JNU Mathematics
PhD Entrance Que. Paper

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ENTRANCE EXAMINATION, 2018

Ph.D. IN
MATHEMATICAL SCIENCES

[Field of Study Code : MATH (897)]

Time Allowed : 3 hours

Maximum Marks : 100

INSTRUCTIONS FOR CANDIDATES

(i) All questions are compulsory.

(ii) There are 10 questions in **Section—A** and 8 questions in each of **Sections B** and **C**.

(iii) Answers must be written in the space provided after each question. Answers that are written elsewhere will not be evaluated.

(iv) A question in **Section—A** may have more than one true option. Your answer will be considered correct provided it consists only of all the true options. A correct answer will be awarded +**2** marks and an incorrect/incomplete answer or an unattempted question will get **0** mark.

(v) Each question in **Section—B** carries 4 marks; each question in **Section—C** carries 6 marks. Answers to all the questions in **Sections B** and **C** must be **justified with mathematical reasoning**.

(vi) In the following, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ denote, respectively, the set of natural numbers, the set of integers, the set of rational numbers, the set of real numbers and the set of complex numbers.

(vii) $\mathbb{C}^n$, $\mathbb{R}^n$ and its subsets are assumed to have the usual topology arising from the usual Euclidean norm unless mentioned otherwise.

(viii) For sets A and B, let $A \setminus B := \{x \in A \mid x \notin B\}$. For $a, b \in \mathbb{R}$, let $[a, b] := \{x \in \mathbb{R} \mid a \le x \le b\}$, $(a, b] := [a, b] \setminus \{a\}$ and $(a, b) := [a, b] \setminus \{a, b\}$.

(ix) Extra pages are attached at the end of the question paper for Rough Work.

(x) **Use of calculator, smart phone, mobile phone or any other digital device is strictly not permitted.**

ENTRANCE EXAMINATION, 2018

Ph.D. IN
MATHEMATICAL SCIENCES

SUBJECT . . . .
(Field of Study/Language)

FIELD OF STUDY CODE . . . .

NAME OF THE CANDIDATE . . . .

. . . .

REGISTRATION NO. | | | | | |

CENTRE OF EXAMINATION . . . .

. . . .

DATE . . . .

(Signature of Candidate)

(Signature of Invigilator)

(Signature and Seal of Presiding Officer)

For Official Use Only

Not to be filled in by the candidate

Final Grading Table

Total of **Section—A**	
Total of **Section—B**	
Total of **Section—C**	
Grand Total	

For Official Use Only

Not to be filled in by the candidate

Grading Table for Section—A

Question Nos.	Marks Awarded	Question Nos.	Marks Awarded
A1		A2	
A3		A4	
A5		A6	
A7		A8	
A9		A10	
		Total (Out of 20)	

SECTION—A

Each question carries 2 marks.

A1. What is the number of zeroes at the end of (2017)!?

Answer :

A2. Five friends went to *A* to *Z* ice-cream shop that was selling 26 flavours of ice cream to have one scoop of ice cream each. One friend asked everyone which flavour of ice-cream they wanted and went to the counter to place the order. How many distinct orders of ice creams can be placed?

Answer :

A3. Let *A*, *B*, *C* and *D* be finite sets and $\varphi : A \to B$, $\psi : B \to D$, $\theta : A \to C$ and $\eta : C \to D$ be maps such that $\psi \circ \varphi = \eta \circ \theta$. Consider the following statements :

(a) φ is surjective

(b) *B* and *D* have the same number of elements

(c) θ is injective

(d) *A* has at least as many elements as in Image (ψ)

Which of the above statements are correct?

Answer :

A4. Is $\mathbb{Z}/899\mathbb{Z}$ an integral domain? Provide your answer in **Yes** or **No** below.

Answer :

A5. Suppose *G* is a group in which every proper subgroup is cyclic. Does this imply that *G* is cyclic? Provide your answer in **Yes** or **No** below.

Answer :

A6. Let $S \subset \mathbb{R}^2$ be defined by $S := \{(t\cos(t), t\sin(t)) \mid 0 < t \leq 2\pi\}$. Consider the following statements :

(a) *S* is bounded

(b) *S* contains all its limit points

(c) *S* is connected

(d) *S* is not path-connected

Which of the above statements are correct?

Answer :

A7. Let X be a normed linear space. Which of the following statements are always true?

(a) X is locally compact with respect to the norm topology

(b) Every linear map $\varphi : X \to \mathbf{C}$ is continuous

(c) X is homeomorphic to its open unit ball $B := \{x \in X \mid \|x\| < 1\}$

(d) X is not homeomorphic to its closed unit ball $\bar{B} := \{x \in X \mid \|x\| \leq 1\}$

Answer :

A8. Let $E \subset \mathbb{R}$ be a Lebesgue measurable set and let m denote the Lebesgue measure. Which of the following statements are true?

(a) If $m(E) = \infty$, then E is dense in $\mathbb{R}$

(b) If $m(E) = 0$, then E has empty interior

(c) If $m(E) = 0$, then $E \subset \mathbb{Q}$

(d) If $m(E) = \infty$, then $E \cap \mathbb{Q}^c \neq \varnothing$

Answer :

A9. What is the value of $\lim_{x \to 0+} \left(1 + \frac{1}{x}\right)^x$?

Answer :

A10. The series

$$\sum_{n=1}^{\infty} \frac{z^n}{n}$$

(a) converges for all z with $|z| = 1$, except at $z = 1$

(b) converges for all z with $|z| = 1$, except at $z = 1, -1$

(c) diverges for all z with $|z| = 1$

(d) diverges for all z with $|z| = 1$, except at $z = -1$

Pick the correct options.

Answer :

SECTION—B

Each question carries 4 marks.

B1. Show that $\sqrt[3]{5\sqrt{2}+7}+\sqrt[3]{5\sqrt{2}-7}=2\sqrt{2}$. (*Hint* : First show that the given number is a root of the polynomial $x^3-3x-10\sqrt{2}$.)

B2. For $x \in (0, \infty)$, define $f(x) := x^{x+1}$ and $g(x) := (x+1)^x$. Determine the values of x for which $f(x) > g(x)$.

B3. Let $\mathcal{T}$ be a collection of subsets of $\mathbb{Z}$ containing $\varnothing$, $\mathbb{Z}$ and non-empty subsets U of $\mathbb{Z}$ which are unions of sets of the form

$$S(a, b) := \{an + b \mid n \in \mathbb{Z}\}$$

for $a, b \in \mathbb{Z}$ and $a \neq 0$. Show that $\mathcal{T}$ defines a topology on $\mathbb{Z}$. Prove or disprove that $\mathcal{T}$ is Hausdorff.

B4. Let X and Y be sets, $\varphi : X \to \mathbb{N}$ and $\psi : \mathbb{N} \to Y$ be maps such that their composite $\psi \circ \varphi : X \to Y$ is surjective. Prove or disprove the following assertion :

X and Y are both countable.

B5. Let $f : \mathbf{R}^3 \to \mathbf{R}^3$ be a map such that $||f(x) - f(y)|| = ||x - y||$ for all $x, y \in \mathbf{R}^3$. Prove or disprove the following assertion :

f maps every closed ball homeomorphically onto its image.

B6. Let X and Y be Banach spaces and suppose that X is finite dimensional. Let $T : X \to Y$ be a linear map. Prove or disprove the following assertion :

If B is a bounded set in X, then $\overline{T(B)}$ is compact in Y.

B7. Let $E \subseteq [0, 1]$ be Lebesgue measurable with $m(E) = 1$ and $f : [0, 1] \to \mathbb{R}$ be a continuous function. Prove or disprove the following assertion :

If $f(E)$ is a singleton, then so is $f([0, 1])$.

B8. Let $f : \mathbf{C} \to \mathbf{C}$ be an entire function such that $|f(z)| < |e^z|$ for all z and $f(0) = \frac{1}{2}$. Find the value of $f(\pi)$.

SECTION—C

Each question carries 6 marks.

C1. Let S be a finite subset of $\mathbb{R}^2$ of the form $A \times B$, where $A, B \subset \mathbb{R}$. Suppose W is an open set in $\mathbb{R}^2$ that contains S. Show that there exist open sets U and V in $\mathbb{R}$ such that $S \subset U \times V \subseteq W$.

C2. Let $f : \mathbb{R} \to [0, \infty)$ be a measurable function such that $0 < \int f\, dm < \infty$. Consider $\mu : \mathcal{L} \to [0, \infty]$ given by $\mu(E) := \int_E f\, dm$, where m is the Lebesgue measure on the σ-algebra $\mathcal{L}$ of Lebesgue measurable sets in $\mathbb{R}$.

(a) Show that μ is a finite measure on $\mathcal{L}$, i.e., $\mu(\mathbb{R}) < \infty$ and for any countable mutually disjoint collection $\{E_n\}$ in $\mathcal{L}$, $\mu(\cup_n E_n) = \Sigma_n \mu(E_n)$.

(b) If $E \in \mathcal{L}$ has $m(E) = 0$, then show that $\mu(E) = 0$.

C3. Let $X = C([0, 1])$ be the complex Banach space consisting of complex-valued continuous functions on $[0, 1]$ with the sup norm and Y be the subspace consisting of polynomials of degree less than or equal to 5. Show that for each coset $x + Y \in X/Y$, there exists an element $z \in x + Y$ such that $||z|| = \inf\{||x + y||: y \in Y\}$.

