Linear Algebra

(Handwritten Classroom Study Material)

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Matrix:-

Matrix is a rectangular array of $m \times n$ in which we can arrange mn well-defined elements.

$$A = [a_{ij}] \ , \quad \begin{matrix} i = 1, 2, \dots, m \\ j = 1, 2, \dots, n \end{matrix}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$

↳ Determinant is a scalar which is associated with a square matrix and is denoted by $|A|$ or $\det(A)$.

↳ $|A| = 0$, then A is singular matrix

$|A| \neq 0$, then A is non-singular matrix.

Q:- Find rank of

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \\ 2 & 3 & 1 & 1 \\ 0 & 2 & -1 & -1 \end{bmatrix} \underset{\substack{R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1}}{\sim} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & -1 & -5 & -7 \\ 0 & 2 & -1 & -1 \end{bmatrix}$$

$$\underset{\substack{R_1 \to R_1 + 2R_2 \\ R_3 \to R_3 - R_2 \\ R_4 \to R_4 + 2R_2}}{\sim} \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & -3 & -4 \\ 0 & 0 & -5 & -7 \end{bmatrix} \underset{R_4 \to 3R_4 - 5R_3}{\sim}$$

$$\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & -3 & -4 \end{bmatrix}$$

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$\therefore$ Rank $(A) = 4$.

Eigen Value of a square matrix A

A scalar λ is said to be eigen value of a square matrix A if $AX = \lambda X$, where $X \neq 0$ and X is known as the eigen vector of matrix A corresponding to the eigen value λ.

A.M :- (Algebraic multiplicity)

~~Algebraic multiplicity~~ of ~~an~~ eigen ~~value~~

No. of repeatation of an eigen value λ is known as the AM of λ.

G.M (Geometric multiplicity) :-

The no. of linearly independent eigen vectors corresponding to an eigen value λ is known as the GM of λ.

* GM can not exceed AM. [GM $\leq$ AM]

* $AX = \lambda X$, $X \neq 0$

Put $\Rightarrow |(A - \lambda I) X| = 0$

$\Rightarrow |A - \lambda I| \, |X| = 0$

$\Rightarrow |A - \lambda I| = 0 \rightarrow$ Characteristic equation.

Q:- $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$, Find the EV and eigen vector of this matrix.

Put $|A - \lambda I| = 0$ or $\lambda^3 - \text{trace}(A)\lambda^2 + (|M_{11}| + |M_{22}| + |M_{33}|$
$- |A| = 0$

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$$\begin{vmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow -\lambda\left(6-5\lambda+\lambda^2\right) - 2\left(-2+\lambda\right) = 0$$

$$\Rightarrow -6\lambda + 5\lambda^2 - \lambda^3 + 4 - 2\lambda = 0$$

$$\Rightarrow \lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$$

$$\Rightarrow \lambda^3 - \lambda^2 - 4\lambda^2 + 4\lambda + 4\lambda - 4 = 0$$

$$\Rightarrow \lambda^2(\lambda-1) - 4\lambda(\lambda-1) + 4(\lambda-1) = 0$$

$$\Rightarrow (\lambda-1)\left(\lambda^2-4\lambda+4\right) = 0$$

$$\Rightarrow (\lambda-1)(\lambda-2)^2 = 0$$

$$\Rightarrow \lambda = 1, 2, 2$$

For $\lambda = 1$

$$AX = \lambda X$$

$$\Rightarrow AX = X$$

$$\Rightarrow \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -2x_3 \\ x_1+2x_2+x_3 \\ x_1+3x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -x_1-2x_3 \\ x_1+x_2+x_3 \\ x_1+2x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

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$x_1 + 2x_3 = 0$

$\Rightarrow x_1 = -2x_3$

$x_1 + x_2 + x_3 = 0$

$\Rightarrow x_2 - x_3 = 0$

$\Rightarrow x_2 = x_3$

$$\therefore X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2x_3 \\ x_3 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} x_3$$

choose $x_3 = 1$ $\therefore X = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$

or

$$\begin{pmatrix} -1 & 0 & -2 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow -x_1 - 2x_3 = 0$

$x_2 - x_3 = 0$

Choose $x_3 = 1$

$\therefore x_1 = -2$

$x_2 = 1$

$$\therefore X = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

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For $\lambda = 2$

$$\begin{pmatrix} -2 & 0 & -2 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow \quad -2x_1 - 2x_3 = 0$

$x_1 + x_3 = 0$

$x_1 + x_3 = 0$

$\Rightarrow \quad x_1 + x_3 = 0$

~~$x_1 + x_3 = 0$~~

$\Rightarrow \quad x_1 = -x_3$

Choose $x_2 = 0$

$$X = \begin{pmatrix} -x_3 \\ 0 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} x_3$$

Let us choose $x_3 = 1$

$$\therefore X = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Again choose $x_2 = 1$

$$X = \begin{pmatrix} -x_3 \\ 1 \\ x_3 \end{pmatrix} = \cancel{\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} x_3}$$

Choose $x_3 = 0$

$$\therefore X = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Null Space of A

The set $S = \{X \in \mathbb{R}^n \mid AX = 0\}$ is known as null space of A.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 2 & 4 & 6 \end{pmatrix}$$

$$\approx \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \approx \begin{pmatrix} 1 & 2 & 3 \\ 0 & -2 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

$\text{Rank}(A) = 2$

$\text{Nullity}(A) = \text{No. of Column} - \text{Rank}(A)$

$= 3 - 2 = 1.$

To find the null space of A, put

$$AX = 0$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 2 & 4 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 2 & 3 \\ 0 & -2 & -2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow \quad x_1 + 2x_2 + 3x_3 = 0$

$-2x_2 - 2x_3 = 0$

$\Rightarrow x_2 = -x_3$

$\therefore x_1 + 2(-x_3) + 3(x_3) = 0$

$\Rightarrow x_1 = -x_3$

$\therefore X = (-x_3, -x_3, x_3)$

$= x_3(-1, -1, 1)$

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Put $x_3 = \alpha$, $\alpha \in \mathbb{R}$

$X = \alpha(-1,-1,1)$

$N(A) = \{\alpha(-1,-1,1)\}, \alpha \in \mathbb{R}$

System of Linear equation

$AX = b$

Consistent :-

A system of Linear equation $AX = b$, $A = [a_{ij}]_{m\times n}$, $b = (b_1, b_2, \dots, b_m)^T$, $X = (x_1, x_2, \dots, x_n)^T$ is said to be consistent (solvable) if $\boxed{\text{rank}(A) = \text{rank}(C)}$, where $C = \underset{\substack{\downarrow \\ \text{Augmented matrix}}}{[A:b]}$, otherwise system will be inconsistent.

Homogeneous System ($AX=0$)	Solution
1) $\text{rank}(A) = \text{rank}(C) = r = m \ (m<n)$	Infinite solution
2) $\text{rank}(A) = \text{rank}(C) = r = m = n$	Trivial solution
3) $\text{rank}(A) = \text{rank}(C) = r = n < m$	unique solution

Non-homogeneous system ($AX=b$)	Solution
1) $\text{rank}(A) = \text{rank}(C) = r = m < n$	Infinite solution
2) $\text{rank}(A) = \text{rank}(C) = r = m = n$	Unique solution
3) $\text{rank}(A) = \text{rank}(C) = r = n < m$	Unique solution
4) $\text{rank}(A) \neq \text{rank}(C)$	No solution

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Classification of Matrices:-

Hermitian Property:-

A square matrix A is said to be hermitian if

$a_{ij} = \overline{a_{ji}}$

Every symmetric matrix is hermitian.

Ex:- $A = \begin{pmatrix} 1 & 1-i \\ 1+i & 2 \end{pmatrix}$

$a_{12} = 1-i = \overline{(1+i)} = \overline{a_{21}}$

Skew-hermitian matrix:-

$a_{ij} = -\overline{a_{ij}}$

Diagonal matrix:-

$A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$, $a, b, c \in \mathbb{R}$.

Scalar matrix:-

$A = \begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$, $k \in \mathbb{R}$.

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Orthogonal matrix :-

A n-square matrix A is said to be orthogonal if $AA^T = I = A^TA$.

Unitary matrix :-

$$AA^* = A^*A = I$$

$$A^* = \overline{A}^T = \overline{A^T}$$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$$

$$A + A^T = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 5 & 2 \end{bmatrix} \leftarrow \text{symmetric matrix}$$

$$A - A^T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \leftarrow \text{skew-symmetric matrix.}$$

Similar matrices :-

~~Two square matrices A and B are said to be~~

Let A and B are square matrices. The matrix A is said to be similar to the matrix B if ~~$P^{-1}AP = B$~~ there exists an invertible matrix P such that $P^{-1}AP = B$.

1→ If two matrices A and B are similar, then

$$\det(A) = \det(B).$$

But converse is not true.

2→ Characteristic polynomial will be same, but converse not true.

3→ Minimal polynomial will be same.

Minimal polynomial $m(t)$ is the lowest degree monic polynomial such that $m(A) = 0$.

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* $a_0 t^n + a_1 t^{n-1} + \ldots\ldots + a_n$, $a_0 \neq 0$

It is monic polynomial if $a_0 = 1$.

↳ If minimal polynomial and characteristic polynomial of two matrices are same, then these two matrices are same.

* $A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

Put $\det(A - \lambda I) = 0$

$\Rightarrow \begin{vmatrix} 1-\lambda & 1 \\ -2 & 4-\lambda \end{vmatrix} = 0$

$\Rightarrow (1-\lambda)(4-\lambda) + 2 = 0$

$\Rightarrow 4 - 5\lambda + \lambda^2 + 2 = 0$

$\Rightarrow \lambda^2 - 5\lambda + 6 = 0$

$\Rightarrow (\lambda - 2)(\lambda - 3) = 0$

$m(\lambda) = (\lambda - 2)(\lambda - 3)$

$\det(B - \lambda I) = 0$

$\Rightarrow (2-\lambda)(3-\lambda) = 0$

$\Rightarrow (\lambda - 2)(\lambda - 3) = 0$

$m(\lambda) = (\lambda - 2)(\lambda - 3)$

∴ Both A and B are similar

* Diagonalisable matrices :-

Any square matrix A is said to be diagonalisable if there exist an invertible matrix P such that $P^{-1}AP = D$.

$A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$

Characteristic polynomial,

$\Delta t = (t - 2)(t - 3)$

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$t = 2, 3$

For $t=2$

$$\begin{pmatrix} -1 & 1 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow -x_1 + x_2 = 0$$

$$\Rightarrow x_1 = x_2$$

$$\therefore X = \begin{pmatrix} 1 \\ 1 \end{pmatrix} x_1$$

$$\therefore X = [(1,1)]$$

For $t=3$

$$\begin{pmatrix} -2 & 1 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow -2x_1 + x_2 = 0$$

$$\Rightarrow x_2 = 2x_1$$

$$\therefore X = \begin{pmatrix} x_1 \\ 2x_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} x_1$$

$$\therefore X = [(1,2)]$$

$$P = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

$$P^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

$$P^{-1}AP = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 4 & -2 \\ -3 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

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VECTOR SPACES

Vector space:-

A non-empty set V over the field $\mathbb{F}$ under the addition and scalar multiplication is said to be a vector spac if it satisfy the following properties:-

(1) If $u, v \in V \Rightarrow u+v \in V$

(2) If $u, v, w \in V$, then $(u+v)+w = u+(v+w)$

(3) Existence of identity under addition

i.e. $u+0 = u = 0+u$

(4) Existence of inverse

$u+(-u) = 0 = (-u)+u$

(5) $u+v = v+u$

(6) If α is a scalar and $u \in V$, then $\alpha u \in V$.

