian, the above argument also hold true.

For illustration, suppose q_1 is absent in H, so we that $\frac{\partial H}{\partial q_1} = 0$ ——— (5)

then the Hamilton's canonical equation of motion gives $\dot{p}_1 = -\frac{\partial H}{\partial q_1} = 0$ ——— (6)

$\therefore$ p_1 = constant $= c_1$ (say)

Now replying p_1 by c_1 in the right hand side of the equation (Hamilton's) and excluding the two equations corresponding to $r = 1$ from the $2n$ equation of Hamilton.

we are left with $(2n-2)$ canonical equation for the $(2n-2)$ quantities $q_2, q_3, \cdots q_n, p_2, \cdots p_n$ under these conditions, we say that q_1 is an ignorable coordinate.

Example 6.15| Consider the Kepler's Problem or planetary motion as discussed and show that the generalized momenta p_θ is constant.

Solⁿ:— The lagrangian form is

$$L = T - V = \frac{1}{2} m v^2 + \frac{m\mu}{r} \quad \text{——— (1)}$$

is term of plane rectangular coordinates.

$$L = \frac{1}{2} (m) (\dot{x}^2 + \dot{y}^2) + \frac{m\mu}{\sqrt{x^2+y^2}} \quad \text{——— (2)}$$

This gives no indication that an ignorable coordinate actually exists.

But the same lagrangian when expressed in polar coordinates, we have

$$L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) + \frac{m\mu}{r} \quad \text{——— (3)}$$

from this expression, it is evident that L doesnot contain θ.

Thus $\frac{\partial L}{\partial \theta} = 0$.

$\therefore \theta$ is ignorable coordinate
The generalized momenta, from

$$p_\theta = \frac{\partial L}{\partial \dot\theta} = mr^2\dot\theta = \text{Constant}.$$

The Routhian Function

Let us consider a simple holonomic system whose configuration is described through n independent generalized coordinates $q_1, q_2, \ldots q_n$. Suppose that first k generalized coordinates are ignorable. This means that the Lagrangian L is a function of $q_{k+1}, q_{k+2}, \ldots\ldots, q_n, \dot q_1, \dot q_2, \ldots\ldots, \dot q_n$ and t.

In previous we showed that the generalized momenta associated with an ignorable coordinate is constant and therefore, it follows at once

$$p_1 = c_1,\ p_2 = c_2, \ldots\ p_k = c_k.$$

Now let us construct a Routhian function. $R(q_{k+1}, q_{k+2}, \ldots, q_n, \dot q_1, \dot q_2, \ldots \dot q_n, c_1, c_2, \ldots c_k, t)$ defined by the relation

$$R = L - \sum_{f=1}^{k} c_f \dot q_f \quad \text{——— (1)}$$

Now let us form an arbitrary variation of all the variables in the Routhian function, then we will have,

$$\delta R = \delta L - \sum_{f=1}^{k} \delta c_f \dot{q}_f - \sum_{f=1}^{k} c_f \delta \dot{q}_f \quad \text{———} (2)$$

where we should note that c's are regarded as variables.

$$\delta R = \sum_{f=k+1}^{n} \frac{\partial L}{\partial q_f} \delta q_f + \sum_{f=1}^{n} \frac{\partial L}{\partial \dot{q}_f} \delta \dot{q}_f - \sum_{f=1}^{k} \delta c_f \dot{q}_f - \sum_{f=1}^{k} c_f \delta \dot{q}_f \quad \text{———} (3)$$

But $p_f = c_f = \dfrac{\partial L}{\partial \dot{q}_f}$ ——— (4).

using this expression for c_f into eq^n (3)

$$\delta R = \sum_{f=k+1}^{n} \frac{\partial L}{\partial q_f} \delta q_f + \sum_{f=1}^{n} \frac{\partial L}{\partial \dot{q}_f} \delta \dot{q}_f - \sum_{f=1}^{k} \delta c_f \dot{q}_f - \sum_{f=1}^{k} \frac{\partial L}{\partial \dot{q}_f} \delta \dot{q}_f .$$

$$\Rightarrow \delta R = \sum_{f=k+1}^{n} \frac{\partial L}{\partial q_f} \delta q_f + \sum_{f=k+1}^{n} \frac{\partial L}{\partial \dot{q}_f} \delta \dot{q}_f - \sum_{f=1}^{k} \delta c_f \dot{q}_f \quad \text{———} (5)$$

on the other hand,

$$\delta R = \sum_{f=k+1}^{n} \frac{\partial R}{\partial q_f} \delta q_f + \sum_{f=k+1}^{n} \frac{\partial R}{\partial \dot{q}_f} \delta \dot{q}_f + \sum_{f=1}^{k} \frac{\partial R}{\partial c_f} \delta c_f \quad \text{———} (6)$$

Comparing eq^n (5) and (6) we get

$$\left.\begin{aligned}\frac{\partial L}{\partial q_f} &= \frac{\partial R}{\partial q_f}\\ \frac{\partial L}{\partial \dot{q}_f} &= \frac{\partial R}{\partial \dot{q}_f}\end{aligned}\right\} \quad \text{—— (7)} \quad \text{for } f = k+1, \dots, n$$

and $$-\frac{\partial R}{\partial c_f} = \dot{q}_f \quad , f = 1, 2, \dots k \quad \text{—— (8)}$$

Substituting (7) into Lagrange's equation of motion for a simple system.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_f}\right) - \frac{\partial L}{\partial q_f} = 0 \quad , f = 1, 2, \dots n \quad \text{—— (9)}$$

we get,

$$\boxed{\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{q}_f}\right) - \frac{\partial R}{\partial q_f} = 0} \quad , f = \cancel{k+1}\ k+2, \dots, n \quad \text{—— (10)}$$

which can be regarded as Routhian equation of motion for a system with $(n-k)$ degrees of freedom.

These are $(n-k)$ second order differential equation in the non-ignorable variables/ coordinates.

Thus, the Routhian procedure eliminates the ignorable coordinates from the equations of the ~~n~~ motion.

If the sol^n of eq^n (10) can be found for $(n-k)$ non-ignorable coordinates, then we can integrate (8) and get expression for ignorable coordinates in the form.

$$q_f = -\int \frac{\partial R}{\partial c_f} dt \quad , f = 1, 2, \ldots k \quad \text{———} (4)$$

Example 6.17/ use the Routhian function method and find the equation of motion in the case of planetary motion (Kepler's Problem) for the non-ignorable coordinates r.

Solⁿ:— The Lagrangian for planetary motion is given as

$$L = T - V = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{m\mu}{r} \quad \text{———} (1)$$

we can notice that θ is the ignorable coordinate and the corresponding generalized momenta p_θ is constant.

i.e $$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} = c \quad \text{———} (2)$$

$$\therefore \dot{\theta} = \frac{c}{mr^2} \quad \text{———} (3)$$

Now, we construct the Routhian function R as

$$R = L - c\dot{\theta} = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{m\mu}{r} - c\dot{\theta}$$

using eqⁿ (3), we get

$$R = \frac{1}{2} m\left(\dot{r}^2 + \frac{r^2 c^2}{m^2 r^4}\right) + \frac{m\mu}{r} - \frac{c \cdot c}{mr^2}$$

$$\Rightarrow R = \frac{1}{2} m\left(\dot{r}^2 + \frac{c^2}{m^2 r^2}\right) + \frac{m\mu}{r} - \frac{c^2}{mr^2}$$

$$\Rightarrow R = \frac{1}{2} m\dot{r}^2 + \frac{1}{2}\frac{c^2}{mr^2} + \frac{m\mu}{r} - \frac{c^2}{mr^2}$$

$$\Rightarrow R = \frac{1}{2} m\dot{r}^2 - \frac{c^2}{2mr^2} + \frac{m\mu}{r}$$

$$\Rightarrow R = \frac{1}{2} m\left(\dot{r}^2 - \frac{c^2}{m^2 r^2}\right) + \frac{m\mu}{r} \quad \text{——— (4)}$$

differentiating eq^n (4) we get

$$\frac{\partial R}{\partial r} = \frac{1}{2} m\left(0 - \frac{c^2}{m^2 \cancel{\,}}\left(\frac{-2}{r^3}\right)\right) - \frac{m\mu}{r^2}$$

$$\left.\begin{aligned} \Rightarrow \frac{\partial R}{\partial r} &= \frac{c^2}{mr^3} - \frac{m\mu}{r^2} \\ \& \ \frac{\partial R}{\partial \dot{r}} &= m\dot{r} \\ \frac{d}{dt}\left(\frac{\partial R}{\partial \dot{r}}\right) &= m\ddot{r} \end{aligned}\right\} \quad \text{——— (5)}$$

Hence, the resulting Routhian Equation for the non-ignorable coordinate r, is

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{r}}\right) - \frac{\partial R}{\partial r} = m\ddot{r} - \frac{c^2}{mr^3} + \frac{m\mu}{r^2} = 0 \quad \text{——— (6)}$$

which is a second-order differential equation involving only r, and can be integrated.

ch-7

Hamiltonian Methods.

formation of a dynamical system is through Hamiltonian.

for a dynamical system with n degrees of freedom

→ the lagrangian formulation given n, second order differential equation.

while

→ The Hamiltonian formulation gives 2n, first order differential equations.

The Hamiltonian formulation has several advantages over that of the lagrangian.

They are :—

1.) Hamiltonian formulation provides deeper insight into the behaviour of a dynamical system.

2) Hamiltonian formulation proves superior in the indirect integration of the equations of the motion, by means of suitable transformations.

3). The transformation used in Hamiltonian mechanics are known as contact transformation, which is characterized by a single function, namely the generating function. If a generating function can be found, any dynamical problem can be reduced to one of differentiations and elimination.

Hamiltonian's Principle :

Natural Motion :

Consider a dynamical system with n- degrees of freedom and a Hamiltonian $H(q_f, p_f, t)$, $f = 1, 2, \ldots n$. A motion is said to be natural, if the Hamiltonian of the physical system considered satisfied the following :

$$\dot{q}_f = \frac{\partial H}{\partial p_f}, \quad \dot{p}_f = -\frac{\partial H}{\partial q_f} \quad \text{---(1)}$$

$$\text{and } \dot{H} = \frac{\partial H}{\partial t} \quad \text{---(2)}$$

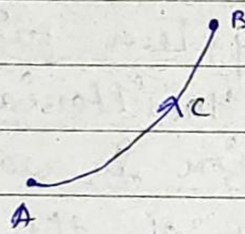
Space of Events :

A point in a representative space of $(n+1)$ dimensions, i.e in E_{n+1}, is denoted by a set of numbers $(q_1, q_2, \ldots q_n, t)$. Such a point corresponds to a configuration of the system at certain time t. We may refer to this point as an event and the space E_{n+1} is called Space of ~~E~~ Events.

Any motion of the system, whether it is natural or not can be described by taking $q's$ as functions of time t.

For example : If $n=2$, the geometrical motion in E_3 is a curve C given by the set of equations.

$$q_1 = q_1(t) \quad \text{and} \quad q_2 = q_2(t).$$



← motion represented by a curve C in E_3 space of events.

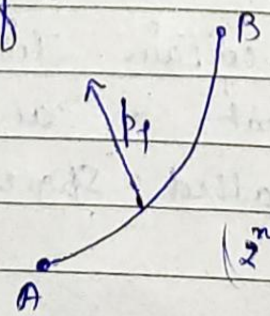
Action for an arbitrary motion :

we can describe a motion from an event A to an event B by writing

$$q_f = q_f(u) \quad , \quad p_f = p_f(u) \quad , \quad t = t(u) \qquad \text{———} (3)$$

where u is a parameter which runs from $u = u_1$ at A to $u = u_2$ at B. These are $(2n+1)$ functions of a parameter u.

Imagine a curve C in E_{n+1} with $p's$ assigned, i.e, we may think of them as defining a momentum vector p_f attached at each point of C.



← Curve of motion with attached momentum.

(2^{nd} dog)

Thus, we may say that eq^n (3) defines a curve c with momentum.

Now, we define action along c as an integral

$$S = \int_{u_1}^{u_2} \left(\sum_{\rho=1}^{n} p_\rho \frac{dq_\rho}{du} - H \frac{dt}{du} \right) du \quad \text{--- (4)}$$

or briefly,

$$S = \int_A^B \left(\sum_{\rho=1}^{n} p_\rho \, dq_\rho - H\,dt \right) = \int_A^B (p\,dq - H\,dt) \quad \text{--- (5)}$$

where the integration is carried out along c.

In eq^n (5), we have suppressed summation. Finally, of course with the understanding that it actually exists.

Hamilton's Principle:

The integral of Action $S = \int_A^B (p\,dq - H\,dt)$ has a stationary value for the natural motion when compared with the adjacent motions having the same end event. (Here q denotes the set of n quantities q_ρ and p denotes p_ρ)

Proof: The integral of action is defined as

$$S = \int_A^B (p\,dq - H\,dt) \quad \text{--- (1)}$$

It's variation is

$$\delta S = \int_A^B [\delta p\, dq + p\,\delta(dq) - \delta H\, dt - H\,\delta(dt)].$$

Interchanging d and δ, we have

$$\delta S = \int_A^B \left[\delta p\, dq + p\, d(\delta q) - \delta H\, dt - H\, d(\delta t) \right]$$

Integrating the second and last term using by parts rule;

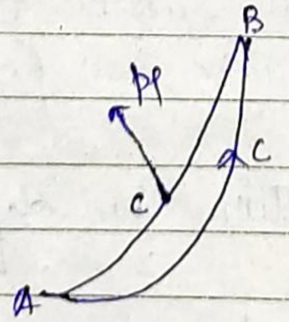
$$\delta S = \int_A^B \delta p\, dq + \left[p\, \delta q \right]_A^B - \int_A^B \delta q\, dp - \int_A^B \delta H\, dt - \left[H\, \delta t \right]_A^B + \int_A^B dH \cdot \delta t$$

$$\delta S = \left[p \cdot \delta q - H \cdot \delta t \right]_A^B + \int_A^B \left(\delta p\, dq - \delta q\, dp - \delta H\, dt + \delta t\, dH \right) \quad \text{--- (2)}$$

Now, consider the two curves in E_{n+L} with the same end event A and B with attached momenta (2^{nd} fig) i.e at A and B, $\delta t = 0$, $\delta q = 0$

$\therefore$ Eqn (2) becomes

$$\delta S = \int_A^B \left(\delta p\, dq - \delta q\, dp - \delta H\, dt + \delta t\, dH \right) \quad \text{--- (3)}$$



Hamilton's Principle.

But $\delta H(q, p, t) = \dfrac{\partial H}{\partial q} \delta q + \dfrac{\partial H}{\partial p} \delta p + \dfrac{\partial H}{\partial t} \delta t$ --- (4)

Substituting eqⁿ (4) in (3)

$$\delta S = \int_A^B \left[\delta p \, dq - \delta q \, dp - \left(\frac{\partial H}{\partial q} \delta q + \frac{\partial H}{\partial p} \delta p + \frac{\partial H}{\partial t} \delta t \right) dt + \delta t \, dH \right]$$

$$\Rightarrow \delta S = \int_A^B \left[\delta p \left(dq - \frac{\partial H}{\partial p} dt \right) - \delta q \left(dp - \frac{\partial H}{\partial q} dt \right) + \delta t \left(dH - \frac{\partial H}{\partial t} dt \right) \right]$$

$$\Rightarrow \delta S = \int_A^B \left[\delta p \left(q - \frac{\partial H}{\partial p} \right) - \delta q \left(\dot{p} + \frac{\partial H}{\partial q} \right) + \delta t \left(\dot{H} - \frac{\partial H}{\partial t} \right) \right] dt \quad \text{——(5)}$$

Now using Hamilton's canonical equation of motion for a natural motion described by

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}, \quad \dot{H} = \frac{\partial H}{\partial t} \quad \text{in (5)}$$

we get $\delta S = 0$

$$\Rightarrow \delta S = \delta \int_A^B (p \, dq - H \, dt) = 0 \quad \text{——(6)}$$

whatever may be the variations $\delta p, \delta q, \delta t$.

Thus, S has a stationary value for a natural motion when compared with arbitrary adjecent motions for the same end events.

Conversely.

If S has a stationary value for arbitrary variations $\delta p, \delta q, \delta t$ except for end events, then c represent a natural motion.

Proof: It is given that $\delta S = 0$ for any arbitrary variations $\delta p, \delta q, \delta t$.

Suppose, we choose these variations as

$$\left.\begin{aligned} \delta q &= -\left(\dot{p} + \frac{\partial H}{\partial q}\right) f \cdot \delta v \\ \delta p &= \left(\dot{q} - \frac{\partial H}{\partial p}\right) f \cdot \delta v \\ \delta t &= \left(\dot{H} - \frac{\partial H}{\partial t}\right) f \cdot \delta v \end{aligned}\right\} \quad \text{--- (7)}$$

where $\delta v > 0$ and f is any arbitrary functions along the curve c such that $f \geq 0$ and $f = 0$ at the end events.

Substituting (7) in (5), we get that

$$\delta S = \int_A^B \left[\left(\dot{q} - \frac{\partial H}{\partial p}\right) f \cdot \delta v \left(\dot{q} - \frac{\partial H}{\partial p}\right) + \left(\dot{p} + \frac{\partial H}{\partial q}\right) f \cdot \delta v \left(\dot{p} + \frac{\partial H}{\partial q}\right) + \left(\dot{H} - \frac{\partial H}{\partial t}\right) f \cdot \delta v \left(\dot{H} - \frac{\partial H}{\partial t}\right) \right] dt$$

$$\Rightarrow \delta S = \int_A^B \left[\left(\dot{q} - \frac{\partial H}{\partial p}\right)^2 f + \left(\dot{p} + \frac{\partial H}{\partial q}\right)^2 f + \left(\dot{H} - \frac{\partial H}{\partial t}\right)^2 f \right] \delta v \, dt \quad \text{--- (8)}$$

How $\delta S = 0$, only if

$$\dot{q} - \frac{\partial H}{\partial p} = 0, \quad \dot{p} + \frac{\partial H}{\partial q} = 0, \quad \dot{H} - \frac{\partial H}{\partial t} = 0 \quad \text{——} (9)$$

on C.

which of course, are Hamilton's canonical Equations.

Therefore, C is a natural path and hence the converse is proved.

It is also possible to bring out a connection between the Hamiltonian H and the Lagrangian L through Hamilton's principle.

<u>Euler – Lagrange Equations (Relation b/w H & L) :–</u>

Hamilton's Principle states that :

$$\delta S = \delta \int_A^B (p\,dq - H\,dt) = 0 \quad \text{——} (1)$$

By the definition of Hamiltonian function H

$$H = \dot{q}p - L \quad \text{——} (2)$$

Also, from Hamilton's Principle

$$\delta S = \delta \int_A^B (p\,dq - H\,dt) = 0$$

$$= \delta \int_A^B \left(p\frac{dq}{dt} - H\right) dt = 0$$

$$= \delta \int_A^B (p\dot{q} - H)\, dt = 0$$

$$= \delta \int_A^B L\, dt = 0$$

$\therefore$ Hamilton's Principle can also be stated in the form

$$\delta S = \delta \int_A^B (\dot{q}p - H)\, dt = \delta \int L\, dt = 0 \quad \text{——— (3)}$$

Thus alternatively, Hamilton's Principle can also be stated as

$$\delta \int_A^B L\, dt = 0 \quad \text{——— (4)}$$

as $L = L(q, \dot{q})$ ——— (5)

$$\delta \int_A^B L\, dt = \int_A^B \delta L\, dt = \int_A^B \left(\frac{\partial L}{\partial \dot{q}} \delta \dot{q} + \frac{\partial L}{\partial q} \delta q \right) dt \quad \text{——— (6)}$$

$$= \cancel{\int_A^B \frac{\partial L}{\partial \dot{q}} \delta q \left(\frac{dq}{dt} \right) dt}$$

$$= \int_A^B \frac{\partial L}{\partial \dot{q}} \delta \left(\frac{dq}{dt} \right) dt + \int_A^B \frac{\partial L}{\partial q} \delta q\, dt$$

$$= \int_A^B \frac{\partial L}{\partial \dot{q}} d(\delta q) + \int_A^B \frac{\partial L}{\partial q} \delta q\, dt$$

Evaluating the first integral using by paths

$$\delta\int_A^B L\,dt = \left[\frac{\partial L}{\partial \dot{q}}\delta q\right]_A^B - \int_A^B \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\cdot \delta q\,dt + \int_A^B \frac{\partial L}{\partial q}\delta q\,dt$$

using (4) $\delta\int_A^B L\,dt = 0$

$$\Rightarrow \left[\frac{\partial L}{\partial \dot{q}}\delta q\right]_A^B - \int_A^B \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\delta q\,dt + \int_A^B \frac{\partial L}{\partial q}\delta q\,dt = 0$$

$\because$ $\delta q = 0$ at A and B, the first term vanishes, and we are left with

$$\int_A^B \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\right]\delta q\,dt = 0 \quad \text{---} (7)$$

This expression is true for all δq and δt, consquently, we have

$$\boxed{\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0} \quad \text{---} (8)$$

This is called the Euler - Lagrange Equation associated with the variational eq^n

$$\delta\int_A^B L\,dt = 0$$

Theorem 7.1 If the Hamiltonian $H(q_f, p_f, t)$ is not an explicit function of time, then it is a constant of the motion.

Proof: Given $H = H(q_f, p_f, t)$, its time rate of change is

$$\frac{dH}{dt} = \sum_{f=1}^{n} \frac{\partial H}{\partial q_f} \dot{q}_f + \sum_{f=1}^{n} \frac{\partial H}{\partial p_f} \dot{p}_f + \frac{\partial H}{\partial t}$$

using Hamilton's canonical equation

i.e. $\dot{p}_f = -\frac{\partial H}{\partial q_f}$ & $\dot{q}_f = \frac{\partial H}{\partial p_f}$

the above expression reduced to

$$\frac{dH}{dt} = \frac{\partial H}{\partial t}$$

Now, if it is not an explicit function of time, then $\frac{\partial H}{\partial t} = 0$

$$\Rightarrow \frac{dH}{dt} = 0 \quad \text{———} (1)$$

$$\Rightarrow H = \text{constant}$$

Here, H is constant of the motion.

7.2.1 Hamilton's Principle for a Conservative System.

In conservative system for which $\frac{\partial H}{\partial t} = 0$ Hamilton's variational Principle can be stated as $\delta \int p\, dq = 0$

for variations from a natural motion, provided that the end-configurations are fixed and H has, in the varied motion, the same constant value which it has in the natural motion.

Proof: Consider

$$\delta \int_A^B p\, dq = \int_A^B (\delta p\, dq + p\, \delta dq)$$

$$= \int_A^B [\delta p\, dq + p\, d(\delta q)] \quad \text{(interchanging } d \text{ \& } \delta)$$

$$= \int_A^B \delta p\, dq + [p\, \delta q]_A^B - \int_A^B dp\, \delta q$$

$$= [p\, \delta q]_A^B + \int_A^B (\delta p\, dq - \delta q\, dp) \quad \text{—— (1)}$$

Since C_1 is a natural motion, Hamilton's canonical equation hold true and

$$\therefore \dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}, \quad \frac{\partial H}{\partial t} = 0$$

i.e $$dq = \frac{\partial H}{\partial p}\, dt, \quad dp = -\frac{\partial H}{\partial q}\, dt \quad \text{—— (2)}$$

$\therefore$ using (2) in (1)

$$\delta\int_A^B p\,dq = [p\,\delta q]_A^B + \int_A^B \left[\delta p\left(\frac{\partial H}{\partial p}dt\right) - \delta q\left(-\frac{\partial H}{\partial q}dt\right)\right]$$

$$\Rightarrow \delta\int_A^B p\,dq = [p\,\delta q]_A^B + \int_A^B \left[\delta p\left(\frac{\partial H}{\partial p}\right) + \delta q\left(\frac{\partial H}{\partial q}\right)\right]dt \quad \text{--- (3)}$$

Now, using the chain rule of partial differentiation and also ~~nathe~~ noting that $\frac{\partial H}{\partial t} = 0$, for a conservative system.

$\therefore$ Eq^n (3) can be written as

$$\delta\int p\,dq = [p\,\delta q]_A^B + \int_A^B \delta H\,dt \quad \text{--- (4)} \quad \left[\because H \text{ is function of } q, p, t \text{ and } \frac{\partial H}{\partial t} = 0\right]$$

Since $\delta q = 0$ at A and B, the end-events, the first term on the right-hand side of eq^n (4) vanishes.

Also, since at H is constant along c for any natural motion, $\delta H = 0$, thereby the second term also vanishes and Hence eq^n (4) finally becomes

$$\delta\int_A^B p\,dq = 0 \quad \text{--- (5)}$$

7.2.2 Principle of Least Action :

For a simple dynamical system $\delta \int T\,dt$ has a stationary value for the natural motion compared with the adjacent motions having the same end-events, provided H has in the varied motion the same constant value which it has in the natural motion.

Proof: For a simple dynamical system, the Hamiltonian is defined as $H = T + V$ —(1)

and the generalized momenta

$$p = \frac{\partial H}{\partial \dot{q}} = \frac{\partial T}{\partial \dot{q}}$$

Since V is not a function of $\dot{q}$, then

$$\int p\,dq = \int \frac{\partial T}{\partial \dot{q}}\,dq = \int \frac{\partial T}{\partial \dot{q}} \frac{dq}{dt}\,dt = \int \dot{q} \frac{\partial T}{\partial \dot{q}}\,dt \quad (2)$$

Since T is homogeneous quadratic function in $\dot{q}$'s. we have, using Euler's theorem

$$\dot{q} \frac{\partial T}{\partial \dot{q}} = 2T \quad —(3)$$

$$\therefore \int p\,dq = \int 2T\,dt = 2\int T\,dt \quad —(4) \text{ (using (3) in (2))}$$

But, the Hamilton's variational principle states, that $\delta \int p\,dq = 0$

using this in (4), we get, $\delta^2 \int T\,dt = 0$

$$\boxed{\delta \int T\,dt = 0}$$

7.3/ Characteristic function and Hamilton – Jacobi Equation

This following Hamilton's Principle for natural motion. Let us consider two events A and B and Action for varion motions which gives us stationary value,

Thus, we have,

$$\delta \int_A^B (p\,dq - H\,dt) = 0 \qquad \text{——— (1)}$$

On the other hand; Hamilton's canonical equations for the natural motion are known ~~to~~ be

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = \frac{-\partial H}{\partial q} \qquad \text{——— (2)}$$

These canonical equations suggest that we can construct the natural motion for initial data, called Cauchy data (q, p, t). The solution of the differential equation is called <u>Cauchy Problem</u>.