C4. Consider the polynomial $f(x) = x^4 + 1$ over the field $\mathbb{Z}/5\mathbb{Z}$.

(a) Is it irreducible? Justify.

(b) Find the splitting field of the polynomial of f over $\mathbb{Z}/5\mathbb{Z}$.

C5. Consider the curve E given by the equation $y^2 = x^3 + Ax + B$, where A and B are integers. Let $P = (a, b)$ and $Q = (c, d)$ be points on E such that $a, b, c, d \in \mathbb{Q}$, with $a \neq c$. Let L be the line joining P and Q. Show that E and L intersect in some point $R = (e, f)$, where $e, f \in \mathbb{Q}$.

C6. (a) Exhibit a 4×4 matrix over $\mathbf{C}$ whose eigenvalues over $\mathbb{R}$ are ± 1 and whose eigenvalues over $\mathbf{C}$ are ± 1 and $\pm i$.

(b) Exhibit three non-conjugate matrices over $\mathbb{Z}/7\mathbb{Z}$ of size 3×3 for which -2 is the only eigenvalue.

C7. Find and classify the extreme values, if any, of the function $f:\mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = y^2 - x^3$.

C8. Find all the poles of the function $f(z) = \frac{z}{\sin z - 1}$ and compute $\int_C f(z)\,dz$, where $C : [0, 2\pi] \to \mathbf{C}$ is the curve defined as $C(\theta) = 2e^{-i\theta}$.

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ENTRANCE EXAMINATION, 2017

Ph.D. IN MATHEMATICAL SCIENCES

[Field of Study Code : MATH (897), MATJ (811)]

Time Allowed : 3 hours

Maximum Marks : 100

INSTRUCTIONS FOR CANDIDATES

(i) All questions are compulsory. There are 10 questions in **Section—A** and 8 questions in each of **Sections—B** and **C**.

(ii) The answers for **Section—A** must be written in the space provided in the answer table.

(iii) The answers for **Sections—B** and **C** must be written in the space provided after each question. Answers written in any other place will not be evaluated.

(iv) Each question in **Section—A** has *only one* correct answer among the choices (a), (b), (c), (d). Each correct answer will be awarded +2 marks. Each incorrect answer or unattempted question will be given 0 marks.

(v) Each question in **Section—B** carries 4 marks; each question in **Section—C** carries 6 marks.

(vi) Answers to all the questions in **Sections—B** and **C** must be **justified with mathematical reasoning**, or else no marks will be awarded.

(vii) In the following $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbf{C}$ denote the set of natural numbers, integers, rational numbers, real numbers and complex numbers respectively. For sets A and B, $A \setminus B := \{x \in A \mid x \notin B\}$. Subsets of $\mathbb{R}^n$ are assumed to have the usual topology unless mentioned otherwise.

(viii) For $a, b \in \mathbb{R}$, by $[a, b]$ we denote the interval of all the real numbers between a and b including the end points, by $(a, b]$, the set $[a, b] \setminus \{a\}$ and by (a, b), the set $(a, b] \setminus \{b\}$.

(ix) Extra pages are attached at the end of the question paper for Rough Work.

(x) Use of calculator is not permitted.

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ENTRANCE EXAMINATION, 2017

Ph.D. IN MATHEMATICAL SCIENCES

SUBJECT . . . .
(Field of Study/Language)

FIELD OF STUDY CODE . . . .

NAME OF THE CANDIDATE . . . .

. . . .

REGISTRATION NO. | | | | | |

CENTRE OF EXAMINATION . . . .

. . . .

DATE . . . .

....
(Signature of Candidate)

....
(Signature of Invigilator)

....
(Signature and Seal of Presiding Officer)

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For Official Use Only

Not to be filled in by the candidate

Total of **Section—A**	
Total of **Section—B**	
Total of **Section—C**	
Grand Total	

SECTION—A

1. Let $f(x) = x^4 - 2 \in \mathbb{Q}[x]$ and let E be the splitting field of the polynomial f over $\mathbb{Q}$. Then the extension degree $[E : \mathbb{Q}]$ is equal to

(a) 4!

(b) 4

(c) 8

(d) 16

2. Let A be an $n \times n$ matrix with real entries. Pick the true statement from below :

(a) $A^{2n} = 0$ implies $A^n = 0$

(b) If all eigenvalues of A are 1, then A is diagonalizable

(c) $A^k = 0$ for some $k \geq 2$ implies $A = 0$

(d) If A is upper triangular and if $A \neq 0$, then A is invertible

3. The order of the permutation $(1\ 12\ 8\ 10\ 4)(2\ 13)(5\ 11\ 7)(6\ 9)$ is equal to

(a) 13!

(b) 12!

(c) 60

(d) 30

4. Let $f : X \to Y$ be a function. Let $U \subseteq X$ and $V \subseteq Y$. Choose the correct relations from below :

(a) $f^{-1}(f(U)) \subseteq U$ and $f(f^{-1}(V)) \subseteq V$

(b) $f^{-1}(f(U)) \supseteq U$ and $f(f^{-1}(V)) \subseteq V$

(c) $f^{-1}(f(U)) \subseteq U$ and $f(f^{-1}(V)) \supseteq V$

(d) $f^{-1}(f(U)) \supseteq U$ and $f(f^{-1}(V)) \supseteq V$

5. Let $\theta := (\sqrt[3]{2+\sqrt{5}} + \sqrt[3]{2-\sqrt{5}})$ be a root of the polynomial $x^3 + 3x - 4$, with the convention that for a real number x, $\sqrt[3]{x}$ denotes the real cube root of x. Then

(a) $\theta = 1$

(b) $\theta > 1 \cdot 2$

(c) $\theta < 1$

(d) None of the above

6. Let $f : \mathbf{C} \to \mathbf{C}$ be defined by $f(z) = |z|^2$. Then

(a) f is complex differentiable everywhere

(b) f is nowhere complex differentiable

(c) f is complex differentiable only at points $z \in \mathbb{R}$

(d) f is complex differentiable only at $z = 0$

7. Let X, Y be Banach spaces with $\dim(X) < \infty$. Let $T : X \to Y$ be a linear map and suppose $B \subseteq X$ is bounded and convex. Pick the false statement from below :

(a) $\overline{T(B)}$ is compact

(b) $\overline{T(B)}$ is neither bounded nor compact

(c) $\overline{B}$ is compact

(d) $T(B)$ is path connected

8. Let X be a complete metric space and let $A \subseteq X$. Then A is compact, if

(a) A is closed and bounded

(b) for every $\varepsilon > 0$, A can be covered by finitely many open balls of radius ε

(c) A is closed and for every positive integer n, A can be covered by finitely many balls of radius $\frac{1}{n}$

(d) there exists a continuous function $f : X \to \mathbb{R}$ such that $f(A)$ is compact

9. Suppose $A \subseteq \mathbb{R}$ has positive finite Lebesgue measure. Consider the following statements :

[S1] A contains an open set.

[S2] A is a bounded set.

Which of the following is correct?

(a) Both [S1] and [S2] are true

(b) [S1] is true but [S2] is false

(c) [S2] is true but [S1] is false

(d) Neither [S1] nor [S2] is true

10. Let X be a compact metric space and let $f : X \to X$ be a continuous injective map. Which of the following is true?

(a) $f(X)$ is dense in X

(b) X and $f(X)$ are homeomorphic

(c) There exists $x \in X$ such that $f(x) = x$

(d) The diameter of $f^n(X)$ decreases to 0 as n tends to infinity, where $f^k := f \circ f^{k-1}$ for $k > 1$ and $f^1 := f$

SECTION—B

Each question carries 4 marks.

B1. Let V be a real vector space and $f, g : V \to \mathbb{R}$ be two linear maps. Suppose $\ker(f) \subseteq \ker(g)$. Prove that $g = c \cdot f$ for some constant $c \in \mathbb{R}$.

B2. Let G be a finite group and let H be a proper subgroup of G. Prove that $G \neq \bigcup_{g \in G} gHg^{-1}$.

(Hint: Use the normalizer of H in G.)

B3. Using Chinese Remainder Theorem (if necessary), find the cardinality of the ring $\frac{\mathbb{Z}[\sqrt{-1}]}{\langle 2+\sqrt{-1}\rangle^3}$.

B4. Let $f : \mathbf{C} \to \mathbf{C}$ be an entire function such that

$$f(z + (1 - i)) = f(z) = f(z + (1 + i)) \text{ for all } z.$$

Suppose $f(1/2) = 2$. Compute the value of $f(2 + i)$.

B5. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as

$$f(x) = \begin{cases} |x|^{\frac{1}{|x|}} & \text{if } x \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Compute the following limits :

$$\lim_{x \to 0-} f(x), \quad \lim_{x \to 0+} f(x) \text{ and } \lim_{x \to \infty} f(x).$$

B6. Let $E := \bigcup_{\alpha \in \mathbb{R}\setminus\mathbb{Q}} \mathbb{R} \times \{\alpha\} \subseteq \mathbb{R}^2$. Prove or disprove that E has infinite Lebesgue measure in $\mathbb{R}^2$.

B7. Let $X := C([0, 1])$ be the vector space of continuous real-valued functions on the closed interval $[0, 1]$, equipped with the sup norm. Is X finite dimensional? Let B be the closed unit ball in X. Is B compact? Is B path connected? Justify your answers.