(7) $\alpha(u+v) = \alpha u + \alpha v$, $\alpha \in \mathbb{F}$, $u, v \in V$

(8) $(\alpha+\beta)u = \alpha u + \beta u$, $\alpha, \beta \in \mathbb{F}$, $u \in V$

(9) $\alpha(\beta u) = (\alpha\beta)\cdot u = \beta\cdot(\alpha u)$, $\alpha, \beta \in \mathbb{F}$, $u \in V$

(10) $1\cdot u = u$, $u \in V$.

Ex:-1 Set of real numbers,

$V = \mathbb{R}$, $\mathbb{F} = \mathbb{R}$.

Ex:-2 $\mathbb{R}(\mathbb{C})$ is not a vector space.

$u \in \mathbb{R}$

But $\alpha u \notin \mathbb{R}$, $\alpha \in \mathbb{C}$

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Ex:-3 $P_n(x)$ = Set of all polynomials of degree $\leq n$.

$\mathbb{F} = \mathbb{R}$.

Let $p(x), q(x) \in P_n(x)$

$p(x) = a_0x^n + a_1x^{n-1} + \dots + a_n$

$q(x) = b_0x^n + b_1x^{n-1} + \dots + b_n$

$P_n(x)(\mathbb{R})$ is a V.S.

Ex:-4 $S = \{$ The set of polynomials of degree $n \}$, $\mathbb{F} = \mathbb{R}$

Not satisfying closure property.

So $S(\mathbb{R})$ is not a vector space.

let $p(x) = a_0x^n + a_1x^{n-1} + \dots + a_n$

$q(x) = -a_0x^n + b_1x^{n-1} + \dots + b_n$

$p+q = c_1x^{n-1} + \dots + c_n \notin S$.

Ex:-5 $\mathbb{R}^+(\mathbb{R})$

$u+v = u \cdot v$

$\alpha u = u^\alpha$

~~$0+v = 0 \cdot v = 0 \notin \mathbb{R}^+$~~

(1) $u \in \mathbb{R}^+$

$v \in \mathbb{R}^+$

$u+v = u \cdot v \in \mathbb{R}^+$

(2) $(u+v)+w = uvw$

$u+(v+w) = uvw$

(3) $u \in \mathbb{R}^+$

$1 \in \mathbb{R}^+$

$u+1 = u \cdot 1 = u$.

(4) $u \in \mathbb{R}^+$

$\frac{1}{u} \in \mathbb{R}^+$

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$u+\frac{1}{u} = u\cdot\frac{1}{u} = 1$

5) $u+v = uv = vu = v+u$.

6) $u \in \mathbb{R}^+$

$\alpha \in \mathbb{R}$

$\alpha u = u^\alpha \in \mathbb{R}^+$

7) $\alpha(u+v) = \alpha(uv) = (uv)^\alpha = u^\alpha v^\alpha$

$\alpha u + \alpha v = u^\alpha + v^\alpha = u^\alpha v^\alpha$

8) $(\alpha+\beta)u = u^{\alpha+\beta}$

$\alpha u + \beta u = u^\alpha + u^\beta = u^\alpha u^\beta = u^{\alpha+\beta}$

$(\alpha+\beta)u = u^{\alpha+\beta}$
$= u^\alpha \cdot u^\beta$
$= \alpha u \cdot \beta u$
$= \alpha u + \beta u$

9) $\alpha(\beta u) = \alpha(u^\beta) = u^{\beta\alpha}$

$(\alpha\beta)u = u^{\alpha\beta}$

$\beta(\alpha u) = \beta(u^\alpha) = u^{\alpha\beta}$

10) $1\cdot u = u^1 = u$

$\therefore$ $\mathbb{R}^+(\mathbb{R})$ is a vector space.

Ex-6:- $\mathbb{R}^2(\mathbb{R})$

$\alpha u = \alpha(x,y) = (2\alpha x, \alpha y)$ & normal addition.

$\therefore$ $\mathbb{R}^2(\mathbb{R})$ is an abelian group under addition

but $1\cdot u = 1(x,y) = (2x, y) \neq u$

$\therefore$ $\mathbb{R}^2(\mathbb{R})$ is not a V.S.

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Ex:-7 $AX = b$

$A_{m\times n}$, $m<n$

Sol^n set of this system is a V.S or not with $\overset{\text{gen.}}{\wedge}$ add^n & mult^n.

Here X can not be a null vector.

So identity doesn't exists.

∴ It is not a V.S.

Closure property also not hold. As $X_1, X_2 \in S$ $A(X_1+X_2) = AX_1 + AX_2$
$= b+b = 2b$

Ex:-8 $AX = b = 0$

Sol^n of this system form a V.S.

*(Associative property)

Ex:-9 $AX = b$

$A_{m\times n}$, $m=n$

Sol^n set of this system is not a V.S.

Ex:-10 S = set of matrices of order m×n , $\mathbb{F}$ $\mathbb{R}$.
is VS under add^n & scalar mul^n

Ex:-11 S = set of symmetric matrice ✓
" diagonal "
" upper triangular "
" lower triangular " form V.S.

Ex:-12 Set of fun's, cont. fun^n, derivable fun's forms VS.

$C^1[a,b]$ → One time differentiable cont. fun^n in $[a,b]$

$C^2[a,b]$ → Two times " " " ".

$$C[a,b] \supseteq C^1[a,b] \supseteq C^2[a,b] \supseteq \dots \supseteq C^\infty[a,b] \supseteq P_n(x)$$

All these are Vector Spaces.

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Subspace:-

A non-empty subset W of $\overset{\text{a VS}}{\wedge}$ V is said to be a subspace of V if W is itself a vectorrspace under the addition and scalar multiplication defined on V.

* $u, v \in W$ then $\alpha u + \beta v \in W$.

Ex:-1 $V = C[a,b]$

$W = C'[a,b]$

$\alpha f + bg \in W$

Ex:-2 $\mathbb{R}^2(\mathbb{R})$

$S = \{$ ~~$(x+1, y) : (x,y) \in \mathbb{R}^2$~~ $\}$ $\{(x,y) \in \mathbb{R}^2 : (x+1, y)\}$

~~$\alpha(x_1, y_1) + \beta(x_2, y_2) = (\alpha x_1, \alpha y_1) + (\beta x_2, \beta y_2)$~~

~~$= (\alpha x_1 + 1, \alpha y_1) + (\beta x_2 + 1, \beta y_2)$~~

~~$= (\alpha x_1 + \beta x_2 + 2, \alpha y_1 + \beta y_2)$~~

$= (\alpha x_1 + \beta x_2 + 1 + 1, \alpha y_1 + \beta y_2) \in S$

$\alpha u + \beta v = \alpha(x_1 + 1, y_1) + \beta(x_2 + 1, y_2)$

$= (\alpha x_1 + \alpha, \alpha y_1) + (\beta x_2 + \beta, \beta y_2)$

$= (\alpha x_1 + \beta x_2 + \alpha + \beta, \alpha y_2 + \beta y_2) \notin S$.

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~~**Ex:-3**~~

~~$S_1 = \{(x,y) \in \mathbb{R}^2 : x = y\}$~~

~~$S_2 = \{(x_1, x_2) \in \mathbb{R}^2 : \frac{x_1}{x_2} = 2\}$~~

~~Let $u, v \in S_2$~~

~~$\Rightarrow u = (2x_1, x_2)$~~

~~$v = (2y_1, y_2)$~~

~~$\alpha u + \beta v = \alpha(2x_1, x_2) + \beta(2y_1, y_2)$~~

~~$= (2\alpha x_1 + 2\beta y_1, \alpha x_2 + \beta y_2)$~~

~~$= (2\alpha x_1 + 2\beta y_1, (\alpha x_2 + \beta y_2)) \in S_2$~~

Ex:-3 $S_1 = \{(x,y) \in \mathbb{R}^2 : x = y\}$

$S_2 = \{(x_1, x_2) \in \mathbb{R}^2 : \frac{x_1}{x_2} = 2\}$

Let $u, v \in S_2$

$u = (2x_1, x_1)$

$v = (2y_1, y_1)$

$\alpha u + \beta v = (2(\alpha x_1 + \beta y_1), \alpha x_1 + \beta y_1) \in S_2$ for, $\alpha, \beta \neq 0$.

But $(0,0) \notin S_2$ as $\frac{0}{0} \neq 2$.

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Ex:-4 $S = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 + x_3 \geq 0\}$

$u = (x_1, x_2, x_3)$

$v = (y_1, y_2, y_3)$

$u+v = (x_1+y_1, x_2+y_2, x_3+y_3)$,

$(x_1+y_1)+(x_2+y_2)+(x_3+y_3) \geq 0$.

$\alpha u = \alpha(x_1+x_2+x_3)$

let α -ve

$\alpha x_1 + \alpha x_2 + \alpha x_3 < 0$.

$\therefore$ S is not a subspace.

Dt:- 20.01.2020

Linear combination of Vectors :-

Let $u_1, u_2, \ldots, u_n$ are n-vectors in a vectorspace V and $\alpha_1, \alpha_2, \ldots, \alpha_n$ are scalars, then

$\alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_n u_n$ is known as the linearcombination of $u_1, u_2, \ldots, u_n$.

$\alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_n u_n = 0$ is known as the trivial linear combination.

↳ If in the trivial linear combination of $u_1, u_2, \ldots, u_n$ all scalars $\alpha_1, \ldots, \alpha_n$ are zero, then $u_1, u_2, \ldots, u_n$ are known as linearly independent otherwise vectors are ld

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Ex:- Check LI or LD

$S = \{(1,1,1), (1,1,-1), (1,2,3)\}$

Put $\alpha_1(1,1,1) + \alpha_2(1,1,-1) + \alpha_3(1,2,3) = 0$

$\Rightarrow (\alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + 2\alpha_3, \alpha_1 - \alpha_2 + 3\alpha_3) = 0$

$\Rightarrow \alpha_1 + \alpha_2 + \alpha_3 = 0$ ——— (i)

$\alpha_1 + \alpha_2 + 2\alpha_3 = 0$ ——— (ii)

$\alpha_1 - \alpha_2 + 3\alpha_3 = 0$ ——— (iii)

Adding (i) & (iii) $2\alpha_1 + 4\alpha_3 = 0$

$\Rightarrow \alpha_1 = -2\alpha_3$

$\alpha_1 + \alpha_2 + 2\alpha_3 = 0$

$\Rightarrow \alpha_2 = 0$

$(\alpha_1, \alpha_2, \alpha_3) = (-2, 0, 1)\alpha_3$

Subtracting (i) & (iii)

$2\alpha_3 = 0$

$\Rightarrow \alpha_3 = 0$

✗ $\alpha_1 = -\alpha_2$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & -1 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & -2 & 2 \end{pmatrix}$$

rank = 3 = no. of vectors.

$\therefore$ S is LI.

Span of S :-

Let S be any non-empty subset of a v.s V, then the span of S is the set of all finite linearcombinations of S. and it is denoted by [S].