Hamilton's Characteristic function:

Let C be the curve in E_{n+1}, representating a natural motion ~~at~~ and also let A and B be two event on it. Then, the action on C from A to B is a function of these event. Thus,

$$S = S(q^*, t^*, q, t) \qquad \text{——— (1)}$$

where (q^*, t^*) refers to the initial event A and (q, t) to the final event B.

Here, q^*, q stand for n generalized coordinates.

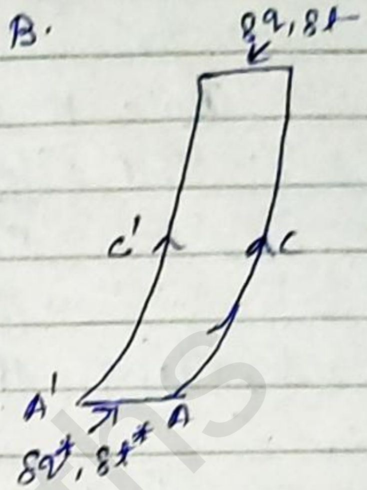


fig → Variation from one natural motion to another.

Now, consider an adjecent natural motion represented by the curve c' with endevents A' and B'.

Let us displacement from A to A' be $(\delta q^*, \delta t^*)$ to another, and that from B to B' be $(\delta q, \delta t)$. The integral of action as

$$S = \int_A^B (p\,dq - H\,dt)$$

its variation is obtained as

$$\delta S = \int_A^B [\delta p\, dq + p\,\delta(dq) - \delta H\, dt - H\,\delta(dt)]$$

Interchanging d and δ.

$$\delta S = \int_A^B [\delta p\, dq + p\,d(\delta q) - \delta H\, dt - H\, d(\delta t)]$$

using integration by parts, in 2nd & 4th term

$$\delta S = \int_A^B \delta p\, dq + [p\,\delta q]_A^B - \int_A^B dp\,\delta q - \int_A^B \delta H dt - [H\,\delta t]_A^B + \int_A^B dH\,\delta t.$$

$$\delta S = \left[p\,\delta q - H\,\delta t\right]_A^B + \int_A^B \left(\delta p\, dq - \delta q\, dp - \delta H dt + \delta t\, dH\right) \quad \text{---(2)}$$

But $H = H(q, p, t)$

$$\delta H = \frac{\partial H}{\partial q}\delta q + \frac{\partial H}{\partial p}\delta p + \frac{\partial H}{\partial t}\delta t$$

Inserting this value of δH in (2), we get

$$\delta S = \left[p\, dq - H\,\delta t\right]_A^B + \int_A^B \left[\delta p\, dq - \delta q\, dp - \left(\frac{\partial H}{\partial q}\delta q + \frac{\partial H}{\partial p}\delta p + \frac{\partial H}{\partial t}\delta t\right)dt + \delta t\, dH\right]$$

$$\Rightarrow \delta S = \left[p\, dq - H\,\delta t\right]_A^B + \int_A^B \left[\delta p\left(dq - \frac{\partial H}{\partial p}dt\right) - \delta q\left(dp + \frac{\partial H}{\partial q}dt\right) + \delta t\left(-\frac{\partial H}{\partial t}dt + dH\right)\right]$$

$$\Rightarrow \delta S = \left[p\, dq - H\,\delta t\right]_A^B + \int_A^B \left[\delta p\left(\dot{q} - \frac{\partial H}{\partial p}\right)dt - \delta q\left(\dot{p} + \frac{\partial H}{\partial q}\right)dt + \delta t\left(\dot{H} - \frac{\partial H}{\partial t}\right)dt\right]$$

$$\Rightarrow \delta S = \left[p\, dq - H\,\delta t\right]_A^B + \int_A^B \left[\delta p\left(\dot{q} - \frac{\partial H}{\partial p}\right) - \delta q\left(\dot{p} - \frac{\partial H}{\partial q}\right) + \delta t\left(\dot{H} - \frac{\partial H}{\partial t}\right)\right] dt$$

Since C represents natural motion, the integral vanishes and we are left with

$$\delta S = \left[p\,\delta q - H\,\delta t\right]_A^B$$

as,

$$\delta S = p\,\delta q - H\,\delta t - p^*\,\delta q^* + H^*\,\delta t^* \quad \text{--- (3)}$$

Here H^* represents the value of H at A etc. Noting that $(2n+2)$ infinitesimals $(\delta q, \delta t; \delta q^*, \delta t^*)$ are arbitrary and independent, the partial derivatives of S with respect to the arguments a given eq^n (1) gives us

$$\frac{\partial S}{\partial q} = p, \quad \frac{\partial S}{\partial t} = -H, \quad \frac{\partial S}{\partial q^*} = -p^*, \quad \frac{\partial S}{\partial t^*} = H^* \quad \text{--- (4)}$$

Here the function S is called the Hamilton's characteristic function. which characterizes the dynamical system.

Suppose for a dynamical system, the Hamiltonian is given by

$$H = H(q, p, t)$$

and 2nd eq^n of (4) gives,

$$\frac{\partial S}{\partial t} + H(q, p, t) = 0$$

Replacing p by $\frac{\partial S}{\partial q}$ i.e; using first eq^n of (4) the above eq^n is

$$\frac{\partial S}{\partial t} + H\left(q, \frac{\partial S}{\partial q}, t\right) = 0 \quad \text{--- (5)}$$

$$\Rightarrow \frac{\partial S}{\partial t} + H\left(q_1, q_2, \ldots q_n, \frac{\partial S}{\partial q_1}, \frac{\partial S}{\partial q_2}, \ldots, \frac{\partial S}{\partial q_n}, t\right) = 0 \quad \text{--- (6)}$$

This is called Hamilton's-Jacobi Equation, which is a partial ~~deri~~ differential equation, satisfied by the characteristic functions.

Complete - Integrals of the Hamilton - Jacobi Equation:

Let the Hamilton - Jacobi equation for a typical dynamical system be $\frac{\partial S}{\partial t} + H\left(q, \frac{\partial S}{\partial q}, t\right) = 0$ —(1)

For a moment, let us forget how this eq^n has been ~~divided~~ derived and simply treat it as a partial differential equation in one independent variable S and (n+1) independent variables q and t.

The derivative in eq^n (1) are only of first order. However, the derivatives occur in second ~~order~~ higher degrees. Therefore, eq^n(1) is in general a first order non - linear partial differential equation.

In view of the above observations, its complete solution must involve (n+1) independent constants.

Let us assume that the Hamilton characteristic function of the form

$$S = S(q, t, a) + C \qquad \text{—(2)}$$

is a complete integral of eq^n (1)

where a stands for a set of n constants $(a_1, a_2, \dots a_n)$ and c for a single constant.

Introducing $\frac{\partial S}{\partial q} = \beta$, and $\frac{\partial S}{\partial a} = -b$ ——— (3)

where b stands for a new set of n constants. we shall prove the following Jacobi's theorem which will help us determine the natural motion for a given dynamical system.

Theorem 7.2/ Jacobi's Theorem

Let $S(q, a, t)$ be any complete integral of the Hamilton — Jacobi Equation

$$\frac{\partial S}{\partial t} + H\left(q, \frac{\partial S}{\partial q}, t\right) = 0$$

then each motion with associated momenta given by $\frac{\partial S}{\partial q} = \beta$ and $\frac{\partial S}{\partial a} = -b$.

is a natural motion satisfying the canonical equations.

Proof: The Hamilton — Jacobi Equation for a dynamical system is

$$\frac{\partial S}{\partial t} + H\left(q, \beta = \frac{\partial S}{\partial q}, t\right) = 0 \quad \text{——— (1)}$$

Let $S = S(q, t, a) + c$ ——— (2)

be the complete integral of eq^n (1), which contain $(2n+2)$ independent quantities

$q_1, q_2, \dots, q_n, t, a_1, a_2, \dots, a_n, c$.

Differentiating eq^n (1) partially w.r.t a_ρ and q_ρ, respectively, we get

$$\frac{\partial^2 S}{\partial a_\rho \partial t} + \sum_{r=1}^{n} \frac{\partial H}{\partial p_r} \frac{\partial^2 S}{\partial a_\rho \cdot \partial q_r} = 0 \quad \text{——} (3)$$

and $$\frac{\partial^2 S}{\partial q_\rho \partial t} + \frac{\partial H}{\partial q_\rho} + \sum_{r=1}^{n} \frac{\partial H}{\partial p_r} \frac{\partial^2 S}{\partial q_\rho \partial q_r} = 0 \quad \text{——} (4)$$

further, we are given $\frac{\partial S}{\partial a} = -b$ ——(5)

which represent n equation.

Now, differentiating eq^n(5) w.r.t we get

$$\frac{\partial^2 S}{\partial t \partial a_\rho} + \sum_{r=1}^{n} \frac{\partial^2 S}{\partial q_r \cdot \partial q_\rho} \cdot \frac{\partial q_r}{dt} = 0 \quad \text{——} (6)$$

This equation represents n linear equations and can be solved to get $\dot{q}_1, \dot{q}_2, \cdots, \dot{q}_n$

It may be noted that eq^n (3) and eq^n (6) are precisely of the same form and therefore eq^n(3) can be solved to get

$$\frac{\partial H}{\partial p_1}, \frac{\partial H}{\partial p_2}, \cdots, \frac{\partial H}{\partial p_n}$$

Thus we have the first Hamilton's canonical equation $\dot{q}_\rho = \frac{\partial H}{\partial p_\rho}$ ——(7) (on comparing (3) & (6))

Also, we note that $\frac{\partial S}{\partial q} = \beta$ ——(8)

differentiating (8) partially w.r.t, we get

$$\frac{\partial^2 s}{\partial q_\rho \partial t} + \sum_{r=1} \frac{\partial^2 s}{\partial q_\rho \partial q_r} \frac{dq_r}{dt} = \frac{d\dot{p}_\rho}{dt} \qquad (9)$$

Compairing (4) and (9), we get the second canonical eq^n

$$\frac{dp_\rho}{dt} = -\frac{\partial H}{\partial q_\rho}$$

$$\boxed{\dot{p}_\rho = -\frac{\partial H}{\partial q_\rho}} \qquad (10)$$

Thus we find that eq^n (7) and eq^n (10) constitute Hamilton's canonical equations and hence each motions described by the characteristic function S is a natural motion.

Example 7.1/ Kepler's Problem

Suppose that a particle of mass m is attracted by an inverse - square gravitational force to a fixed point O, and the particle is moving in a plane whose position in polar coordinate is given by (r, θ). Discuss the motion of the particle.

Sol^n:- The kinetic Energy of the system is found to be $T = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2)$ (1)

while the potential energy is given by
$$V = -\int f\,dr.$$

From the given data, $f = -\dfrac{m\mu}{r^2}$

$$\therefore V = \int \frac{m\mu}{r^2}\,dr$$

$$\Rightarrow V = -\frac{m\mu}{r} \quad \text{——(2)}$$

Taking r, θ as generalized coordinates, the Lagrangian for the given system is

$$L = T - V = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{m\mu}{r} \quad \text{——(3)}$$

The generalized momenta is given by
$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}, \quad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \quad \text{——(4)}$$

However, the Hamiltonian is obtained as
$$H = T + V = H(q, p, t)$$

$\therefore$ from eq^n (1), (2) and (4), we have

$$H = \frac{1}{2} m\left(\frac{p_r^2}{m^2} + r^2\left(\frac{p_\theta}{mr^2}\right)^2\right) - \frac{m\mu}{r}$$

$$\Rightarrow H = \frac{1}{2} m\left(\frac{p_r^2}{m^2} + \frac{r^2 p_\theta^2}{m^2 r^4}\right) - \frac{m\mu}{r}$$

$$\Rightarrow \quad H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) - \frac{m\mu}{r} \qquad \text{—— (5)}$$

But H should be a function of r, θ, p_r, p_θ and t. However, we notice that θ is absent in H and therefore it is ~~call~~ ignorable coordinate

To write down the Hamilton – Jacobi – Equation for the given problem, we have to substitute $\frac{\partial S}{\partial r}$ and $\frac{\partial S}{\partial \theta}$ for p_r and p_θ in H and thus obtain.

$$\frac{\partial S}{\partial t} + H\left(q, p = \frac{\partial S}{\partial q}, t\right) = 0$$

$$\Rightarrow \quad \frac{\partial S}{\partial t} + \frac{1}{2m}\left[\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial S}{\partial \theta}\right)^2\right] - \frac{m\mu}{r} = 0 \qquad \text{—— (6)}$$

This being a non-linear first order partial differential equation, we use separations of variables method to find the complete integral of eq^n (6). Thus we assume tentatively,

$$S = -a_1 t + a_2 \theta + f(r) \qquad \text{—— (7)}$$

where a_1 and a_2 are arbitrary constants and $f(r)$ is a function of r alone.

Substituting eq^n (7) in eq^n (6), we get at once

$$-a_1 + \frac{1}{2m}\left[\left(\frac{df}{dr}\right)^2 + \frac{1}{r^2}(a_2)^2\right] - \frac{m\mu}{r} = 0$$

$$\Rightarrow -a_1 + \frac{1}{2m}\left(\frac{\partial f}{\partial r}\right)^2 + \frac{a_2^2}{2mr^2} - \frac{m\mu}{r} = 0$$

Solving for $f(r)$, we get

$$\frac{1}{2m}\left(\frac{df}{dr}\right)^2 = a_1 - \frac{a_2^2}{2mr^2} + \frac{m\mu}{r}$$

$$\Rightarrow \left(\frac{df}{dr}\right)^2 = 2ma_1 - \frac{a_2^2}{r^2} + \frac{2m^2\mu}{r}$$

$$\Rightarrow \frac{df}{dr} = \sqrt{2ma_1 + \frac{2m^2\mu}{r} - \frac{a_2^2}{r^2}}$$

$\Rightarrow$ On integrating w.r.t r, we get,

$$f(r) = \int\sqrt{2ma_1 + \frac{2m^2\mu}{r} - \frac{a_2^2}{r^2}}\,dr + c \quad \text{———(8)}$$

which contains three arbitrary constants.

In order to determine the natural motion of the of the given system $\frac{\partial s}{\partial a} = -b$.

In this example we have a_1 and a_2 as constants and therefore obtain after differentiating under integral sign.

$$-b_1 = \frac{\partial s}{\partial a_1} = -t + m\int\frac{dr}{\sqrt{2ma_1 + \frac{2m^2\mu}{r} - \frac{a_2^2}{r^2}}} \quad \text{———(10)}$$

and $$-b_2 = \frac{\partial s}{\partial a_2} = 0 - a_2\int\frac{dr}{r^2\sqrt{2ma_1 + \frac{2m^2\mu}{r} - \frac{a_2^2}{r^2}}} \quad \text{———(11)}$$

Here, eq^n (10) is a relations connecting r and t, which when solved for r gives us r in terms of t.

Similarly, eq^n (11) gives us θ in terms of t.

The corresponding associated momenta is found to be

$$p_r = \frac{\partial S}{\partial r} = \sqrt{2ma_1 + \frac{2m^2\mu}{r} - \frac{a_2^2}{r^2}} \quad \text{——— (12)}$$

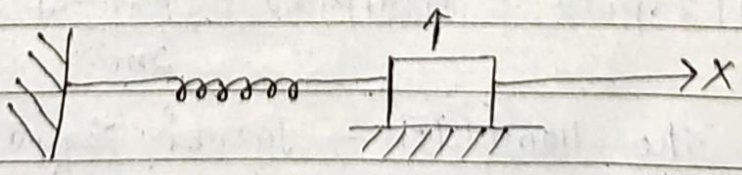
and
$$p_\theta = \frac{\partial S}{\partial \theta} = a_2 \quad \text{——— (13)}$$

Noting that

$$H = -\frac{\partial S}{\partial t} = a_1 \quad \text{——— (14)}$$

from eq^n (13), we conclude that the constant a_2 corresponds to that of constant angular momentum $mr^2\dot{\theta}$ of the system, while eq^n(14) indicates that the constant a_1 corresponds to that ~~of~~ of the total energy of the system.

Example 7.2/ Discuss the problem of Harmonic oscillator, i.e. simple mass – spring system using the Hamilton – Jacobi Method.



Solⁿ :- This being a natural system, its kinetic energy is given by

$$T = \frac{1}{2} m\dot{x}^2 \quad \text{---------(1)}$$

While the potential energy V of this system is $V = \frac{1}{2} kx^2$ ---------(2)

Taking x as the generalized coordinates q, the lagrangian for the given system is

$$L = T - V = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} kx^2$$

Then the generalized momenta is

$$p_x = \frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \quad \text{---------(3)}$$

The Hamiltonian of the system from eqⁿ(1) to (3)

$$H = T + V$$

$$H = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} kx^2$$

$$\Rightarrow H = \frac{1}{2} m \frac{p_x^2}{m^2} + \frac{1}{2} kx^2 \quad \text{(using 3)}$$

$$\Rightarrow H = \frac{1}{2} \frac{p_x^2}{m} + \frac{1}{2} kx^2$$

or, $H(q, p, t) = H(x, p, t) = \dfrac{p_x^2}{2m} + \dfrac{kx^2}{2}$ ---------(4)

Now, the Hamilton - Jacobi Equation for the system is $\dfrac{\partial S}{\partial t} + H(q, p, t) = \dfrac{\partial S}{\partial t} + \dfrac{p_x^2}{2m} + \dfrac{1}{2} kx^2 = 0.$

Substituting $\frac{\partial S}{\partial x}$ for p_x in H, we obtain

$$\frac{\partial S}{\partial t} + \frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 + \frac{1}{2}kx^2 = 0 \quad \text{——— (5)}$$

This is a non-linear partial differential equation of first order.

To get its complete integral using the variables separable method, we assume tentatively

$$S = -a_1 t + f(x) \quad \text{——— (6)}$$

where a_1 is an arbitrary constant and $f(x)$ is a function of x only. Substituting eqⁿ(6) into eqⁿ (5) we get

$$-a_1 + \frac{1}{2m}\left(\frac{df}{dx}\right)^2 + \frac{1}{2}kx^2 = 0$$

$$\Rightarrow \frac{1}{2m}\left(\frac{df}{dx}\right)^2 = a_1 - \frac{1}{2}kx^2$$

$$\Rightarrow \frac{df}{dx} = \sqrt{2ma_1 - mkx^2}$$

On integrating w.r.t x we get

$$\Rightarrow f = \int\sqrt{2ma_1 - mkx^2}\,dx + c$$

$$\Rightarrow f = m\int\sqrt{\frac{2ma_1 - mkx^2}{m^2}}\,dx + c$$

$$\Rightarrow f = m\int\sqrt{\frac{2a_1}{m} - \frac{kx^2}{m}}\,dx + c$$

Let $\frac{k}{m} = w^2$, then we can rewrite the above eq^n as

$$f(x) = m\int \sqrt{\frac{2aw^2}{k} - w^2x^2} \cdot dx + C$$

which can be recast as

$$f(x) = mw\int\sqrt{\frac{2a_1}{k} - x^2}\, dx + C$$

$$f(x) = mw\int\sqrt{\alpha^2 - x^2}\, dx + C \quad \text{———} (7)$$

$$\text{where } \alpha^2 = \frac{2a_1}{k} = \frac{2a_1}{mw^2} \quad \text{———} (8)$$

Substituting eq^n (7) in eq^n(6) we get

$$S = -a_1 t + mw\int\sqrt{\alpha^2 - x^2}\, dx + C \quad \text{———} (9)$$

To determine the natural motion of the system, we may note that, there is only one constant a_1 in this problem.

Now,

$$-b_1 = \frac{\partial S}{\partial a_1} = -t + \frac{1}{w}\int_{x_0}^{x_1} \frac{dx}{\sqrt{\alpha^2 - x^2}}$$

$$\Rightarrow t - b = \frac{1}{w}\left[\cos^{-1}\left(\frac{x}{a}\right)\right]_{x_0}^{x_1} \quad \text{———} (10)$$

The corresponding associated momenta is obtained from eq^n (9) as

$$p_x = \frac{\partial S}{\partial x} = m\omega\sqrt{\alpha^2 - x^2} \quad \text{———} (11).$$

Special Transformations :-

Point Transformation :

If there exist a transformation from one set of generalized coordinates $q_1, q_2, \ldots, q_n$ to another set of generalized coordinates $Q_1, Q_2, \ldots Q_n$ such a transformation is called point transformation.

For instance, a point transformation is described by a set of equations

$$Q_i = Q_i(q_1, q_2, \ldots, q_n) \quad \text{———} (1)$$

The inverse transformation is given by

$$q_i = q_i(Q_1, Q_2, \ldots Q_n) \quad \text{———} (2)$$

If we substitute (2) into the lagrangian function $L(\vec{q}, \dot{\vec{q}}, t)$, we obtain a lagrange function. $L(\vec{Q}, \dot{\vec{Q}}, t)$. It means that the lagrangian is invariant under point transformation.

Similarly,

Substituting eqn (2) into lagrange's equations

$$\frac{d}{dt}\left[\frac{\partial L(\vec{q}, \dot{\vec{q}}, t)}{\partial \dot{q}_p}\right] - \frac{\partial L(\vec{q}, \dot{\vec{q}}, t)}{\partial q_p} = 0 \quad \text{———} (3).$$

we get

$$\frac{d}{dt}\left[\frac{\partial L(\vec{Q},\dot{\vec{Q}},t)}{\partial \dot{Q}_\rho}\right] - \frac{\partial L(\vec{Q},\dot{\vec{Q}},t)}{\partial Q_\rho} = 0 \quad \text{——— (4)}$$

which also indicates that the lagrange's equations are invariant under point Transformation

This property of invariance is not preserved in the Hamiltonian formulation.

Theorem 7.3| Suppose the configuration of a dynamical system is defined by the generalized coordinates $q_1, q_2, \ldots, q_n$ whose behaviour under the action of a given force is described by the Hamiltonian $H(\vec{q},\vec{p},t)$ with the generalized momenta $\vec{p}$. If a new set of generalized coordinates $Q_1, Q_2, \ldots Q_n$ are introduced which are related to $\vec{q}$ by the point Transformation

$$Q_\rho = Q_\rho(\vec{q}, t) \quad \text{——— (1)}$$

then the corresponding momenta $\vec{P}$ and the Hamiltonian K are given by

$$P_j = \sum_{\rho=1}^{n} p_\rho \left(\frac{\partial q_\rho(\vec{Q},t)}{\partial Q_j}\right) \quad \text{——— (2)}$$

and $$K = H - \sum_{\rho=1}^{n} p_\rho \frac{\partial q_\rho(\vec{Q},t)}{\partial t} \quad \text{——— (3)}$$

Proof: ~~for~~ a dynamical system with n degrees of freedom, the Hamiltonian is defined as

$$H = \sum_{f=1} \dot{q}_f p_f - L \quad \text{or} \quad L = \sum \dot{q}_f p_f - H$$

we have already observed the Lagrangean is invariant under point transformation.

$\therefore$ we can write

$$\sum p_f \dot{q}_f - H = \sum P_f \dot{Q}_f - K$$

$$\Rightarrow \sum_{f=1}^{n} p_f \frac{dq_f}{dt} - H = \sum P_f \frac{dQ_f}{dt} - K$$

$$\Rightarrow \sum_{f=1}^{n} p_f\, dq_f - H\,dt = \sum P_f\, dQ_f - K\,dt \qquad \text{———} (4).$$

But from calculus, we have,

$$dq_f = \sum_{j=1}^{n} \frac{\partial q_f(\vec{Q},t)}{\partial Q_j} dQ_j + \frac{\partial q_f(\vec{Q},t)}{dt} dt \qquad \text{———} (5)$$

Substituting eq^n (5) in eq^n (4)

$$\sum_{f=1}^{n} p_f \left[\sum_{j=1}^{n} \frac{\partial q_f(\vec{Q},t)}{\partial Q_j} dQ_j + \frac{\partial q_f(\vec{Q},t)}{\partial t} dt \right] - H\,dt$$
$$= \sum_{j=1}^{n} P_j\, dQ_j - K\,dt$$

$$\Rightarrow \sum_{f=1}^{n} \sum_{j=1}^{n} p_f \frac{\partial q_f(\vec{Q},t)}{\partial Q_j} dQ_j + \sum_{f=1}^{n} p_f \frac{\partial q_f(\vec{Q},t)}{\partial t} - H\,dt$$
$$= \sum_{j=1}^{n} P_j\, dQ_j - K\,dt \qquad \text{———} (6).$$

Equating the coefficient of dQ_j and dt on both side of eqⁿ (6) we get

$$P_j = \sum_{f=1}^{n} p_f \frac{\partial q_f(\vec{Q},t)}{\partial Q_j}$$

and $$K = H - \sum_{f=1}^{n} p_f \frac{\partial q_f(\vec{Q},t)}{\partial t}$$

Note : It may be noted that if the point transformation from $\vec{q}$ to $\vec{Q}$ is independent of time, then $K = H$, while if the transformation depend on time then $K \neq H$.

Ex 7.3 | The configuration of a known dynamical system is described by the generalized coordinate q and its behaviour by the Hamiltonian function $H(q,p,t) = ap^2 + bp(q+t)^2$, where a and b are constants. A point transformation is made to a new generalized coordinate $Q = q+t$. find the corresponding Hamiltonian function $K(Q,P,t)$.

Solⁿ:- we have $Q = q+t$

$$\Rightarrow q = Q - t \Rightarrow \frac{\partial q}{\partial Q} = 1, \quad \frac{\partial q}{\partial t} = -1$$

from previous theorem 7.3

$$P_j = \sum_{f=1} p_f \frac{\partial q_f(\vec{Q},t)}{\partial Q_j}$$

$$\Rightarrow P = p\,\frac{\partial q(Q,t)}{\partial Q}$$

$$\Rightarrow P = p(1) \Rightarrow P = p \qquad \text{———}(1).$$

Also,

$$K = H - \sum_{i=1}^{n} p_i \frac{\partial q_i(\vec{Q},t)}{\partial t} \Rightarrow K = H - p\left(\frac{\partial q(Q,t)}{\partial t}\right)$$

$$\Rightarrow K = H - p(-1)$$

$$\Rightarrow K = H + p.$$

$$\because H = ap^2 + bp(q+t)^2.$$

$$\therefore K = H + p = ap^2 + bp(q+t)^2 + p \qquad \text{———}(2).$$

using (1) $(P = p)$ in (2)

$$K = ap^2 + bp(q+t)^2 + p$$

$$K = aP^2 + bPQ^2 + P \qquad (\because p = P,\ Q = q+t)$$

Canonical Transformation

For solving many complex problems in mechanics it is often found to be advantageous to transform from one set of generalized coordinates $q_1, q_2, \dots q_n$ and generalized momenta $p_1, p_2, \dots p_n$ to another set of variables $Q_1, Q_2, \dots Q_n$ and $P_1, P_2, \dots P_n$.