B8. Let V be a vector space of finite dimension and let $\varphi : V \to V$ be a linear transformation. Prove that $\mathrm{Im}(\varphi^m) \cap \ker(\varphi^m) = \{0\}$ for some integer m.

SECTION—C

Each question carries 6 marks.

C1. Let V be the vector space of polynomials of degree less than or equal to 3 with real coefficients. Let $D : V \to V$ be the differential operator defined by $D(a_0 + a_1 x + a_2 x^2 + a_3 x^3) := a_1 + a_2 x + a_3 x^2$. Find the trace, characteristic polynomial and eigenvalues of D.

C2. Let D_{2n} be the dihedral group generated by x, y with relations $x^2 = y^n = e$ and $yxy = x$, where $n \geq 2$ is an integer.

(a) Prove that D_{2n} can be considered as a subgroup of $GL_2(\mathbb{R})$ for every $n \geq 2$.

(b) Prove that D_{2n} can be considered as a subgroup of the permutation group S_n for every $n \geq 3$.

C3. Let $a_1, a_2, \ldots, a_n$ be distinct integers for a positive integer n. Let $f(x) = \prod_{i=1}^{n}(x - a_i) - 1$.

Prove that f is an irreducible polynomial in $\mathbb{Z}[x]$.

C4. Find the poles of the function $f(z) = \frac{z}{\cos(z) - 1}$ and compute

$$\int_C f(z)\, dz,$$

where $C: [0, 2\pi] \to \mathbf{C}$ is the circle defined by $C(\theta) := 10e^{i\theta}$.

C5. Show that $2 < (1+\frac{1}{n})^n < 3$ for all $n \geq 2$. Find the first digit and the last three digits of $(101)^{100}$.

C6. For each $0 < r \leq \frac{2}{\pi}$, consider the subset of $\mathbb{R}^2$ given by

$$S_r := \left\{\left(x, \sin\left(\frac{1}{x}\right)\right) : x \in \left(0, \frac{2}{\pi}\right]\right\} \cup ([0, r] \times \{1, -1\})$$

(a) Describe the set of limit points of S_r in $\mathbb{R}^2$ without providing justification.

(b) Prove or disprove that $\overline{S_r}$ is path connected for every $0 < r \leq \frac{2}{\pi}$.

C7. Let c_0 denotes the Banach space of complex sequences converging to 0 with the norm given by $||(x_1, \ldots, x_n, \ldots)|| = \sup|x_n|$. Define $T : c_0 \to c_0$ by

$$T(x_1, \ldots, x_n, \ldots) = \left(x_1, \frac{x_2}{2}, \ldots, \frac{x_n}{n}, \ldots\right).$$

Compute $||T||$. Show that T is not an open map. Prove directly (without using Open Mapping Theorem) that any non-zero linear functional on c_0 is an open map.

C8. (a) Let A be a set. Suppose there is a surjective map $f : A \to \mathbb{N}$. Show that there is an injective map $g : \mathbb{N} \to A$. Is the converse true? Justify.

(b) Write the negation of the following statement :
For each $\varepsilon > 0$, there is a positive integer N such that $n \geq N$ implies $|f_n(x) - f(x)| < \varepsilon$ for all x.

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ENTRANCE EXAMINATION, 2016

Pre-Ph.D./Ph.D.
Mathematical Sciences

[Field of Study Code : MATP (160)]

Time Allowed : 3 hours

Maximum Marks : 70

INSTRUCTIONS FOR CANDIDATES

(i) All questions are compulsory.

(ii) **Section—A** consists of 5 objective-type questions. The answers must be written in the space provided in the **Answer Table for Section—A**. For each question in **Section—A**, *one and only one* from (a), (b), (c), (d) provides the correct answer. Each correct answer will be awarded +3 marks. Each wrong answer will be given −1 mark. An unattempted question will not get any marks.

(iii) There are 5 questions in **Section—B** and have short answers. Please write only the answers on the blank line provided after each question. Note that proofs or any other details should not be written. Please Do Not write anything else other than the answers on the provided line. Please do all your rough work on blank pages appropriately marked towards the end of the question paper. Please note that answers written in any other place will not be evaluated. Each complete and correct answer will be awarded +3 marks. Wrong answer will get 0 mark. Partially correct answer will get partial marks. An unattempted question will not get any marks.

(iv) Questions in **Section—C** are subjective type and each question carries 4 marks. Answers to all the questions in **Section—C** must be **justified with mathematical reasoning and details**. Only complete and correct answers will get full marks.

(v) Throughout, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ will respectively denote the set of natural numbers, the set of rational integers, the set of rational numbers, the set of real numbers and the set of complex numbers respectively. Subsets of $\mathbb{R}^n$ are assumed to have the usual topology unless mentioned otherwise.

(vi) Throughout, for real numbers a, b with $a \leq b$, by $[a, b] \subset \mathbb{R}$ we will denote the interval consisting of all the real numbers between a and b including the end points a and b. In order to exclude the end points $\{a, b\}$ from the interval, the notation (a, b) is used.

(vii) Pages at the end of the question paper are for rough work.

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ENTRANCE EXAMINATION, 2016

Pre-Ph.D./Ph.D.
Mathematical Sciences

SUBJECT . . . .
(Field of Study/Language)

FIELD OF STUDY CODE . . . .

NAME OF THE CANDIDATE . . . .

. . . .

REGISTRATION NO. | | | | |

CENTRE OF EXAMINATION . . . .

. . . .

DATE. . . . .

....
(Signature of Candidate)

....
(Signature of Invigilator)

....
(Signature and Seal of Presiding Officer)

/91-A

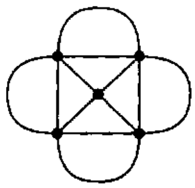
FOR OFFICIAL USE ONLY

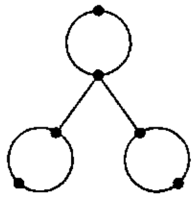
Not to be filled in by the candidate

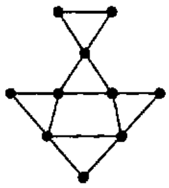
Total of **Section—A**	
Total of **Section—B**	
Total of **Section—C**	
Grand Total	

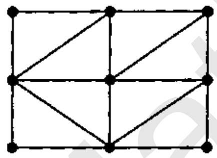
SECTION—A

1. We say that a finite graph has ***property E*** if it can be drawn without lifting the pen from paper and without tracing any edge twice. Consider the following graphs :

(i)


(ii)


(iii)


(iv)


Which of the following is necessarily true?

(a) Both graphs *(i)* and *(ii)* have property *E*.

(b) Graphs *(i)* and *(iv)* do not have property *E*. However graph *(i)* acquires property *E* after adding an edge.

(c) Only graph *(iii)* has property *E* and no other graph has property *E*.

(d) Graph *(iii)* has property *E* but the modified graph obtained by adding an edge to graph *(iii)*, does not have property *E*.

2. Let $f : \mathbf{C} \to \mathbf{C}$ be an analytic function. Consider the following statements :

(S1) $|f(z)| > 0$ for all $z \in \mathbf{C}$ implies f is a constant function.

(S2) $|f(z)| > 1$ for all $z \in \mathbf{C}$ implies f is a constant function.

(S3) $|f(z)| \in \mathbb{Q}$ for all $z \in \mathbf{C}$ implies f is a constant function.

Which of the following options is correct?

(a) (S1) and (S2)

(b) Only (S2)

(c) (S1) and (S3)

(d) (S3) and (S2)

3. Let $A \in M_4(\mathbb{C})$ be a 4×4 matrix such that $A^{10} = 0$. Let I_4 be the 4×4 identity matrix. Which one of the following is true for any such A?

(a) At least three of the eigenvalues of A are 0.

(b) $A^3 = 0$.

(c) A is diagonalizable.

(d) There exists λ such that $\lambda I_4 + A$ is diagonalizable.

4. Consider the following statements :

(S1) The polynomial $x^2 + 1$ is reducible over $\mathbb{F}_{31}$.

(S2) The polynomial equation $x^7 - x = 0$ has 7 solutions over $\mathbb{F}_{49}$.

(S3) The polynomial $x^2 + 1$ is reducible over $\mathbb{F}_{17}$.

(S4) The polynomial $x^{13} + 12x + 7$ does not have solution over $\mathbb{Z}/13\mathbb{Z}$.

Which of the following is true?

(a) Only (S2) and (S4) are true.

(b) Only (S4) is true.

(c) Only (S1) is false.

(d) (S3) is true and (S2) is false.

5. Let $I := (\sqrt{2}, \sqrt{3}) \cap \mathbb{Q}$. We equip $\mathbb{Q}$ with the subspace topology from $\mathbb{R}$. Which of the following is false about $I \subset \mathbb{Q}$?

(a) I is compact in $\mathbb{Q}$.

(b) I is closed in $\mathbb{Q}$.

(c) I is bounded.

(d) I is not connected in $\mathbb{Q}$.

SECTION—B

6. Let $\mathbb{F}$ be a field with $16807 = 7^5$ elements. What is the cardinality of the set given by $\{\alpha^7 - \beta^7 \mid \alpha, \beta \in \mathbb{F}\}$?