* Prove that [S] is a subspace of V.

Proof:- Let $S = \{u_1, u_2, \ldots, u_n\}$

$[S] = [u_1, u_2, \ldots, u_n]$

let $u = \alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_n u_n$

$v = \beta_1 u_1 + \beta_2 u_2 + \ldots + \beta_n u_n$

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$\therefore$ Now $u+v = (\alpha_1 u_1 + \dots + \alpha_n u_n) + (\beta_1 u_1 + \dots + \beta_n u_n)$

$$= (\alpha_1+\beta_1)u_1 + \dots + (\alpha_n+\beta_n)u_n$$

$$= \gamma_1 u_1 + \dots + \gamma_n u_n \in [S]$$

$$\alpha u = \alpha(\alpha_1 u_1 + \dots + \alpha_n u_n)$$

$$= \alpha\alpha_1 u_1 + \dots + \alpha\alpha_n u_n$$

$$= c_1 u_1 + \dots + c_n u_n \in [S].$$

$\therefore$ $[S]$ is a subspace of v.s. V. ☐

Theorem:-

If S is a non-empty subset of a vectorspace V, then $[S]$ is the smallest subspace of V containing S.

Proof:-

(i) $[S]$ is subspace of V.

(ii) $S \subseteq [S]$

(iii) $[S] \subseteq W$

Let $\{u_1, u_2, \dots, u_n\} = S$

$S \subseteq W$ (Assume)

$\Rightarrow \{u_1, u_2, \dots, u_n\} \subseteq W$

$\Rightarrow \alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n \in W$

$\Rightarrow [S] \subseteq W.$

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* Let X_1 and X_2 be two subspaces of a v.s V, then the sum of X_1 and X_2 is defined as,

$$X_1 + X_2 = \{u+v \mid u \in X_1 \text{ and } v \in X_2\}$$

* $X_1 + X_2$ is a subspace

let $u_1+v_1 \in X_1+X_2$

$u_2+v_2 \in X_1+X_2$

Now, $\alpha(u_1+v_1)+\beta(u_2+v_2) = \alpha u_1 + \alpha v_1 + \beta u_2 + \beta v_2$

$= (\alpha u_1 + \beta u_2) + (\alpha v_1 + \beta v_2)$

$\in X_1 + X_2$ $\left[\because \alpha u_1 + \beta u_2 \in X_1 \text{ \& } \alpha v_1 + \beta v_2 \in X_2\right]$

$\therefore$ X_1+X_2 is a subspace of V.

* Prove that $[X_1 \cup X_2]$ is a subspace of V.

* Show that $X_1+X_2 = [X_1 \cup X_2]$

1) Let $X_1 = \{u_1, u_2, \dots, u_n\}$

$X_2 = \{v_1, v_2, \dots, v_n\}$

$X_1 \cup X_2 = \{u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n\}$

$[X_1 \cup X_2] = [u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n]$

Let $u, v \in [X_1 \cup X_2]$ such that

$u = \alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n + \beta_1 v_1 + \beta_2 v_2 + \dots + \beta_n v_n$

$v = c_1 u_1 + c_2 u_2 + \dots + c_n u_n + d_1 v_1 + d_2 v_2 + \dots + d_n v_n$,

where $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n, c_1, \dots, c_n, d_1, \dots, d_n$ are scalars.

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Now,

$$\alpha u+\beta v=\alpha(\alpha_1u_1+\alpha_2u_2+\dots+\alpha_nu_n+\beta_1v_1+\beta_2v_2+\dots+\beta_nv_n)$$
$$+\beta(c_1u_1+c_2u_2+\dots+c_nu_n+d_1v_1+d_2v_2+\dots+d_nv_n)$$

$$=\alpha\alpha_1u_1+\alpha\alpha_2u_2+\dots+\alpha\alpha_nu_n+\alpha\beta_1v_1+\alpha\beta_2v_2+\dots+\alpha\beta_nv_n$$
$$+\beta c_1u_1+\beta c_2u_2+\dots+\beta c_nu_n+\beta d_1v_1+\beta d_2v_2+\dots+\beta d_nv_n$$

~~$=p_1u_1+p_2u_2+\dots+p_nu_n+q_1v_1+q_2v_2+\dots+q_nv_n$~~

$$=(\alpha\alpha_1+\beta c_1)u_1+(\alpha\alpha_2+\beta c_2)u_2+\dots+(\alpha\alpha_n+\beta c_n)u_n$$
$$+(\alpha\beta_1+\beta d_1)v_1+(\alpha\beta_2+\beta d_2)v_2+\dots+(\alpha\beta_n+\beta d_n)v_n$$

$$=p_1u_1+p_2u_2+\dots+p_nu_n+q_1v_1+q_2v_2+\dots+q_nv_n$$

$$\in[X_1\cup X_2]$$

$\therefore$ $[X_1\cup X_2]$ is a subspace of V. □

2) Let $X_1=\{u_1,u_2,\dots,u_n\}$

$X_2=\{v_1,v_2,\dots,v_n\}$

Given that X_1 and X_2 are subspaces of V.

$\because$ X_1 is a subspace of V

so, $u_i+u_j\in X_1$ $\forall$ $1\le i,j\le n$

and $\alpha_iu_i\in X_1$ $\forall$ $1\le i\le n$.

$\because$ X_2 is a subspace of V

so $v_i+v_j\in X_2$ $\forall$ $1\le i,j\le n$

and $\beta_iv_i\in X_2$ $\forall$ $1\le i\le n$.

$\alpha_iu_i\in X_1$, $\beta_iv_i\in X_2$ $\forall$ $1\le i\le n$

$\Rightarrow$ $\alpha_iu_i+\beta_iv_i\in X_1+X_2$ $\forall$ $1\le i\le n$

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$\Rightarrow X_1 + X_2 = \{\alpha_1 u_1 + \beta_1 v_1 \ , \ \alpha_2 u_2 + \beta_2 v_2 , \cdots , \alpha_n u_n + \beta_n v_n\}$

$X_1 \cup X_2 = \{u_1, \cdots, u_n, v_1, \cdots, v_n\}$

$[X_1 \cup X_2] = \alpha_1 u_1 + \cdots + \alpha_n u_n + \beta_1 v_1 + \cdots + \beta_n v_n$

$= \alpha_1 u_1 + \beta_1 v_1 + \alpha_2 u_2 + \beta_2 u_2 + \cdots + \alpha_n u_n + \beta_n v_n$

$\in X_1 + X_2$

P Kalika Maths

Direct sum of two subspaces :-

Let U and W are two subspaces of a vectorspace V then the direct sum of U and W is denoted by $U \oplus W$ and each element of this set is uniquely expressible.

$v = u + w$, $u \in U$ and $w \in W$.

Ex:- Let $V = \mathbb{R}^3$

$U = xy$ - plane

$W = yz$ - plane

$U \cap W = y\text{-axis} \neq \{0\}$

So $U \oplus W$ is not defined.

Ex:- $U = x$ - axis $\quad V = \mathbb{R}^2$

$W = y$ - axis

$\boxed{U \cap W = \{0\}}$

So we can define $U \oplus W$.

Theorem:-

Let U and W be two subspaces of V and $Z = U + W$.

Then $Z = U \oplus W$ iff the following conditions satisfied.

Any vector $z \in Z$ can be expressed uniquely as the sum

$z = u + w$, $u \in U$, $w \in W$

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Proof:-

N.P:- Let $Z = U \oplus W$

Let z is not uniquely expressible as the sum of the elements of U and W

$$z = u_1 + \omega_1 \quad \text{and} \quad z = u_2 + \omega_2$$

$$\Rightarrow u_1 + \omega_1 = u_2 + \omega_2$$

$$\Rightarrow u_1 - u_2 = \omega_2 - \omega_1 \quad , u_1 - u_2 \in U$$

$$\omega_2 - \omega_1 \in W$$

$$U \cap W = \{0\}$$

$$\Rightarrow u_1 - u_2 = 0 = \omega_2 - \omega_1$$

$$\Rightarrow u_1 = u_2$$

$$\omega_1 = \omega_2 .$$

which is a contradiction.

So z is uniquely expressible as the sum of the elements of U and W.

which proves the N.P.

S.P:- Suppose that z is uniquely expressible as the sum

$$z = u + w \quad , u \in U \quad , w \in W$$

Let $U \cap W \neq \{0\}$

Let $v \in U \cap W$

$$\Rightarrow v \in U \wedge v \in W$$

Since $Z = U + W$

$$v \in Z$$

$$v = 0 + v \quad , 0 \in U , v \in W$$

$$= v + 0 \quad , v \in U , 0 \in W$$

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∴ z is not uniquely expressible which is a contradiction.

Theorem:- Dt:- 22.01 20

In a vector space V suppose $\{v_1, v_2, \ldots, v_n\}$ is an ordered set of vectors with $v_1 \neq 0$. The set is LD iff one of the vectors ~~v_2, v_3~~ ...... $v_k \in [v_1, v_2, \ldots, v_{k-1}]$ for some $k = 2, 3, \ldots$

Proof:- S.P Let $v_k \in [v_1, v_2, \ldots, v_{k-1}]$

Claim:- $\{v_1, v_2, \ldots, v_k\}$ is LD.

$v_k \in [v_1, v_2, \ldots, v_{k-1}]$

$\Rightarrow v_k = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_{k-1} v_{k-1}$

$\Rightarrow v_k$ is the linear combination of $v_1, v_2, \ldots, v_{k-1}$

Hence $\{v_1, v_2, \ldots, v_{k-1}, v_k\}$ is LD.

N.P Let the set $\{v_1, \ldots, v_k, \ldots, v_n\}$ is LD.

Assume that $S_1 = \{v_1\}$

$S_2 = \{v_1, v_2\}$

$\vdots$

$S_{k-1} = \{v_1, v_2, \ldots, v_{k-1}\}$

$S_k = \{v_1, v_2, \ldots, v_k\}$

$\vdots$

$S_n = \{v_1, v_2, \ldots, v_n\}$

∴ S_1 is LI.

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Suppose S_{k-1} is LI and S_k is LD.

$\because$ S_k is LD then $\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_{k-1} v_{k-1} + \alpha_k v_k = 0$ ——— (1)

If $\alpha_k = 0$, then S_k is LI which is a contradiction.

$\Rightarrow \alpha_k \neq 0$.

$$eq(1) \Rightarrow v_k = -\frac{\alpha_1}{\alpha_k} v_1 - \frac{\alpha_2}{\alpha_k} v_2 - \dots\dots - \frac{\alpha_{k-1}}{\alpha_k} v_{k-1}$$

$$\Rightarrow v_k \in [v_1, v_2, \dots\dots, v_{k-1}] \quad , \ 2 \le k \le n$$

which proves the N.P.

□

* An infinite subset S of a vector space V is said to LI if every finite subset of S is LI.

Ex:- $\{1, x, x^2, \dots, x^n, \dots\}$

Q: HW

$V = \mathbb{R}^3$

$W_1 = \{(x,y,z) \in \mathbb{R}^3, \ x+y=0\}$

$W_2 = \{(x,y,z) \in \mathbb{R}^3, \ 2x+z=0\}$

Check that W_1, W_2 and $W_1 \cap W_2$ are subspaces or not.