A Transformation of the type

$$\vec{Q} = \vec{Q}(\vec{q}, \vec{p}) \quad , \quad \vec{P} = \vec{P}(\vec{q}, \vec{p}) \qquad \text{---(1)}$$

is called a canonical transformation, if and only if there exists a function of the type $f(\vec{q}, \vec{p})$ satisfying the property

$$df(\vec{q}, \vec{p}) = \sum_{f=1}^{n} p_f \, dq_f - \sum_{f=1}^{n} P_f \, dQ_f(\vec{q}, \vec{p}) \qquad \text{---(2)}$$

However, a time-dependent transformation.

$$\vec{Q} = \vec{Q}(\vec{q}, \vec{p}, t) \quad , \quad \vec{P} = \vec{P}(\vec{q}, \vec{p}, t) \qquad \text{---(3)}$$

is said to be a canonical transformation, if and only if any one of the following three equivalent properties are found to be true —

(i) The transformation is canonical in the sense of eqⁿ (2) $df(\vec{q}, \vec{p}) = \sum_{f=1}^{n} p_f \, dq_f - \sum_{f=1}^{n} P_f \, dQ_f(\vec{q}, \vec{p})$

for every value of time.

(ii) There exists a function $f(\vec{q}, \vec{p}, t)$ such that for every arbitrary fixed time, say $t = t_0$ satisfying

$$df(\vec{q}, \vec{p}, t_0) = \sum_{\rho=1}^{n} p_\rho \, dq_\rho - \sum_{\rho=1}^{n} P_\rho(\vec{q}, \vec{p}, t_0) \, dQ_\rho(\vec{q}, \vec{p}, t_0) \quad \text{---(4)}$$

where,

$$df(\vec{q}, \vec{p}, t_0) = \sum_{\rho=1}^{n} \frac{\partial f(\vec{q}, \vec{p}, t_0)}{\partial q_\rho} dq_\rho + \sum_{\rho=1}^{n} \frac{\partial f(\vec{q}, \vec{p}, t_0)}{\partial p_\rho} dp_\rho \quad \text{---(5)}$$

and,

$$dQ_\rho(\vec{q}, \vec{p}, t_0) = \sum_{j} \frac{\partial Q_\rho(\vec{q}, \vec{p}, t_0)}{\partial q_j} dq_j + \sum_{j} \frac{\partial Q_\rho(\vec{q}, \vec{p}, t_0)}{\partial p_j} dp_j \quad \text{---(6)}$$

(iii) There exists a function $f(\vec{q}, \vec{p}, t)$ such that

$$df(\vec{q}, \vec{p}, t) = \sum_{\rho} p_\rho \, dq_\rho - \sum_{\rho} P_\rho(\vec{q}, \vec{p}, t) \, dQ_\rho(\vec{q}, \vec{p}, t) + R(\vec{q}, \vec{p}, t) \quad \text{---(7)}$$

where $R = \sum P_\rho(\vec{q}, \vec{p}, t) \dfrac{\partial Q_\rho(\vec{q}, \vec{p}, t)}{\partial t} + \dfrac{\partial f(\vec{q}, \vec{p}, t)}{\partial t}$ ---(8)

The equivalence of (i) and (ii) is obvious. The equivalence of (ii) and (iii) follows from Taylor's expansion.

$$df(\vec{q},\vec{p},t) = df(\vec{q},\vec{p},t_0) + \frac{\partial f(\vec{q},\vec{p},t)}{\partial t}\Big|_{t=t_0} \quad \text{——(9)}$$

and

$$dQ_f(\vec{q},\vec{p},t) = dQ_f(\vec{q},\vec{p},t_0) + \frac{\partial Q_f(\vec{q},\vec{p},t)}{\partial t}\Big|_{t=t_0} \quad \text{——(10)}$$

7.5.1/ Lagrange Brackets.

Let u and v be any two members of the set of variables $q_1, q_2, \dots q_n$ and $p_1, p_2, \dots p_n$; then the lagrange bracket associated with the transformation

$$\vec{Q} = \vec{Q}(\vec{q},\vec{p},t) \quad , \quad \vec{P} = \vec{P}(\vec{q},\vec{p},t) \quad \text{——(1)}$$

denoted by $[u,v]$ is defined as

$$[u,v] = \sum_i \left(\frac{\partial Q_i}{\partial u}\frac{\partial P_i}{\partial v} - \frac{\partial Q_i}{\partial v}\frac{\partial P_i}{\partial u} \right) \quad \text{——(2)}$$

Theorem 7.4/ The transformation $\vec{Q} = \vec{Q}(\vec{q},\vec{p},t)$, $\vec{P} = \vec{P}(\vec{q},\vec{p},t)$ —(1) is a canonical transformation if and only if the lagrange bracket $[q_i, q_j]$, $[q_i, p_j]$, $[p_i, p_j]$ satisfy $[q_i, q_j] = 0$, $[q_i, p_j] = \delta_{ij}$, $[p_i, p_j] = 0$ —(2) where δ_{ij} is a kronecker delta with the property,

$$\delta_{ij} = \begin{cases} 1, & i=j \\ 0 & i \neq j \end{cases} \quad \text{——(3)}$$

<u>Proof</u> for simplicity,

let us assume that t is a constant, then the transformation (1) is canonical if and only if, there exists a function f, such that,

$$df = \sum_{\rho} p_\rho \, dq_\rho - \sum_{\rho} P_\rho \, dQ_\rho$$

$$\Rightarrow df = \sum_{\rho} p_\rho \, dq_\rho - \sum_{\rho} P_\rho \left(\sum_j \frac{\partial Q_\rho}{\partial q_j} dq_j + \sum_j \frac{\partial Q_\rho}{\partial p_j} dp_j \right)$$

$$\Rightarrow df = \sum_j \left(p_j - \sum_k P_k \frac{\partial Q_k}{\partial q_j} \right) dq_j + \sum_j \left(- \sum_k P_k \frac{\partial Q_k}{\partial p_j} \right) dp_j \quad \text{—— (4)}$$

Now, we define $A_j = p_j - \sum_k P_k \dfrac{\partial Q_k}{\partial q_j}$ —— (5)

and $B_j = - \sum_k P_k \dfrac{\partial Q_k}{\partial p_j}$ —— (6)

So that eq^n (4) becomes,

$$df = \sum_j A_j \, dq_j + \sum_j B_j \, dp_j \quad \text{—— (7)}$$

But the necessary and sufficient conditions that there exist a function f such that eq^n(7) is a perfect differential are given by

$$\cancel{\frac{\partial A_j}{\partial q_j}} \quad \frac{\partial A_j}{\partial q_i} = \frac{\partial A_i}{\partial q_j} \quad \text{—— (8)}$$

$$\frac{\partial A_j}{\partial p_i} = \frac{\partial B_i}{\partial q_j} \quad \text{—— (9)}$$

$$\frac{\partial B_i}{\partial p_i} = \frac{\partial B_i}{\partial p_j} \quad \text{———} (10)$$

Substituting eqⁿ (5) into eqⁿ (8), we get

$$\frac{\partial}{\partial q_i}\left(p_j - \sum_k p_k \frac{\partial Q_k}{\partial q_j}\right) = \frac{\partial}{\partial q_j}\left(p_i - \sum_k p_k \frac{\partial Q_k}{\partial q_i}\right)$$

on carrying out the differentiation we get

$$\Rightarrow \frac{\partial p_j}{\partial q_i} - \sum_k \frac{\partial p_k}{\partial q_i}\frac{\partial Q_k}{\partial q_j} - \sum_k p_k \frac{\partial^2 Q_k}{\partial q_i \partial q_j} = \frac{\partial p_i}{\partial q_j} - \sum_k \frac{\partial p_k}{\partial q_j}\frac{\partial Q_k}{\partial q_i} - \sum_k p_k \frac{\partial^2 Q_k}{\partial q_j \partial q_i}$$

$$\Rightarrow -\sum_k \frac{\partial p_k}{\partial q_i}\frac{\partial Q_k}{\partial q_j} - \sum_k p_k \frac{\partial^2 Q_k}{\partial q_i \partial q_j} = -\sum_k \frac{\partial p_k}{\partial q_j}\frac{\partial Q_k}{\partial q_i} - \sum_k p_k \frac{\partial^2 Q_k}{\partial q_j \partial q_i}$$

$$\Rightarrow -\sum_k \frac{\partial p_k}{\partial q_i}\frac{\partial Q_k}{\partial q_j} = -\sum_k \frac{\partial p_k}{\partial q_j}\frac{\partial Q_k}{\partial q_i}$$

$$\Rightarrow \sum_k \frac{\partial p_k}{\partial q_j}\frac{\partial Q_k}{\partial q_i} - \sum_k \frac{\partial p_k}{\partial q_i}\frac{\partial Q_k}{\partial q_j} = 0$$ (Transforming R.H.S to L.H.S)

$$\Rightarrow \sum_k \frac{\partial Q_k}{\partial q_i}\frac{\partial p_k}{\partial q_j} - \sum_k \frac{\partial Q_k}{\partial q_j}\frac{\partial p_k}{\partial q_i} = 0 \quad \text{———} (11)$$

which is just the lagrange's bracket $[q_i, q_j]$ and thus we obtained $[q_i, q_j] = 0$ ——— (12)

Similarly, substituting (5) into (9) and (6) into (0)
we get, $[q_i, p_j] = \delta_{ij}$ ——— (A)

and $[p_i, p_j] = 0$ ——— (B)

→ Lagrange bracket doesnot hold commutative law.

$$[u, v] = \sum_i \left(\frac{\partial Q_i}{\partial u} \frac{\partial P_i}{\partial v} - \frac{\partial Q_i}{\partial v} \frac{\partial P_i}{\partial u} \right)$$

$$= - \sum_i \left(\frac{\partial Q_i}{\partial v} \frac{\partial P_i}{\partial u} - \frac{\partial Q_i}{\partial u} \frac{\partial P_i}{\partial v} \right)$$

$$= -[v, u]$$

Properties of Canonical Transformation

P_1 : If the Transformation $\vec{Q} = \vec{Q}(\vec{q}, \vec{p}, t)$, $\vec{P} = \vec{P}(\vec{q}, \vec{p}, t)$ is a canonical transformation, the Jacobian of the transformation is equal ± 1.

i.e

$$J\left(\frac{\vec{Q}, \vec{P}}{\vec{q}, \vec{p}}\right) = \pm 1 \qquad \text{——— (1)}$$

$$J\left(\frac{\vec{Q},\vec{P}}{\vec{q},\vec{p}}\right) = \begin{vmatrix} \frac{\partial Q_1}{\partial q_1} & \cdots & \frac{\partial Q_n}{\partial q_1} & \frac{\partial P_1}{\partial q_1} & \cdots & \frac{\partial P_n}{\partial q_1} \\ \vdots & & & & & \\ \frac{\partial Q_1}{\partial q_n} & \cdots & \frac{\partial Q_n}{\partial q_n} & \frac{\partial P_1}{\partial q_n} & \cdots & \frac{\partial P_n}{\partial q_n} \\ \frac{\partial Q_1}{\partial p_1} & \cdots & \frac{\partial Q_n}{\partial p_1} & \frac{\partial P_1}{\partial p_1} & \cdots & \frac{\partial P_n}{\partial p_1} \\ \vdots & & & & & \\ \frac{\partial Q_1}{\partial p_n} & \cdots & \frac{\partial Q_n}{\partial p_n} & \frac{\partial P_1}{\partial p_n} & \cdots & \frac{\partial P_n}{\partial p_n} \end{vmatrix}$$

On performing the following sequences of operation on the determenant

(i) multiplying the bottom set of n rows by (-1)

(ii) multiplying the right hand set of n columns by (-1).

(iii) ~~multiply~~ interchanging the top set of n rows with the bottom set of n rows, we see –

$$J\left(\frac{\vec{Q},\vec{P}}{\vec{q},\vec{p}}\right) = \begin{vmatrix} \frac{\partial Q_1}{\partial q_1} & \cdots & \frac{\partial Q_n}{\partial q_1} & -\frac{\partial P_1}{\partial q_1} & \cdots & -\frac{\partial P_n}{\partial q_1} \\ \vdots & & & & & \\ \frac{\partial Q_1}{\partial q_n} & \cdots & \frac{\partial Q_n}{\partial q_n} & -\frac{\partial P_1}{\partial q_n} & \cdots & -\frac{\partial P_n}{\partial q_n} \\ -\frac{\partial Q_1}{\partial p_1} & \cdots & -\frac{\partial Q_n}{\partial p_1} & \frac{\partial P_1}{\partial p_1} & \cdots & \frac{\partial P_n}{\partial p_1} \\ \vdots & & & & & \\ -\frac{\partial Q_1}{\partial p_n} & \cdots & -\frac{\partial Q_n}{\partial p_n} & \frac{\partial P_1}{\partial p_n} & \cdots & \frac{\partial P_n}{\partial p_n} \end{vmatrix} \quad \text{((ii) is applied here)}$$

$$J\left(\frac{\vec{Q},\vec{P}}{\vec{q},\vec{p}}\right) = \begin{vmatrix} -\frac{\partial Q_1}{\partial p_1} & \cdots & -\frac{\partial Q_n}{\partial p_1} & \frac{\partial P_1}{\partial p_1} & \cdots & \frac{\partial P_n}{\partial p_1} \\ \vdots \\ -\frac{\partial Q_1}{\partial p_n} & \cdots & -\frac{\partial Q_1}{\partial p_n} & \frac{\partial P_1}{\partial p_n} & \cdots & \frac{\partial P_n}{\partial p_n} \\ \frac{\partial Q_1}{\partial q_1} & \cdots & \frac{\partial Q_n}{\partial q_1} & -\frac{\partial P_1}{\partial q_1} & \cdots & -\frac{\partial P_n}{\partial q_1} \\ \vdots \\ \frac{\partial Q_1}{\partial q_n} & \cdots & \frac{\partial Q_n}{\partial q_n} & -\frac{\partial P_1}{\partial q_n} & \cdots & -\frac{\partial P_n}{\partial q_n} \end{vmatrix}$$ ((iii) is applied here)

performing further the following series of operation in order, i.e;

(i) interchanging the left hand n columns with the right hand n columns.

(ii) Taking the transpose of the resultant, the value of the determinant remains unchanged and the result is.

$$J\left(\frac{\vec{Q},\vec{P}}{\vec{q},\vec{p}}\right) = \begin{vmatrix} \frac{\partial P_1}{\partial p_1} & \cdots & \frac{\partial P_n}{\partial p_1} & -\frac{\partial Q_1}{\partial p_1} & \cdots & -\frac{\partial Q_n}{\partial p_1} \\ \vdots \\ \frac{\partial P_1}{\partial p_n} & \cdots & \frac{\partial P_n}{\partial p_n} & -\frac{\partial Q_1}{\partial p_n} & \cdots & -\frac{\partial Q_n}{\partial p_n} \\ -\frac{\partial P_1}{\partial q_1} & \cdots & -\frac{\partial P_n}{\partial q_1} & \frac{\partial Q_1}{\partial q_1} & \cdots & \frac{\partial Q_n}{\partial q_1} \\ \vdots \\ -\frac{\partial P_1}{\partial q_n} & \cdots & -\frac{\partial P_n}{\partial q_n} & \frac{\partial Q_1}{\partial q_n} & \cdots & \frac{\partial Q_n}{\partial q_n} \end{vmatrix}$$ ((i) applied)

$$J\left(\frac{\vec{Q},\vec{P}}{\vec{q},\vec{p}}\right) = \begin{vmatrix} \frac{\partial P_1}{\partial p_1} & \cdots & \frac{\partial P_1}{\partial p_n} & -\frac{\partial P_1}{\partial q_1} & \cdots & -\frac{\partial P_1}{\partial q_n} \\ \vdots & & & & & \\ \frac{\partial P_n}{\partial p_1} & \cdots & \frac{\partial P_n}{\partial p_n} & -\frac{\partial P_n}{\partial q_1} & \cdots & -\frac{\partial P_n}{\partial q_n} \\ -\frac{\partial Q_1}{\partial p_1} & \cdots & -\frac{\partial Q_1}{\partial p_n} & \frac{\partial Q_1}{\partial q_1} & \cdots & \frac{\partial Q_1}{\partial q_n} \\ \vdots & & & & & \\ -\frac{\partial Q_n}{\partial p_1} & \cdots & -\frac{\partial Q_n}{\partial p_n} & \frac{\partial Q_n}{\partial q_1} & \cdots & \frac{\partial Q_n}{\partial q_n} \end{vmatrix} \quad \text{(ii) applied.}$$

Multiplying matrix (1) & (2) we get

1^{st} row and 1^{st} column elt.

$$= \frac{\partial Q_1}{\partial q_1}\frac{\partial P_1}{\partial p_1} + \cdots + \frac{\partial Q_n}{\partial q_1}\frac{\partial P_n}{\partial p_1} - \frac{\partial P_1}{\partial q_1}\frac{\partial Q_1}{\partial p_1} - \cdots - \frac{\partial P_n}{\partial q_1}\frac{\partial Q_n}{\partial p_1}$$

$$= \sum_k \left(\frac{\partial Q_k}{\partial q_1}\frac{\partial P_k}{\partial p_1} - \frac{\partial P_k}{\partial q_1}\frac{\partial Q_k}{\partial p_1}\right) = [q_1, p_1].$$

n^{th} row and n^{th} column.

$$= \frac{\partial Q_1}{\partial q_n}\frac{\partial P_1}{\partial p_n} + \cdots + \frac{\partial Q_n}{\partial q_n}\frac{\partial P_n}{\partial p_n} - \frac{\partial P_1}{\partial q_n}\frac{\partial Q_1}{\partial p_n} - \cdots - \frac{\partial P_n}{\partial q_n}\frac{\partial Q_n}{\partial p_n}$$

$$= \sum_k \left(\frac{\partial Q_k}{\partial q_n}\frac{\partial P_k}{\partial p_n} - \frac{\partial P_k}{\partial q_n}\frac{\partial Q_k}{\partial p_n}\right) = [q_n, p_n]$$

n^{th} row & 1^{st} column.

$$= \frac{\partial Q_1}{\partial q_n}\frac{\partial P_1}{\partial p_1} + \cdots + \frac{\partial Q_n}{\partial q_1}\frac{\partial P_n}{\partial p_1} - \frac{\partial P_1}{\partial q_n}\frac{\partial Q_1}{\partial p_1} - \cdots - \frac{\partial P_n}{\partial q_n}\frac{\partial Q_n}{\partial p_1}$$

$$= \sum_k \left(\frac{\partial Q_k}{\partial q_n}\frac{\partial P_k}{\partial p_1} - \frac{\partial P_k}{\partial q_n}\frac{\partial Q_k}{\partial p_1}\right) = [q_n, p_1].$$

1st row & nth column.

$$\frac{\partial Q_1}{\partial q_1}\frac{\partial P_1}{\partial p_n}+\cdots+\frac{\partial Q_n}{\partial q_1}\frac{\partial P_n}{\partial p_n}-\frac{\partial P_1}{\partial q_1}\frac{\partial Q_1}{\partial p_n}-\cdots-\frac{\partial P_n}{\partial q_1}\frac{\partial Q_n}{\partial p_n}$$

$$=\sum_k\left(\frac{\partial Q_k}{\partial q_1}\frac{\partial P_k}{\partial p_n}-\frac{\partial P_k}{\partial q_1}\frac{\partial Q_k}{\partial p_n}\right)=[q_1,p_n]$$

$(n+1)^{th}$ row & $(n+1)^{th}$ column

$$=-\frac{\partial Q_1}{\partial p_1}\frac{\partial P_1}{\partial q_1}+\cdots-\frac{\partial Q_n}{\partial p_1}\frac{\partial P_n}{\partial q_1}+\frac{\partial P_1}{\partial p_1}\frac{\partial Q_1}{\partial q_1}+\cdots+\frac{\partial P_n}{\partial p_1}\frac{\partial Q_n}{\partial q_1}$$

$$=\sum_k\left(-\frac{\partial Q_k}{\partial p_1}\frac{\partial P_k}{\partial q_1}+\frac{\partial P_k}{\partial p_1}\frac{\partial Q_k}{\partial q_1}\right)$$

$$=\sum_k\left(\frac{\partial Q_k}{\partial q_1}\frac{\partial P_k}{\partial p_1}-\frac{\partial P_k}{\partial q_1}\frac{\partial Q_k}{\partial p_1}\right)=[q_1,p_1]$$

$2n^{th}$ row & $2n^{th}$ column

$$-\frac{\partial Q_1}{\partial p_n}\frac{\partial P_1}{\partial q_n}-\cdots-\frac{\partial Q_n}{\partial p_n}\frac{\partial P_n}{\partial q_n}+\frac{\partial P_1}{\partial p_n}\cdot\frac{\partial Q_1}{\partial q_n}+\cdots+\frac{\partial P_n}{\partial p_n}\frac{\partial Q_n}{\partial q_n}$$

$$=\sum_k\left(-\frac{\partial Q_k}{\partial p_n}\frac{\partial P_k}{\partial q_n}+\frac{\partial P_k}{\partial p_n}\frac{\partial Q_k}{\partial q_n}\right)$$

$$=\sum_k\left(\frac{\partial Q_k}{\partial q_n}\frac{\partial P_k}{\partial p_n}-\frac{\partial P_k}{\partial q_n}\frac{\partial Q_k}{\partial p_n}\right)=[q_n,p_n]$$

1st row & $(n+1)^{th}$ column.

$$-\frac{\partial Q_1}{\partial q_1}\frac{\partial P_1}{\partial q_1}-\cdots-\frac{\partial Q_n}{\partial q_1}\frac{\partial P_n}{\partial q_1}+\frac{\partial P_1}{\partial q_1}\frac{\partial Q_1}{\partial q_1}+\cdots+\frac{\partial P_n}{\partial q_1}\frac{\partial Q_n}{\partial q_1}$$

$$=\sum_k\left(-\frac{\partial Q_k}{\partial q_1}\frac{\partial P_k}{\partial q_1}+\frac{\partial P_k}{\partial q_1}\frac{\partial Q_k}{\partial q_1}\right)$$

$$=\sum_k\left(\frac{\partial Q_k}{\partial q_1}\frac{\partial P_k}{\partial q_1}-\cancel{\partial Q_k}\frac{\partial P_k}{\partial q_1}\frac{\partial Q_k}{\partial q_1}\right)=[q_1,q_1]$$

1^{st} row and $(2n)^{th}$ column.

$$-\frac{\partial Q_1}{\partial q_1}\frac{\partial P_1}{\partial q_n} - \cdots - \frac{\partial Q_n}{\partial q_1}\frac{\partial P_n}{\partial q_n} + \frac{\partial P_1}{\partial q_1}\frac{\partial Q_1}{\partial q_n} + \cdots + \frac{\partial P_n}{\partial q_1}\frac{\partial Q_n}{\partial q_n}.$$

$$= \sum_k \left(-\frac{\partial Q_k}{\partial q_1}\frac{\partial P_k}{\partial q_n} + \frac{\partial P_k}{\partial q_1}\frac{\partial Q_k}{\partial q_n} \right)$$

$$= \sum_k \left(\frac{\partial Q_k}{\partial q_n}\frac{\partial P_k}{\partial q_1} - \frac{\partial P_k}{\partial q_n}\frac{\partial Q_k}{\partial q_1} \right) = [q_n, q_1].$$

Similarly, after finding all the terms we get

$$\left[J\left(\frac{\vec{Q},\vec{P}}{\vec{q},\vec{p}}\right)\right]^2 = \begin{vmatrix} [q_1,p_1] & \cdots & [q_1,p_n] & [q_1,q_1] & \cdots & [q_n,q_1] \\ \vdots & & & & & \\ [q_n,p_1] & \cdots & [q_n,p_n] & [q_1,q_n] & \cdots & [q_n,q_n] \\ [p_1,p_1] & \cdots & [p_1,p_n] & [q_1,p_1] & \cdots & [q_n,p_1] \\ \vdots & & & & & \\ [p_n,p_1] & \cdots & [p_n,p_n] & [q_1,p_n] & \cdots & [q_n,p_n] \end{vmatrix}$$

$$\Rightarrow \left[J\left(\frac{\vec{Q},\vec{P}}{\vec{q},\vec{p}}\right)\right]^2 = \begin{vmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{vmatrix} = 1.$$

$$\therefore \quad J\left(\frac{\vec{Q},\vec{P}}{\vec{q},\vec{p}}\right) = 1^{2} = \pm 1$$

P_2: If the Transformation $\vec{Q} = \vec{Q}(\vec{q}, \vec{p}, t)$, $\vec{P} = \vec{P}(\vec{q}, \vec{p}, t)$ is a canonical transformation, then the inverse transformation $\vec{q} = \vec{q}(\vec{Q}, \vec{P}, t)$ and $\vec{p} = (\vec{Q}, \vec{P}, t)$ is also a ~~cor~~ canonical transformation.

Verification

The Transformation $\vec{Q} = \vec{Q}(\vec{q}, \vec{p}, t)$, $\vec{P} = \vec{P}(\vec{q}, \vec{p}, t)$ possesses inverse if the Jacobian satisfies the condition

$$J\left(\frac{\vec{Q}, \vec{P}}{\vec{q}, \vec{p}}\right) \neq 0.$$

In fact, we have verified the truth of this condition under P_1.

$\therefore$ the function $f(\vec{q}, \vec{p}, t)$ exists satisfying

$$df(\vec{q}, \vec{p}, t_0) = \sum_f p_f\, dq_f - \sum_f P_f(\vec{q}, \vec{p}, t_0)\, dQ_f(\vec{q}, \vec{p}, t_0) \quad \text{---(3)}$$

if we substitute the inverse transformation into eq^n (3), we find that the function $-f(\vec{Q}, \vec{P}, t_0)$ satisfies the relation

$$d[-f(\vec{Q}, \vec{P}, t_0)] = \sum_f P_f\, dQ_f - \sum_f p_f(\vec{Q}, \vec{P}, t_0)\, dq_f(\vec{Q}, \vec{P}, t_0) \quad \text{---(4)}$$

which shows that the inverse transformation is also a canonical transformation.