Answer : ____________________

7. Which of the following groups are isomorphic?

(a) $G_1 = \mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/10\mathbb{Z}$

(b) $G_2 = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/15\mathbb{Z}$

(c) $G_3 = \mathbb{Z}/20\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$

(d) $G_4 = \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$

(e) $G_5 = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/30\mathbb{Z}$

Answer : ____________________

8. Find the number of Hausdorff topologies on the set $X = \{x_1, x_2, x_3, x_4, x_5\}$.

Answer : ____________________

9. Let a, b, r be three integers such that $a^{10} = 31b + r$. What are the possible values for $r \pmod{31}$?

Answer : ____________________

10. Consider the sequence given by $9, \sqrt{9}, \sqrt{9+\sqrt{9}}, \sqrt{9+\sqrt{9+\sqrt{9}}}, \cdots$. Find its lim sup.

Answer : ____________________

SECTION—C

11. Consider the series

$$S := \sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}}$$

Prove or disprove that S converges.

12. Define $f:\mathbb{R}\to\mathbb{R}$ as follows $f(x)=\dfrac{\sin x}{x}$ for $x\neq 0$ and $f(0)=1$. Calculate the following integral providing the details of your calculation :

$$\lim_{M\to+\infty}\int_{-M}^{M} f(x)\,dx$$

13. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a field isomorphism.

(a) Prove that ϕ is continuous.

(b) Prove that ϕ is the identity map.

14. Let n be a positive integer and let S be a subset of $\mathbb{R}^n$.

(a) Suppose for each $s \in S$ there exists some open set U_s containing s such that $U_s \cap S$ is a countable set. Prove that S is a countable set.

(b) Suppose S is an infinite set and for each $s \in S$ there exists some open set U_s containing s such that $U_s \cap S$ is a finite set. Prove that S is not a bounded set.

15. Let $\mathcal{H}$ be a Hilbert space. Let K and L be closed vector subspaces of $\mathcal{H}$ such that K and L are isomorphic. Prove or disprove that $K^{\perp}$ is isomorphic to $L^{\perp}$.

16. Let V be a non-zero finite dimensional vector space over a field k. We say that a vector subspace W of V is ***proper*** if $\dim(W) < \dim(V)$. If k is an infinite field, prove that V cannot be written as a union of finitely many proper vector subspaces of V.

17. Let A and B be two $n \times n$ matrices with coefficients in a field $\mathbb{F}$. Prove that

$$\text{rank}(A) + \text{rank}(B) - n \leq \text{rank}(AB) \leq \min\{\text{rank}(A), \text{rank}(B)\}$$

18. Let $\mathbb{F} := \mathbb{Z}/23\mathbb{Z}$. Let U be the subgroup of upper triangular matrices inside $G := GL(2, \mathbb{F})$, the group of invertible 2×2 matrices over $\mathbb{F}$.

(a) Exhibit a non-Abelian subgroup of order 253 inside G.

(b) Let $SU := \{A \in U \mid \det(A) = 1\}$. Calculate the order of SU.

19. Let V be a finite dimensional vector space over a field $\mathbb{F}$ of dimension n over $\mathbb{F}$. Let $f : V \to V$ be an endomorphism of V, i.e., a linear map from V to V. Show that if with respect to all bases, f is represented by the same matrix A, i.e., $A = \Phi^{-1} f \Phi$ for all isomorphisms $\Phi : \mathbb{F}^n \simeq V$, then there exists $\lambda \in \mathbb{F}$ such that $f = \lambda\, Id_V$, where Id_V is the identity map from V to V.

20. We say that two fields $\mathbb{F}$ and $\mathbb{F}'$ are isomorphic if there is a one-one onto isomorphism from $\mathbb{F}$ to $\mathbb{F}'$. If a system of linear equations with coefficients in a field $\mathbb{F}$ has exactly 121 solutions, what are the possible candidates for the field $\mathbb{F}$ up to isomorphism? Provide all the details.

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ENTRANCE EXAMINATION, 2015

Pre-Ph.D./Ph.D.
Mathematical Sciences

[Field of Study Code : MATP (160)]

Time Allowed : 3 hours

Maximum Marks : 70

INSTRUCTIONS FOR CANDIDATES

(i) All questions are compulsory.

(ii) For Section—A, the answers must be written in the space provided in the **Answer Table for Section—A**. For questions from Section—B, please write your answers in the space provided after each question. Please note that answers written in any other place will not be evaluated.

(iii) For each question in Section—A, *one and only one* of the four choices provides the correct answer. Each correct answer will be awarded +3 marks. Each wrong answer will be given −1 mark. An unattempted question will not get any marks.

(iv) Questions in Section—B are subjective type and each question carries 5 marks.

(v) Answers to all the questions in Section—B must be **justified with mathematical reasoning and details as required.** Only complete and correct answers will get full marks.

(vi) Throughout, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ will denote the set of natural numbers, the set of integers, the set of rational numbers, the set of real numbers and the set of complex numbers respectively. Subsets of $\mathbb{R}^n$ are assumed to have the usual topology unless mentioned otherwise.

(vii) Throughout, for real numbers a, b with $a \leq b$, by $[a, b] \subset \mathbb{R}$ we will denote the interval consisting of all the real numbers between a and b including the end points a and b. In order to exclude the end points $\{a, b\}$ from the interval, the notation (a, b) is used.

(viii) Extra pages are attached at the end of the question paper for rough work.

/91-B

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ENTRANCE EXAMINATION, 2015

Pre-Ph.D./Ph.D.
Mathematical Sciences

SUBJECT . . . .
(Field of Study/Language)

FIELD OF STUDY CODE . . . .

NAME OF THE CANDIDATE . . . .

. . . .

REGISTRATION NO. | | | | | |

CENTRE OF EXAMINATION . . . .

. . . .

DATE. . . . .

(Signature of Candidate)

(Signature of Invigilator)

(Signature and Seal of Presiding Officer)

/91-B

FOR OFFICIAL USE ONLY

Not to be filled in by the candidate

Total of Section—A	
Total of Section—B	
Grand Total	

SECTION—A

1. Let λ be the Lebesgue measure on $\mathbb{R}$ and $E \subset \mathbb{R}$ be a Lebesgue measurable set with $\lambda(E) > 0$. Let E° be the interior of E and $\bar{E}$ the closure of E in $\mathbb{R}$. Let $-E := \{-x \mid x \in E\}$. Which of the following is true?

(a) $\lambda(E) = \lambda(-E + b)$ for every $b \in \mathbb{R}$

(b) $\lambda(a + E) = \lambda(\bar{E})$ for every $a \in \mathbb{R}$

(c) $\lambda(E) = \lambda(E^\circ)$

(d) E contains an open interval

2. Consider the polynomials given by $P(x) := x^{2015} + 5x + 14$ and $Q(x) := x^{2015} + 5x - 14$. Which of the following is true?

(a) $P \cdot Q$ has more than 5 real roots

(b) P has exactly one real root

(c) Q has more than 3 real roots

(d) P has 2015 real roots

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Consider the statements :

[S1] f takes compact sets to compact sets.

[S2] f takes connected sets to connected sets.

[S3] f is continuous.

[S4] f takes bounded sets to bounded sets.

Which of the following is true?

(a) [S1] $\Rightarrow$ [S3]

(b) [S2] $\Rightarrow$ [S3]

(c) [S3] $\Rightarrow$ [S4]

(d) None of the above

4. Let $V := \mathbb{Q}(\sqrt{3})$ and $W := \mathbb{Q}(\sqrt{5})$. Consider the statements :

[S1] $V \simeq W$ as groups under addition.

[S2] $V \simeq W$ as vector spaces over $\mathbb{Q}$.

[S3] $V \simeq W$ as fields.

Which of the following is true?

(a) [S1] but not [S2]

(b) [S2] but not [S3]

(c) [S1] and [S3]

(d) [S3]

5. Let f, g be non-negative continuous real-valued functions on [0, 1]. Suppose $\int_0^1 f(x)\,g(x)\,dx = 0$. Which of the following is false?

(a) $f^{-1}(0) \cup g^{-1}(0) = [0, 1]$

(b) $f^{-1}(0) \cap g^{-1}(0) \neq \phi$

(c) $\int_0^1 f(x)^2\, g(x)^3\, dx = 0$

(d) Either $f \equiv 0$ or $g \equiv 0$

6. Consider the following statement :

(*) Let X be a topological space. Let $f : X \to X$ be any continuous map. Then there exists $x_0 \in X$ such that $f(x_0) = x_0$.

For which topological space among the topological spaces listed below, does the above statement (*) hold true?

(a) $X := \{z \in \mathbb{C} \mid |z| \leq 2\}$

(b) $X := (-2, 2)$

(c) $X := \mathbb{R}^2$

(d) $X := \{z \in \mathbb{C} \mid |z| = 2\}$

/91-B

7. Suppose G is isomorphic to $\mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$. Which one of the following is true?

(a) Any subgroup of G is of the form $H \times K$ for some subgroup H of $\mathbb{Z}/6\mathbb{Z}$ and some subgroup K of $\mathbb{Z}/4\mathbb{Z}$

(b) G has a cyclic subgroup of order 12

(c) G has a cyclic subgroup of order 8

(d) G has no cyclic subgroup other than the trivial

8. Let λ be the Lebesgue measure on $\mathbb{R}$ and $f, g \in C(\mathbb{R})$, the space of continuous functions from $\mathbb{R}$ to $\mathbb{C}$. Let $A := \{x \in \mathbb{R} \mid f(x) \neq g(x)\}$. Suppose f and g are equal almost everywhere with respect to λ, i.e., $\lambda(A) = 0$. Which one of the following is true?