P Kalika Maths

- **Basis of a V.S V :-**

A subset B of a V.S V is said to be a basis for V if (i) B is LI.

(ii) $[B] = V$ i.e. B generates V.

Th

S **Ex:-** $V = \mathbb{R}^3$

V $B = \{(1,0,0), (0,1,0), (0,0,1)\}$

Pr **A:-** $\alpha(1,0,0) + \beta(0,1,0) + \gamma(0,0,1) = 0$

$\Rightarrow (\alpha, \beta, \gamma) = 0$

$\Rightarrow \alpha = 0, \beta = 0, \gamma = 0$

$\therefore B$ is LI.

* Now, $x_1(1,0,0) + x_2(0,1,0) + x_3(0,0,1) = (x_1, x_2, x_3)$, $x_1, x_2, x_3 \in \mathbb{R}$.

$\therefore$ B generates $\mathbb{R}^3$

Let $(x,y,z) \in \mathbb{R}^3$

$(x,y,z) = \alpha(1,0,0) + \beta(0,1,0) + \gamma(0,0,1)$

$= (\alpha, \beta, \gamma)$

$\therefore \alpha = x, \beta = y, \gamma = z$.

$\therefore$ ~~B genera~~ Every elements of $\mathbb{R}^3$ can be represented as linear combination of $\{(1,0,0), (0,1,0), (0,0,1)\}$

P Kalika Maths

∴ B generates R

i.e. $[B] = V$.

∴ B is a basis.

Q:- Check that $\{(1,1,1), (1,2,3), (1,0,0)\}$ is basis or not.

* $P_n(x)$

Standard basis for $P_n(x)$, $B = \{1, x, x^2, \ldots, x^n\}$.

~~tet~~

* Dimension of a vector space V :-

The no. of elements in a basis of a v.s. V is called the dimension of V.

Ex:- Set of matrices of order 2×3

$$\begin{bmatrix} a & b & e \\ c & d & f \end{bmatrix}$$

$$\left\{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right\}$$

* Check that $\left\{\begin{pmatrix} 1 & 0 & 1 \\ 1 & -1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 3 & 4 \\ -2 & 1 & 4 \end{pmatrix}\right\}$ is LI or not.

$$\begin{pmatrix} 1 & 0 & 1 & 1 & -1 & 1 \\ 2 & 3 & 4 & -2 & 1 & 4 \end{pmatrix}$$ Find the rank.

$r < 2 \rightarrow$ LD.

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Dimension of symmetric matrix $= n + \frac{n^2-n}{2}$

$$= \frac{2n+n^2-n}{2} = \frac{n^2+n}{2} = \frac{n(n+1)}{2}$$

* Find the dimension of skew-symmetric matrix.

Q:- Check that $B = \{(1,1,1),(1,2,3),(1,0,0)\}$ is a basis or not.

A:- Now,

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 0 & 0 \end{vmatrix} = 3-2 = 1 \neq 0.$$

$\therefore$ B is LI.

Let $(x,y,z) \in \mathbb{R}^3$ and

$$(x,y,z) = \alpha(1,1,1) + \beta(1,2,3) + \gamma(1,0,0)$$
$$= (\alpha+\beta+\gamma,\ \alpha+2\beta,\ \alpha+3\beta)$$

Comparing both sides, we get,

$$\alpha+\beta+\gamma = x \longrightarrow \text{(i)}$$
$$\alpha+2\beta = y \longrightarrow \text{(ii)}$$
$$\alpha+3\beta = z \longrightarrow \text{(iii)}$$

eq(iii) $-$ eq(ii) $\Rightarrow \beta = z-y$

eq(i) $\Rightarrow \alpha+\beta+\gamma = x$

$\Rightarrow \alpha + z - y + \gamma = x$

$\Rightarrow \alpha + \gamma = x+y-z \longrightarrow$ (iv)

eq(ii) $\Rightarrow \alpha = y - 2\beta$

$= y - 2z + 2y = 3y - 2z$

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eq(iv) $\Rightarrow \gamma = x+y-z-\alpha$

$= x+y-z-(3y-2z)$

$= x+y-z-3y+2z$

$= x-2y+z$.

$\therefore \alpha = 3y-2z$

$\beta = z-y$

$\gamma = x-2y+z$

$\therefore$ Every elements of $\mathbb{R}^3$ can be represented as linear combination of $\{(1,1,1),(1,2,3),(1,0,0)\}$

$\therefore [B] = \mathbb{R}^3$.

$\therefore B$ is a basis for $\mathbb{R}^3$.

Q:- $V = \mathbb{R}^3$

$W_1 = \{(x,y,z) \in \mathbb{R}^3 \;,\; x+y=0\}$

$W_2 = \{(x,y,z) \in \mathbb{R}^3 \;,\; 2x+z=0\}$

Check that W_1, W_2 and $W_1 \cap W_2$ are subspaces or not.

A:- Let $u_1, u_2 \in W_1$ such that

$u_1 = (x_1, y_1, z_1) \;,\; x_1+y_1=0$

$u_2 = (x_2, y_2, z_2) \;,\; x_2+y_2=0$

Now, $u_1+u_2 = (x_1+x_2, y_1+y_2, z_1+z_2),\; x_1+x_2+y_1+y_2=0$.

~~$x_1+x_2=0 \;,\; y_1+y_2=0$~~

$\alpha u_1 = (\alpha x_1, \alpha y_1, \alpha z_1),\; \alpha x_1+\alpha y_1 = 0$

$\therefore u_1+u_2 \in W_1 \;\; \forall u_1, u_2 \in W_1$

$\alpha u_1 \in W_1 \;\; \forall \alpha \in F,\; u_1 \in W_1$

$\therefore W_1$ is a subspace of V.

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Now, let $v_1, v_2 \in W_2$ such that

$v_1 = (a_1, b_1, c_1) \in \mathbb{R}^3$, $2a_1 + c_1 = 0$

$v_2 = (a_2, b_2, c_2) \in \mathbb{R}^3$, $2a_2 + c_2 = 0$

$v_1 + v_2 = \left(a_1 + a_2, b_1 + b_2, c_1 + c_2\right)$, $2a_1 + c_1 + 2a_2 + c_2 = 0$

$\Rightarrow 2(a_1 + a_2) + (c_1 + c_2) = 0$

$\therefore v_1 + v_2 \in W_2$

$\alpha v_1 = \alpha(a_1, b_1, c_1) = (\alpha a_1, \alpha b_1, \alpha c_1)$, $\alpha(2a_1 + c_1) = 0$

$\Rightarrow 2(\alpha a_1) + \alpha c_1 = 0$

$\therefore \alpha v_1 \in W_2$

$\therefore W_2$ is a subspace of $\mathbb{R}^3$.

Let $u, v \in W_1 \cap W_2$

$\Rightarrow u, v \in W_1 \wedge u, v \in W_2$

$\Rightarrow \alpha u + \beta v \in W_1 \wedge \alpha u + \beta v \in W_2$ [$\because W_1$ & W_2 are subspaces of V]

$\Rightarrow \alpha u + \beta v \in W_1 \cap W_2$

$\therefore W_1 \cap W_2$ is a subspace of V.

□

Q:- Find the dimension of a skew-symmetric matrix.

A:- Let us consider a $n \times n$ skew-symmetric matrix. In this matrix all the diagonal elements are zero and $a_{ij} = -a_{ji}$, $j \neq i$

$\therefore$ Dimension $= \dfrac{n^2 - n}{2}$.

P Kalika Maths

Co-ordinate of a vector :-

Let $B = \{v_1, v_2, \dots, v_n\}$ be an ordered basis of a v.s V, then any vector $v \in V$ can be written as,

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$$

The vector $(\alpha_1, \alpha_2, \dots, \alpha_n)$ is known as the co-ordinate vector of v relative to the ordered basis B and it is denoted by $[v]_B$.

$V = \mathbb{R}^4$

* $S = \{(1,1,1,0), (1,2,3,4), (1,1,1,-1), (0,0,0,1), (0,0,1,0), (1,-1,-2,-3)\}$

First Find the rank.

The rank must be 4.

And exclude the vector which gives zero rows.

* Casting out Algorithm

$$\begin{matrix} 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{matrix}$$

$$\begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \\ \end{array}$$
$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 & 0 & -1 \\ 1 & 3 & 1 & 0 & 1 & -2 \\ 0 & 4 & -1 & 1 & 0 & -3 \end{pmatrix} \sim \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & -2 \\ 0 & 2 & 0 & 0 & 1 & -3 \\ 0 & 4 & -1 & 1 & 0 & -3 \end{pmatrix} \sim \begin{matrix} R_3 \to R_3 - 2R_2 \\ R_4 \to R_4 - 4R_2 \end{matrix} \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 & 0 & 5 \end{pmatrix}$$

$$\sim \begin{pmatrix} \boxed{1} & 1 & 1 & 0 & 0 & 1 \\ 0 & \boxed{1} & 0 & 0 & 0 & -2 \\ 0 & 0 & \boxed{-1} & 1 & 0 & 5 \\ 0 & 0 & 0 & 0 & \boxed{1} & 1 \end{pmatrix}$$

$\therefore v_4$ is the linear combination of prev.(3) vectors.

$B = \{v_1, v_2, v, v_5\}$

P Kalika Maths

* Find a homogeneous system whose solution set W is spanned by $\{(1,-2,0,3),\ (1,-1,-1,4),\ (1,0,-2,5)\}$

A:-

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & x \\ -2 & -1 & 0 & y \\ 0 & -1 & -2 & z \\ 3 & 4 & 5 & t \end{array}\right) \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 + 2R_1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & x \\ 0 & 1 & 2 & y+2x \\ 0 & -1 & -2 & z \\ 0 & 1 & 2 & t-3x \end{array}\right)$$

$$\xrightarrow[R_4 \to R_4 - R_2]{R_3 \to R_3 + R_2} \underbrace{\left(\begin{array}{ccc|c} 1 & 1 & 1 & x \\ 0 & 1 & 2 & y+2x \\ 0 & 0 & 0 & z+y+2x \\ 0 & 0 & 0 & t-5x-y \end{array}\right)}_{M}$$

with the first three columns A.

$$\text{rank}(A) = \text{rank}(M)$$

$$2x+y+z=0$$
$$t-5x-y=0$$

Theorem:-

Let U, ~~be~~ ~~and W,~~ be ~~an~~ ~~two~~ subspace of a vectorspace V which is finite dimensional then $\dim(U) \leq \dim(V)$.

* $\boxed{\dim(U+W) = \dim(U) + \dim(W) - \dim(U \cap W)}$.

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* 48)(b) No. of free variables = dimension

03.02.2020

Theorem:-

If U and W are two ~~finite~~ subspaces of a finite vector space V, then

$$\dim(U+W) = \dim(U) + \dim(W) - \dim(U \cap W).$$

Proof:- Let $\dim(V) = n$

$\dim(U) = m$

$\dim(W) = p$

& $\dim(U \cap W) = r$.

$\therefore m \le n,\ p \le n,\ r \le n$.

Let $S_1 = \{v_1, v_2, \dots, v_r\}$ be the basis of $U \cap W$.

To get the basis of U, we can extend S_1 ~~i.e.~~ as

$S_2 = \{v_1, v_2, \dots, v_r, u_{r+1}, u_{r+2}, \dots, u_m\}$

Similarly we can get the basis of W.