P_3: If the transformations $\vec{q}'' = \vec{q}''(\vec{q}', \vec{p}', t)$, $\vec{p}'' = \vec{p}''(\vec{q}', \vec{p}', t)$ and $\vec{q}' = \vec{q}'(\vec{q}, \vec{p}, t)$, $\vec{p}' = \vec{p}'(\vec{q}, \vec{p}, t)$ are canonical transformation i.e the transformation.

$\vec{q}'' = \vec{q}''(\vec{q}, \vec{p}, t)$ and $\vec{p}'' = \vec{p}''(\vec{q}, \vec{p}, t)$ is also a canonical transformation.

P_4: The set of canonical transformation obeys the associative law of multiplication.

P_5: Property P_2 to P_4 shows that the canonical transformation is a group.

Example 7.4/ Show that the transformation $Q = \log\left(\frac{1}{q}\sin p\right)$, $P = q\cot p$ is a canonical transformation and hence find the function f!

Sol^n:- To show that the given transformation is canonical, the necessary and sufficient conditions are that the Lagrange bracket satisfy the following relations:

$$[q, q] = [p, p] = 0, \quad [q, p] = 1 \quad \text{—— (1)}$$

from the given data

$$\frac{\partial Q}{\partial q} = \frac{1}{\frac{1}{q}\sin p}\left(\frac{-1}{q^2}\sin p\right) = -\frac{q}{\sin p} \times \frac{\sin p}{q^2} = -\frac{1}{q}$$

$$\frac{\partial Q}{\partial p} = \frac{1}{\frac{1}{q}\sin p}\left(\frac{1}{q}\cos p\right) = \frac{q}{\sin p}\cdot\frac{\cos p}{q} = \cot p.$$

$$\frac{\partial P}{\partial q} = \cot p$$

$$\frac{\partial P}{\partial p} = -q\,\mathrm{cosec}^2 p.$$

$$[q,q] = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial q} - \frac{\partial Q}{\partial q}\frac{\partial P}{\partial q} = \frac{-1}{q}\cot p - \left(\frac{-1}{q}\right)\cot p = 0.$$

$$[p,p] = \frac{\partial Q}{\partial p}\cdot\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\cdot\frac{\partial P}{\partial p} = \cot p(-q\,\mathrm{cosec}^2 p) - \cot p(-q\,\mathrm{cosec}^2 p)$$
$$= 0.$$

$$[q,p] = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial P}{\partial q}\frac{\partial Q}{\partial p} = \frac{-1}{q}(-q\,\mathrm{cosec}^2 p) - \cot^2 p$$
$$= \mathrm{cosec}^2 p - \cot^2 p = 1.$$

$\therefore$ the given Transformation is a canonical Transformation.

The function f is given by

$$df = p\,dq - P\,dQ$$
$$= p\,dq - q\cot p\; d\left[\log\left(\frac{1}{q}\sin p\right)\right]$$
$$= p\,dq - q\cot p\,\frac{1}{\frac{1}{q}\sin p}\left(\frac{1}{q}\cos p\,dp + \sin p\left(\frac{-1}{q^2}\right)dq\right)$$

$$= p\,dq - q\cot p \,\frac{q}{q}\cot p\,dp + q\cot p\,\frac{q}{\sin p}\,\frac{\sin p}{q^2}\,dq$$

$$= p\,dq - q\cot^2 p\,dp + \cot p\,dq$$

$$= (p+\cot p)\,dq - q(\operatorname{cosec}^2 p - 1)\,dp$$

$$= (p+\cot p)\,dq - q\operatorname{cosec}^2 p\,dp + q\,dp$$

$$= p\,dq + \cot p\,dq - q\operatorname{cosec}^2 dp + q\,dp$$

$$df = (p\,dq + q\,dp) + (\cot p\,dq - q\operatorname{cosec}^2 p\,dp)$$

$$df = d(pq) + d(q\cot p)$$

on integrating both side, we get —

$$f = pq + q\cot p.$$

Example 7.5| Show that the Transformation $Q = p^2+q;\ P = p+t$ is canonical and hence find the function $f(q,p,t)$ such that

$$df(\vec{q},p,t_0) = p\,dq - P(q,p,t_0)\,dQ(q,p,t_0).$$

Sol^n:— To show that the given Transformation is canonical, the necessary and sufficient condition is that the lagrange bracket satisfy:

$$[q,q] = [p,p] = 0, \quad [q,p] = 1$$

from, the given data, we get

$$Q = p^2+q, \quad P = p+t$$

$$\frac{\partial Q}{\partial q} = 1, \quad \frac{\partial Q}{\partial p} = 2p, \quad \frac{\partial P}{\partial q} = 0, \quad \frac{\partial P}{\partial p} = 1.$$

$$[q, q] = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial q} - \frac{\partial P}{\partial q}\frac{\partial Q}{\partial q} = 1\times 0 - 0\times 1 = 0$$

$$[p, p] = \frac{\partial Q}{\partial p}\frac{\partial P}{\partial p} - \frac{\partial P}{\partial p}\frac{\partial Q}{\partial p} = 2p(1) - (1)\cdot 2p = 0.$$

$$[q, p] = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial P}{\partial q}\frac{\partial Q}{\partial p} = 1\times 1 - 0\times 2p = 1$$

Hence, the given transformation is canonical and there exist a function $f(q, p, t)$ such that

$$df(q, p, t_0) = p\,dq - P(q, p, t_0)\,dQ(q, p, t_0)$$

using the given data,

$$df(q, p, t_0) = p\,dq - (p + t_0)(2p\,dp + dq)$$

$$= p\,dq - 2p(p + t_0)\,dp - (p + t_0)\,dq$$

$$= (p - p - t_0)\,dq - 2p(p + t_0)\,dp$$

$$= -t_0\,dq - 2p(p + t_0)\,dp.$$

Integrating the above result at a time $t = t_0$ along any path from some refrences point (q_0, p_0) to (q, p) we get

$$f(q, p, t_0) - f(q_0, p_0, t_0) = -t_0 q - \frac{2}{3}p^3 - p^2 t_0$$

$$f(q, p, t_0) = -q t_0 - \frac{2}{3}p^3 - p^2 t_0 + f(q_0, p_0, t_0).$$

7.5.2 Poisson Brackets

Suppose we are given two functions f and g of the dynamical variables $\vec{q}, \vec{p}$ and time t, then the Poisson Bracket expression for the two functions is defined as

$$(f,g) = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right)$$

An important property of Poisson bracket is Jacobi's identity, namely,

$$(f,(g,h)) + (g,(h,f)) + (h,(f,g)) = 0$$

where f, g and h are functions of $\vec{q}$ and $\vec{p}$.

To verify this identity, let us recall the Hamilton canonical equations for a dynamical system.

i.e $\dot{q}_i = \frac{\partial H}{\partial p_i}$, $\dot{p}_i = -\frac{\partial H}{\partial q_i}$ ——— (1)

using the Poisson Bracket notation and (1), we have

$$(q_i, H) = \sum_{k=1}^{n} \left(\frac{\partial q_i}{\partial q_k} \frac{\partial H}{\partial p_k} - \frac{\partial q_i}{\partial p_k} \frac{\partial H}{\partial q_k} \right) = \frac{\partial H}{\partial p_i} \quad \text{——— (2)}$$

and

$$(p_i, H) = \sum_{k=1}^{n} \left(\frac{\partial p_i}{\partial q_k} \frac{\partial H}{\partial p_k} - \frac{\partial p_i}{\partial p_k} \cdot \frac{\partial H}{\partial q_k} \right) = -\frac{\partial H}{\partial q_i} \quad \text{——— (3)}$$

from (1), (2) and (3) the canonical eq^n can be written as $\dot{q}_i (q_i, H)$, $p_i = (p_i, H)$ ——— (4)

Since f and g are functions of $\vec{q}, \vec{p}$ and t, we have from calculus,

$$\frac{df}{dt} = \sum_i \left(\frac{\partial f}{\partial q_i} \dot{q}_i + \frac{\partial f}{\partial p_i} \dot{p}_i \right) + \frac{\partial f}{\partial t}$$

$$\frac{df}{dt} = \sum_i \left(\frac{\partial f}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial H}{\partial q_i} \right) + \frac{\partial f}{\partial t} \quad \text{(using (4))}$$

$$\frac{df}{dt} = (f, H) + \frac{\partial f}{\partial t} \quad \text{——— (5)}$$

Similarly, we can show that

$$\frac{dg}{dt} = (g, H) + \frac{\partial g}{\partial t} \quad \text{——— (6)}$$

If f and g are not explicit function of time, then

$$(f, H) + \frac{\partial f}{\partial t} = 0 \quad \text{——— (7)}$$

$$(g, H) + \frac{\partial g}{\partial t} = 0 \quad \text{——— (8)}$$

utilizing the result (5), we can also write

$$\frac{d}{dt}(f, g) = ((f, g), H) + \frac{\partial}{\partial t}(f, g) = 0 \quad \text{(from 7)}$$

$$\Rightarrow \frac{d}{dt}(f, g) = ((f, g), H) + \left(\frac{\partial f}{\partial t}, g\right) + \left(f, \frac{\partial g}{\partial t}\right) = 0 \quad \text{——— (9)}$$

$$\Rightarrow ((f,g),H) + \left(\frac{\partial f}{\partial t}, g\right) + \left(f, \frac{\partial g}{\partial t}\right) = 0.$$

$$\Rightarrow ((f,g),H) + (-(f,H),g) + (f, -(g,H)) = 0 \quad (\text{using } (7) \,\&\, (8))$$

writing in reverse order

$$\Rightarrow (f, -(g,H)) + (-(f,H),g) + ((f,g),H) = 0.$$

$$\Rightarrow -(f,(g,H)) + ((H,f),g) + ((f,g),H) = 0.$$

$$\Rightarrow -(f,(g,H)) - (g,(H,f)) - (H,(f,g)) = 0.$$

$$\Rightarrow (f,(g,H)) + (g,(H,f)) + (H,(f,g)) = 0 \quad \text{—— (10)}$$

H is a function of q's and p's the Jacobi's identity follows —

It can be shown that the Lagrange's and Poisson bracket are reciprocal Quantity.

Ex-7.6) If q_1 and q_2 are generalized coordinates and p_1 and p_2 are the corresponding generalized momenta, then find the Poisson bracket (X,Y) where $X = q_1^2 + q_2^2$ and $Y = 2p_1 + p_2$.

Sol^n:— Here $X = q_1^2 + q_2^2$, $Y = 2p_1 + p_2$.

$$(X, Y) = \cancel{\partial H}\left(\frac{\partial X}{\partial q_1}\frac{\partial Y_1}{\partial p_1} - \frac{\partial X}{\partial p_1}\frac{\partial Y}{\partial q_1}\right) + \left(\frac{\partial X}{\partial q_2}\frac{\partial Y}{\partial p_2} - \frac{\partial X}{\partial p_2}\frac{\partial Y}{\partial p_2}\right)$$

$$= (2q_1 \times 2 - 0 \times 0) + (2q_2 \times 1 - 0 \times 0)$$

$$= 4q_1 + 2q_2.$$

Small Oscillations

Lagrange's Method:

Let the position of the system be defined by n independent co-ordinates $q_1, q_2, \ldots, q_n$ and let these be so choisen that they vanish in the position of equilibrium. Then if the system makes small oscillations about this position, the coordinate will remain small throughout the motion and so will the velocities $\dot{q}_1, \dot{q}_2, \ldots \dot{q}_n$.

The kinetic energy T is given by

$$2T = a_{11}\dot{q}_1^2 + 2a_{12}\dot{q}_1\dot{q}_2 + a_{22}\dot{q}_2^2 + \cdots + a_{nn}\dot{q}_n^2 \quad \text{--- (1)}$$

where the a's are functions of the coordinates.

we may suppose that these functions are expanded in powers of the co-ordinates, and as in small oscillations, we only required the dynamical equations to be corrected to the first order of the small quantities which they contain, we may neglect the variable parts of these coefficients and taken them to be constants.

On this hypothesis T does not contain the co-ordinates explicity and

$$\frac{\partial T}{\partial q_r} = 0 \quad (r = 1, 2, \ldots n) \quad \text{--- (2)}$$

Let V be the potential energy in a position which is only a slight departure from the equilibrium position.

Then V is a function of the co-ordinate, which can be expanded in powers of the co-ordinates thus.

$$2V = 2V_0 + 2c_1 q_1 + 2c_2 q_2 + \cdots + 2c_n q_n + c_{11} q_1^2 + 2c_{12} q_1 q_2 + \cdots + c_{nn} q_n^2 + \cdots \quad \text{--- (3)}$$

But since the potential energy is stationary in the position of equilibrium, we must have

$$\frac{\partial V}{\partial q_1} = \frac{\partial V}{\partial q_2} = \cdots = \frac{\partial V}{\partial q_n} = 0 \quad \text{--- (4)}$$

when the co-ordinates vanish,

so that $c_1 = c_2 = \cdots = c_n = 0$.

and $2V = 2V_0 + c_{11} q_1^2 + 2c_{12} q_1 q_2 + \cdots + c_{nn} q_n^2$ --- (5)

it being sufficient for our purpose to retain the quadratic term and neglect all higher powers.

Further, if the position where were one of unstable equilibrium the system would not oscillate about it.

we may therefore assume that the position is one of stable equilibrium, so that V_0 represents a minimum value of V and the quadratic terms in (5) must be positive for all values of the co-ordinates. The quadratic expression (1) for the kinetic energy is also essentially positive for all values of the velocities and the conditions for this are that the discriminant

$$\begin{vmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ - & - & - & - \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{vmatrix}$$

and the like determinants obtained by erasing the first row and column and repeating this process shall all be positive.

we have a like set of condition for the coefficients in (5)

The typical lagranges equation formed from (1) and (5)

taking account of (2) is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right) + \frac{\partial V}{\partial q_r} = 0 \qquad (r = 1, 2, \ldots, n) \quad \text{——(6)}$$

and this gives a set of n equations —

$$\left.\begin{aligned} a_{11}\ddot{q}_1 + a_{21}\ddot{q}_2 + \cdots + a_{n1}\ddot{q}_n + c_{11}q_1 + c_{21}q_2 + \cdots + c_{n1}q_n &= 0 \\ a_{12}\ddot{q}_1 + a_{22}\ddot{q}_2 + \cdots + a_{n2}\ddot{q}_n + c_{12}q_1 + c_{22}q_2 + \cdots + c_{n2}q_n &= 0 \\ \text{— — — — — — — — —} & \\ a_{1n}\ddot{q}_1 + a_{2n}\ddot{q}_2 + \cdots + a_{nn}\ddot{q}_n + c_{1n}q_1 + c_{2n}q_2 + \cdots + c_{nn}q_n &= 0 \end{aligned}\right\} \quad —(7)$$

where.

These are lagrange's equations for the small oscillations and since there are n equations of the second order their complete solution will contain $2n$ arbitrary constants.

Take as a trial solution.

$$q_1 = M_1 \sin(pt+\epsilon), \quad q_2 = M_2 \sin(pt+\epsilon), \ldots \ldots q_n = M_n \sin(pt+\epsilon) \quad —(8)$$

By substituting in (7), we get

$$\left.\begin{aligned} (a_{11}p^2 - c_{11})M_1 + (a_{21}p^2 - c_{21})M_2 + \cdots + (a_{n1}p^2 - c_{n1})M_n &= 0 \\ (a_{12}p^2 - c_{12})M_1 + (a_{22}p^2 - c_{22})M_2 + \cdots + (a_{n2}p^2 - c_{n2})M_n &= 0 \\ \text{— — — — — — — —} & \\ (a_{1n}p^2 - c_{1n})M_1 + (a_{2n}p^2 - c_{2n})M_2 + \cdots + (a_{nn}p^2 - c_{nn})M_n &= 0 \end{aligned}\right\} \quad —(9)$$

In order that (9) may have a solution in which not all of $M_1, M_2, \ldots, M_n$ are zero, we must have —

$$\begin{vmatrix} a_{11}p^2-c_{11} & a_{21}p^2-c_{21} & \cdots & a_{n1}p^2-c_{n1} \\ a_{12}p^2-c_{12} & a_{22}p^2-c_{22} & \cdots & a_{n2}p^2-c_{n2} \\ - & - & - & - \\ a_{1n}p^2-c_{1n} & a_{2n}p^2-c_{2n} & \cdots & a_{nn}p^2-c_{nn} \end{vmatrix} = 0 \quad \text{—(10)}$$

This is an equation of the n^{th} degree in p^2, and it will be the roots are all real and positive provided V is essential positive. we shall confine our attention to the case in which these roots are distinct.

Thus, there appear to be $2n$ values for p of the form $\pm p_1, \pm p_2, \ldots \pm p_n$; but we need not concern ourselves with the negative values, because a positive and negative pair only imply exponential terms in the solution of the form.

$$A e^{ipt} + B e^{-ipt}$$

which we rearrange in the form $M \sin(pt+\epsilon)$.

Hence we confine our considerations to the positive roots.

Taking any one root p_1 and substitute it in (9), we have a set of n equations for the corresponding values of the ~~rations~~ ratios of $M_1, M_2, \ldots M_n$; and they give.

$$\frac{M_1}{I_1(p_r)} = \frac{M_2}{I_2(p_r)} = \cdots = \frac{M_n}{I_n(p_r)} = \alpha_r \quad (r = 1, 2 \cdots n) \qquad (11)$$

where $I_1 I_2, \dots I_n$ denote the minars of any row or column in (10). Equation (10) may be called the Period or frequency Equations, as it determines the periods.

$\frac{2\pi}{p_1}, \frac{2\pi}{p_2}, \dots \frac{2\pi}{p_n}$ of the oscillations.

Finally, since the equations (7) are linear, we get the complete solution by summing for each co-ordinate the n ~~d~~ different solution of type (8)

Thus,

$$\left.\begin{aligned}
q_1 &= \alpha_1 I_1(p_1)\sin(p_1 t + \epsilon_1) + \alpha_2 I_1(p_2)\sin(p_2 t + \epsilon_2) + \cdots \\
&\quad \cdots + \alpha_n I_1(p_n)\sin(p_n t + \epsilon_n) \\
q_2 &= \alpha_1 I_2(p_1)\sin(p_1 t + \epsilon_1) + \alpha_2 I_2(p_2)\sin(p_2 t + \epsilon_2) + \cdots \\
&\quad \cdots + \alpha_n I_2(p_n)\sin(p_n t + \epsilon_n) \\
&\cdots\cdots\cdots \\
q_n &= \alpha_1 I_n(p_1)\sin(p_1 t + \epsilon_1) + \alpha_2 I_n(p_2)\sin(p_2 t + \epsilon_2) + \cdots \\
&\quad \cdots + \alpha_n I_n(p_n)\sin(p_n t + \epsilon_n)
\end{aligned}\right\} \qquad (12)$$

where $\alpha_1, \alpha_2, \dots, \alpha_n, \epsilon_1, \epsilon_2, \dots \epsilon_n$ are the 2n arbitrary constants, which depend on the initial conditions.

If V were not a minimum in the position of equilibrium it would be possible for the equation in p^2 to have negative roots and this ~~motion~~ would give rise to terms in the solution of (7) of the form $Ae^{pt} + Be^{-pt}$, where p is real, implying finite departure from the equilibrium position and instability.

We notice from (11) that the solution fails if all the first minors of the determinant in (10) are zero.

Denoting the determinant by Δ, it is easy to show that in this case

$$\frac{d\Delta}{d(p^2)} = 0$$

so that the equation $\Delta = 0$ has equal roots.

10.21 The Potential Energy:

In any dynamical problem it may be convenient to separate the forces which depend only on the configuration from other extraneous forces. Then using V for the potential energy of the former forces and Q_r for the generalized component of the latter ~~of~~ type, the Lagrange equations are of the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right) - \frac{\partial T}{\partial q_r} + \frac{\partial V}{\partial q_r} = Q_r \quad \text{---} (1)$$

Hence in any position of equilibrium, we must have

$$Q_r = \frac{\partial V}{\partial q_r} \quad \text{———} (2)$$

It follows that in any conservation system, the extraneous forces which would keep it at rest in any configuration are given by relation of the form (2).

In such cases ~~as~~ as are under consideration in this, V is a quadratic function of the coordinates, we have

Since V is homogeneous,

$$2(V - V_0) = q_1 \frac{\partial V}{\partial q_1} + q_2 \frac{\partial V}{\partial q_2} + \cdots + q_n \frac{\partial V}{\partial q_n}$$

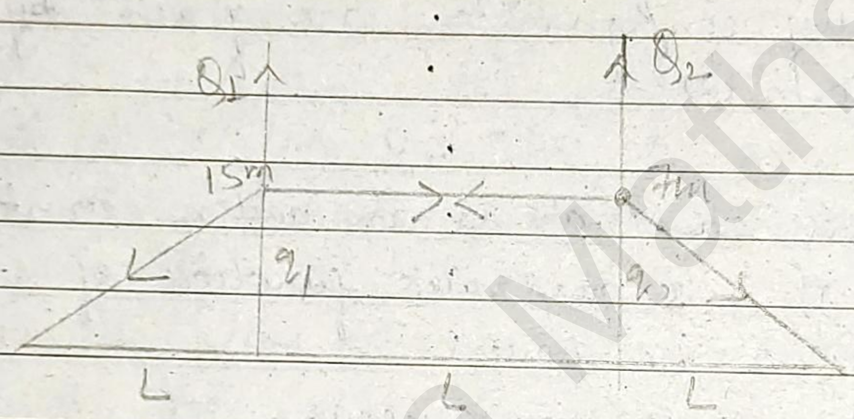
$$\Rightarrow 2(V - V_0) = q_1 Q_1 + q_2 Q_2 + \cdots + q_n Q_n \quad \text{———} (3).$$

10.22 Example: A light string of length $3l$ is stretched horizontally between two fixed points gravity is neglected. Masses 15m, 7m are attached to be points of trisection. The tension in equilibrium is lml. The particle of mass 15m is drawn aside a distance 'a' the other remaining undisplayed, and both are simultaneously released. Prove that in the subsequent motion the displacement of the particle of mass 7m is

$$\frac{15}{26}\, a \left\{ \cos\sqrt{\frac{3\lambda}{35}}\, t - \cos\sqrt{\frac{\lambda}{3}}\, t \right\}$$

Sol^n:— Let q_1, q_2 denote the small displacement of ~~ther~~ the particles of masses $15m$, $7m$ at time t, then:

$$2T = 15m\dot{q}_1^2 + 7m\,\dot{q}_2^2 \qquad \text{——— (1)}$$



we may neglect variations in the ~~tem~~ tension due to the slight extension of the string as they would only introduce terms of a higher order in q_1, q_2 than those which we retain.

To find V we may proceed and let Q_1, Q_2 denote forces which would keep the particles at rest in the displaced position i.e;

$$Q_1 = \lambda m l \left(\frac{q_1}{l} + \frac{q_1 - q_2}{l} \right) = \lambda m (2q_1 - q_2).$$

$$Q_2 = \lambda m l \left(\frac{q_2 - q_1}{l} + \frac{q_2}{l} \right) = \lambda m (2q_2 - q_1)$$

and then $2V = q_1 Q_1 + q_2 Q_2$.

$$= q_1 \lambda m(2q_1 - q_2) + q_2 \lambda m(2q_2 - q_1)$$
$$= \lambda m(2q_1^2 - q_1q_2) + \lambda m(2q_2^2 - q_1q_2)$$
$$= \lambda m(2q_1^2 - 2q_1q_2 + 2q_2^2)$$
$$= 2\lambda m(q_1^2 - q_1q_2 + q_2^2) \quad \text{——— (2)}$$

or we might write down the work done by the tension when small increments are made in the co-ordinate ; thus ;

$$-\delta V = \lambda ml\left(\frac{q_2 - q_1}{l} - \frac{q_1}{l}\right)\delta q_1 + \lambda ml\left(\frac{q_1 - q_2}{l} - \frac{q_2}{l}\right)\delta q_2$$

$$\Rightarrow -\delta V = \lambda m(q_2 - 2q_1)\,\delta q_1 + \lambda m(q_1 - 2q_2)\,\delta q_2 \quad \text{——— (3)}$$

and if we adopt this method there is no need to be find more than the form of δV since $\frac{\partial V}{\partial q_1}$, $\frac{\partial V}{\partial q_2}$ are the actual functions required in the equations of motion.

The lagrange equations are then

$$15\,\ddot{q}_1 - \lambda(q_2 - 2q_1) = 0 \quad \text{and}$$
$$7\,\ddot{q}_2 - \lambda(q_1 - 2q_2) = 0$$

we note that these equations might have been written down at once as the equations of motion of the two particles.

To solve them, we put

$$q_1 = M\cos(pt+\epsilon)$$
$$q_2 = N\cos(pt+\epsilon)$$

and we get

$$\left.\begin{aligned}(2\lambda-15p^2)M-\lambda N&=0\\ -\lambda M+(2\lambda-7p^2)N&=0\end{aligned}\right\} \quad\text{——(4)}$$

The period equation is then

$$(2\lambda-15p^2)(2\lambda-7p^2)-\lambda^2=0 \quad\text{——(5)}$$

with roots $p_1^2 = \frac{3\lambda}{35}$ and $p_2^2 = \frac{\lambda}{3}$ ——(6)

But from (4) $\frac{M}{\lambda} = \frac{N}{2\lambda-15p^2}$,

and substituting the two values of p^2 in succession gives two sets of corresponding values for M, N which may write.

$$\frac{M_1}{7} = \frac{N_1}{5} = \alpha_1 \quad\text{——(7)}$$

$$\frac{M_2}{1} = \frac{N_2}{-3} = \alpha_2 \quad\text{——(8)}$$

Hence, the complete solution of the differential equations is

$$\left.\begin{aligned}q_1 &= 7\alpha_1\cos(p_1t+\epsilon_1)+\alpha_2\cos(p_2t+\epsilon_2)\\ q_2 &= 5\alpha_1\cos(p_1t+\epsilon_1)-3\alpha_2\cos(p_2t+\epsilon_2)\end{aligned}\right\} \quad\text{——(9)}$$

where $\alpha_1, \alpha_2, \epsilon_1, \epsilon_2$ are arbitrary constants and p_1^2, p_2^2 have the values (6).