(a) $A = \phi$

(b) $A \neq \phi$ and A is finite

(c) A is countably infinite set

(d) A is uncountable

9. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Suppose T has non-zero distinct eigenvalues. For a scalar $\lambda \in \mathbb{R}$, consider the linear transformation $\lambda I_n + T$, where I_n is the $n \times n$ identity matrix. Which one of the following is not necessarily true?

(a) T is invertible

(b) There exists $\lambda \in \mathbb{R}$ such that $\lambda I_n + T$ is invertible

(c) There exists $\lambda \in \mathbb{R}$ such that $\lambda I_n + T$ is diagonalizable

(d) $\lambda I_n + T$ is invertible for all but finitely many $\lambda \in \mathbb{R}$

10. Let $M_n(\mathbb{R})$ and $M_n(\mathbb{C})$ denote the set of $n \times n$ matrices with real coefficients and complex coefficients respectively. Let $GL_n(\mathbb{R})$ and $GL_n(\mathbb{C})$ be the invertible matrices in $M_n(\mathbb{R})$ and $M_n(\mathbb{C})$ respectively. Which one of the following topological spaces is connected?

(a) $GL_1(\mathbb{R})$

(b) The set of matrices $M \in M_2(\mathbb{R})$ having trace 0

(c) $GL_2(\mathbb{R})$

(d) $\mathbb{C} \setminus p([0, 1])$, where $p : [0, 1] \to \mathbb{C}$ is a continuous differentiable non-constant function satisfying $p(0) = p(1)$

11. Which one of the following functions from $\mathbb{C}$ to $\mathbb{C}$ is bounded?

(a) $\sin^3(z)$

(b) $\dfrac{\sin(z)}{\cos(z)+3}$

(c) $\dfrac{1}{8z^{16}+4z^8+2z^2+1}$

(d) $\sin^2(2z)+(\cos^2(z)-\sin^2(z))^2+e^{i|z|}$

12. Let V be a vector space of dimension 6 over the finite field $\mathbb{Z}/7\mathbb{Z}$. Let W_1 and W_2 be two vector subspaces of V. Let $\dim(W_1)=4$ and $\dim(W_2)=3$. Which one of the following is necessarily true?

(a) $\dim(W_1 \cap W_2) \geq 1$

(b) $\dim(W_1 + W_2) = 4$

(c) $\dim(W_1 \cap W_2) = 3$

(d) $W_1 \cup W_2 = V$

13. The last two digits of $3^{3^{49}}$ are

(a) 49

(b) 53

(c) 27

(d) 61

14. Let $A=(a_{i,j})$ be a 15×15 matrix such that $a_{i,j}=1$ for all i, j. Let $r(A)$ denote the spectral radius of A, i.e., $r(A) := \sup\{|\lambda| : \lambda \text{ eigenvalue of } A\}$. Which of the following is true?

(a) $r(A)=0$

(b) $0 < r(A) < 10$

(c) $r(A)=10$

(d) $r(A) > 10$

15. Consider the following statements :

[S1] Countable union of countable sets is countable.

[S2] Product of countably many finite non-empty sets is countable.

[S3] Product of finitely many countable sets is countable.

Which of the following is true?

(a) [S1] and [S2]

(b) [S2] and [S3]

(c) [S1] and [S3]

(d) None of the above

SECTION—B

16. Let $A \in M_n(\mathbb{C})$ be an $n \times n$ matrix such that $A^{2015} = 0 \in M_n(\mathbb{C})$. Prove that $\lambda I_n + A$ is an invertible matrix if and only if $\lambda \neq 0$. Here $\lambda \in \mathbb{C}$ and I_n is the $n \times n$ identity matrix.

17. Let N be a positive integer. Let T be a linear operator on $\mathbb{R}^N$. Suppose 0 is a limit point of the set $\left\{\|T^n\| : n \in \mathbb{N}\right\}$. Show that $\|T^n\| \to 0$ as $n \to \infty$.

18. Consider the Möbius transformation $\phi : \mathbf{C} \to \mathbf{C}$ given by $\phi(z) = \frac{5z+i}{3z-1}$. Find the set $\phi^{-1}(\mathbb{R})$.

19. Let G be the group of 2×2 invertible matrices with coefficients in the finite field $\mathbb{Z}/13\mathbb{Z}$. Prove or disprove that G has a 13-Sylow subgroup providing details.

20. Let f and g be real-valued continuous functions defined on $\mathbb{R}$ as

$$f(x) = \begin{cases} \log_2 x & \text{if } x \geq 1 \\ 0 & \text{if } x \leq 1 \end{cases}$$

and $g(x) = 6 - x$ for all $x \in \mathbb{R}$.

(a) Find the points of intersection of the graph of f and the graph of g.

(b) Find the area of the bounded region enclosed by the graph of f, the graph of g and the X-axis.

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ENTRANCE EXAMINATION, 2013

Pre-Ph.D./Ph.D.
Mathematical Sciences

[Field of Study Code : MATP (160)]

Time Allowed : 3 hours

Maximum Marks : 70

INSTRUCTIONS FOR CANDIDATES

(i) All questions are compulsory.

(ii) For Section—A, the answers must be written in the space provided in the answer table. For Section—B, Section—C and Section—D, answers are to be written in the space given after each question. Answer written in any other place will not be evaluated. Additional pages are provided at the end for rough work.

(iii) For each question in Section—A, *exactly one* of the four choices [(a), (b), (c), (d)] is the correct answer. Each correct answer will be awarded +3 marks. Each wrong answer will be given −1 mark. If a question is not attempted, then no marks will be awarded for it.

(iv) Questions in Section—B have short answers and each question carries 2 marks.

(v) Answers to all the questions in Section—C and Section—D must be **justified with mathematical reasoning**, or else they will be considered **invalid**. Each question in Section—C carries 3 marks. The question in Section—D carries 6 marks.

(vi) In the following, the symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, integers, rational numbers, real numbers and complex numbers, respectively. Subsets of $\mathbb{R}^n$ are assumed to have the usual topology unless mentioned otherwise. For $x \in \mathbb{C}$, $|x|$ denotes the absolute value of x.

(vii) The notation $|S|$ is used to denote the cardinality of a finite set S.

ENTRANCE EXAMINATION, 2013

Pre-Ph.D./Ph.D.
Mathematical Sciences

SUBJECT . . . .
(Field of Study/Language)

FIELD OF STUDY CODE . . . .

NAME OF THE CANDIDATE . . . .

. . . .

REGISTRATION NO. | | | | | |

CENTRE OF EXAMINATION . . . .

. . . .

DATE. . . . .

(Signature of Candidate)

(Signature of Invigilator)

(Signature and Seal of Presiding Officer)

Not to be filled in by the candidate

Q. No.	Marks
B1	
B2	
Total—B	
C1	
C2	
C3	
C4	
Total—C	
D1	

Total of Section—A	
Total of Section—B	
Total of Section—C	
Total of Section—D	
Grand Total	

Answer table for Section—A

Question No.	Answer	Question No.	Answer
1.		9.	
2.		10.	
3.		11.	
4.		12.	
5.		13.	
6.		14.	
7.		15.	
8.		16.	

SECTION—A

1. Which of the following rings is a field?

(a) $\mathbb{Z}/57\mathbb{Z}$

(b) $(\mathbb{Z}/3\mathbb{Z})\times(\mathbb{Z}/3\mathbb{Z})$

(c) $\mathbb{R}[x]/\langle x^2-2\rangle$

(d) $\mathbb{Q}[x]/\langle x^2+2\rangle$

2. For a finite group G

(a) there does not exist any group homomorphism $\varphi: G\to\mathbb{Z}$

(b) there is a unique group homomorphism $\varphi: G\to\mathbb{Z}$

(c) there are infinitely many group homomorphisms $\varphi: G\to\mathbb{Z}$

(d) there are exactly $|G|$ group homomorphisms $\varphi: G\to\mathbb{Z}$

3. Let R be a subring of $\mathbb{C}$ containing $\mathbb{Q}$. Suppose $\pi, \sqrt{3}\in R$. Which of the following is ***not*** necessarily true?

(a) $\sqrt{3}/\pi\in R$

(b) $\pi/\sqrt{3}\in R$

(c) $[(\pi+1)^2-(\pi-1)^2]/(\pi\sqrt{3})\in R$

(d) $(\sqrt{3}\pi^2-7)/(\sqrt{3}+1)\in R$

4. Let X be a set and let B and C be some fixed subsets of X. If for any subset A of X, $A\subseteq C$ implies $A\subseteq B$, which of the following statements is true?