$S_3 = \{v_1, v_2, \dots, v_r, w_{r+1}, w_{r+2}, \dots, w_p\}$

Now, we have to prove that the set,

$B = \{v_1, v_2, \dots, v_r, u_{r+1}, \dots, u_m, w_{r+1}, \dots, w_p\}$ is a basis for U+W.

$$\sum_{i=1}^{r} \alpha_i v_i + \sum_{i=r+1}^{m} \beta_i u_i + \sum_{i=r+1}^{p} \gamma_i w_i = 0$$

$$\Rightarrow \sum_{i=1}^{r} \alpha_i v_i + \sum_{i=r+1}^{m} \beta_i u_i = -\sum_{i=r+1}^{p} \gamma_i w_i \quad \text{———} (A)$$

We can see in eq(1) LHS is lying in U and RHS lying in W.

So we can conclude that this element is lie in $U \cap W$ from eq(A).

$$\Rightarrow -\sum_{i=r+1}^{p} \gamma_i w_i = \sum_{i=1}^{r} \delta_i v_i$$

$$\Rightarrow \cancel{\sum_{i=1}^{r} \alpha_i v_i + \sum_{i=r+1}^{m} \beta_i u_i = \sum_{i=1}^{r} \delta_i v_i}$$

$$\Rightarrow \sum_{i=1}^{r} \delta_i v_i + \sum_{i=r+1}^{p} \gamma_i w_i = 0$$

Since S_3 is the basis of W,

$\Rightarrow \gamma_i = 0 , \; i = r+1, \dots, p$

$\delta_i = 0 , \; i = 1, \dots, r$

Now from eq(A).

$$\sum_{i=1}^{r} \alpha_i v_i + \sum_{i=r+1}^{m} \beta_i u_i = 0 \quad [\because \gamma_i = 0 , \; i = r+1, \dots, p]$$

$\Rightarrow \alpha_i = 0 , \; i = 1, \dots, r$

$\beta_i = 0 , \; i = r+1, \dots, m$

$\because$ S_2 is a basis of U.

$\therefore$ Set B is LI.

HW

Now it remains to show that $[B] = U + W$

P Kalika Maths

(i) $[B] \subseteq U+W$

Let $x \in [B]$

$$\Rightarrow x = \alpha_1 v_1 + \dots + \alpha_r v_r + \beta_{r+1} u_{r+1} + \dots + \beta_m u_m + \gamma_{r+1} w_{r+1} + \dots + \gamma_p w_p$$

$$\in U+W$$

$\therefore [B] \subseteq U+W$. ——— (1)

(ii) $U+W \subseteq [B]$

Let $y \in U+W$

$$\Rightarrow y = a_1 v_1 + \dots + a_r v_r + b_{r+1} u_{r+1} + \dots + b_m u_m + \dots + c_1 v_1 + \dots + c_r v_r + d_{r+1} w_{r+1} + \dots + d_p w_p$$

$$= (a_1 + c_1) v_1 + \dots + (a_r + c_r) v_r + b_{r+1} u_{r+1} + \dots + b_m u_m + d_{r+1} w_{r+1} + \dots + d_p w_p$$

$$= \delta_1 v_1 + \dots + \delta_r v_r + b_{r+1} u_{r+1} + \dots + b_m u_m + d_{r+1} w_{r+1} + \dots + d_p w_p$$

$$\in [B]$$

$\therefore U+W \subseteq [B]$ ——— (2)

From (1) and (2), we get $U+W = [B]$.

P Kalika Maths

LINEAR TRANSFORMATION

↳ Let U and V are two vector spaces over the same field. Then the mapping $T : U \longrightarrow V$ is a said to be a linear map or linear transformation/linear operator if

$$T(\alpha u_1 + \beta u_2) = \alpha T(u_1) + \beta T(u_2) \quad \forall \; u_1, u_2 \in U$$

& α, β are scalars.

Ex:-

$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$T(x, y) = (x + y, y)$$

Let $u_1 = (x_1, y_1)$

$u_2 = (x_2, y_2)$

$$\Rightarrow T(u_1) = (x_1 + y_1, y_1)$$

$$T(u_2) = (x_2 + y_2, y_2)$$

$$T(\alpha u_1 + \beta u_2) = T(\alpha(x_1, y_1) + \beta(x_2, y_2))$$

$$= T(\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2)$$

$$= (\alpha x_1 + \beta x_2 + \alpha y_1 + \beta y_2, \alpha y_1 + \beta y_2)$$

$$\alpha T(u_1) + \beta T(u_2) = \alpha(x_1 + y_1, y_1) + \beta(x_2 + y_2, y_2)$$

$$= (\alpha x_1 + \alpha y_1, \alpha y_1) + (\beta x_2 + \beta y_2, \beta y_2)$$

$$= (\alpha x_1 + \alpha y_1 + \beta x_2 + \beta y_2, \alpha y_1 + \beta y_2)$$

$$\therefore T(\alpha u_1 + \beta u_2) = \alpha T(u_1) + \beta T(u_2)$$

$\therefore$ T is linear.

P Kalika Maths

Ex:- $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$

① $T(x,y,z) = (x+z,\ z+y)$ (Linear)

② $T(x,y,z) = (x+1,\ y+z)$ (Not linear)

③ $T(x,y,z) = (xy,\ z)$ (Non-linear)

④ $T(x,y,z) = (x+y+z,\ 1)$ (Non-linear)

Identify which is linear?

2) $u_1 = (x_1, y_1, z_1)$

$u_2 = (x_2, y_2, z_2)$

$$T(\alpha u_1 + \beta u_2) = T\left(\alpha(x_1,y_1,z_1) + \beta(x_2,y_2,z_2)\right)$$

$$= T(\alpha x_1 + \beta x_2,\ \alpha y_1 + \beta y_2,\ \alpha z_1 + \beta z_2)$$

$$= (\alpha x_1 + \beta x_2 + 1,\ \alpha y_1 + \beta y_2 + \alpha z_1 + \beta z_2)$$

$$\alpha T(u_1) + \beta \cancel{f} T(u_2) = \alpha(x_1+1,\ y_1+z_1) + \beta(x_2+1,\ y_2+z_2)$$

$$= (\alpha x_1 + \alpha,\ \alpha y_1 + \alpha z_1) + (\beta x_2 + \beta,\ \beta y_2 + \beta z_2$$

$$= (\alpha x_1 + \beta x_2 + \alpha + \beta,\ \alpha y_1 + \alpha z_1 + \beta y_2 + \beta z_2$$

$\therefore T(\alpha u_1 + \beta u_2) \neq \alpha T(u_1) + \beta T(u_2)$

P Kalika Maths

4) Let $u_1 = (x_1, y_1, z_1)$

$u_2 = (x_2, y_2, z_2)$

$$T(\alpha u_1) = T(\alpha x_1, \alpha y_1, \alpha z_1) = (\alpha x_1 + \alpha y_1 + \alpha z_1, 1)$$

$$\neq \alpha T(u_1) = \alpha(x_1 + y_1 + z_1, 1) = (\alpha x_1 + \alpha y_1 + \alpha z_1, \alpha)$$

Results:-

Let $T: U \to V$ be a linear map, then

(i) $T(0_U) = 0_V$ (converse need not true)

(ii) $T(-u) = -T(u)$

(iii) $T(\alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n) = \alpha_1 T(u_1) + \dots + \alpha_n T(u_n)$.

(i) Let $u \in U$

$\Rightarrow -u \in U$

$u + (-u) = 0$

$$T(0) = T(u + (-u)) = T(u) + T(-u)$$
$$= T(u) - T(u)$$
$$= 0$$

(ii) $T(-u) = T\{(-1)\cdot u\} = (-1)T(u)$ [$\because$ T is linear]

$= -T(u)$.

(iii) $T(\alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n) = T\{\alpha_1 u_1 + (\alpha_2 u_2 + \dots + \alpha_n u_n)\}$

$= T(\alpha_1 u_1) + T(\alpha_2 u_2 + \dots + \alpha_n u_n)$

$= \alpha_1 T(u_1) + T\{(\alpha_2 u_2) + (\alpha_3 u_3 + \dots + \alpha_n u_n)\}$

$= \alpha_1 T(u_1) + T(\alpha_2 u_2) + T(\alpha_3 u_3 + \dots + \alpha_n u_n)$

$= \alpha_1 T(u_1) + \alpha_2 T(u_2) + T(\alpha_3 u_3 + \dots + \alpha_n u_n)$

$\vdots$

continuing in this way we get

$= \alpha_1 T(u_1) + \alpha_2 T(u_2) + \dots + \alpha_n T(u_n)$

P Kalika Maths

04.02.2020

$= V = P_3$

$U = \{ p \in P_3 \mid p(2) = 0 \}$

$W = \{ p \in P_3 \mid p'(1) = 0 \}$

Find, $U, W, U \cap W, U + W$.

HW

Q:- U and W are two distinct subspaces of a V.S V and $\dim(V) = 7$, $\dim(U) = 5$, $\dim(W) = 4$. Then find the all possible dimension of $U \cap W$.

Result:-

A linear transformation T is completely determined by its values on the elements of a basis.

If $B = \{u_1, u_2, \dots, u_n\}$ is a basis of U and $v_1, v_2, \dots, v_n$ vectors (not necessarily distinct) in V, then $\exists$ a unique linear transformation, $T : U \to V$ st

$T(u_i) = v_i \quad , i = 1, 2, \dots, n$ ——— (A)

* Let $u \in U$, then u can be uniquely expressed as

$u = \alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n$

$T(u) = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$

Now, we will prove that (i) T is linear

(ii) T satisfies the eq(A)

(iii) T is unique.

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(i) Let $u' = \alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n$

$u'' = \beta_1 u_1 + \beta_2 u_2 + \dots + \beta_n u_n$

$$T(\alpha u' + \beta u'') = T\left(\alpha \sum_{i=1}^{n} \alpha_i u_i + \beta \sum_{i=1}^{n} \beta_i u_i\right)$$

$$= T\left(\sum_{i=1}^{n} (\alpha\alpha_i + \beta\beta_i) u_i\right)$$

$$= \sum_{i=1}^{n} (\alpha\alpha_i + \beta\beta_i) v_i$$

$$= \sum_{i=1}^{n} \alpha\alpha_i v_i + \sum_{i=1}^{n} \beta\beta_i v_i$$

$$= \alpha \sum_{i=1}^{n} \alpha_i v_i + \beta \sum_{i=1}^{n} \beta_i v_i$$

$$= \alpha T(u') + \beta T(u'').$$

$\therefore$ T is linear.

* $u' + u'' = (\alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n) + (\beta_1 u_1 + \beta_2 u_2 + \dots + \beta_n u_n)$

$= (\alpha_1 + \beta_1) u_1 + (\alpha_2 + \beta_2) u_2 + \dots + (\alpha_n + \beta_n) u_n$

$T(u' + u'') = (\alpha_1 + \beta_1) v_1 + (\alpha_2 + \beta_2) v_2 + \dots + (\alpha_n + \beta_n) v_n$

$= (\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n) + (\beta_1 v_1 + \dots + \beta_n v_n)$

$= T(u') + T(u'')$

$\alpha u' = \alpha\alpha_1 u_1 + \alpha\alpha_2 u_2 + \dots + \alpha\alpha_n u_n$

$T(\alpha u') = \alpha\alpha_1 v_1 + \alpha\alpha_2 v_2 + \dots + \alpha\alpha_n v_n$

$= \alpha(\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n)$

$= \alpha T(u')$.