The initial conditions are $q_1 = a$, $q_2 = 0$, $\dot{q}_1 = 0$, $\dot{q}_2 = 0$. The vanishing of the velocities shows that $\epsilon_1 = \epsilon_2 = 0$, then we have

$$a = 7\alpha_1 + \alpha_2$$

and $0 = 5\alpha_1 - 3\alpha_2$

so that, $\alpha_1 = \frac{3a}{26}$ and $\alpha_2 = \frac{5}{26} a$

and $q_2 = \frac{15a}{26} \left\{ \cos\sqrt{\frac{3\lambda}{35}}\, t - \cos\sqrt{\frac{\lambda}{3}}\, t \right\}$.

10.3 Normal mode:

When we have two homogeneous quadratic functions of any number of variables, one of which is positive for all values of the variables, then by a real linear Transformation, we may clear both expressions of the terms containing products of the variables and at the same time change the coefficients of the square terms in one of the expressions so that each is equal to unity.

Suppose that we Transform the co-ordinates by linear relations.

$$\left.\begin{aligned} q_1 &= b_{11}\xi_{11} + b_{21}\xi_{12} + \cdots + b_{n1}\xi_{1n} \\ q_2 &= b_{12}\xi_{12} + b_{22}\xi_{12} + \cdots + b_{n2}\xi_{1n} \\ &- - - - - - - - - \end{aligned}\right\} \quad (1)$$

where the b's are constants; then the velocities will be transformed by precisely the same set of relations; and by the theorem quoted above, a real transformation can be made so that the kinetic and potential energies are transformed to

$$2T = \dot{\xi}_{11}^2 + \dot{\xi}_{12}^2 + \cdots + \dot{\xi}_{1n}^2 \quad (2)$$

$$2V = \lambda_1 \xi_{11}^2 + \lambda_2 \xi_{12}^2 + \cdots + \lambda_n \xi_{1n}^2 \quad (3)$$

where the λ's are constants.

These give rise to lagrange equations

$$\ddot{\xi}_{11} + \lambda_1 \xi_{11} = 0, \quad \ddot{\xi}_{12} + \lambda_2 \xi_{12} = 0, \quad \cdots \quad \ddot{\xi}_{1n} + \lambda_n \xi_{1n} = 0 \quad (4)$$

with solutions

$$\xi_{11} = A_1 \sin(p_1 t + \epsilon_1), \quad \xi_{12} = A_2 \sin(p_2 t + \epsilon_2)$$

$$- - -, \quad \xi_{1n} = A_n \sin(p_n t + \epsilon_n) \quad (5)$$

where, $p_1^2, p_2^2, \cdots, p_n^2 = \lambda_1, \lambda_2, \cdots, \lambda_n \quad (6)$

and the A's and ϵ's are $2n$ arbitrary constants.

In fact, the determinantal equation in this case is simply

$$\begin{vmatrix} p^2-\lambda_1 & 0 & \cdots & 0 \\ 0 & p^2-\lambda_2 & \cdots & 0 \\ - & - & - & - \\ 0 & 0 & \cdots & p^2-\lambda_n \end{vmatrix} = 0 \quad \text{---(7)}$$

and it is shown in the theory of the linear transformation of the variables that the roots of this determinant are not affected by the transformation, i.e; the periods are the same for all systems of co-ordinates.

It is obvious that if we substitute (5) in (1) we get values for $q_1, q_2, \ldots q_n$ of exactly the same types as in the general solution.

The new co-ordinates $\xi_1, \xi_2, \ldots \xi_n$ are called the Normal or Principle co-ordinates. When the initial condition are such that all the normal co-ordinates but one are constant throughout the motion, the system is said to be vibrating in a normal mode.

In such a case, say if all the normal co-ordinates are constant save ξ_r, the original coordinates. $q_1, q_2, \ldots q_n$ will all be proportional to $\sin(p_s t + \epsilon)$ and bear constant ratios to one another.

It happens therefore that the general oscillalations of a system can be resolved into certain primary oscillation — the ~~mo~~ normal modes — and that if the system is stable for the normal modes it is stable for general small oscillations.

This may be spoken of as "The principle of the coexistence or super-position of small oscillations".

Further, physical properties of the normal modes are that when the system is vibrating in a normal mode the motion is simple periodic, passing twice through the equilibrium position in every complete oscillation, and the velocity of every particle of the system vanishes at the same instant twice in every complete oscillation.

To find the normal co-ordinates in terms of the original co-ordinates, we have only to substitute $\epsilon_{11}, \epsilon_{12}, \ldots$ for $\sin(p_1 t + \epsilon_1)$, $\sin(p_2 t + \epsilon_2) \ldots$ and solve them for the ϵ_1's in terms of the q's.

Dynamics of a System of Particles and Rotating Coordinate Systems.

(•) Motion of a System of Particles:

Consider a system of n particles, with masses $m_1, m_2, \ldots, m_n$ moving with velocities $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}$ in a Newtonian frame. Then,

The linear momentum $\vec{p}$ of the system is defined as a vector

$$\vec{p} = \sum_{i=1}^{n} m_i \vec{v_i} \longrightarrow (2.1)$$

differentiating with respect to time t, we get,

$$\dot{\vec{p}} = \sum_{i=1}^{n} m_i \dot{\vec{v_i}} = \sum_{i=1}^{n} m_i \vec{a_i} \longrightarrow (2.2)$$

where, $\vec{a_i}$ is the acceleration of the i^{th} particle.

Ignoring the inertial forces, let $\vec{F_i}$ be the external force acting on the i^{th} particle then, we can write eq^n (2.2) as

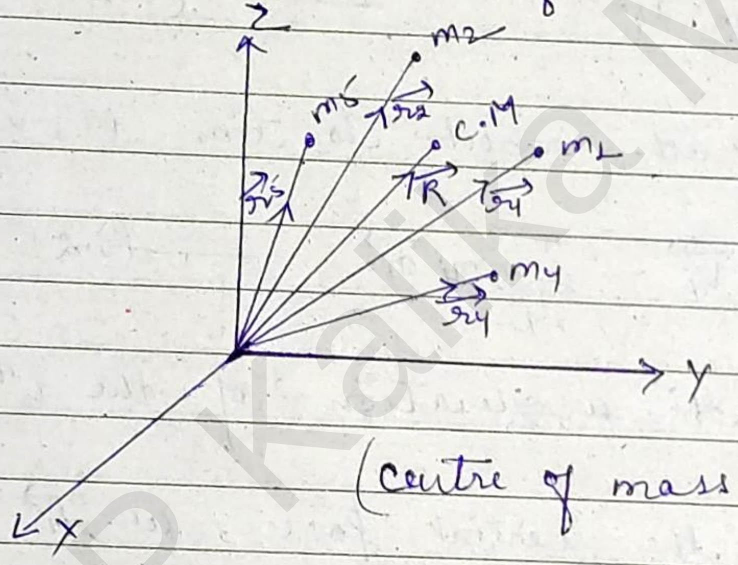
$$\dot{\vec{p}} = \sum_{i=1}^{n} \vec{F_i} = \vec{F} \longrightarrow (2.3)$$

where $\vec{F}$ is the vector sum of all the forces. Physically eq^n (2.3) means that "the rate of change of linear momentum of a system is equal to the vector sum of all external forces acting on the system."

This is the "Principle of linear momentum" of a system of n particles.

(•) Centre of Mass and Centre of Gravity:

Consider a system of n particles of masses $m_1, m_2, \ldots m_n$ whose position vectors are denoted by $\vec{r_1}, \vec{r_2}, \ldots \vec{r_n}$ with respect to the origin O.



(Centre of mass of a system)

Then the position vector of the centre of mass of the system, denoted by C.M is defined as

$$\vec{R} = \frac{1}{M} \sum_{i=1}^{n} m_i \vec{r_i} \qquad \text{——— (2.4)}$$

where M is the total mass of the system.

Physically,

The C.M of the system can be thought of as a weighted average position of the system of particles.

(•) In the special case of a uniform gravitational field the centre of mass coincides with the centre of gravity.

Now, we shall establish the law of ~~a~~ centre of mass of a system of particles in the following steps.

The position vector of the centre of mass of the system, is denoted by

$$\vec{R} = \frac{1}{M}\sum_{i=1}^{n} m_i \vec{r_i}$$

$$\Rightarrow M\vec{R} = \sum_{i=1}^{n} m_i \vec{r_i}$$

without loss of generality, differentiating w.r.t 't' yields an expression for the velocity of the centre of mass of the system as

$$M\vec{V} = \sum_{i=1}^{n} m_i \vec{v_i} \qquad \text{———} (2.5)$$

differentiating once again, the above equation gives us

$$M\dot{\vec{V}} = M\vec{a} = \sum_{i=1}^{n} m_i \dot{\vec{v}_i} = \sum_{i=1}^{n} m_i \vec{a_i} = \vec{F} = \dot{\vec{p}} \quad \text{—} (2.6)$$

Showing that the centre of mass of the system moves like a particle whose mass is equal to the total mass of the system, acted upon by an external force equal to the vector sum of all the external forces acting on the system. This is the 'law of motion of the centre of mass of the system.'

Physically eq^n (2.5) means that—
The linear momentum particle moving with the centre of mass is equal to the linear momentum of the system under consideration.

Ex - 2.1) Find the position, velocity and acceleration of the centre of mass of the following system of a particles.

Particle	Mass (Kg)	Position (m)	Velocity (m/sec)	Acceleration m/sec²
1.	0.5	$2\hat{i} - 3\hat{j}$	$-4\hat{k}$	$3\hat{j} + 5\hat{k}$
2	0.3	$-2\hat{i} + 4\hat{j}$	$-\hat{i} + 3\hat{j}$	$4\hat{i} - 3\hat{j}$
3	0.4	$-\hat{i} - 2\hat{j} - \hat{k}$	$4\hat{i} - 3\hat{j}$	$\hat{i} + \hat{j} + \hat{k}$

Solⁿ (i) The total mass of the system is given by

$$M = \sum_{i=1}^{3} m_i = m_1 + m_2 + m_3 = 0.5 + 0.3 + 0.4 = 1.2.$$

Let $\vec{R}$ be the position vector of the centre of mass of the given system, then

$$\sum_{i=1}^{n} m_i \vec{r_i} = m_1 \vec{r_1} + m_2 \vec{r_2} + m_3 \vec{r_3}.$$
$$= (0.5)(2\hat{i} - 3\hat{j}) + (0.3)(-2\hat{i} + 4\hat{j}) + (0.4)(-\hat{i} - 2\hat{j} - \hat{k})$$
$$= \hat{i} - 1.5\hat{j} - 0.6\hat{i} + 1.2\hat{j} - 0.4\hat{i} - 0.8\hat{j} - 0.4\hat{k}$$
$$= -1.1\hat{j} - 0.4\hat{k}.$$

∴ The position of the centre of mass of the given system is

$$\vec{R} = \frac{1}{M} \sum_{i=1}^{3} m_i \vec{r_i} = \frac{1}{1.2}(-1.1\hat{j} - 0.4\hat{k})$$
$$\vec{R} = (-0.92\hat{j} - 0.33\hat{k})\, m.$$

(ii) $$\sum_{i=1}^{3} m_i v_i = m_1 v_1 + m_2 v_2 + m_3 v_3.$$
$$= (0.5)(-4\hat{k}) + (0.3)(-\hat{i} + 3\hat{j}) + (0.4)(4\hat{i} - 3\hat{j})$$
$$= -2\hat{k} - 0.3\hat{i} + 0.9\hat{j} + 1.6\hat{i} - 1.2\hat{j}$$
$$= 1.3\hat{i} - 0.3\hat{j} - 2\hat{k}.$$

The velocity of the centre of mass of the system can be computed from the formula

$$\vec{V} = \frac{1}{M} \sum_{i=1} m_i \vec{v_i}.$$

$$= \frac{1}{1.2} \left(1.3\hat{i} - 0.3\hat{j} - 2\hat{k}\right)$$

$$\vec{V} = \left(1.08\,\hat{i} - 0.25\,\hat{j} - 1.67\hat{k}\right) m/s.$$

(iii) The acceleration of centre of mass of the system is given by

$$\vec{a} = \frac{1}{M} \sum_{i=1} m_i \vec{a_i}$$

$$\vec{a} = \frac{1}{1.2} \left(m_1 \vec{a_1} + m_2 \vec{a_2} + m_3 \vec{a_3}\right).$$

$$= \frac{1}{1.2} \left[(0.5)(-3\hat{j} + 5\hat{k}) + (0.3)(4\hat{i} - 3\hat{j}) + (0.4)(\hat{i} + \hat{j} + \hat{k})\right]$$

$$= \frac{1}{1.2} \left[-1.5\hat{j} + 2.5\hat{k} + 1.2\,\hat{i} - 0.9\hat{j} + 0.4\hat{i} + 0.4\hat{j} + 0.4\hat{k}\right]$$

$$= \frac{1}{1.2} \left(1.6\,\hat{i} - 2\hat{j} + 2.9\,\hat{k}\right)$$

$$= \left(1.33\,\hat{i} - 1.67\,\hat{j} + 2.42\,\hat{k}\right) m/s^2.$$

<u>Ex-2.21</u>. Find the resultant external forces acting on the system of particles in Ex-2.1. Also find the angular momentum vector with respect to the origin of the given system.

Solⁿ: (i) The resultant external forces acting on the system of particles can be ~~computs~~ compared as

$$\vec{F} = \sum_{i=1}^{3} m_i \vec{a_i} = (1.6\hat{i} - 2\hat{j} + 2.9\hat{k}) \text{ kg m/s}^2.$$

(ii) The total Angular momentum vector of the given system can be obtained,

$$\vec{H} = \sum_{i=1}^{3} \vec{r_i} \times m_i \vec{v_i}$$

$$\vec{H} = \vec{r_1} \times m_1\vec{v_1} + \vec{r_2} \times m_2\vec{v_2} + \vec{r_3} \times m_3\vec{v_3}.$$

$$= (2\hat{i} - 3\hat{j}) \times (-2\hat{k}) + (-2\hat{i} + 4\hat{j}) \times (-0.3\,\hat{i} + 0.9\hat{j})$$
$$+ (-\hat{i} - 2\hat{j} - \hat{k}) \times (1.6\,\hat{i} - 1.2\,\hat{j}).$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -3 & 0 \\ 0 & 0 & -2 \end{vmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 4 & 0 \\ -0.3 & 0.9 & 0 \end{vmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & -2 & -1 \\ 1.6 & -1.2 & 0 \end{vmatrix}.$$

$$= (6-0)\hat{i} + \hat{j}(0+4) + \hat{k}(0) + \hat{i}(0) + \hat{j}(0) + \hat{k}(-1.8+1.2)$$
$$+ \hat{i}(0-1.2) + \hat{j}(-1.5+0) + \hat{k}(1.2+3.2).$$

$$= 6\hat{i} + 4\hat{j} - 0.6\hat{k} - 1.2\hat{i} - 1.6\hat{j} + 4.4\hat{k}$$

$$\vec{H} = (4.8\,\hat{i} + 2.4\hat{j} + 3.8\,\hat{k}) \text{ kg m/s}^2.$$

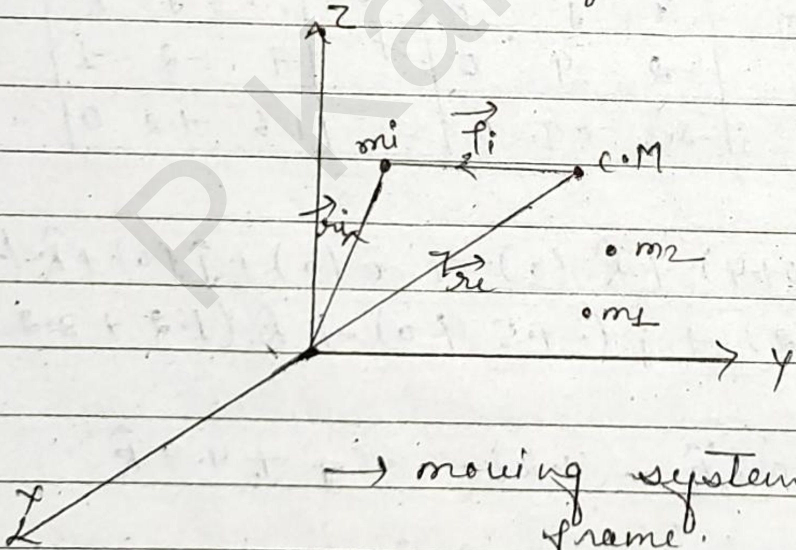
2.2 Principle of Angular Momentum

The generalization of the principle of angular momentum of a system is explained in following theorem:

Theorem 2.1 The angular momentum of a moving system about a point O is the sum of

(a) The angular momentum about O of a particle moving with the centre of mass, whose mass is equal to that of the total mass of the system, and

(b) The angular momentum of the system about the centre of mass.



→ moving system in a Newtonian frame.

Proof: Consider a system of n particles with masses $m_1, m_2, \ldots m_n$, whose position vectors are $\vec{r_1}, \vec{r_2}, \ldots \vec{r_n}$ with reference to a fixed point O, in a Newtonian frame at a given instant (in fig). Then the angular momentum of the particle m_i about O is $\vec{h_i} = \vec{r_i} \times m_i \dot{\vec{r_i}}$ ——— (1)

Hence, the total angular momentum of the system is found by summing the angular momenta of ~~each~~ individual particles.

$$\vec{H} = \sum_{i=1}^{n} \vec{h_i} = \sum_{i=1}^{n} \vec{r_i} \times m_i \dot{\vec{r_i}} \quad \text{——— (2)}$$

from fig $\vec{r_i} = \vec{r_c} + \vec{f_i}$

using this relation, we can shift the refrence point 'O' to the centre of mass.

$\therefore$ (2) becomes

$$\vec{H} = \sum_{i=1}^{n} (\vec{r_c} + \vec{f_i}) \times m_i (\dot{\vec{r_c}} + \dot{\vec{f_i}})$$

$$= \sum_{i=1}^{n} \left(\vec{r_c} \times m_i \dot{\vec{r_c}} + \vec{r_c} \times m_i \dot{\vec{f_i}} + \vec{f_i} \times m_i \dot{\vec{r_c}} + \vec{f_i} \times m_i \dot{\vec{f_i}} \right)$$

By the def^n of centre of mass of the system

$$\vec{r_c} = \frac{1}{M} \sum_{i=1}^{n} m_i \vec{r_i}$$

where M is the total mass of the system.

$$\Rightarrow M\vec{r_c} = \sum_{i=1}^{n} m_i \vec{r_i}$$

$$= \sum_{i=1}^{n} m_i (\vec{r_c} + \vec{\rho_i})$$

$$= \sum_{i=1}^{n} m_i \vec{r_c} + \sum_{i=1}^{n} m_i \vec{\rho_i}$$

$$\Rightarrow M\vec{r_c} = M\vec{r_c} + \sum_{i=1}^{n} m_i \vec{\rho_i}$$

$$\Rightarrow \sum_{i=1}^{n} m_i \vec{\rho_i} = 0 \quad \text{and} \quad \sum_{i=1}^{n} m_i \dot{\vec{\rho_i}} = 0$$

Hence, the 2^{nd} and 4^{th} term on the R.H.S of eqn (3) vanishes and eqn (3) reduced to

$$\vec{H} = \sum_{i=1}^{n} \vec{r_c} \times m_i \dot{\vec{r_c}} + \sum_{i=1}^{n} \vec{\rho_i} \times m_i \dot{\vec{r_c}}$$

$$\Rightarrow \vec{H} = \vec{r_c} \times M\dot{\vec{r_c}} + \vec{H_c} \qquad \text{——— (4)}$$

Hence proved.

Establish the Principle of angular momentum.

→ By the formula of Total angular momentum of the system.

$$\vec{H} = \sum_{i=1}^{n} \vec{h_i} = \sum_{i=1}^{n} \vec{r_i} \times m_i \dot{\vec{r_i}} \qquad \text{——— (1)}$$

differentiating (1) w.r.t time 't' we get —

$$\dot{\vec{H}} = \sum_{i=1}^{n} \vec{r_i} \times m_i \ddot{\vec{r_i}} + \sum_{i=1}^{n} \dot{\vec{r_i}} \times m_i \dot{\vec{r_i}}$$

$$= \sum_{i=1}^{n} (\vec{r_i} \times m_i \ddot{\vec{r_i}})$$

$$= \sum_{i=1}^{n} (\vec{r_i} \times m_i a_i)$$

$$\therefore \dot{\vec{H_0}} = \sum_{i=1}^{n} \vec{r_i} \times \vec{F_i} = \vec{G_0} \quad \text{——— (2)}$$

where $\vec{F_i}$ are the external forces and $\vec{G}$ is the total moment about O due to external force acting on the system.

Thus the principle of angular momentum can be summed up as follows —

for a system of particles, the time rate of angular momentum about a fixed point O, is equal to the total moment about O of the external forces acting on the system.

This fundamental relation (2) still holds when the frame of reference OXYZ is replaced by another frame with origin at the centre of mass C, of the system of particles.

Thus $\boxed{\dot{\vec{H_C}} = \vec{G_C}}$

Relative Motion of Two Particles.

Consider two particles A and B moving along a st. line with ~~&~~ respect to O

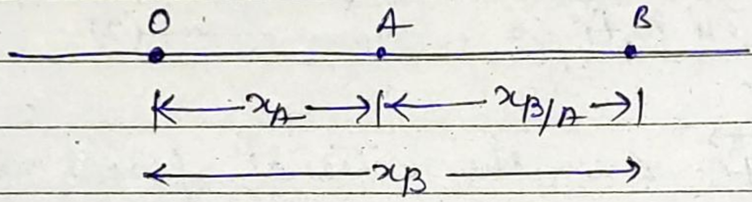


fig: a Simple system.

Let ~~us~~ the coordinates of A and B be denoted by x_A and x_B.

Then, the difference $x_B - x_A$, define the relative position of B w.r.t A and is denoted by $x_{B/A}$. Thus, we may write

$$x_{B/A} = x_B - x_A \quad \text{or} \quad x_B = x_A + x_{B/A}$$

The sign for $x_{B/A}$ is positive if B is to the right of A and negative if B is to the left of A.

If we designate the relative velocity of B with respect to A by $V_{B/A}$ then on the differentiating the above eqn, we get

$$\dot{x}_{B/A} = \dot{x}_B - \dot{x}_A$$

$$\Rightarrow v_{B/A} = v_B - v_A \quad \text{or} \quad v_B = v_A + v_{B/A}$$

Similarly, the rate of change of $v_{B/A}$ is called relative acceleration of B with respect to A and is denoted by $a_{B/A}$.

$$\Rightarrow \dot{v}_{B/A} = \dot{v}_B - \dot{v}_A$$

$$\Rightarrow a_{B/A} = a_B - a_A \Rightarrow a_B = a_A + a_{B/A}$$

2.3) Motion of a Rigid Body.

→ A rigid body is defined as the one in which the distance between any two particles is fixed. Such a body in general has a motion which is a superposition of both translation and rotation.

Translation Motion

A body is said to have translation motion, if the position of any line inscribed on the body remains parallel to its original position during its motion. It may be noted that during translation motion, all particles of a rigid body will have the same velocity with respect to some reference frame, at any instant of time. In fact, the velocity vector may change from time to time.

Ex: Consider a situation as shown in fig. In either case, the motion is translational, as the line PQ inscribed on the body remains parallel to its original position at all instants of time.

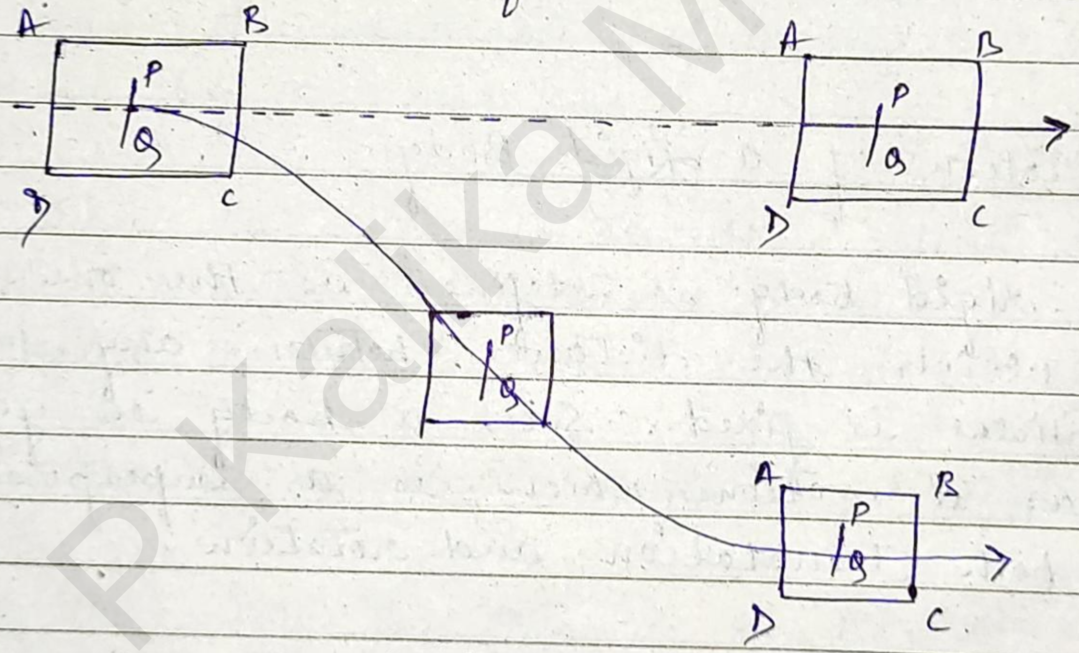


fig: Translational Motion.

Rotational Motion :

A body is ~~sat~~ said to have rotational motion, if the body rotates such that along some straight line, all the particles of the body have zero velocity with respect to some reference frame. Such a line of stationary particles is called the axis of rotation.

During rotation, the distance of any point of the body from the axis of rotation remains unaltered.