(a) $C\subsetneq B$

(b) $B\subsetneq C$

(c) $C\subseteq B$

(d) $B\subseteq C$

5. Let $f : X \to Y$ be a surjective map. Which of the following is necessarily true? (In the following, Id_S stands for the identity map on the set S)

(a) There exists $g : Y \to X$ such that $g \circ f = \mathrm{Id}_X$

(b) There exists a unique $g : Y \to X$ such that $g \circ f = \mathrm{Id}_X$

(c) There exists $g : Y \to X$ such that $f \circ g = \mathrm{Id}_Y$

(d) There exists a unique $g : Y \to X$ such that $f \circ g = \mathrm{Id}_Y$

6. Let ~ be some equivalence relation on $\mathbb{R}$. We are told that under this relation, $r \sim (r+1)$ for every $r \in \mathbb{R}$. We can now definitely conclude that

(a) the number of equivalence classes is infinite

(b) the number of equivalence classes is finite

(c) $(-\pi) \sim \pi$

(d) $(\pi - \frac{7}{2}) \sim (\pi + \frac{7}{2})$

7. Let V be a non-trivial inner product space over $\mathbb{R}$. For vectors $v, w \in V$, we say $v \sim w$ if $\langle v, w \rangle = 0$. Then the relation ~ is

(a) symmetric but neither reflexive nor transitive

(b) transitive but neither reflexive nor symmetric

(c) an equivalence relation (reflexive, symmetric and transitive)

(d) symmetric and transitive, but not reflexive

8. Let $A \in \mathrm{SL}_3(\mathbb{R})$ be a matrix such that $Av = v$ for some $v \neq 0$ in $\mathbb{R}^3$. Which of the following statements about A is necessarily true?

(a) A is a rotation

(b) A is the identity map

(c) A is diagonalizable

(d) None of the above

9. A box contains 4 blue and 3 green balls. Two balls are drawn out together at random from the box. What is the probability that the two balls are of different colours?

(a) 5/7

(b) 4/7

(c) 3/7

(d) 2/7

10. Which of the following is a complex analytic (holomorphic) function on the complex plane $\{x + iy \mid x,\ y \in \mathbb{R},\ i = \sqrt{-1}\}$?

(a) $3(x^2 - y^2) + 2ixy$

(b) $(x^3 - 3xy^2 - 3x) - i(y^3 - 3x^2y - 3y)$

(c) $(x^3 - 3xy^2 + 3x) - i(y^3 - 3x^2y - 3y)$

(d) $(x^3 + xy^2 + 3x) + i(y^3 + x^2y + 3y)$

11. For a complex analytic (holomorphic) function f on $\mathbf{C}$, consider the following conditions :

[C1] $(\operatorname{Re} f)(z) > 0$

[C2] $|f(z)| \in \mathbb{Z}$ for all $z \in \mathbf{C}$

[C3] $f(z) = i$ if $z = 1 + \frac{1}{n} + i$ for all $n \in \mathbb{N}$, where $i = \sqrt{-1}$

Which of the above conditions implies/imply that f is a constant function?

(a) All of [C1], [C2] and [C3]

(b) Both [C2] and [C3], but not [C1]

(c) Only [C2]

(d) Only [C3]

12. The series $\sum_{n=0}^{\infty} \frac{(-1)^n (n!)^2}{(2n)!}$ is

(a) absolutely convergent

(b) divergent

(c) conditionally convergent

(d) bounded but not convergent

13. For any pair of non-negative real numbers x, y, consider the following inequalities :

[I1] $\sqrt{x^2 + y^2} \geq \frac{1}{\sqrt{2}}(x + y)$

[I2] $\sqrt{x^2 + y^2} \leq x + y$

Which of the following is true?

(a) Only [I1] holds

(b) Only [I2] holds

(c) Neither [I1] nor [I2] holds

(d) Both [I1] and [I2] hold

14. Consider two maps $d_i : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$, $i = 1, 2$ defined as follows :

$$d_1((x_1, y_1), (x_2, y_2)) := |x_1 - x_2| + 3|y_1 - y_2|$$

$$d_2((x_1, y_1), (x_2, y_2)) := \frac{1}{3} \max\{|x_1 - x_2|, |y_1 - y_2|\}$$

Which of the following is true?

(a) d_1 is a metric on $\mathbb{R}^2$, but d_2 is not

(b) d_2 is a metric on $\mathbb{R}^2$, but d_1 is not

(c) Both d_1 and d_2 are metrics on $\mathbb{R}^2$

(d) Neither d_1 nor d_2 is a metric on $\mathbb{R}^2$

15. Consider the following statements about the closed interval $X = [0, 1]$:

[S1] Every infinite sequence in X has a limit point.

[S2] X has a subset which is connected but not path connected.

[S3] X has the finite intersection property.

[S4] X is a complete metric space.

Which of the above statements about X are true?

(a) Only [S1] and [S3]

(b) Only [S1], [S3] and [S4]

(c) Only [S2] and [S4]

(d) All of the above

16. Let $A_1 \supseteq A_2 \supseteq \cdots$ be a countable family of nonempty connected subsets of $\mathbf{R}^2$. Suppose $A := \bigcap_{n \geq 1} A_n$ is a nonempty set. Which of the following statements is necessarily true?

(a) A is always connected

(b) A is connected if each A_n is path connected

(c) A is connected if each A_n is closed

(d) A is connected if each A_n is compact

SECTION—B

B1. Prove or disprove :

The multiplicative groups $\mathbb{R}^{\times} = \mathbb{R} \setminus \{0\}$ and $\mathbb{C}^{\times} = \mathbb{C} \setminus \{0\}$ are isomorphic.

B2. Find with reason the flaw, if any, in the following sequence of arguments :

$$\frac{(-1)}{(64)} = \frac{1}{(-64)}$$

Step 1. $\Rightarrow \dfrac{\sqrt{-1}}{\sqrt{64}} = \dfrac{\sqrt{1}}{\sqrt{-64}}$, square roots taken on both sides

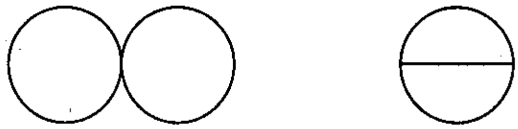
Step 2. $\Rightarrow \dfrac{i}{8} = \dfrac{1}{8i}$, $\sqrt{-1} = i$ is used

Step 3. $\Rightarrow \dfrac{i}{1} = \dfrac{1}{i}$, $\dfrac{1}{8}$ cancelled from both sides

Step 4. $\Rightarrow i^2 = 1$, multiplied by i on both sides

SECTION—C

C1. Prove that the figure of eight and the figure of theta (as shown below) are not homeomorphic as subsets of $\mathbb{R}^2$:



C2. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function such that $\int_0^1 f(x)\, x^n\, dx = 0$ for all $n \geq 0$. Prove that $f \equiv 0$.

C3. Let X and Y be two finite sets and $f : X \to Y$ be a map. Prove that

$$|X| = \sum_{y \in Y} |f^{-1}(y)|$$

C4. For a fixed $n \in \mathbb{N}$, let $X = \{1, 2, \cdots, n\}$. Let μ be a measure on X defined by $\mu(\{a\}) = n - a$ for every $a \in X$. Find a non-constant real-valued function f on X such that $\int_X f \, d\mu = 0$.

SECTION—D

D1. Let $E = \mathbb{R}^{[3]}[x]$ be the real vector space of real polynomials of degree less than or equal to 3, with an inner product defined by

$$\langle P, Q \rangle := \int_{-1}^{1} P(t)\, Q(t)\, dt$$

Consider the map $\sigma : E \to E$ defined by $(\sigma P)(x) = P(-x)$ for all $P \in E$. Prove that σ is a linear operator on E and that $\langle \sigma P, \sigma Q \rangle = \langle P, Q \rangle$. Find the eigenvalues and eigenvectors of σ. Is σ diagonalizable?

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Total Pages : 17

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ENTRANCE EXAMINATION, 2012

Pre-Ph.D./Ph.D.
Mathematical Sciences

[Field of Study Code : MATP (160)]

Time Allowed : 3 hours

Maximum Marks : 70

INSTRUCTIONS FOR CANDIDATES

(i) All questions are compulsory.

(ii) The answer must be written in the space provided in the answer table for Section—A and in the space given after each question of Section—B and of Section—C. Answer written in any other place will not be evaluated.

(iii) For each question in Section—A, *one and only one* of the four choices given is the correct answer. Indicate the correct answer for (a), (b), (c) or (d). Each correct answer will be awarded +3 marks. Each wrong answer will be given −1 mark. If a question is not attempted, then no marks will be awarded for it.

(iv) Questions in Section—B have short answers and each question carries 2 marks.

(v) Answers to all the questions in Section—C must be **justified with mathematical reasoning**, or else they will be considered **invalid**. They carry 4 marks each.

(vi) In the following, $\mathbf{N}$, $\mathbb{Z}$, $\mathbf{Q}$, $\mathbb{R}$ and $\mathbf{C}$ denote the set of natural numbers, integers, rational numbers, real numbers and complex numbers respectively. Subsets of $\mathbf{R}^n$ are assumed to have the usual topology unless mentioned otherwise.

(vii) By $[a, b] \subset \mathbb{R}$ we denote the interval of all the real numbers between a and b including the end points. In order to exclude the end points $\{a, b\}$ from the interval, the notation (a, b) is used.

(viii) Extra pages are attached at the end of the question paper for rough work.