$\therefore$ T is linear.

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(ii) $u = \alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n$

Let $\alpha_1 = 1, \ \alpha_2 = \dots = \alpha_n = 0$

$u = u_1$

$T(u_1) = v_1$

$\vdots$

(iii) Let T is not unique.

Let $\exists$ another map $F \neq T$ such that

$F(u) = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$

Now, $F(u) - T(u) = (\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n) - (\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n)$

$= 0$

$\Rightarrow F(u) = T(u)$

P Kalika Maths

Assign:-

Can we defined a linear transformation in different field.

Q:- $T : \mathbb{R}^2(\mathbb{R}) \longrightarrow \mathbb{R}^3(\mathbb{R})$

$T(1,1) = (1,2,3)$

$T(1,-1) = (1,0,0)$, If possible find T.

$(x,y) \in \mathbb{R}^2$

$(x,y) = \alpha(1,1) + \beta(1,-1)$

$\alpha + \beta = x$

$\alpha - \beta = y$

$\Rightarrow \alpha = \frac{x+y}{2}$, $\beta = \frac{x-y}{2}$

$$T(x,y) = T\left(\frac{x+y}{2}(1,1) + \frac{x-y}{2}(1,-1)\right)$$

$$= \frac{x+y}{2} T(1,1) + \frac{x-y}{2} T(1,-1)$$

$$= \frac{x+y}{2}(1,2,3) + \frac{x-y}{2}(1,0,0)$$

$$= \left(x,\ x+y,\ \frac{3}{2}(x+y)\right)$$

* If $T(1,2) = (1,0,0)$, then T doesn't exist

* $T : P_3 \longrightarrow P_3$

$T(1+x) = 1+x$

$T(2+x) = x+3x^2$

$T(x^2) = 0$

A:- $\{1+x,\ 2+x,\ x^2\}$ is LI.

$$a_0 + a_1x + a_2x^2 + a_3x^3 = \alpha(1+x) + \beta(2+x) + \gamma x^2$$

$$= (\alpha+2\beta) + (\alpha+\beta)x + \gamma x^2$$

$\Rightarrow$ $a_3 = 0$

$a_2 = \gamma$

$a_1 = \alpha + \beta$

$a_0 = \alpha + 2\beta$

$\Rightarrow$ $\beta = a_0 - a_1$

$\Rightarrow$ $\alpha = a_1 - a_0 + a_1 = 2a_1 - a_0$

$$\therefore \quad a_0 + a_1x + a_2x^2 + a_3x^3 = (2a_1 - a_0)(1+x) + (a_0 - a_1)(2+x) + a_2x^2$$

$$T(a_0 + a_1x + a_2x^2 + a_3x^3) = (2a_1 - a_0)(1+x) + (a_0 - a_1)(x+3x^2) + a_2$$

Consider $\{1+x, 2+x, x^2, x^3\}$

The set is LI.

Assume $T(x^3) = 0$

$$\begin{vmatrix} 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1(1) - 1\begin{vmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 1 - 2 = -1 \neq 0$$

$\therefore$ The set is LI.

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Now, $a_0 + a_1x + a_2x^2 + a_3x^3 = \alpha(1+x) + \beta(2+x) + \gamma x^2 + \delta x^3$

$$= (\alpha + 2\beta) + (\alpha+\beta)x + \gamma x^2 + \delta x^3$$

$a_3 = \delta$

$a_2 = \gamma$

$a_1 = \alpha + \beta$

$a_0 = \alpha + 2\beta$

$\Rightarrow \beta = a_0 - a_1$

$\alpha = a_1 - a_0 + a_1 = 2a_1 - a_0$

$\therefore \ a_0 + a_1x + a_2x^2 + a_3x^3 = (2a_1 - a_0)(1+x) + (a_0 - a_1)(2+x) + a_2x^2 + a_3x^3$

$$T(a_0 + a_1x + a_2x^2 + a_3x^3) = (2a_1 - a_0)(1+x) + (a_0 - a_1)(x + 3x^2) + 0 + 0$$

$$= 2a_1 - a_0 + (2a_1 - a_0 + a_0 - a_1)x + (a_0 - a_1)3x^2$$

$$= (2a_1 - a_0) + a_1 x + 3(a_0 - a_1)x^2$$

P Kalika Maths

Q:- $V = P_3$

$U = \{ P \in P_3 \mid P(2) = 0 \}$

$W = \{ P \in P_3 \mid P'(1) = 0 \}$

Find, U, W, U∩W, U+W.

A:- $V = \{ P \mid \text{degree}(P) \le 3 \}$

$= \{ a_0 + a_1x + a_2x^2 + a_3x^3 \mid a_0, a_1, a_2, a_3 \in \mathbb{R} \}$

$U = \{ P \in P_3 \mid P(2) = 0 \}$

$= \{ a_0 + a_1x + a_2x^2 + a_3x^3 \mid a_0 + 2a_1 + 4a_2 + 8a_3 = 0 \}$

$= \{ (-2a_1 - 4a_2 - 8a_3) + a_1x + a_2x^2 + a_3x^3 \mid a_1, a_2, a_3 \in \mathbb{R} \}$

$= \{ a_1(x-2) + a_2(x^2-4) + a_3(x^3-8) \mid a_1, a_2, a_3 \in \mathbb{R} \}$

$\dim(U) = 3$

$W = \{ P \in P_3 \mid P'(1) = 0 \}$

$= \{ a_0 + a_1x + a_2x^2 + a_3x^3 \mid a_1 + 2a_2 + 3a_3 = 0 \}$

$= \{ a_0 + (-2a_2 - 3a_3)x + a_2x^2 + a_3x^3 \mid a_0, a_2, a_3 \in \mathbb{R} \}$

$= \{ a_0 + a_2(x^2 - 2x) + a_3(x^3 - 3x) \mid a_0, a_2, a_3 \in \mathbb{R} \}$

$\dim(W) = 3$

$U \cap W = \{ P \in P_3 \mid P(2) = 0,\ P'(1) = 0 \}$

$= \{ a_0 + a_1x + a_2x^2 + a_3x^3 \mid a_0 + 2a_1 + 4a_2 + 8a_3 = 0,$
$a_1 + 2a_2 + 3a_3 = 0,\ a_0, a_1, a_2, a_3 \in \mathbb{R} \}$

Now, $a_1 = -2a_2 - 3a_3$

$a_0 + 2(-2a_2 - 3a_3) + 4a_2 + 8a_3 = 0$

$\Rightarrow a_0 - \cancel{4a_2} - 6a_3 + \cancel{4a_2} + 8a_3 = 0$

$\Rightarrow a_0 = -2a_3$

$\therefore\ U \cap W = \{-2a_3 + (-2a_2 - 3a_3)x + a_2x^2 + a_3x^3 \mid a_2, a_3 \in \mathbb{R}\}$

$= \{a_2(x^2 - 2\cancel{a_2}x) + a_3(x^3 - 3x - 2) \mid a_2, a_3 \in \mathbb{R}\}$

$\dim(U \cap W) = 2$.

Now, $U + W = \{a(x-2) + b(x^2-4) + c(x^3-8) + d + e(x^2-2x) + f(x^3-3x) \mid a, b, c, d, e, f \in \mathbb{R}\}$

$\dim(U+W) = \dim(U) + \dim(W) - \dim(U \cap W)$

$= 3 + 3 - 2$

$= 4$

$\therefore$ The set $\{(x-2), (x^2-4), (x^3-8), 1, (x^2-2x), (x^3-3x)\}$ is LD.

Now
$$\begin{array}{c}1\\2\\3\\4\\5\\6\end{array}\begin{pmatrix}-2 & 1 & 0 & 0\\ -4 & 0 & 1 & 0\\ -8 & 0 & 0 & 1\\ 1 & 0 & 0 & 0\\ 0 & -2 & 1 & 0\\ 0 & -3 & 0 & 1\end{pmatrix} \xrightarrow{R_4 \leftrightarrow R_1} \begin{array}{c}4\\2\\3\\1\\5\\6\end{array}\begin{pmatrix}1 & 0 & 0 & 0\\ -4 & 0 & 1 & 0\\ -8 & 0 & 0 & 1\\ -2 & 1 & 0 & 0\\ 0 & -2 & 1 & 0\\ 0 & -3 & 0 & 1\end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_4}$$

$$\begin{array}{c}4\\1\\3\\2\\5\\6\end{array}\begin{pmatrix}1 & 0 & 0 & 0\\ -2 & 1 & 0 & 0\\ -8 & 0 & 0 & 1\\ -4 & 0 & 1 & 0\\ 0 & -2 & 1 & 0\\ 0 & -3 & 0 & 1\end{pmatrix} \xrightarrow[\substack{R_3 \to R_3 + 8R_1\\ R_4 \to R_4 + 4R_1}]{R_2 \to R_2 + 2R_1} \begin{array}{c}4\\1\\3\\2\\5\\6\end{array}\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & -2 & 1 & 0\\ 0 & -3 & 0 & 1\end{pmatrix}$$

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$$\xrightarrow[\sim]{R_3 \leftrightarrow R_4} \begin{array}{c} 4 \\ 1 \\ 2 \\ 3 \\ 5 \\ 6 \end{array} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -2 & 1 & 0 \\ 0 & -3 & 0 & 1 \end{pmatrix} \underset{\substack{R_5 \to R_5 + 2R_2 \\ R_6 \to R_6 + 3R_2}}{\sim} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\underset{R_5 \to R_5 - R_3}{\sim} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \underset{R_6 \to R_6 - R_4}{\sim} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\therefore$ The set $\{(x-2), (x^2-4), (x^3-8), 1\}$ is LI.

$\therefore$ $U+W = \{\alpha(x-2) + \beta(x^2-4) + \gamma(x^3-8) + \delta \mid \alpha, \beta, \gamma, \delta \in \mathbb{R}\}$

2) Given, U and W are two distinct subspaces of v.s V

$\dim(V) = 7$

$\dim(U) = 5$

$\dim(W) = 4$

$\therefore \dim(U \cap W) \leq 4$

$\dim(U+W) \leq \dim(V)$

$\Rightarrow \dim(U+W) \leq 7$

$\dim(U+W) = \dim(U) + \dim(W) - \dim(U \cap W)$

$\Rightarrow \dim(U \cap W) = \dim(U) + \dim(W) - \dim(U+W)$

$\geq 5 + 4 - 7$

$= 2$

$\therefore\ 2 \le \dim(U\cap W) \le 4$.

05.02.2020

Nullity of T :-

Let $T: U \to V$ be a linear map over the same field, then null space (Kernel) of T is defined as,

$$N(T) = \{u \in U : T(u) = 0\} = Ker(T)$$

$$R(T) = \{v \in V : T(u) = v\}$$

* N(T) is a subspace of U.

Let $u_1, u_2 \in N(T)$

$\Rightarrow u_1, u_2 \in U$ st $T(u_1) = 0$, $T(u_2) = 0$

$\Rightarrow \alpha u_1 + \beta u_2 \in U$

$$T(\alpha u_1 + \beta u_2) = \alpha T(u_1) + \beta T(u_2) \quad [\because T \text{ is linear}]$$
$$= \alpha\cdot 0 + \beta\cdot 0$$
$$= 0$$

$\therefore\ \alpha u_1 + \beta u_2 \in N(T)$.