Angular velocity :

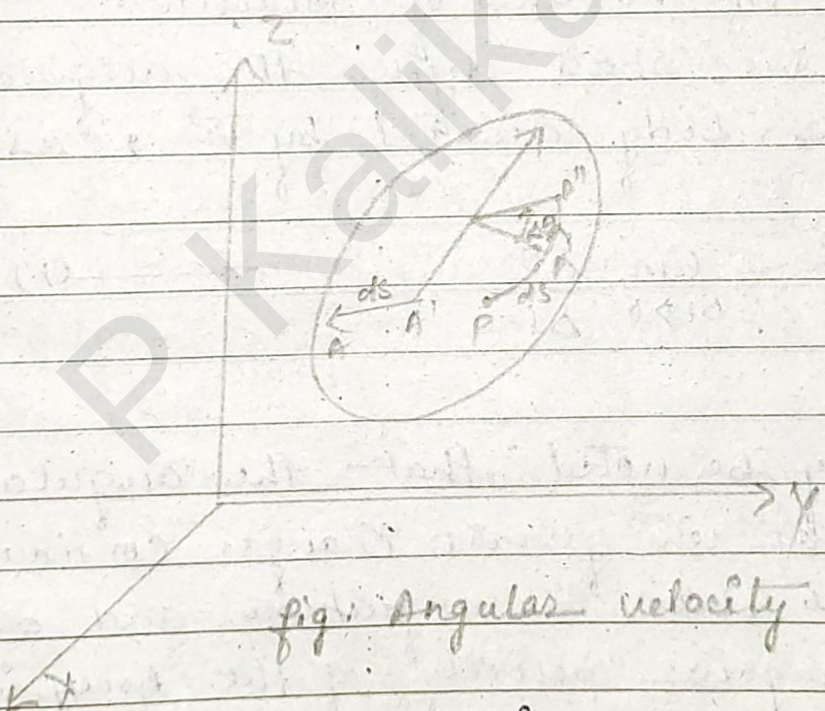


fig: Angular velocity of a body.

Consider the motion of rigid body during an infinitesimal interval δt, as show in fig.

During this interval, suppose, we

have a translation displacement ds of all points in the body together with a rotational displacement $\Delta\theta$ about an axis through the base point A" fixed in the body. The order of performing the translation and rotation is immaterial.

In fig, we have shown that translation occurs first, followed by rotation. Thus a typical point P moves to P' and the base point A moves to A', each undergoes the same displacement ds, then follow rotation by an amount $\Delta\theta$ moving P' to P". However A' does not move as it is on the axis of rotation.

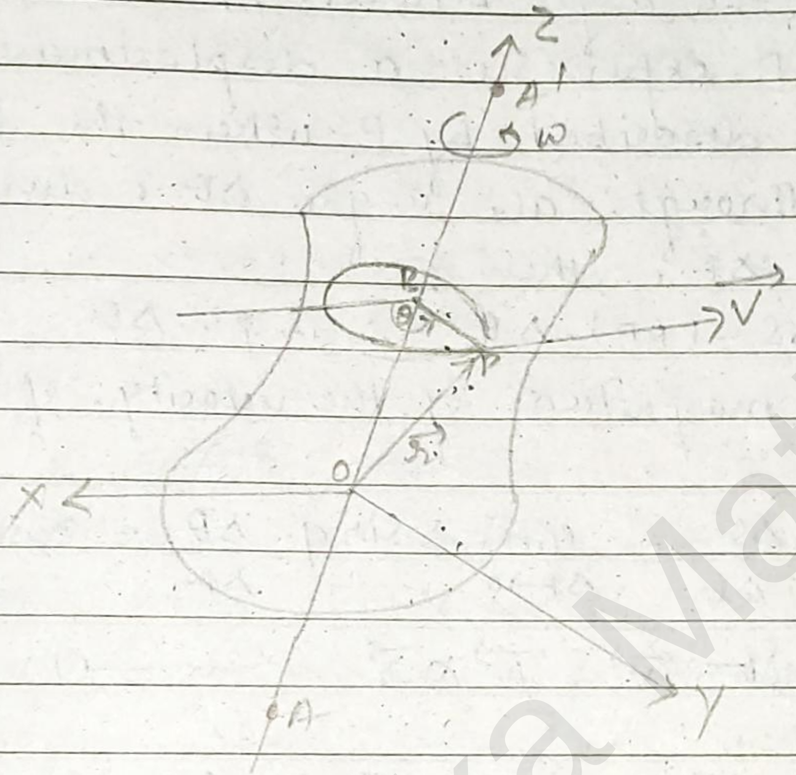
Now, we shall define the angular velocity of the body denoted by $\vec{\omega}$, as

$$\vec{\omega} = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} \quad \text{——— (1)}$$

It may be noted that the angular velocity vector in general changes continuously with time both in magnitude and direction. Also, the angular velocity of the body is different in different frames of reference.

Therefore, while specifying the angular velocity of a body, one has to specify the reference frame too, otherwise the inertial frame is assumed.

2.3.2| Rigid body Rotation about a fixed axis.



Consider a rigid body which is constrained to rotate about a fixed axis AA', that is fixed in an inertial frame $OXYZ$ (as in fig.)

For convenince, we shall assume that the origin is on AA', and z-axis coincides with AA'.

Let $\vec{v}$ be the absolute velocity of a typical point P of the body, whose position vector is $\vec{r}$ w.r.t ~~ent~~ inertial frame, and $\vec{\omega}$, the angular velocity of the body. Then $\vec{\omega}$ has a fixed direction and the path described by the point P is a circle of radius $r \sin\phi$, with centre at B.

Here, ϕ is the angle between AA' and $\vec{r}$.

If P experiences a displacement Δs of the arc described by P when the body rotates through an angle $\Delta\theta$, during the interval Δt, then

$$\Delta s = (BP) \cdot \Delta\theta = r\sin\phi \cdot \Delta\theta$$

and the magnitude of the velocity of P is

$$|\vec{v}| = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{\Delta t \to 0} r\sin\phi \cdot \frac{\Delta\theta}{\Delta t} = r\sin\phi \cdot \omega$$

$\therefore$ we write $\vec{v} = \vec{\omega} \times \vec{r}$ ——— (1)

Here, the vector $\vec{\omega} = \omega\hat{k} = \dot{\theta}\hat{k}$ is the angular velocity of the body.

The acceleration $\vec{a}$ of P can be obtained by diff. eqn (1) w.r.t time t.

Thus, $\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}(\vec{\omega} \times \vec{r}) = \frac{d\vec{\omega}}{dt} \times \vec{r} + \vec{\omega} \times \frac{d\vec{r}}{dt}$

If we define the vector $\frac{d\vec{\omega}}{dt}$ by $\vec{\alpha}$, called angular acceleration of the body

$\therefore$ above eqn becomes.

$$\vec{a} = \vec{\alpha} \times \vec{r} + \vec{\omega} \times \vec{v} \qquad \left(\because \vec{v} = \frac{d\vec{r}}{dt}\right).$$

$$\Rightarrow \vec{a} = \vec{\alpha} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) \quad \text{—(2)} \quad [\because \vec{v} = \vec{\omega} \times \vec{r}]$$

we also have,

$$\Rightarrow \vec{\alpha} = \alpha \hat{k} = \frac{d\vec{\omega}}{dt} \hat{k} = \dot{\vec{\omega}} \hat{k} = \ddot{\theta} k$$

In eq^n(2), the term $\vec{\omega} \times (\vec{\omega} \times \vec{r})$, a vector that is directed ~~radilly~~ radially inwards from point P towards the instantaneous axis of rotation and perpendicular to it, it is called Centripetal acceleration, while the term $\vec{\alpha} \times \vec{r} = \dot{\vec{\omega}} \times \vec{r}$ is called the Tangential acceleration.

~~Kine~~ Kinematics of A Rigid Body Motion.

(•) Moment of Inertia :

Consider a particle of mass m. Situated at a distance r from a line about which about the moment of intertia (M.I) of the particle to be found: then the moment of inertia of the particle about a line is defined as the product of the mass of the particle and the square of its perpendicular distance from the line. Symbolically, it can be written as

$$I = mr^2 \quad \text{———} (1)$$

This defn can be readily generalized for a system of particles mi,

$$I = \sum_{i=1}^{n} m_i r_i^2 \quad \text{———} (2)$$

Where $m_i \longrightarrow$ masses of n particles situated at distance r_i respectively from the line.

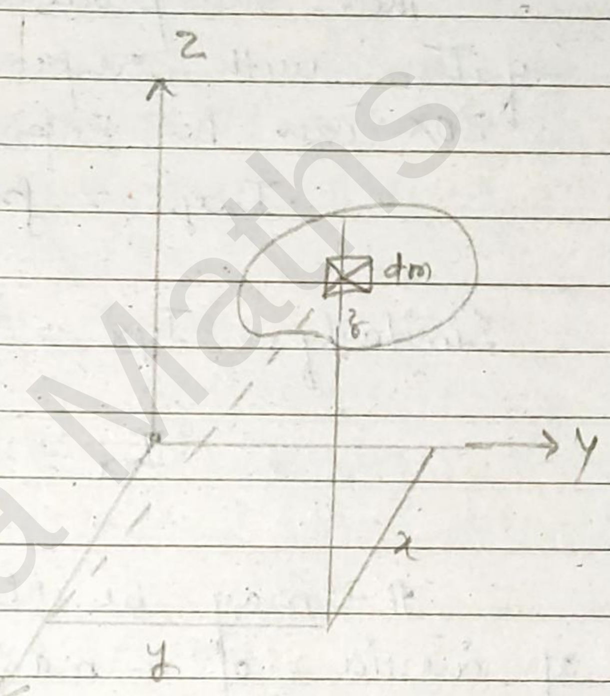
For a continuous distribution of matter, the M.I of any body with respect to a line is defined as $I = \int r^2 dm$. ——— (3)

Here, dm is the mass of an infinitesimal element and r is the distance from the line about which the M.I is to be found.

for Illustration

Consider a rigid body,
The M.I of the system about OX denoted by I_{xx} is defined as

$$I_{xx} = \int (y^2 + z^2)\, dm \quad \text{—(1)}$$



Similarly, the M.I of the system about OY and OZ is defined as

$$I_{yy} = \int (z^2 + x^2)\, dm \quad \text{—(2)}$$

&

$$I_{zz} = \int (x^2 + y^2)\, dm \quad \text{—(3)}.$$

The dimension of the moment of ~~and~~ inertia can be easily seen from the definition as ML^2 and it is measured as $kg-m^2$ in the SI of or MKS system. CGS system — $g-cm^2$.

Product of Inertia :

The product of inertia (P.I) of an element of mass dm with respect to the pair of orthogonal coordinate planes is defined as the 'Product of the mass of the element and the coordinate distances from the planes to the element considered.

For ex: The product of inertia of the element with respect to YOZ and ZOX planes is defined as $dI_{xy} = xy\, dm$.

Then the product of inertia of the total system with respect to the planes YOZ and ZOX can be expressed as

$$I_{xy} = \int xy\, dm \quad \text{——— (1)}$$

Similarly, $$I_{yz} = \int yz\, dm \quad \text{——— (2)}$$

$$I_{zx} = \int zx\, dm \quad \text{——— (3)}$$

It may be observed that the product of inertia of a mass has the dimensions ML^2. The M.I of a body is always positive, negative or zero, since the two coordinate planes can have independent signs. If the coordinate have the same sign, it will be positive and will be negative for coordinates with opposite sign.

<u>Centre of Mass (for a continuous system)</u>

For a rigid body (continuous body), if we mass density of the body is designated by $\rho(r)$, the position vector $\vec{R}$ of the centre of mass is defined by the

$$\vec{R} = \frac{1}{M}\int_V \rho(r)\,\vec{r}\,dv \quad , \text{ where } M \to \text{total mass.}$$

given by,

$$M = \int_V \rho(r)\,dv \quad \text{and } dv = dx\,dy\,dz.$$

Ex1. To locate the centre of mass of a hemisphere of uniform density, we may proceed as follows:

"from the symmetry of the body, it is clear that the centre of mass lies along the axis of symmetry which is perpendicular to the base of the hemisphere and passes through its centre.

Calling this axis as z-axis:

we find, the z-component of the position vector of the centre of mass as

$$z = \frac{\int \rho z\,dv}{\int \rho\,dv}.$$

using spherical coordinate, $z = r\cos\theta$,

$dv = r^2 dr \sin d\theta\, d\phi$ we obtain the z-component of the centre of mass as.

$$z = \frac{\int_0^{2\pi}\int_0^{\pi/2}\int_0^{R} r^3 \cos\theta \sin\theta \,dr\, d\theta\, d\phi}{\int_0^{2\pi}\int_0^{\pi/2}\int_0^{R} r^2 \sin\theta\, dr\, d\theta\, d\phi}.$$

$$= \frac{\left[\frac{r^4}{4}\right]_0^R \left[\phi\right]_0^{2\pi} \frac{1}{2}\left[-\frac{\cos 2\theta}{2}\right]_0^{\pi/2}}{\left[\frac{r^3}{3}\right]_0^R \left[\phi\right]_0^{2\pi} \left[-\cos\theta\right]_0^{\pi/2}}$$

$$= \frac{\frac{R^4}{4}\, 2\pi \left(-\frac{1}{4}\right)(\cos\pi - \cos 0)}{\frac{R^3}{3}\, 2\pi\, (-1)\left(\cos\frac{\pi}{2} - \cos 0\right)}$$

$$= \frac{\frac{R}{4} \times \frac{1}{4}\,(-2)}{\frac{1}{3}\,(-1)}$$

$$z = \frac{3R}{8}$$

Theorem 3.1/ Parallel Axis Theorem

The moment of inertia (M.I) of a system about any line L, is equal to the sum of the M.I of the same system about an axis through its centre of mass (C.M) parallel to L and the M.I about L of a particle whose mass is equal to the total mass of the system placed at its centre of mass.

Proof: Consider a body whose centre of mass is located at the origin O' of the primed coordinate system, that is at the point (x_c, y_c, z_c) related to unprimed system, (fig)

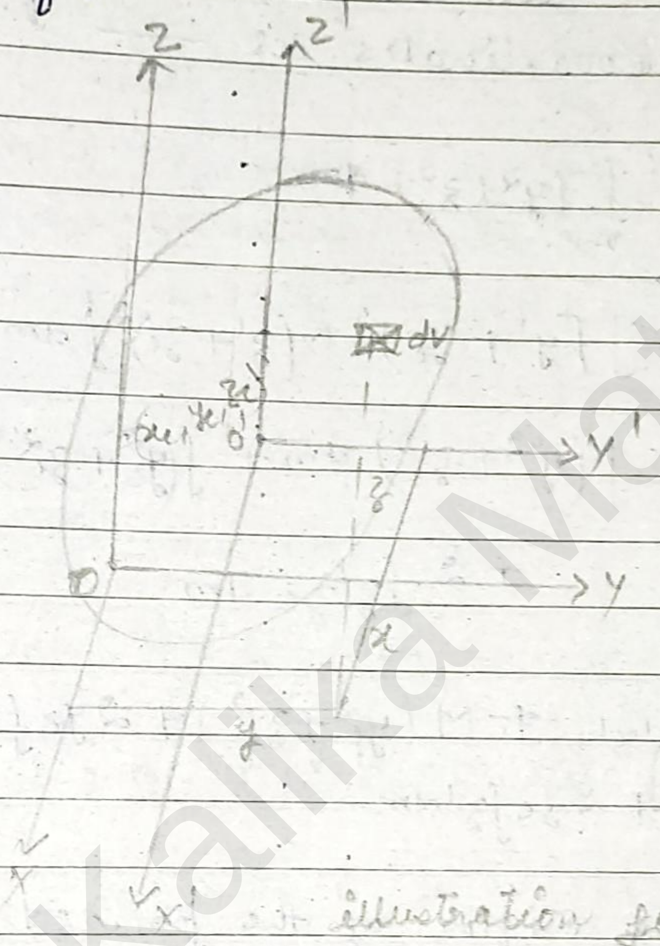


illustration for parallel axis theorem.

Also, let dv be an infinitesimal volume element which is located at (x, y, z) relative to the ~~up~~ unprimed system & at (x', y', z') in the primed system. Then ~~p~~ these coordinates are related through the following expression:

$$x = x' + x_c \quad ; \quad y = y' + y_c \quad , \quad z = z' + z_c \quad \text{——— (1)}$$

Now,

The M.I of the system about the x-axis can be written in terms of primed ~~continuous~~ coordinates as —

$$I_{xx} = \int_V (y^2+z^2)\, dm.$$

$$= \int [(y'+y_c)^2 + (z'+z_c)^2]\, dm.$$

$$= \int (y'^2+z'^2)\, dm + \int (y_c^2+z_c^2)\, dm + 2\int y' y_c\, dm + 2\int z' z_c\, dm.$$

$$I_{xx} = I_{x'x'} + M(y_c^2+z_c^2) + 2y_c \int y'\, dm + 2z_c \int z'\, dm.$$

Since the origin of the primed coordinate system has been chosen to be at the centre of mass, we have

$$\int y'\, dm = 0 = \int z'\, dm.$$

Thus, the above equation becomes

$$I_{xx} = I_{x'x'} + M(y_c^2+z_c^2) \quad \text{——— (2)}$$

Similarly, we can show that

$$I_{yy} = I_{y'y'} + M(z_c^2+x_c^2) \quad \text{——— (3)}$$

& $$I_{zz} = I_{z'z'} + M(x_c^2+y_c^2) \quad \text{——— (4)}$$

Based on similar arguments, we can prove the parallel axis theorem for products of inertia as can be seen in the following steps:

"The product of inertia of the body with respect to the coordinate planes YOZ and ZOX (in fig.) can be written as:

$$I_{xy} = \int xy \, dm$$

$$= \int (x' + x_c)(y' + y_c) \, dm$$

$$= \int (x'y' + x'y_c + y'x_c + x_c y_c) \, dm$$

$$= \int x'y' \, dm + \int x_c y_c \, dm + y_c \int x' \, dm + x_c \int y' \, dm$$

$$I_{xy} = \int x'y' \, dm + \int x_c y_c \, dm \quad \left[\text{as } \int y' dm = 0 = \int x' dm\right]$$

$$\Rightarrow I_{xy} = I_{x'y'} + M x_c y_c \quad \text{——— (5)}$$

IIIy. $\Rightarrow I_{yz} = I_{y'z'} + M y_c z_c$ ——— (6)

$$I_{zx} = I_{z'x'} + M z_c x_c \quad \text{——— (7)}$$

Theorem 3.2/ Perpendicular Axis Theorem.

The moment of ~~cut~~ inertia of a plane body about a perpendicular axis is equal to the sum of the M.I, about the orthogonal axis in the plane.

Proof: (This theorem is useful for computing the M.I of plane distribution of matter).

Let us consider a rectangular frame of reference OXYZ. If there is a distribution of matter in the plane $z=0$ and if we denote the M.I of this matter about three coordinate axes by I_{xx}, I_{yy} and I_{zz} respectively, then we have

$$I_{xx} = \int(y^2+z^2)\,dm = \int y^2\,dm \quad \text{---}(1)$$

$$I_{yy} = \int(z^2+x^2)\,dm = \int x^2\,dm \quad \text{---}(2)$$

$$I_{zz} = \int(x^2+y^2)\,dm \quad \cancel{= \int} \quad \text{---}(3)$$

$$\therefore \quad I_{zz} = \int(x^2+y^2)\,dm = \int x^2\,dm + \int y^2\,dm = I_{xx} + I_{yy}$$

$$\boxed{I_{zz} = I_{xx} + I_{yy}}$$

This result is known as Perpendicular axis theorem.

Radius of Gyration :

Divide the value of I_{zz} by the mass and then taking the square root of that Quotient, we get the Radius of Gyration ~~is~~ about z-axis.

Thus, $$k_z = \sqrt{\frac{I_{zz}}{m}}$$

Similarly, the Radius of Gyration can be defined about other axes.

Example - 3.1

Q/ Find the moment of inertia of:

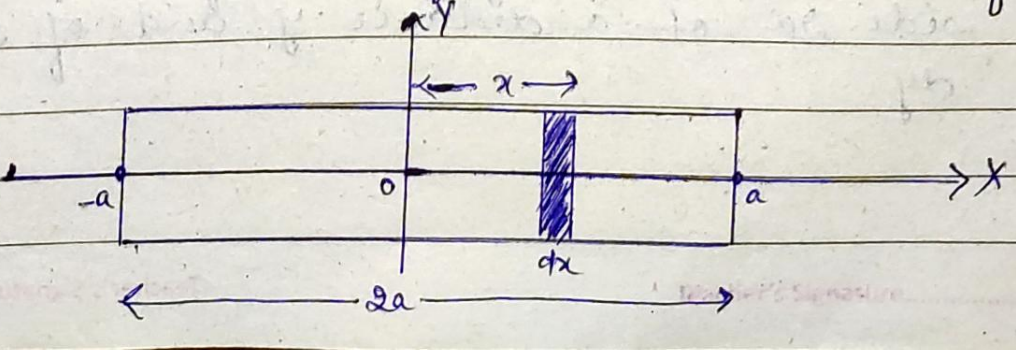
(a) A uniform rod of mass M, length 2a about a line through its centre and perpendicular to its length.

(b) A uniform rectangular plate of mass M and edges of lengths 2a, 2b about a line passing through its centre :

(i) Parallel to the sides 2a and 2b and

(ii) ~~Per~~ Perpendicular to its plane.

Sol^n: — (a) Suppose the rod is aligned along the X-axis with the origin at the centre of a rod,



The mass of the element dx is

$$dm = \cancel{A\,dx\,(\rho)}\ A\,dx\,\rho$$

where A is area of cross-section

$\rho \to$ density.

Then the M.I of the element about OY is

$$I = \int_{-a}^{a} A\rho\, x^2\,dx = A\rho\left[\frac{x^3}{3}\right]_{-a}^{a} = A\rho\,\frac{2a^3}{3}$$

$$I = \frac{1}{3}(A\rho\, 2a)\, a^2 = \frac{1}{3} Ma^2$$

where M is mass of the rod.

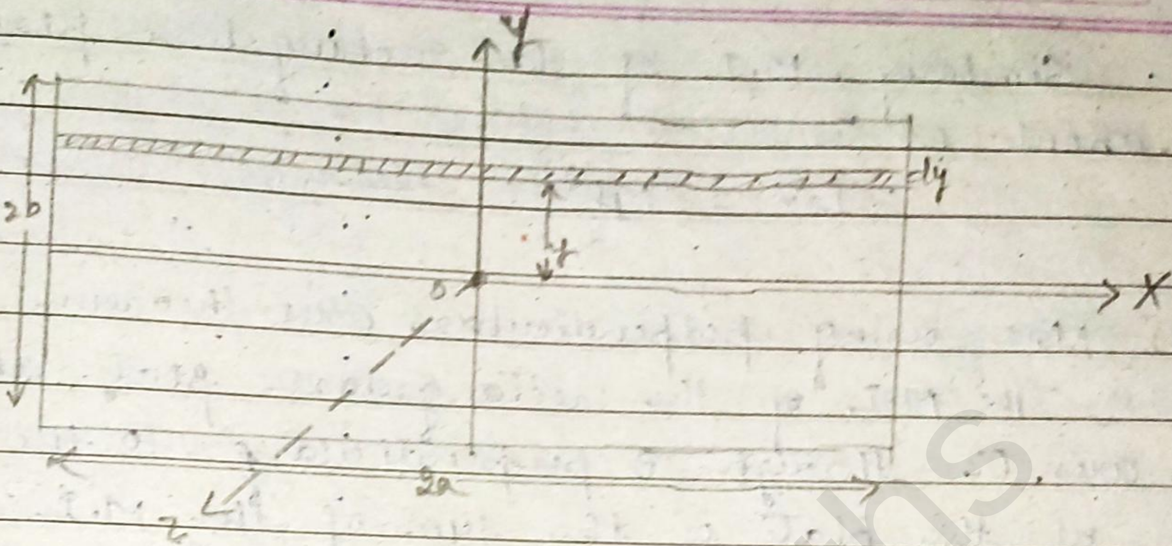
(b) (i) Let M be the mass of the plate and let the dimensions of the given plate $2a \times 2b$.

Then, the area of the plate is equal to $4ab$.

Let the density i.e mass per unit area of the plate, be given by

$$\rho = \frac{M}{4ab} \qquad \text{——— (1)}$$

Now, Consider an elementary strip parallel to side $2a$ at a distance y and of the width dy.



Then, the area of the strip is $2a\,dy$ and hence,

$$\text{Mass} = 2a\,dy\,\rho = 2a\,dy\frac{M}{4ab} = \frac{M}{2b}\,dy.$$

using the result of the first part,
The M.I of the strip about OY is

$$I_{OY} = \frac{M}{2b}dy\,\frac{a^2}{3}.$$

Hence, the M.I of the plate about OY is

$$I_{OY} = \frac{M}{2b}\frac{a^2}{3}\int_{-a}^{b}dy = \frac{M}{2b}\frac{a^2}{3}\left[y\right]_{-b}^{b}$$

$$\Rightarrow \quad I_{OY} = \frac{M}{2b}\frac{a^2}{3}\cdot 2b.$$

$$\Rightarrow \quad I_{OY} = \frac{Ma^2}{3} \qquad \text{—— (2)}$$

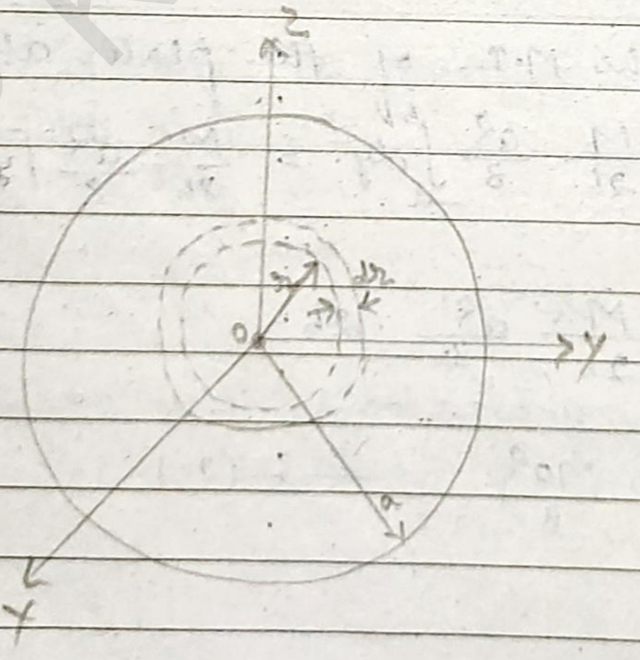
Similarly, M.I of the rectangular plate about OX is

$$I_{OX} = \frac{Mb^2}{3} \quad \text{——— (3)}$$

(ii) Now, using perpendicular axis theorem,
The M.I of the rectangular plate about an axis OZ through O perpendicular to the plane of the plate is the sum of the M.I of the plate about OX and the M.I of the plate about OY.

Thus, required $M.I = \frac{M}{3}(a^2+b^2)$ ——— (4).

Ex 3.2) Find the moment of inertia of a uniform circular disc of mass M and radius 'a' about a line through its centre and perpendicular to its plane.