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SECTION—A

1. Let G be a finite group and let $O(G)$ denote the order of G. Which of the following is true?

(a) $O(G) = 8 \Rightarrow G$ is non-Abelian

(b) $O(G) = 21 \Rightarrow G$ is Abelian

(c) $O(G) = 9 \Rightarrow G$ is Abelian

(d) $O(G) = 6 \Rightarrow G$ has a unique proper normal subgroup

2. For the closed unit disk D in $\mathbf{R}^2$, which of the following subsets has positive Lebesgue measure?

(a) S^1, the unit circle

(b) $D \setminus \{(x, y) \in D \mid x \in \mathbf{R} \setminus \mathbf{Q}\}$

(c) $D \setminus \{(x, y) \in D \mid x \notin \mathbf{Q},\ y \notin \mathbf{Q}\}$

(d) $D \setminus \{(x, y) \in D \mid x^2 + y^2 \in \mathbf{Q}\}$

3. Let A and B be linear operators on a finite-dimensional vector space V over $\mathbf{R}$ such that $AB = (AB)^2$. If BA is invertible, then which of the following is true?

(a) $BA = AB$ on V

(b) $\mathrm{Tr}(A)$ is non-zero

(c) 0 is an eigenvalue of B

(d) 1 is an eigenvalue of A

4. The number of elements of order 6 in $\mathbb{Z}/12\mathbb{Z}$ is

(a) 2

(b) 3

(c) 4

(d) 6

5. Let $A \in M_{2\times4}(\mathbb{R})$ and $B \in M_{4\times2}(\mathbb{R})$. Then the rank of BA is

(a) at least 1

(b) at most 2

(c) exactly 3

(d) at least 4

6. Let X be the set of roots of unity in $\mathbb{C}$. Let $S(X)$ be the set of all sequences of elements in X. Which of the following subsets of $S(X)$ is countable?

(a) The set A of all $(x_n) \in S(X)$ such that (x_n) is an eventually constant sequence

(b) The set B of all $(x_n) \in S(X)$ such that $x_n = 1$ whenever n is a prime number

(c) The set C of all $(x_n) \in S(X)$ such that each x_n is a 26th root of unity

(d) The set D of all $(x_n) \in S(X)$ such that $x_{2n} = 1$ for all $n \geq 1$

7. Let $\mathbb{F}$ be a field with 729 elements. How many distinct proper subfields does $\mathbb{F}$ contain?

(a) 2

(b) 3

(c) 1

(d) 4

8. Two cards are drawn one by one at random from a standard deck of 52 cards. What is the probability that the first card is an ace or the second card is an ace?

(a) $\frac{30}{221}$

(b) $\frac{31}{221}$

(c) $\frac{32}{221}$

(d) $\frac{33}{221}$

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9. Let $G = A_4$ be the alternating group of permutations on 4 symbols. Let $n = \max\{o(g) \mid g \in G\}$, where $o(g)$ denotes the order of an element $g \in G$. Then the value of n is

(a) 12

(b) 4

(c) 3

(d) 6

10. Let E be a subset of $\mathbb{R}$. Which of the following statements are equivalent?

(S1) E is connected.

(S2) E is path connected.

(S3) E is closed.

(S4) E is convex.

(S5) E is uncountable.

(a) S3 and S5

(b) S2, S3 and S4

(c) S1, S2 and S4

(d) S4 and S5

11. Let X be a metric space and $E \subset X$. Then which of the following are equivalent?

(S1) E is closed and bounded.

(S2) E is compact.

(S3) E is bounded.

(S4) E has finite intersection property.

(S5) Every infinite subset of E has a limit point in E.

(S6) E is closed.

(a) S1 and S2

(b) S1, S3 and S6

(c) S1 and S5

(d) S2, S4 and S5

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12. Let f be a holomorphic (complex analytic) function on $\mathbf{C}$ such that $f(x) = x^2$ for all $x \in \mathbb{R}$ with $x \geq 0$. Then which of the following is/are correct?

(S1) $\operatorname{Re} f(z) = (\operatorname{Re} z)^2$ for all $z \in \mathbf{C}$

(S2) $f(z) = z^2$ on $\mathbf{C}$

(S3) f is real valued

(S4) The Cauchy-Riemann equations of f have a zero of order 2

(a) S3 and S4

(b) S1

(c) S2

(d) None of the above

13. For a commutative ring R, let $R[X]$ denote the polynomial ring in one variable over R. Let I be the ideal $\langle 5, X^2 + 3X - 1 \rangle$ in $\mathbb{Z}[X]$. Then which of the following hold(s) for the ring $\mathbb{Z}[X]/I$?

(S1) It is isomorphic to $(\mathbb{Z}/5\mathbb{Z})[X]$.

(S2) It has 5 zero divisors.

(S3) It is a finite field.

(S4) It has no element of finite order.

(a) S1 and S2

(b) S3

(c) S4

(d) None of the above

14. Let $f : (0, 1) \to [0, 1]$ be a continuous function. Which of the following is/are true?

(S1) f is uniformly continuous.

(S2) f has a fixed point.

(S3) f is a differentiable function.

(S4) There is a continuous function $\tilde{f} : [0, 1] \to [0, 1]$ such that $\tilde{f} = f$ on $(0, 1)$.

(a) S1 and S2

(b) S3

(c) S4

(d) None of the above

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15. In the set $M_2(\mathbb{R})$, which of the following sets contain a singular matrix?

$X = \{A \mid A^2 = \mathrm{Id}\}$
$Y = \{A \mid {}^tA = A\}$
$Z = \{A \mid {}^tAA = A\,{}^tA = \mathrm{Id}\}$
$W = \{A \mid AB = BA, \text{ for all } B \in M_2(\mathbb{R})\}$
$T = \{A \mid A \text{ has at least one eigenvalue which is not real}\}$

(a) X

(b) Y and W

(c) $Z \cap W$

(d) T

16. Let $f : \mathbb{R} \to \mathbb{R}$ is such that f is open. Then which of the following is/are true?

(S1) f is one to one.

(S2) f is onto.

(S3) f has exactly two zeros.

(S4) f is bounded.

(a) S1

(b) S2 and S3

(c) S4

(d) None of the above

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SECTION—C

C1. Find all complex polynomials $p \in \mathbb{C}[z]$ such that $p(z_1 + z_2) = p(z_1) + p(z_2)$ for all $z_1, z_2 \in \mathbb{C}$.

C2. Let (X, d) be a compact metric space. Let $f : X \to X$ be such that $d(f(x), f(y)) = d(x, y)$ for all $x, y \in X$. Prove that f is onto.

C3. Define $f : \mathbf{R} \to \mathbf{R}$ as follows :

$$f(x) = \begin{cases} x^2 (\log_e x)^2 & \text{if } x \in (0, 1) \\ 0 & \text{otherwise} \end{cases}$$

Find a point x such that $f(x)$ attains a local maximum value. Find the area enclosed between the x-axis and the curve of $f(x)$.

C4. Let $f : [0, 1] \to \mathbb{R}$ be a continuous map such that $f(x) = f(x^3)$ for all $x \in [0, 1]$. Find the value of $f(1) - f(0)$.

SECTION—B

B1. Write the logical opposite (that is, negation) of the following statement :

"If E is a non-empty set, then it is finite."

B2. Let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -12 & 35 \\ 0 & -6 & 17 \end{pmatrix}$$

Find a matrix P such that $P^{-1}AP$ is diagonal.

B3. Let X be a non-empty set with a relation ~ on it. Suppose ~ is both symmetric and transitive. Prove or disprove that ~ is an equivalence relation.

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ENTRANCE EXAMINATION, 2011

Pre-Ph.D./Ph.D.
Mathematical Sciences

[Field of Study Code : MATP (160)]

Time Allowed : 3 hours

Maximum Marks : 70

INSTRUCTIONS FOR CANDIDATES

(i) This question paper consists of two Parts—Part A and Part B.

(ii) All questions are compulsory. Answers should be written in the space following each question.

(iii) Extra pages are attached at the end of the question paper for rough work.

(iv) Answers to all questions must be justified with mathematical reasoning, or else they will be considered invalid.

(v) In the following $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, integers, rational numbers, real numbers and complex numbers respectively. Any subset of $\mathbb{R}^n$, $n \in \mathbb{N}$, is assumed to have the usual topology wherever applicable, unless otherwise mentioned.

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PART—A

Answer all questions. Each question carries 6 marks

A1. *(a)* For a metric space (X, d), let $d^2(x, y) := (d(x, y))^2$, for all $x, y \in X$. Is (X, d^2) also a metric space?

(b) Recall that any non-zero $x \in \mathbb{Q}$ can be written as $x = 2^r \cdot (a/b)$, where a and b are both odd integers and $r \in \mathbb{Z}$. Let $\|x\| = 2^{-2r}$, for such an x. Define the function $d_0 : \mathbb{Q} \times \mathbb{Q} \to \mathbb{R}$ as

$$d_0(x, y) = \begin{cases} \|x - y\| & \text{if } x \neq y, \\ 0 & \text{otherwise.} \end{cases}$$

Is $(\mathbb{Q}, d_0)$ a metric space?

A2. Let U be a bounded open disk in $\mathbb{C}$ and let $\mathscr{H}(U)$ denote the ring of complex analytic functions on U.

(a) Is $\mathscr{H}(U)$ an integral domain?

(b) Give an example of a maximal ideal in $\mathscr{H}(U)$.

(c) The ring $\mathscr{H}(U)$ is also a vector space over $\mathbb{C}$. Is it finite dimensional?

A3. Let H be a Hilbert space over $\mathbb{C}$ and let $BL(H)$ denote the space of bounded linear operators on H. Let $S = \{A \in BL(H) \mid A^6 = I\}$, where I denotes the identity operator on H. Let T be an operator in S.