$\therefore$ N(T) is a subspace of U.

* R(T) is a subspace of V.

Proof:- Let $v_1, v_2 \in R(T)$

$\Rightarrow v_1, v_2 \in V$ st $T(u_1) = v_1$, $T(u_2) = v_2$

$\alpha v_1 + \beta v_2 \in V \quad [\because V \text{ is a v.s}]$

~~Data~~ Now, $u_1, u_2 \in U$

$\Rightarrow \alpha u_1 + \beta u_2 \in U$

$$T(\alpha u_1 + \beta u_2) = \alpha T(u_1) + \beta T(u_2) \quad [\because T \text{ is linear}]$$
$$= \alpha v_1 + \beta v_2$$

P Kalika Maths

$\{\because \alpha v_1 + \beta v_2 \in R(T)$

$$\alpha v_1 + \beta v_2 = \alpha T(u_1) + \beta T(u_2)$$
$$= T(\alpha u_1 + \beta u_2)$$

$\therefore \alpha v_1 + \beta v_2 \in R(T)$

***Q:-1** $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$$T(x,y,z) = (0,y,z)$$

Let $u \in \mathbb{R}^3$ st $T(u) = 0$, $u = (x,y,z)$

$$T(x,y,z) = (0,0,0)$$
$\Rightarrow (0,y,z) = (0,0,0) \Rightarrow y = z = 0$, x is arbitary

Let $x = \alpha$

$\therefore \mathrm{Ker}(T) = \{(x,0,0) \in \mathbb{R}^3\}$

$= \{\alpha(1,0,0), \alpha \in \mathbb{R}\}$

$R(T) = \{(0,y,z) \in \mathbb{R}^3 \mid T(x,y,z) = (0,y,z)\}$

$=$ YZ-plane.

Q:-2 $T(x,y,z) = (x+y+z,\ x+y,\ x+y)$

Let $T(x,y,z) = (0,0,0)$

$$\Rightarrow \begin{aligned} x+y+z &= 0 \\ x+y &= 0 \\ x+y &= 0 \end{aligned}$$

$$\Rightarrow \begin{aligned} x+y+z &= 0 \\ x+y &= 0 \end{aligned}$$

$$\Rightarrow \begin{aligned} x+y+z &= 0 \\ x &= -y \end{aligned}$$

P Kalika Maths

$\therefore N(T) = \{\alpha(-1,1,0)\}$

$T(1,0,0) = (1,1,1)$

$T(0,1,0) = (1,1,1)$

$T(0,0,1) = (1,0,0)$

$R(T) = [\{(1,1,1),(1,0,0)\}]$

Q:-(2) $T(1,0,0) = (0,0,0)$

$T(0,1,0) = (0,1,0)$

$T(0,0,1) = (0,0,1)$

$R(T) = [\{(0,1,0),(0,0,1)\}]$

= YZ-plane.

* Let $T: U \to V$ be a linear map, then

(i) $R(T)$ is a subspace of V

(ii) $N(T)$ is a subspace of U.

(iii) T is one-one iff $N(T)$ is a zero subspace of U.

(iv) If $[u_1, u_2, \dots, u_n] = U$, then

$$R(T) = [T(u_1), T(u_2), \dots, T(u_n)]$$

(v) If U is finite dimensional, then $\dim R(T) \leq \dim U$.

(iii) Let T is one-one.

$$\left.\begin{array}{l} \text{i.e. } T(u_1) = T(u_2) \\ \Rightarrow u_1 = u_2 \\ \Rightarrow u_1 - u_2 = 0 \\ \Rightarrow T(u_1 - u_2) = 0 \end{array}\right\} \times$$

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Let $u \in N(T)$

$\Rightarrow u \in U : T(u) = 0_V$

$\Rightarrow T(u) = T(0_v)$

$\Rightarrow u = 0$

$\Rightarrow N(T) = \{0\}$

(iii) T is one-one

Let $u(\neq 0) \in N(T)$

$\Rightarrow u \in U : T(u) = 0_V$

$T(0_v) = 0_V$

$\therefore T(u) = T(0_u)$

$\Rightarrow u = 0_u$

$\therefore N(T) = \{0\}$

S.P:- Let $N(T) = \{0\}$

Claim:- T is one-one

Let $u_1, u_2 \in U$ st

$T(u_1) = T(u_2)$

$\Rightarrow T(u_1 - u_2) = 0$

$\Rightarrow u_1 - u_2 \in N(T)$

$\Rightarrow u_1 - u_2 = 0$

$\Rightarrow u_1 = u_2$

$\therefore$ T is one-one.

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Result:- Let $T: U \to V$ be a linear map, then

(i) If T is one-one and $u_1, u_2, \dots, u_n$ are LI vectors of U, then $T(u_1), T(u_2), \dots, T(u_n)$ are LI.

(ii) If $v_1, v_2, \dots, v_n$ are LI vectors of $R(T)$ and $u_1, u_2, \dots, u_n$ are vectors of U st $T(u_i) = v_i$, $i = 1, 2, \dots, n$ then $u_1, u_2, \dots, u_n$ are LI.

* $\dim(\text{Ker } T) = \text{Nullity}(T)$

$\dim(R(T)) = \text{rank}(T)$

Rank-Nullity Theorem :-

Let $T: U \to V$ be a linear map and U is finite dimensional v.s, then $\dim(U) = \text{rank}(T) + \text{nullity}(T)$

$$\dim(U) = r(T) + n(T)$$

Proof:-

Let $\dim(N(T)) = n(T) = n$

$\dim(U) = p$

Suppose $B = \{u_1, u_2, \dots, u_n\}$ is a basis of $N(T)$

i.e. $T(u_i) = 0$, $i = 1, 2, \dots, n$

To get the basis of U, we can extend the set B

$\{u_1, u_2, \dots, u_n, u_{n+1}, u_{n+2}, \dots, u_p\}$ is the basis of U.

Let $A = \{T(u_{n+1}), T(u_{n+2}), \dots, T(u_p)\}$ is the basis of $R(T)$

i.e. $R(T) = [A]$

$\alpha_{n+1} T(u_{n+1}) + \alpha_{n+2} T(u_{n+2}) + \dots + \alpha_p T(u_p) = 0$

$\Rightarrow T(\alpha_{n+1} u_{n+1} + \alpha_{n+2} u_{n+2} + \dots + \alpha_p u_p) = 0$ [$\because$ T is linear]

$\Rightarrow \alpha_{n+1} u_{n+1} + \alpha_{n+2} u_{n+2} + \dots + \alpha_p u_p \in N(T)$

$\Rightarrow \alpha_{n+1} u_{n+1} + \alpha_{n+2} u_{n+2} + \dots + \alpha_p u_p = \alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n$

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$\Rightarrow \alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n - \alpha_{n+1} u_{n+1} - \dots - \alpha_p u_p = 0$

$\Rightarrow \alpha_i = 0 \ , \ i = 1, \dots, p \qquad [\because \{u_1, u_2, \dots, u_p\}$ is the basis of $U]$

$\Rightarrow$ A is LI.

Now, it remains to show that $R(T) = [A]$.

$$R(T) = [\{T(u_1), T(u_2), \dots, T(u_n), \dots, T(u_p)\}]$$

$$T(u_i) = 0 \ , \ i = 1, \dots, n$$

$$R(T) = [\{T(u_{n+1}), \dots, T(u_p)\}]$$
$$= [A].$$

*Q:- $T : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$

$T(e_1) = (1,1,1)$

$T(e_2) = (1,-1,1)$

$T(e_3) = (1,0,0)$

$T(e_4) = (1,0,1)$

HW Find $T(x,y,z,t) = ?$

$R(T) = [\{(1,1,1), (1,-1,1), (1,0,0), (1,0,1)\}]$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \underset{\substack{R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \\ R_4 \to R_4 - R_1}}{\sim} \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & 0 \end{pmatrix} \overset{\sim}{R_4 \to 2R_4 - R_2}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

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$r(T) = 3$

$\dim(U) = r(T) + n(T)$

$\Rightarrow 4 = 3 + n(T)$

$\Rightarrow n(T) = 1$

Q:- $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$T(e_1) = e_1 + e_2 \cancel{e_3} = (1,1,0)$

$T(e_2) = e_2 + e_3 = (0,1,1)$

$T(e_3) = e_1 + e_2 + e_3 = (1,1,1)$

$r(T) = 3$

$n(T) = 3 - 3 = 0$

$\therefore N(T) = \{0\}$

$\therefore T$ is 1-1

Check T is invertible or not. If yes the find T^{-1}.

$r(T) = \dim(\mathbb{R}^3)$

$\therefore T$ is onto.

Let $(x,y,z) \in \mathbb{R}^3$

$(x,y,z) = \alpha e_1 + \beta e_2 + \gamma e_3$

$\Rightarrow \alpha = x, \ \beta = y, \ \gamma = z$

$$T(x,y,z) = T(xe_1 + ye_2 + ze_3)$$

$$= xT(e_1) + yT(e_2) + zT(e_3)$$

$$= x(e_1+e_2) + y(e_2+e_3) + z(e_1+e_2+e_3)$$

$$= ((x+z)e_1, (x+y+z)e_2, (y+z)e_3)$$

$$= (x+z, x+y+z, y+z)$$

Put $T(x,y,z) = (0,0,0)$

$x + z = 0$

$x + y + z = 0$

$y + z = 0$

$\Rightarrow x = 0, \ y = 0, \ z = 0$

$\therefore N(T) = \{0\}$

$\therefore T$ is one-one.

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$T^{-1}(x_1, x_2, x_3) = (y_1, y_2, y_3)$

$\Rightarrow T(y_1, y_2, y_3) = (x_1, x_2, x_3)$

$\Rightarrow T(y_1e_1 + y_2e_2 + y_3e_3) = (x_1, x_2, x_3)$

$\Rightarrow y_1T(e_1) + y_2T(e_2) + y_3T(e_3) = (x_1, x_2, x_3)$

$\Rightarrow y_1(e_1+e_2) + y_2(e_2+e_3) + y_3(e_1+e_2+e_3) = (x_1, x_2, x_3)$

$\Rightarrow (y_1+y_3 \,,\, y_1+y_2+y_3 \,,\, y_2+y_3) = (x_1, x_2, x_3)$

$\Rightarrow$
$x_1 = y_1+y_3$
$x_2 = y_1+y_2+y_3$
$x_3 = y_2+y_3$

$\Rightarrow$
$y_2 = x_2 - x_1$
$y_1 = x_2 - x_3$
$y_3 = x_1 - x_2 + x_3$

$\therefore T^{-1}(x_1, x_2, x_3) = (x_2 - x_3 \,,\, x_2 - x_1 \,,\, x_1 - x_2 + x_3)$

HW
Thm - 4.5.1 (Page - 127)

HW
$T : U \longrightarrow V$

$L(U,V)$ = Set of all Linear transformation is a V.S.
(Prove it)

Ex:- $T_1 : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$

$T_2 : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$

$$T_1((x,y,z)) = (x+y, y+z)$$

$$T_2((x,y,z)) = (x, y+z)$$

$$\begin{aligned}(T_1+T_2)(x,y,z) &= T_1(x,y,z) + T_2(x,y,z)\\ &= (x+y, y+z) + (x, y+z)\\ &= (2x+y, 2y+2z)\end{aligned}$$

* $T : U \to V$

$S : V \longrightarrow W$

$$\begin{aligned}S \circ T &= ST(u)\\ &= S(v)\\ &= \omega\end{aligned}$$

Nilpotent operator:-

A linear transformation T on a v.s V is said to be nilpotent if $T^n = 0$, where n is a least positive integer. where n is known as the nilpotency index of T.