Solⁿ:— Imagine that we have divided the disc into several rings by drawing a large number of concentric circle.

One such a ring is shown by dotted lines. Let r and $r + dr$ be the radii of the inner and outer circle of a typical ring.

The area of this ring is $2\pi r\,dr$.
It's mass is given by

$$dm = \frac{M}{\pi a^2} \times 2\pi r\,dr = \cancel{2\pi}\,\frac{2M}{a^2} r\,dr$$

Let OZ be perpendicular to the disc through its centre O. Then the M.I of the disc about OZ denoted by I_z is given by

$$I_z = \int_0^a r^2\,dm = \int_0^a r^2 \frac{2M}{a^2}\, r\,dr$$

$$= \frac{2M}{a^2} \int_0^a r^3\,dr$$

$$= \frac{2M}{a^2} \left[\frac{r^4}{4}\right]_0^a$$

$$= \cancel{Ma}\,\frac{Ma^4}{2a^2}$$

$$\therefore \quad I_z = \frac{Ma^2}{2}$$

Recalling, the perpendicular axis theorem, which states that the M.I of a plane body about a perpendicular axis is equal to sum of the M.I's about orthogonal axes in the plane. Then we have

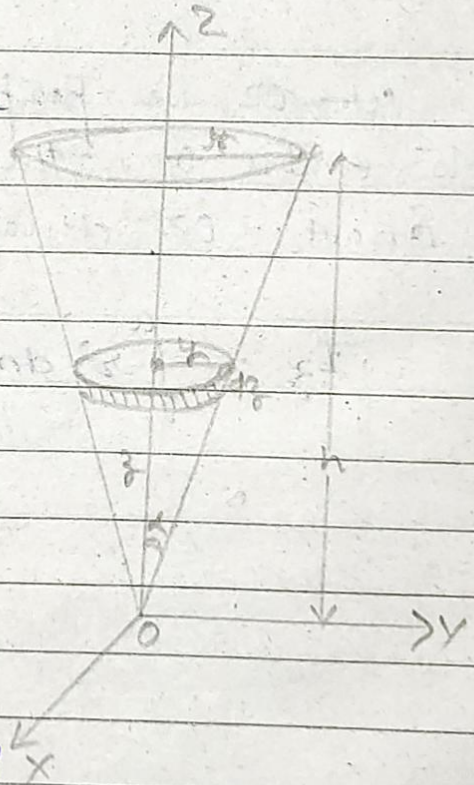
$$I_x = I_y = \frac{1}{2} I_z = \frac{Ma^2}{4}$$

Example – 3.3/
Find the moment of inertia of a circular cone about its axis.

Sol^n:– Imagine that we divide the cone into a large numbers of thin discs of thickness dz, such that the mass of a typical disc is given as

$$dm = \pi y^2 \, dz \, \rho$$

where ρ is the density of the ~~inertia~~ material.



Then,
The M.I of the typical disc about OZ as obtained is given by

$$dI_z = \frac{dm}{2} y^2 .$$

Thus, the M.I of the whole cone about its axis OZ denoted by I_{zz} is

$$I_{zz} = \frac{1}{2}\int_0^h y^2\,dm = \frac{\pi\rho}{2}\int_0^h y^4\,dz.$$

where h is the altitude of the cone.

Let r be the radius of its base.

$$\frac{y}{z} = \frac{r}{h} \quad \text{or} \quad y = \frac{rz}{h}.$$

Consquently, The M.I of the circular cone about its axis is

$$I_{zz} = \frac{\pi}{2}\rho\int_0^h \left(\frac{r}{h}\right)^4 dz \cdot z^4 = \frac{\pi}{2}\rho\left(\frac{r}{h}\right)^4 \int z^4\,dz.$$

$$= \frac{\pi}{2}\rho\left(\frac{r}{h}\right)^4 \left|\frac{z^5}{5}\right|_0^h$$

$$= \frac{\pi}{2}\rho\left(\frac{r}{h}\right)^4 \cdot \frac{h^5}{5}$$

$$I_{zz} = \frac{\pi}{2}\rho\left(\frac{r}{h}\right)^4 \frac{h^5}{5} \qquad \text{——— (1)}$$

It is known as the mass of cone

$$M = \frac{1}{3}\pi r^2 h\rho \qquad \text{——— (2)}.$$

Hence, the required M.I is found to be

$$I_{zz} = \left(\pi r^2 h\rho\right) \cdot \frac{r^2 h^4}{10 h^4}$$

$$I_{zz} = \frac{3Mr^2}{10} \qquad \text{——— (3)}$$

Introducing the semi-vertical angle of the cone, α,

$$r = \cancel{h \tan\alpha}\; h \tan\alpha .$$

Then the (3) becomes

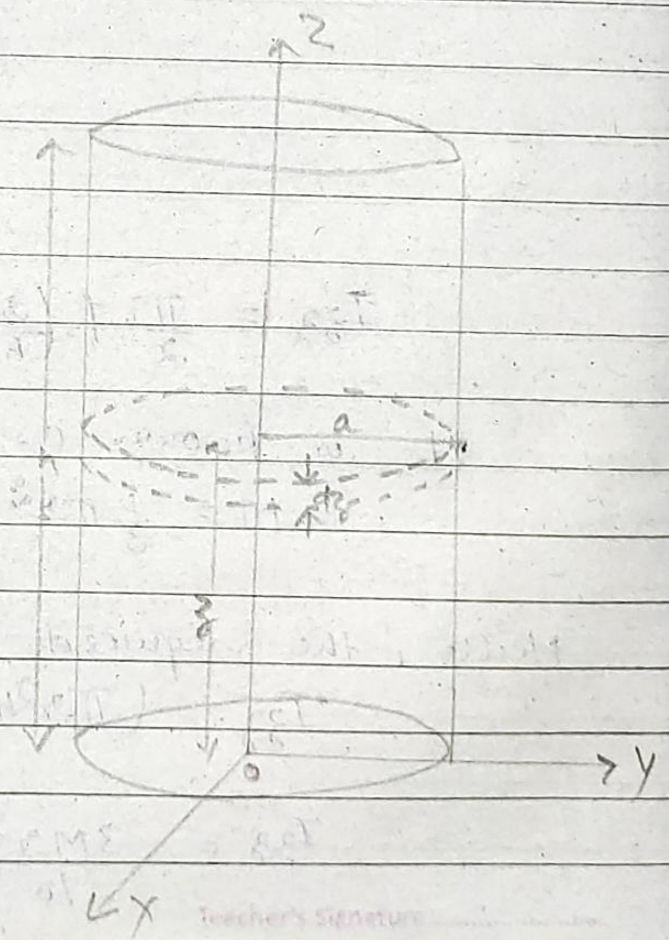
$$I_{zz} = \frac{3M}{10} \cdot (h \tan\alpha)^2$$

$$I_{zz} = \frac{3M}{10} h^2 \tan^2\alpha \quad \text{———} (4) .$$

<u>Ex-3.4</u> find the M.I of a uniform circular cylinder of height h and ~~reo~~ radius a,
(a) with respect to its logitudinal axis,
(b) about an axis through its centre of mass and perpendicular to its axis.

Solⁿ:—

(a) Let us imagine that the cylinder is divided into several discs of radius 'a' and of thickness dz. If ρ is the density of the material, then the mass of the typical disc is found to be



$$dm = \pi a^2 \rho \, dz \quad \text{———} (1)$$

The M.I of the typical disc about OZ denoted by $dI = dm \frac{a^2}{2}$

∴ the M.I of the cylinder about OZ, the axis of the cylinder is

$$I_{zz} = \int_0^h \pi a^2 dz \, \rho \frac{a^2}{2} = \frac{\pi a^4}{2} \rho \int_0^h dz = \frac{\pi a^4 \rho h}{2}$$

$$I_{zz} = \frac{\pi a^4 \rho h}{2} \quad \text{——— (2)}$$

where h is the height of the ~~particle~~ cylinder (or length). But the mass M of the cylinder of height h and radius 'a' is known to be

$$M = \pi a^2 h \rho \quad \text{——— (3)}$$

Thus from eq^n (2)

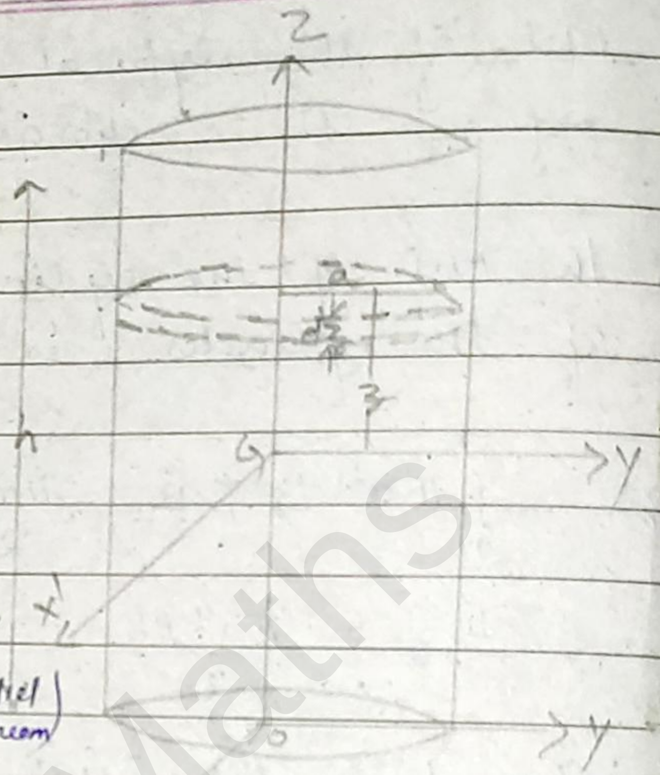
$$I_{zz} = (\pi a^2 h \rho) \frac{a^2}{2} = \frac{Ma^2}{2}$$

$$\therefore \; I_{zz} = \frac{Ma^2}{2} \quad \text{——— (4)}$$

(b) using the perpendicular axis theorem, we know that

M.I of the disc about OX = M.I of the disc about OY

$= \frac{1}{2}$ (M.I of the disc about OZ)



Let G be the centre of mass of the cylinder. If the disc considered is at a distance of z from G, then the M.I of the disc about GY' is

$$\cancel{I_G}\ I_{GY'} = dm\frac{a^2}{4} + dm\, z^2$$

(Parallel axis theorem)

$$= dm\left(\frac{a^2}{4} + z^2\right)$$

Hence, the M.I of the cylinder about GY' is

$$I_{GY'} = \int_{-h/2}^{h/2} \pi a^2 \rho\left(\frac{a^2}{4} + z^2\right) dz$$

$$= 2\int_0^{h/2} \pi a^2 \rho\left(\frac{a^2}{4} + z^2\right) dz$$

$$= 2\pi a^2 \rho \int_0^{h/2} \left(\frac{a^2}{4} + z^2\right) dz$$

$$= 2\pi a^2 \rho \left[\frac{a^2 z}{4} + \frac{z^3}{3}\right]_0^{h/2}$$

$$= 2\pi a^2 \rho \left(\frac{a^2 h}{4\times 2} + \frac{h^3}{2^3\times 3}\right)$$

$$= \pi a^2 \rho \left(\frac{a^2 h}{4} + \frac{h^3}{12}\right)$$

$$= \pi a^2 \rho h \left(\frac{a^2}{4} + \frac{h^2}{12}\right)$$

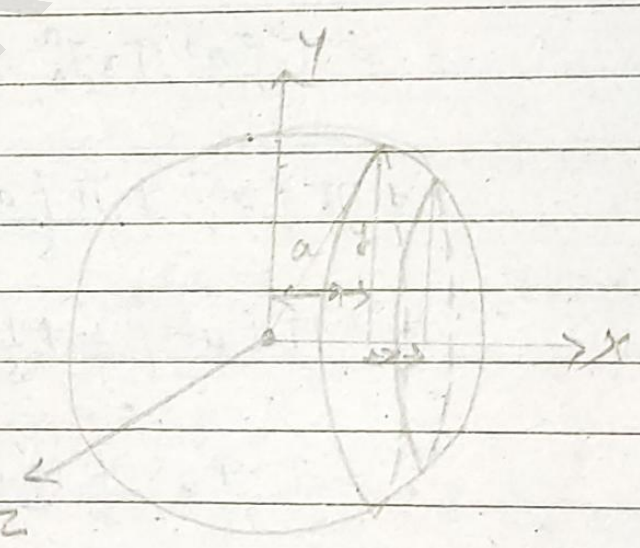
$$= M\left(\frac{3a^2 + h^2}{12}\right) \quad (\because M = \pi a^2 \rho h)$$

$$I_{GY'} = \frac{M}{12}(3a^2 + h^2).$$

Ex-3.5/ Determine the M.I of the mass of a solid homogeneous sphere w.r.t any geometrical axis.

Solⁿ: Imagine that the sphere is split into thin circular discs.

If ρ is the density of the material, then the mass of the disc between planes at distances x and $x + dx$ from the centre of the sphere is given as



$$dm = \pi y^2 dx \, \rho \quad \text{—— (1)}$$

$\therefore$ The M.I of the typical disc considered about the dimeter is $dI = dm \left(\frac{y^2}{2}\right)$ —— (2)

Thus the M.I of the sphere about a geometric axis say OX, is

$$I = \int_{-a}^{a} \pi y^2 \, dx \, \rho \, \frac{y^2}{2}$$

$$= \frac{\pi \rho}{2} \int_{-a}^{a} y^4 \, dx.$$

$$= \frac{\pi \rho}{2} \int_{-a}^{a} (a^2 - x^2) \, dx \quad (\because x^2 + y^2 = a^2)$$

i.e

$$I = \frac{\pi \rho}{2} \, 2 \int_{0}^{a} (a^2 - x^2)^2 \, dx$$

$$= \pi \rho \int_{0}^{a} [a^4 + x^4 - 2a^2 x^2] \, dx$$

$$= \pi \rho a^4 \int_{0}^{a} dx + \pi \rho \int_{0}^{a} x^4 \, dx - 2\pi a^2 \rho \int_{0}^{a} x^2 \, dx.$$

$$= \pi \rho a^4 [x]_0^a + \pi \rho \left[\frac{x^5}{5}\right]_0^a - 2\pi a^2 \rho \left[\frac{x^3}{3}\right]_0^a$$

$$= \pi \rho a^5 + \frac{\pi \rho a^5}{5} - 2\pi a^2 \rho \frac{a^3}{3}.$$

$$= \pi \rho a^5 \left(1 + \frac{1}{5} - \frac{2}{3}\right)$$

$$= \pi \rho a^5 \left(\frac{15 + 3 - 10}{15}\right)$$

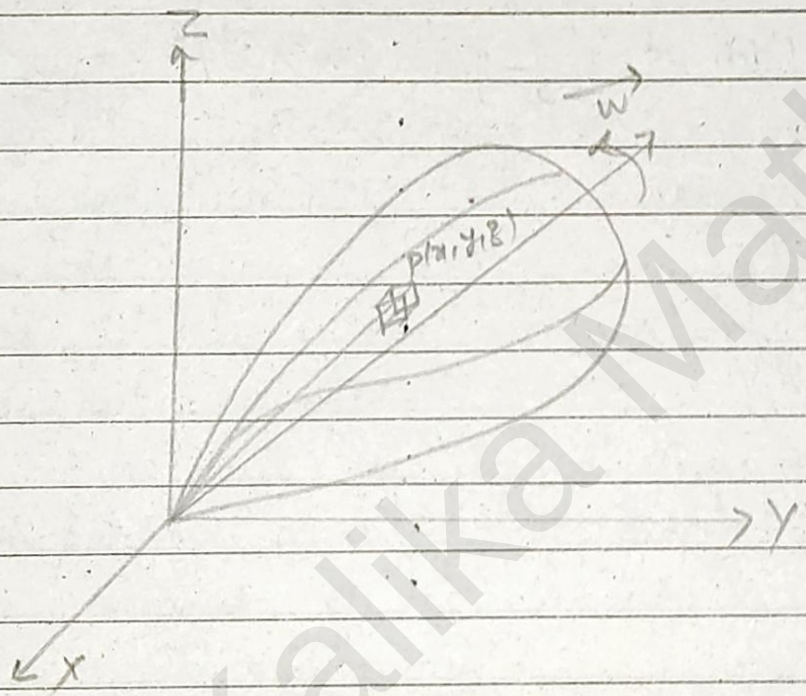
$$= \frac{8}{15} \pi \rho a^5$$

$$= \left(\frac{4}{3} \pi a^3 \rho\right) \frac{2}{5} a^2.$$

$$= M \frac{2a^2}{5}$$

$$\boxed{I = \frac{2}{5} Ma^2}$$ $M \rightarrow$ mass of the sphere.

3.2) Kinetic Energy of A Rigid Body Rotating About A fixed Point.



Let OXYZ be any rectangular system of coordinate axes.

Imagine a rigid body rotating about a fixed point O with angular velocity $\vec{w}$ ~~to~~ at a given instant.

Now, consider a typical particle $P(x,y,z)$ of mass dm. we note that the speed v of this particle P is equal to the magnitude of the velocity given by the expression

$$\vec{v} = \vec{w} \times \vec{r} \quad \text{———— (1)}$$

Then, the kinematic energy of the particle P is

$$T = \frac{1}{2} dm\, v^2 \quad \text{---} (2)$$

Hence, the total rotating kinetic energy T_{rot} of the body at a given instant is

$$T_{rot} = \frac{1}{2}\int dm\, v^2 = \frac{1}{2}\int dm\, |\vec{\omega}\times\vec{r}|^2 \quad \text{---} (3)$$

3.3/ Angular Momentum of a Rigid Body :—

It is known that the general motion of a rigid body passes ~~H~~ both Translational and Rotational Motion. In Rotational motion, angular momentum and Torque will come into play.

Consider the motion of a particle of mass m, moving with velocity $\vec{v}$, relative to some fixed frame of refrence OXYZ.

Then, the linear momentum of the particle is defined as $m\vec{v}$.

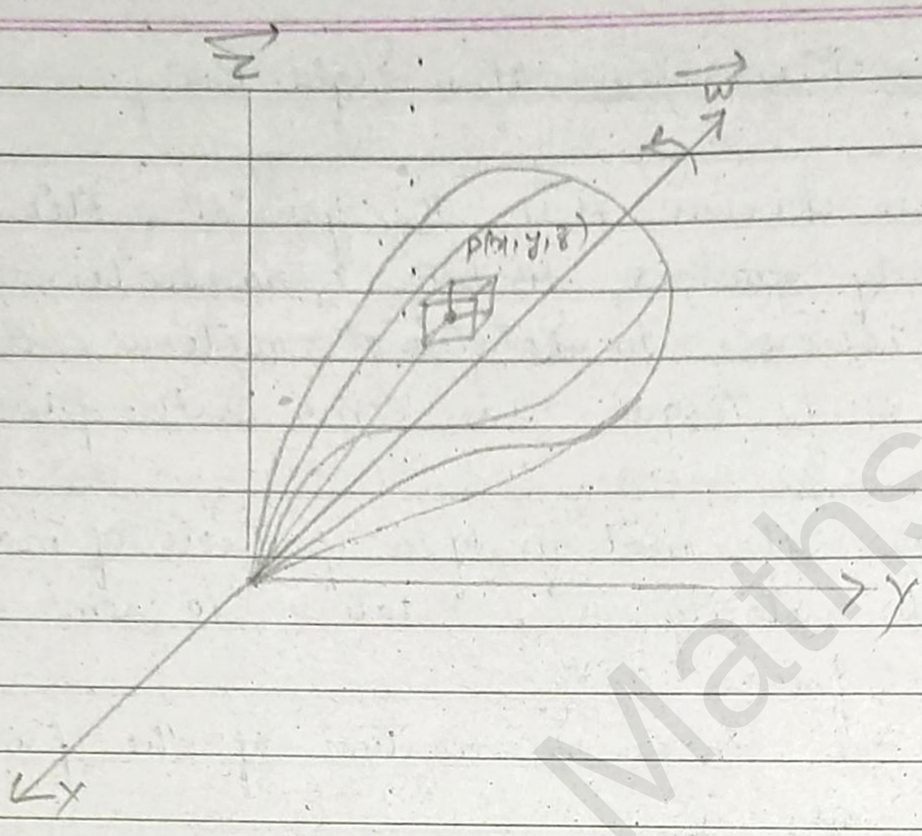
Now, we can define angular momentum $\vec{h}$ of the particle about any point, say O. Thus, the moment of $m\vec{v}$ about O is

$$\vec{h} = \vec{r} \times m\vec{v}$$

where r is the position vector of the particle with respect to O.

→ To extend this concept to a rigid body —

"A rigid body in rotating about a fixed point possesses momentum."



Consider a typical mass dm situated at P. whose position vector $\vec{r}$ relative to the origin O. Then, the angular momentum of the typical mass dm at P about O is

$$d\vec{h} = \vec{r} \times dm\,\vec{v} = \vec{r} \times (\vec{\omega} \times \vec{r})\,dm.$$

Hence, The total angular momentum of the rigid body about O can be found by integrating the momentum of each differential element.

i.e;

$$\vec{H} = \int \vec{r} \times (\vec{\omega} \times \vec{r})\,dm.$$

$$= \int \left[(\vec{r} \cdot \vec{r})\,\vec{\omega} - (\vec{r} \cdot \vec{\omega})\,\vec{r} \right] dm$$

$$= \int (\vec{r} \cdot \vec{r})\,\vec{\omega}\,dm - \int (\vec{r} \cdot \vec{\omega})\,\vec{r}\,dm.$$

In other words,

$$\vec{H} = \int \omega r^2 dm - \int \vec{r}\,(\vec{\omega}\cdot\vec{r})\,dm \quad \text{———} (1).$$

Here, the integral is over the volume of the body. Let $\hat{i}, \hat{j}, \hat{k}$ be the unit vectors along the coordinate axes, then

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \Rightarrow r^2 = x^2 + y^2 + z^2.$$

$$\vec{\omega} = \omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}$$

$$\vec{H} = H_x \hat{i} + H_y \hat{j} + H_z \hat{k}$$

Now eq^n (1) becomes

$$(H_x \hat{i} + H_y \hat{j} + H_z \hat{k}) = \int (\omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k})(x^2+y^2+z^2)\,dm$$
$$- \int (x\hat{i} + y\hat{j} + z\hat{k})(x\omega_x + y\omega_y + z\omega_z)\,dm.$$

$$= \int (x^2+y^2+z^2)\,\omega_x \hat{i}\,dm + \int (x^2+y^2+z^2)\,\omega_y \hat{j}\,dm$$

$$+ \int (x^2+y^2+z^2)\,\omega_z \hat{k}\,dm - \int (x\cdot\omega_x + y\cdot\omega_y + z\cdot\omega_z)\,x\hat{i}\,dm$$

$$- \int (x\cdot\omega_x + y\omega_y + z\omega_z)\,y\hat{j}\,dm - \int (x\omega_x + y\omega_y + z\omega_z)\,z\hat{k}\,dm$$

Now, comparing the components of $\hat{i}, \hat{j}$ and $\hat{k}$ from both side we get

$$H_x = \int (\cancel{x^2} + y^2 + z^2)\,\omega_x\,dm - \int x(\cancel{x\cdot\omega_x} + y\omega_y + z\omega_z)\,dm$$

$$= \omega_x \int (y^2+z^2)\,dm - \omega_y \int xy\,dm - \omega_z \int x\cdot z\,dm.$$

$$H_x = I_{xx}\,\omega_x - I_{xy}\,\omega_y - I_{xz}\,\omega_z$$

Similarly, $H_y = I_{yy}\,\omega_y - I_{yx}\,\omega_x - I_{yz}\,\omega_z$.

$$H_z = I_{zz}\,\omega_z - I_{zx}\,\omega_x - I_{zy}\,\omega_y.$$

Thus, the complete set of components of angular momentum vector of a rotating rigid body can be written in matrix notation as

$$\begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix} = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \quad \text{——(2)}$$

$$\Rightarrow \boxed{\vec{H} = [I]\,\vec{\omega}} \quad \text{——(3)}$$

Here, the matrix $[I]$ is called Inertia Tensor of a rigid body and is a Dyadic Operator which changes angular velocity ~~of~~ to angular momentum.

Thus, we observed that "the angular momentum vector of rotating rigid body does not coincides with its angular velocity vector." It may be noted that the elements of ~~the~~ Dyadic or Inertia Tensor depend on the shape and mass distribution of the body and the coordinate system to which they are refered.

In fact, when a body fixed coordinate system is employed in the study of angular motions of a rigid body such as a missile or a rocket, the element of inertia matrix would be a function of time.

In case $\hat{i}, \hat{j}, \hat{k}$ coincides with the principal axes, then

$$H_x = I^*_{xx}\,\omega_x \;,\; H_y = I^*_{yy}\,\omega_y \;,\; H_z = I^*_{zz}\,\omega_z \quad \text{---(4)}$$

where I^*_{xx}, I^*_{yy} and I^*_{zz} are the principal M.I's of the body.

Suppose, the rigid body is constrained to rotate about a fixed axis, say z-axis, with angular velocity ω, then eqn(2) simply reduces to

$$H_x = -I_{xz}\,\omega \;,\; H_y = -I_{yz}\,\omega \;,\; H_z = I_{zz}\,\omega \quad \text{---(5)}$$

and then the angular momentum vector does not lie along the axis of rotation, unless the latter is a principal axis of inertia.

However, the component H_z along the axis of rotation is equal to the product of the M.I about that axis and angular velocity.

This idea is a fundamental importance to understand the phenomenon of ~~gy~~ Gyroscopic motion.

Ex – 8.3 If a rigid body rotates about a fixed point with an angular velocity $\vec{\omega}$ and has an angular momentum $\vec{H}$, show that the kinetic Energy T is given by $T = \frac{1}{2}\vec{\omega}\cdot\vec{H}$.

Sol:— we know that the kinetic Energy in the given situation is given by

$$T = \frac{1}{2}\int dm\, v^2.$$

where dm is the mass of the typical element of the body which can be written as

$$T = \frac{1}{2}\int dm\,(\vec{v}\cdot\vec{v}) = \frac{1}{2}\int dm\,(\vec{\omega}\times\vec{r})\cdot(\vec{\omega}\times\vec{r})$$

where $\vec{r}$ is the position vector of the typical particle.