(a) Find the spectrum of T.

(b) Is T invertible?

(c) Is S a finite set?

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A4. Let G be a group and let e denote the identity element in G. A subgroup H of G is said to be *characteristic* if any automorphism of G carries H to itself.

(a) Prove that any characteristic subgroup is normal in G.

(b) Prove that the center $Z(G)$ of G is a characteristic subgroup.

(c) Let $S = \{a \in G \mid a^n = e \text{ for some } n \in \mathbf{N}\}$. Prove that the subgroup generated by S is a characteristic subgroup.

A5. Let $f(x) = x^3 - 5 \in \mathbf{Q}[x]$ and let K denote the splitting field of f over $\mathbf{Q}$.

(a) Find $\alpha_1, \ldots, \alpha_n \in \mathbf{C}$ (for some $n \in \mathbf{N}$) such that $K = \mathbf{Q}(\alpha_1, \ldots, \alpha_n)$. What is the value of $[K : \mathbf{Q}]$?

(b) Compute $[L : \mathbb{R}]$; L being the splitting field of f over $\mathbb{R}$.

(c) Find $[K(\sqrt{3}) : \mathbf{Q}(\sqrt{3})]$.

PART—B

Answer **all** questions. Each question carries *4* marks

B1. Let $S = \{f : (0, 1) \to \mathbb{R}\}$ be a set of maps. Write the logical opposite (negation) of the following statements :

(a) All maps in S are continuous and at least one map in S is uniformly continuous.

(b) All maps in S are continuous but exactly one map in S is uniformly continuous.

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B2. For $x \in \mathbb{R}$, let $p(x) = x^2 - 2x - 1$ and $f(x) = x^{1/x}$. Find

(i) $\lim_{x \to 0} f(|x|)$

(ii) $\lim_{x \to \infty} p(f(x))$

B3. Let $S = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 4\}$.

(a) Find the set $A = \{P \in \mathbb{R}^2 \mid d(P, Q) \le 1$ for some $Q \in S\}$.

(b) Is there a continuous map from A onto S?

B4. Let X be a Hausdorff topological space. Let $C \subset X$ be a compact subset. Prove that C is closed. Is the conclusion true even if X is not Hausdorff?

B5. Let $A = S_1 \cup S_2$, where $S_1 = \{(x, y) \in \mathbb{R}^2 \mid (x-1)^2 + y^2 = 1\}$ and $S_2 = \{(x, y) \in \mathbb{R}^2 \mid (x+1)^2 + y^2 = 1\}$. Consider the set $B = \{(x, y) \in \mathbb{R}^2 \mid y^2 = x^2, -2 \le x \le 2\}$.

(a) Are A and B homeomorphic?

(b) Are A and S_1 homeomorphic?

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B6. Consider the sets $A = \{(x, y) \in \mathbb{R}^2 \mid 0 \le x \le 1, y = 0\}$ and $B = \{(x, y) \in \mathbb{R}^2 \mid 2 < x < 3, y = 0\}$. Find a surjective map from $\mathbb{R}^2 \setminus A$ to $\mathbb{R}^2 \setminus B$.

B7. Suppose four cards are drawn one-by-one from a standard deck of 52 cards. What is the probability that the third card is either a King or a Queen?

B8. Let μ be the Lebesgue measure on the interval $X = [2, 3]$. For $x \in X$, define

$$f(x) = \begin{cases} 0 & \text{if } x = \frac{a}{2^n}, \quad a, n \in \mathbb{Z}; \\ x^2 (\log_e x)^2 & \text{otherwise.} \end{cases}$$

(a) Is f Lebesgue measurable?

(b) Is f Lebesgue integrable?

(c) Is f Riemann integrable?

B9. *(a)* Prove that $\mathbb{Z}$ and $\mathbb{Q}$ are not isomorphic as abelian groups.

(b) Give two different reasons to show that $\mathbb{Z}/6\mathbb{Z}$ is not isomorphic to the permutation group S_3 of the set $\{1, 2, 3\}$.

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B10. Let S_3 denote the permutation group of $\{1, 2, 3\}$. To each $\sigma \in S_3$, associate a map $L_\sigma : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $L_\sigma(x_1, x_2, x_3) := (x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)})$.

(a) Show that L_σ is a linear operator on $\mathbb{R}^3$.

(b) Find a common eigenvalue and a common eigenvector of all the σ in S_3.

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ENTRANCE EXAMINATION, 2010

Pre-Ph.D./Ph.D.
Mathematical Sciences

[Field of Study Code : MATP (160)]

Time Allowed : 3 hours

Maximum Marks : 70

INSTRUCTIONS FOR CANDIDATES

(i) This question paper consists of two parts—Part A and Part B.

(ii) All questions are compulsory. Answers should be written in the space following each question.

(iii) Extra pages are attached at the end of the question paper for rough work.

(iv) **Answers to all questions must be justified** with mathematical reasoning, or else they will be considered invalid.

(v) In the following **N**, **Z**, **Q**, **R** and **C** denote the set of natural numbers, integers, rational numbers, real numbers and complex numbers respectively. Any subset of $\mathbf{R}^n$, $n \in \mathbf{N}$, is assumed to have the usual topology wherever applicable.

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PART—A

Answer **all** questions. Each question carries 6 marks

A1. Find the extension degree of F over $\mathbb{Q}$ when

(a) $F = \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{4})$

(b) $F = \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$

(c) $F = \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{6})$

A2. What are the points of intersection of the following two curves in $\mathbb{R}^2$?

$$y = \begin{cases} x \log_e x & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases} \qquad \text{(curve 1)}$$

$$y = x(1-x) \qquad \text{(curve 2)}$$

Compute the area enclosed between these two curves.

A3. Let

$$X = \{(x, y) \in \mathbb{R}^2 \mid 0 < x^2 + y^2 \le 2\} \text{ and } Y = X \setminus \{(1, 0)\}$$

(a) Is X compact?

(b) Is X connected?

(c) Can one find a bijective map from X to Y?

A4. Evaluate

$$\int_0^\infty \frac{x \sin(ax)}{x^2 + b^2} dx$$

where $a, b \in \mathbb{R}$, $b \neq 0$.

A5. Let G be a cyclic group.

(a) Prove that any subgroup of G is cyclic.

(b) Suppose that the order of G is n and $d \ge 1$ is a divisor of n. Prove that G has a unique subgroup of order d.

(c) Suppose that ϕ is the Euler function defined as

$$\phi(m) = \# \{1 \le k \le m \mid \gcd(k, m) = 1\}$$

Prove that $\displaystyle\sum_{d \mid n} \phi(d) = n$

PART—B

Answer **all** questions. Each question carries 4 marks

B1. Construct a group G with the following two properties :

(a) G is uncountable.

(b) Every element of G has finite order.

B2. Let R be a ring with unity and let $x \in R$.

(a) If x has a left inverse y and a right inverse y', then prove that $y = y'$.

(b) Is it necessary that if x has a left inverse in R, then it is invertible?

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B3. Evaluate :

(a) $\lim_{x \to \infty} (x^{-1}e^{2x} - 1)$

(b) $\lim_{x \to \infty} x^{-1} \sin x$

B4. Let $f : [0, 1] \to [0, 1]$ be a continuous function. Show that f has a fixed point, that is, show that there exists $x \in [0, 1]$ such that $f(x) = x$.

B5. Find the value of

$$\lim_{n \to \infty} \int_0^1 x^n \cos \frac{\pi x}{2} \, dx$$

B6. Let $\mu_1, \ldots, \mu_n$ be measures on a measurable space (X, Σ) and let $a_1, \ldots, a_n$ be non-negative real numbers.

(a) Show that for any $E \in \Sigma$, $\mu(E) = \sum_{i=1}^{n} a_i \mu_i(E)$ also defines a measure on (X, Σ).

(b) Find the conditions which ensure that μ is a probability measure.

B7. Let T be a bounded linear normal operator on a Hilbert space. Given that

(a) T is invertible

(b) $\|T\| \leq 1$

(c) $\|T^{-1}\| \leq 1$

show that T is unitary.

B8. Let A be a bounded linear operator on a Hilbert space $\mathscr{H}$ and let sp(A) denote the spectrum of A.

(a) Prove that if $\lambda \in \text{sp}(A)$, then $e^{\lambda} \in \text{sp}(e^{A})$.

(b) Show that for a bounded linear invertible operator B on $\mathscr{H}$, $\text{sp}(A) = \text{sp}(BAB^{-1})$.

B9. Let (X, τ) be a topological space where $X = \{1, 2, 3, 4\}$ and

$$\tau = \{X, \emptyset, \{1, 2\}, \{2\}, \{2, 4\}, \{1, 2, 4\}, \{2, 3, 4\}\}$$

(a) Prove that X is not Hausdorff.

(b) Find the smallest Hausdorff topology on X which contains τ.

B10. Is $\mathbb{R}^2 \setminus (\mathbb{Q} \times \mathbb{Q})$ path-connected?

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1. **Free Maths Study Materials (**https://pkalika.in/2020/04/06/free-maths-study-materials/**)**

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7. **List of Maths Suggested Books** (https://pkalika.in/suggested-books-for-mathematics/)

8. **CSIR-NET Mathematics Details Syllabus (**https://wp.me/p6gYUB-Fc)