Since '$\cdot$' and '$\times$' can be interchanged. we have

$$T = \frac{1}{2}\int dm\,\{(\vec{\omega}\cdot\vec{r})\times(\vec{\omega}\times\vec{r})$$

$$= \frac{1}{2}\int dm\,\{\vec{\omega}\cdot[\vec{r}\times(\vec{\omega}\times\vec{r})]\}$$

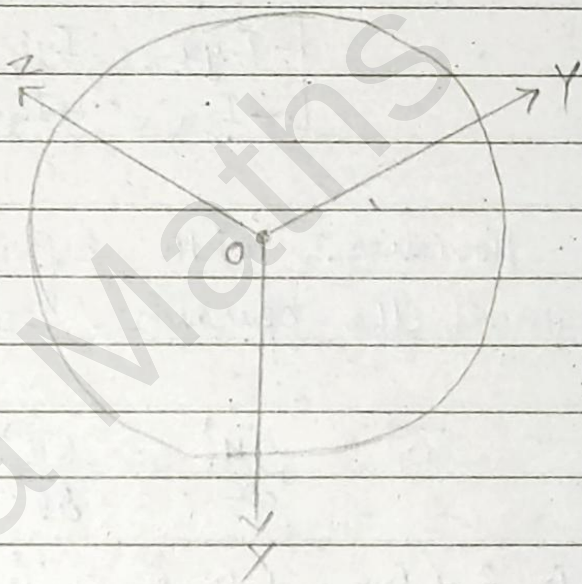
$$= \frac{1}{2}\vec{\omega}\cdot\vec{H}$$

Hence $\boxed{T = \frac{1}{2}\vec{\omega}\cdot\vec{H}}$

Dynamics of a Rigid body motion in Space.

4.1 Euler's Dynamical Equations for a rigid body rotating about a fixed point.

Imagine a rigid body which is constrained to rotate about a fixed point O (may be its centre of Gravity), with angular velocity $\vec{w}$.



(motion of a rigid body in space)

Suppose the origin of the body-fixed coordinate system is O. Let $\hat{i}, \hat{j}, \hat{k}$ be the unit vectors along OX, OY, OZ respectively.

If the components of $\vec{w}$ are w_x, w_y and w_z, then, we write $\vec{w} = w_x \hat{i} + w_y \hat{j} + w_z \hat{k}$ ——(1)

From the principle of angular momentum, which states that "The rate of change of angular momentum of a body about a point, fixed or moving with the centre of mass, is equal to the total moment of the external forces about that point.

Symbolically, we can express it as

$$\frac{d\vec{H}}{dt} = \vec{G} \text{ (Torques)} \quad \text{--- (2)}$$

Also, we know that $\vec{H} = [I]\vec{\omega}$.

$$\vec{H} = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \quad \text{--- (3)}$$

However, with reference to the body fixed frame, the operator relation allows us to write

$$\vec{G} = \frac{d\vec{H}}{dt} = \frac{\delta \vec{H}}{\delta t} + \vec{\omega} \times \vec{H}$$

$$G_x\hat{i} + G_y\hat{j} + G_z\hat{k} = \dot{H}_x\hat{i} + \dot{H}_y\hat{j} + \dot{H}_z\hat{k} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_x & \omega_y & \omega_z \\ H_x & H_y & H_z \end{vmatrix}$$

$$= \dot{H}_x\hat{i} + \dot{H}_y\hat{j} + \dot{H}_z\hat{k} + \hat{i}(\omega_y H_z - \omega_z H_y) - \hat{j}(-\omega_z H_x + \omega_x H_z) + \hat{k}(\omega_x H_y - \omega_y H_x)$$

$$= (I_{xx}\dot{\omega}_x - I_{xy}\dot{\omega}_y - I_{xz}\dot{\omega}_z)\hat{i} + \hat{j}(-I_{yx}\dot{\omega}_x + I_{yy}\dot{\omega}_y - I_{yz}\dot{\omega}_z) + \hat{k}(-I_{zx}\dot{\omega}_x - I_{zy}\dot{\omega}_y + I_{zz}\dot{\omega}_z) +$$

$$= \hat{i}\left[\omega_y(-I_{zx}\omega_x - I_{zy}\omega_y + I_{zz}\omega_z) - \omega_z(-I_{yx}\omega_x + I_{yy}\omega_y - I_{yz}\omega_z)\right]$$

$$- \hat{j}\left[\omega_x(-I_{zx}\omega_x - I_{zy}\omega_y + I_{zz}\omega_z) - \omega_z(I_{xx}\omega_x - I_{xy}\omega_y - I_{xz}\omega_z)\right]$$

$$+\hat{k}\left[\omega_x(-I_{yx}\omega_x + I_{yy}\omega_y - I_{yz}\omega_z) - \omega_y(I_{xx}\omega_x - I_{xy}\omega_y - I_{xz}\omega_z)\right]$$

$$= \left[(I_{xx}\dot{\omega}_x - I_{xy}\dot{\omega}_y - I_{xz}\dot{\omega}_z) + \omega_y(-I_{zx}\omega_x - I_{zy}\omega_y + I_{zz}\omega_z) - (\omega_z)(-I_{yx}\omega_x + I_{yy}\omega_y - I_{yz}\omega_z)\right]\hat{i}$$

$$+ \left[(-I_{zz}\dot{\omega}_x - I_{zy}\dot{\omega}_y + I_{yz}\dot{\omega}_z) - \omega_x(-I_{zx}\omega_x - I_{zy}\omega_y + I_{zz}\omega_z) + \omega_z(I_{xx}\omega_x - I_{xy}\omega_y - I_{xz}\omega_z)\right]\hat{j}$$

$$+ \left[(-I_{zz}\dot{\omega}_x - I_{zy}\dot{\omega}_y + I_{zz}\dot{\omega}_z) + \omega_x(-I_{yx}\omega_x + I_{yy}\omega_y - I_{yz}\omega_z) - \omega_y(I_{xx}\omega_x - I_{xy}\omega_y - I_{xz}\omega_z)\right]\hat{k}$$

$$= \left[I_{xx}\dot{\omega}_x + I_{xy}(\omega_z\omega_x - \dot{\omega}_y) - I_{xz}(\dot{\omega}_z + \omega_x\omega_y) + I_{yz}(\omega_z^2 - \omega_y^2) + (I_{zz} - I_{yy})\omega_y\omega_z\right]i$$

$$+ \hat{j}\left[I_{yy}\dot{\omega}_y + I_{yz}(\omega_x\omega_y - \dot{\omega}_z) - I_{yx}(\dot{\omega}_x + \omega_y\omega_z) + \cancel{I_{zz}}\, I_{xz}(\omega_x^2 - \omega_z^2) + (I_{xx} - I_{zz})\omega_x\omega_z\right]$$

$$+ \hat{k}\left[I_{zz}\dot{\omega}_x + I_{xz}(\omega_y\omega_z - \dot{\omega}_x) - I_{yz}(\dot{\omega}_y + \omega_z\omega_x) + I_{xy}(\omega_y^2 - \omega_x^2) + (I_{yy} - I_{xx})\omega_y\omega_x\right]$$

Now, Equating the coefficients of $\hat{i}, \hat{j}, \hat{k}$ on both sides,

$$G_x = I_{xx}\dot{\omega}_x + I_{xy}(\omega_z\omega_x - \dot{\omega}_y) - I_{xz}(\dot{\omega}_z + \omega_x\omega_y) + I_{yz}(\omega_z^2 - \omega_y^2) + (I_{zz} - I_{yy})\omega_y\omega_z \quad \text{—(4)}$$

$$G_y = \cancel{I_x}\, I_{yy}\,\dot{\omega}_y + I_{yz}(\omega_x \omega_y - \dot{\omega}_z) - I_{yz}(\dot{\omega}_x + \omega_y \omega_z) + I_{xz}(\omega_x^2 - \omega_z^2) + (I_{xx} - I_{zz})\,\omega_x \omega_z \quad \text{——(5)}$$

$$G_z = I_{zz}\,\dot{\omega}_z + I_{xz}(\omega_y \omega_z - \dot{\omega}_x) - I_{yz}(\dot{\omega}_y + \omega_z \omega_x) + I_{xy}(\omega_y^2 - \omega_x^2) + (I_{yy} - I_{xx})\,\omega_y \omega_x \quad \text{——(6)}$$

This set of equations of motion has very important applications, specially in space-related problem. Some of them are presented in the following sections.

These equations are known as Euler's Dynamical Equations of motion for a rigid body rotating about its centre of mass.

Considerable of simplification of these equations can be archieved by allowing the body-fixed coordinates axes to coincide with its principal axes.

In such a case, all the P.I's becomes zero and (4), (5) & (6) becomes

$$\left.\begin{aligned}
I_{xx}\,\dot{\omega}_x - (I_{yy} - I_{zz})\,\omega_y \omega_z &= G_x \\
I_{yy}\,\dot{\omega}_y - (I_{zz} - I_{xx})\,\omega_x \omega_z &= G_y \\
\cancel{I_{zz}\,\dot{\omega}_z - (I_{zz} - I_{xx})\,\omega_x \omega_z} &\cancel{= G_y} \\
I_{zz}\,\dot{\omega}_z - (I_{xx} - I_{yy})\,\omega_x \omega_y &= G_z
\end{aligned}\right\} \text{——(7)}$$

These are Euler's Equations of motion for a rigid body with a fixed point, in a body-fixed frame of reference.

These equations are widely used in solving the rotational motion of a rigid body. These are also non-linear ordinary differential equation and found to be difficult to have analytical solution and therefore, require the use of numerical Techniques, such as Runge-Kutta method of fourth order and a fast electronic computer for their solution.

However, for a given set of initial conditions and assuming that $\vec{G}$ is known function of time, position and velocity of the body. Eq^n (7) can be solved, in principle, to determine the components of $\vec{\omega}$ as a function of time.

Body-fixed Translational Equations

It is convenient to solve both the translation as well as rotational motions of a rigid body, in a body-fixed coordinate system, especially,

When the applied forces can be specified in a body-fixed coordinate system.

In the Newtonian frame of reference,

the Translational equations of motion of a rigid body can be written as

$$\vec{F} = m\dot{\vec{v}} \quad \text{——— (1)}$$

where $\vec{F}$ is the total external force acting on the rigid body and $\vec{v}$ and the velocity of the centre of mass of the body.

In a body-fixed frame, $\vec{v}$ and $\vec{F}$ may be expressed as

$$\vec{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$$
$$\vec{F} = f_x\hat{i} + f_y\hat{j} + f_z\hat{k}$$

Now, following Newton's second law of motion, we have

$$\vec{F} = m\left(\frac{\delta\vec{v}}{\delta t} + \vec{\omega}\times\vec{v}\right)$$

$$= m\left[\left(\dot{v}_x\hat{i} + \dot{v}_y\hat{j} + \dot{v}_z\hat{k}\right) + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_x & \omega_y & \omega_z \\ v_x & v_y & v_z \end{vmatrix}\right] \text{——(2)}$$

$$\vec{f_x}\hat{i} + \vec{f_y}\hat{j} + \vec{f_z}\hat{k} = m\left[(\dot{v}_x\hat{i} + \dot{v}_y\hat{j} + \dot{v}_z\hat{k}) + (\omega_y v_z - \omega_z v_y)\hat{i} + \hat{j}(\omega_z v_x - \omega_x v_z) + \hat{k}(\omega_x v_y - \omega_y v_x)\right]$$

Now, in component form, we also write as

$$f_x = m\left(\frac{dv_x}{dt} + \omega_y v_z - \omega_z v_y\right) \Big\} \text{——— (3)}$$

$$\left.\begin{aligned} f_y &= m\left(\frac{dv_y}{dt} + \omega_z v_x - \omega_x v_z\right) \\ f_z &= m\left(\frac{dv_z}{dt} + \omega_x v_y - \omega_y v_x\right) \end{aligned}\right\} \text{ --- (3)}$$

Here, the components of $\vec{\omega}$ are either given directly or may be obtained from the rotational eqⁿ G_x, G_y, G_z.

Thus, eqⁿ (3) can be solved to get v_x, v_y and v_z as a function of time.

To sum up, the set of eqⁿ (7), (3) together constitute the dynamical equation (previous theorem) of motion of a rigid body in a body-fixed coordinate frame.

Ex-4.1 Show directly from the Euler's dynamical equations of motion (7) eqⁿ that if $\vec{G} = 0$ and $I_{xx} = I_{yy}$, then ω is constant.

Solⁿ:— We are given the data $I_{xx} = I_{yy}$ and therefore the Euler's dynamical equations reduce to

$$I_{xx}\dot{\omega}_x - (I_{yy} - I_{zz})\omega_y\omega_z = 0 \quad \text{--- (1)}$$

$$I_{yy}\dot{\omega}_y - (I_{zz} - I_{xx})\omega_z\omega_x = 0 \quad \text{--- (2)}$$

$$I_{zz}\dot{\omega}_z = 0 \quad \text{--- (3)}$$

Since $I_{zz} \neq 0$, eqⁿ (3) becomes $\dot{\omega}_z = 0$

$$\Rightarrow \omega_z = \text{Constant} \quad \text{--- (4)}$$

Now multiplying the eqⁿ (1) by ω_x and eqⁿ (2) by ω_y and adding we get.

$$[I_{xx}\dot{\omega}_x\omega_x - (I_{yy} - I_{zz})\omega_x\omega_y\omega_z]$$
$$+ [I_{yy}\dot{\omega}_y\omega_y - (I_{zz} - I_{xx})\omega_x\omega_y\omega_z] = 0$$

$$\Rightarrow I_{xx}\dot{\omega}_x\omega_x + I_{yy}\dot{\omega}_y\omega_y - (I_{yy} - I_{zz} + I_{zz} - I_{xx})\omega_x\omega_y\omega_z = 0$$

using the fact $I_{xx} = I_{yy}$, the above eqⁿ reduces to

$$\Rightarrow I_{xx}\dot{\omega}_x\omega_x + I_{xx}\dot{\omega}_y\omega_y - 0 = 0$$

$$\Rightarrow I_{xx}(\dot{\omega}_x\omega_x + \dot{\omega}_y\omega_y) = 0.$$

$\because$ Since $I_{xx} \neq 0$, we have $\dot{\omega}_x\omega_x + \dot{\omega}_y\omega_y = 0$ which can be rewritten as

$$\frac{1}{2}\frac{d}{dx}(\omega_x^2 + \omega_y^2) = 0.$$

$$\Rightarrow \omega_x^2 + \omega_y^2 = \text{Constant} \quad \text{---(5)}$$

Eqⁿ (4) and (5) can be written as

$$\omega_x^2 + \omega_y^2 + \omega_z^2 = \text{Constant}$$

Which means $\omega = |\vec{\omega}| = \omega_x^2 + \omega_y^2 + \omega_z^2 = \text{constant}.$

Ex-4.2/ Imagine that a rigid body is rotating about a fixed point with angular velocity $\vec{\omega}$. Assuming that the coordinate axes coincide with the principal axes, if T stands for kinetic energy and $\vec{G}$ for external torque acting on the body, show that

$$\frac{dT}{dt} = \vec{G} \cdot \vec{\omega}$$

Solⁿ:– Assuming that the body-fixed coordinate axes coincide with the principal axes.

Let $\vec{\omega} = \omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}$

and $\vec{G} = G_x \hat{i} + G_y \hat{j} + G_z \hat{k}$

Then, the expression for the kinetic Energy of a rigid body rotating about a fixed point is given as

$$T = \frac{1}{2}\left(I_{xx}\,\omega_x^2 + I_{yy}\,\omega_y^2 + I_{zz}\,\omega_z^2\right) \quad \text{——(1)}$$

$$\therefore \frac{dT}{dt} = \frac{1}{2}\left(I_{xx}\,2\omega_x\dot{\omega}_x + I_{yy}2\dot{\omega}_y\,\omega_y + I_{zz}\,2\omega_z\dot{\omega}_z\right)$$

$$\frac{dT}{dt} = \left(I_{xx}\,\omega_x\dot{\omega}_x + I_{yy}\,\omega_y\dot{\omega}_y + I_{zz}\,\omega_z\dot{\omega}_z\right) \quad \text{——(2)}$$

By Euler's dynamical equations of motion such as

$$I_{xx}\,\dot{\omega}_x - (I_{yy} - I_{zz})\,\omega_y\omega_z = G_x$$

$$I_{yy}\,\dot{\omega}_y - (I_{zz} - I_{xx})\,\omega_x\,\omega_z = G_y$$

$$I_{zz}\,\dot{\omega}_z - (I_{xx} - I_{yy})\,\omega_x\,\omega_y = G_z$$

So eqⁿ (2) becomes

$$\frac{dT}{dt} = \omega_x\left[G_x + (I_{yy} - I_{zz})\,\omega_y\,\omega_z\right] +$$

$$+ \omega_y [G_y + (I_{zz} - I_{xx}) \omega_x \omega_z]$$

$$+ \omega_z [G_z + (I_{xx} - I_{yy}) \omega_x \omega_y]$$

$$\frac{dT}{dt} = \omega_x G_x + \omega_y G_y + \omega_z G_z + \omega_x \omega_y \omega_z (I_{yy} - I_{zz} + I_{zz} - I_{xx} + I_{xx} - I_{yy})$$

$$\frac{dT}{dt} = \omega_x G_x + \omega_y G_y + \omega_z G_z = \vec{\omega} \cdot \vec{G}$$

$$\Rightarrow \boxed{\frac{dT}{dt} = \vec{\omega} \cdot \vec{G}}$$

Ex – 4.3/ Show that the kinetic energy and angular momentum of the torque-free motion of a rigid body is constant.

Solⁿ:– If the torque $\vec{G}$ is set to zero, then Euler's eqⁿ of motion for a rigid body is

$$\left.\begin{aligned} I_{xx} \dot{\omega}_x - (I_{yy} - I_{zz}) \omega_y \omega_z &= 0 \\ I_{yy} \dot{\omega}_y - (I_{zz} - I_{xx}) \omega_z \omega_x &= 0 \\ I_{zz} \dot{\omega}_z - (I_{xx} - I_{yy}) \omega_x \omega_y &= 0 \end{aligned}\right\} \quad \text{—— (1)}$$

Multiplying these eqⁿ with ω_x, ω_y and ω_z and then adding, we get

$$\Rightarrow \left[I_{xx}\dot{\omega}_x\omega_x - (I_{yy}-I_{zz})\omega_x\omega_y\omega_z\right] + \left[I_{yy}\dot{\omega}_y\omega_y - (I_{zz}-I_{xx})\omega_x\omega_y\omega_z\right] + \left[I_{zz}\dot{\omega}_z\omega_z - (I_{xx}-I_{yy})\omega_x\omega_y\omega_z\right] = 0$$

$$\Rightarrow I_{xx}\dot{\omega}_x\omega_x + I_{yy}\dot{\omega}_y\omega_y + I_{zz}\dot{\omega}_z\omega_z - (I_{yy}-I_{zz}+I_{zz}+I_{xx}-I_{xx}-I_{yy})\omega_x\omega_y\omega_z = 0$$

$$\Rightarrow I_{xx}\dot{\omega}_x\omega_x + I_{yy}\dot{\omega}_y\omega_y + I_{zz}\dot{\omega}_z\omega_z = 0$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}\left(I_{xx}\omega_x^2 + I_{yy}\omega_y^2 + I_{zz}\omega_z^2\right) = 0$$

on integrating, we get

$$\Rightarrow \frac{1}{2}\left(I_{xx}\omega_x^2 + I_{yy}\omega_y^2 + I_{zz}\omega_z^2\right) = \text{constant} \quad \text{—(2)}$$

Also, we know that kinetic Energy of a rigid body rotating about a fixed point is

$$T = \frac{1}{2}\left(I_{xx}\omega_x^2 + I_{yy}\omega_y^2 + I_{zz}\omega_z^2\right) \quad \text{—(3)}$$

$\Rightarrow$ using (3) in (2) we get

$$\Rightarrow \frac{1}{2}\left(I_{xx}\omega_x^2 + I_{yy}\omega_y^2 + I_{zz}\omega_z^2\right) = T = \text{constant}$$

$$\Rightarrow I_{xx}\omega_x^2 + I_{yy}\omega_y^2 + I_{zz}\omega_z^2 = 2T = \text{Constant} \quad \text{—(4)}$$

Hence, the kinetic Energy, T, of a torque free motion of a rigid body is constant.

This is also called the "Energy Integral of the System".

Multiplying eq^n (1) respectively by $I_{xx}\,\omega_x$, $I_{yy}\,\omega_y$ and $I_{zz}\,\omega_z$ and ~~so~~ adding, we get

$$\Rightarrow \left[I_{xx}^2\,\dot{\omega}_x\,\omega_x - (I_{yy}-I_{zz})\,I_{xx}\,\omega_x\,\omega_y\,\omega_z\right]$$
$$+\left[I_{yy}^2\,\dot{\omega}_y\,\omega_y - (I_{zz}-I_{xx})\,I_{yy}\,\omega_x\,\omega_y\,\omega_z\right]$$
$$+\left[I_{zz}^2\,\dot{\omega}_z\,\omega_z - (I_{xx}-I_{yy})\,I_{zz}\,\omega_x\,\omega_y\,\omega_z\right]=0$$

$$\Rightarrow I_{xx}^2\,\dot{\omega}_x\,\omega_x + I_{yy}^2\,\dot{\omega}_y\,\omega_y + I_{zz}^2\,\dot{\omega}_z\,\omega_z$$
$$-\left[I_{yy}I_{xx} - I_{xx}I_{zz} + I_{zz}I_{yy} - I_{xx}I_{yy} + I_{xx}I_{zz} + I_{yy}I_{zz}\right]\cdot\omega_x\,\omega_y\,\omega_z = 0$$

$$\Rightarrow I_{xx}^2\,\dot{\omega}_x\,\omega_x + I_{yy}^2\,\dot{\omega}_y\,\omega_y + I_{zz}^2\,\dot{\omega}_z\,\omega_z = 0$$

Now, integrating this eq^n w.r.t. time, we get

$$(I_{xx}\,\omega_x^2) + (I_{yy}\,\omega_y^2) + (I_{zz}\,\omega_z^2) = \text{constant}$$

also, we know that

$$|\vec{H}|^2 = (I_{xx}\,\omega_x)^2 + (I_{yy}\,\omega_y)^2 + (I_{zz}\,\omega_z)^2$$

Hence

$$(I_{xx}\,\omega_x)^2 + (I_{yy}\,\omega_y)^2 + (I_{zz}\,\omega_z^2)^2 = |\vec{H}|^2 = \text{constant}$$

This equation means that the "Square of the modulus of the angular momentum of a torque - free motion of a rigid body is also ~~cat~~ a constant. This is called the "Angular momentum integral of the system considered."

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Subject wise Marks Weightage of CSIR-NET Examination

(Maximum Marks: 200)

Subject(Mathematics)	Marks Range (Min. ~ Max.)	No. of Que. (Min. ~ Max.)	Important Topics (Note Here)
UNIT-I			
Real Analysis	(45.25 ~ 73.25)	15 ~ 20	
Linear Algebra	(41.25 ~ 75.00)	15 ~ 20	
UNIT-II			
Abstract Algebra	(25.00 ~ 45.25)	6 ~ 8	
Number Theory	(3 ~ 07.75)	1 ~ 2	
Complex Analysis	(25.00 ~ 34.50)	5 ~ 8	
Topolgy	(3 ~ 07.75)	1 ~ 2	
UNIT-III			
Ordinary Differential Equation	(15.50 ~ 25.00)	4 ~ 7	
Partial Differential Eqn.(PDE)	(20.20 ~ 25.00)	4 ~ 7	
Dynamical System	(0 ~ 07.75)	0 ~ 2	
Numerical Analysis(NA)	(3 ~ 12.50)	1 ~ 3	
Calculua of Variation (COV)	(3 ~ 12.50)	1 ~ 3	
Integral Equation(I.E)	(3 ~ 12.50)	1 ~ 3	
Classical Mechanics	(0 ~ 07.75)	0 ~ 2	
UNIT-IV			
Probability & Statistics			
Markov Chain	(3 - 12.50)	1 to 3	
Operation Research(LPP)	(0 ~ 07.75)	0 ~ 2	
TOTAL			

CSIR-NET Exam. Paper Structure(Total Marks = 200)

PARTS	Total Que.	To Attempt	Max. Mark	Negative	Major Part
PART - A	20 (2 Marks)	15	30	0.50 Neg.	General
PART - B	40 (3 Marks)	25	75	0.75 Neg.	Pure Maths
	UNIT I - IV				
PART - C	60 (4.75 Mark)	20	95	No Neg.	Pure Maths
	UNIT I - IV				
UNIT - I	*Real Analysis & Linear Algebra*				
UNIT II-1V	*Complex Anal., Modern Alg.,ODE,PDE*				

*** Prepare Accordingly***

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CSIR-NET Year wise Cut-off (Subject : Mathematics)

	Category	General		EWS		OBC		SC		ST		PwD	
Year		JRF	LS(NET)	JRF	LS(NET)	JRF	LS(NET)	JRF	LS(NET)	JRF	LS(NET)	JRF	LS(NET)
2020	December												
	June	114	102.6	102.76	92.476	101.5	91.35	80.5	72.45	61.76	55.576	57.5	51.75
2019	December	107.3	96.54	96.26	86.64	92.5	83.26	70.26	63.25	55	50	50	50
	June	111.5	100.36	93.26	83.94	97.76	87.98	75.5	67.96	61	54.9	57	50
2018	December	97.26	87.54	-	-	82	73.8	63.76	57.38	50.5	50	50	50
	June	112.5	101.26	-	-	94.76	85.28	74	66.6	55.5	50	50	50
2017	December	96.76	87.08	-	-	81.5	73.36	62.5	56.26	50	50	50.26	50
	June	100.8	90.68	-	-	85.76	77.18	68.26	61.48	50	50	50	50
2016	December	119	107.1	-	-	100	90	78.5	70.66	55.26	50	52	50
	June	109.8	98.78	-	-	94.76	85.28	75.26	67.74	50	50	51.5	50
2015	December	109.8	98.78	-	-	95.5	85.96	77.26	69.54	51.26	50	64	51.08
	June	106.3	95.64	-	-	84.5	81.5	72.24	65.04	51	50	77.26	69.54
2014	December												
	June												

Category	General		EWS		OBC		SC		ST		PwD	
Your Target	119	107.5	103	93	102	92	81	73	62	56	78	70

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2. **BSc/MSc Free Study Materials (**https://pkalika.in/2019/10/14/study-material/**)**

3. **PhD/MSc Entrance Exam Que. Paper: (**https://pkalika.in/que-papers-collection/**)**
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4. **CSIR-NET Maths Que. Paper: (**https://pkalika.in/2020/03/30/csir-net-previous-yr-papers/**)**
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