Ex:- Differential operator

$$D : P_n \to P_n$$

$$D^{n+1} = 0$$

2) $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$$T(x,y,z) = (0,x,y)$$

$$T^2(x,y,z) = T(0,x,y) = (0,0,x)$$

$$T^3(x,y,z) = T(0,0,x) = (0,0,0).$$

∴ nilpotency index of $T = 3$

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Q.1 $T : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$

$$T(e_1) = (1,1,1)$$
$$T(e_2) = (1,-1,1)$$
$$T(e_3) = (1,0,0)$$
$$T(e_4) = (1,0,1)$$

Find $T(x,y,z,t)$?

A:- $\{e_1, e_2, e_3, e_4\}$ is LI.

Let $(x,y,z,t) = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 + \alpha_4 e_4$

$$= (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$$

$$\Rightarrow \alpha_1 = x$$
$$\alpha_2 = y$$
$$\alpha_3 = z$$
$$\alpha_4 = t$$

$\therefore$ $(x,y,z,t) = xe_1 + ye_2 + ze_3 + te_4$

$$T(x,y,z,t) = xT(e_1) + yT(e_2) + zT(e_3) + tT(e_4)$$
$$= x(1,1,1) + y(1,-1,1) + z(1,0,0) + t(1,0,1)$$
$$= (x+y+z+t,\ x-y+t,\ x+y+t)$$

Thm - 4.5.1

Let $T : U \to V$ be a linear map and $\dim(U) = \dim(V) = n$. Then the following statements are equivalent.

(a) T is non-singular (an isomorphism)

(b) T is one-one

(c) T transforms Linearly independent subsets of U into LI subsets of V.

(d) T transforms every basis for U into a basis for V.

(e) T is onto.

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(g) $n(T) = 0$

(h) T^{-1} exists.

Proof:- Given that $T : U \to V$ is a linear map
$\dim(U) = \dim(V) = p$.

(a) ⇒ (b)

Let T is non-singular (an isomorphism)
⇒ T is one-one and onto [By defⁿ of non-singular]
⇒ T is one-one.

(b) ⇒ (c)

Let T is one-one.
Let $u_1, u_2, \dots, u_p$ are LI vectors of U.
⇒ $T(u_1), T(u_2), \dots, T(u_p)$ are LI vectors of V
[By prev. theorem]
∴ T transforms LI subsets of U into LI subsets of V.

(c) ⇒ (d)

Let T transforms LI subsets of U into LI subsets of V.
Now, let $\{u_1, u_2, \dots, u_p\}$ is a basis for U.
Then $\{T(u_1), T(u_2), \dots, T(u_p)\}$ is LI.
But $\dim(V) = \dim(U) = p$
So $\{T(u_1), T(u_2), \dots, T(u_p)\}$ is a basis for V.

(d) ⇒ (e)

Let T transforms every basis for U into a basis for V
Let $\{u_1, u_2, \dots, u_p\}$ is a basis for U. Then
$\{T(u_1), T(u_2), \dots, T(u_p)\}$ is a basis for V.

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$\Rightarrow R(T) = V$

Hence T is onto.

(e) ⇒ (f)

Let T is onto

$\Rightarrow R(T) = V$

$\Rightarrow \dim(R(T)) = \dim(V)$

$\Rightarrow r(T) = p$.

(f) ⇒ (g)

By the rank-nullity theorem,

$\dim(U) = r(T) + n(T)$

$\Rightarrow p = p + n(T)$

$\Rightarrow n(T) = 0$.

(g) ⇒ (h)

Let $n(T) = 0$

$\Rightarrow N(T) = \{0_U\}$

$\Rightarrow$ T is one-one

Again, $\dim(U) = \dim(V)$

So T is onto.

Hence T^{-1} exists.

(h) ⇒ (a)

Let T^{-1} exists.

$\Rightarrow$ T must be one-one and onto.

$\therefore$ T is non-singular.

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Theorem:-

Let $T : U \to V$, The set $L(U,V)$ of all linear transformations from U to V together with the operations of addition and scalar multiplication ~~do~~ is a vectorspace.

Proof:-

Given $L(U,V)$: The set of all linear transformations from U to V.

(i) Let $T_1, T_2 \in L(U,V)$ such that

$T_1(u) = V_1$

$T_2(u) = V_2$

$$(T_1+T_2)(u) = T_1(u) + T_2(u)$$
$$= V_1 + V_2 \in V$$

$\therefore$ $T_1 + T_2$ is a Linear transformation.

(ii) Let $T_1, T_2, T_3 \in L(U,V)$ such that

$T_1(u) = V_1$

$T_2(u) = V_2$

$T_3(u) = V_3$

$$\{(T_1+T_2)+T_3\}(u) = (T_1+T_2)(u) + T_3(u)$$
$$= T_1(u) + T_2(u) + T_3(u)$$
$$= T_1(u) + (T_2+T_3)(u)$$
$$= T_1 + (T_2+T_3)$$

(iii) $\exists$ a zero map $0 : U \to V$ such that

$$(T_1 + 0)(u) = T_1(u) + 0(u)$$
$$= V_1 + 0$$
$$= V_1$$
$$= T_1(u)$$

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(iv) ∃ a map $-T_1 : U \to V$ such that

$$(T_1 + (-T_1))(u) = T_1(u) + (-T_1)(u)$$
$$= T_1(u) - T_1(u)$$
$$= v_1 - v_1$$
$$= 0$$

(v) $$(T_1 + T_2)(u) = T_1(u) + T_2(u)$$
$$= v_1 + v_2$$
$$= v_2 + v_1 = T_2(u) + T_1(u)$$
$$= (T_2 + T_1)(u)$$

(vi) Let $\alpha \in \mathbb{F}$

$$\alpha (T_1 + T_2)(u) = \alpha\{T_1(u) + T_2(u)\}$$
$$= \alpha(v_1 + v_2)$$
$$= \alpha v_1 + \alpha v_2$$
$$= \alpha T_1(u) + \alpha T_2(u)$$

(vii) Let $\alpha, \beta \in \mathbb{F}$

$$(\alpha + \beta) T_1(u) = (\alpha + \beta) v_1$$
$$= \alpha v_1 + \beta v_1$$
$$= \alpha T_1(u) + \beta T_1(u)$$

(viii) $$I \cdot T_1(u) = I(v_1)$$
$$= v_1$$
$$= T_1(u)$$

Since $L(U,V)$ satisfies all the conditions, so $L(U,V)$ is a vectorrspace.

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Proof:-

Let $T_1, T_2 \in L(U,V)$

$$(T_1+T_2)(\alpha u_1 + \beta u_2) = T_1(\alpha u_1+\beta u_2) + T_2(\alpha u_1+\beta u_2)$$
$$= \alpha T_1(u_1) + \beta T_1(u_2) + \alpha T_2(u_1) + \beta T_2(u_2)$$
$$= \alpha(T_1+T_2)(u_1) + \beta(T_1+T_2)(u_2)$$

$\therefore$ $T_1+T_2 \in L(U,V)$.

Q:- $R(T) = \{(x_1, x_2, x_3) \in V_3 \mid 4x_1 - 3x_2 + x_3 = 0\}$

find T.

Q:- $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$T(x,y) = (2x, 3y)$

$S : x^2+y^2 = 1$

$T(S) = ?$

$T^{-1}(S) = ?$

$T(1,0) = (2,0)$

$1 + 0 = 1$

$(2,3)$

A:- Let $T(x,y) = (s,t)$

$\Rightarrow (s,t) = (2x, 3y)$

$\Rightarrow s = 2x \; , \; t = 3y$

$\Rightarrow x = \frac{s}{2} \; , \; y = \frac{t}{3}$.

$T(S) : \left(\frac{s}{2}\right)^2 + \left(\frac{t}{3}\right)^2 = 1$

$: \frac{s^2}{4} + \frac{t^2}{9} = 1$

$T^{-1}(S) : (2x)^2 + (3y)^2 = 1$.

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* $T(x,y) = (x+y,\ 2x+3y)$

$S: x^2+y^2=1$

$(s,t) = (x+y,\ 2x+3y)$

$\Rightarrow s = x+y \Rightarrow 2s = 2x+2y$

$t = 2x+3y$

$\Rightarrow y = t-2s$

$x = s-y = s-t+2s = 3s-t$

$T(s\text{[illegible]}) = (3s-t)^2+(t-2s)^2=1$

$\Rightarrow 9s^2-6st+t^2+t^2-4st+4s^2=1$

$\Rightarrow 13s^2-10st+2t^2=1$

$T^{-1}(S) = (x+y)^2+(2x+3y)^2=1$

$\Rightarrow x^2+2xy+y^2+4x^2+12xy+9y^2=1$

$\Rightarrow 5x^2+14xy+10y^2=1.$

$\therefore T^{-1}(S):\ 5x^2+14xy+10y^2=1.$

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Matrix Representation of Linear Transformation :-

$T : U \rightarrow V$

Let S and S' are the basis of U and V respectively.

$S = \{u_1, \ldots, u_m\}$

$S' = \{v_1, v_2, \ldots, v_n\}$

$[T(u_1)]_{S'}, [T(u_2)]_{S'}, \ldots, [T(u_m)]_{S'}$

$[T(u_1)]_{S'} = [a_{11}, a_{21}, \ldots, a_{n1}]$

$[T(u_2)]_{S'} = [a_{12}, a_{22}, \ldots, a_{n2}]$

$\vdots$

$[T(u_n)]_{S'} = [a_{1m}, a_{2m}, \ldots, a_{nm}]$

$$(T)_{SS'} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}_{n \times m}$$

Ex:- $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$, $T(x, y, z) = (x+y, y-z)$

$S = \{(1,0,0), (0,1,0), (0,0,1)\}$

$S' = \{(1,0), (0,1)\}$

$T(1,0,0) = (1,0) = 1(1,0) + 0(0,1)$ $\quad [(1,0)]_{S'} = (1,0)^T$

$T(0,1,0) = (1,1) = 1(1,0) + 1(0,1)$ $\quad [(1,1)]_{S'} = (1,1)^T$

$T(0,0,1) = (0,-1) = 0(1,0) - 1(0,1)$ $\quad [(0,-1)]_{S'} = (0,-1)^T$

$$(T)_{SS'} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$$

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$$[T]_{SS'}\, u_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = (1\ 0)^T$$

Q:- $T : \mathbb{R}^3 \rightarrow P_2$

$$T(a,b,c) = (a+b) + (b+c)t + (c+a)t^2$$

$$S = \{(1,1,1),\ (1,2,3),\ (0,-1,-1)\}$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{vmatrix} = 1(-2+3) - 1(-1+1)$$
$$= 1 \neq 0$$

$$S' = \{1,\ 1+t,\ 1+t^2\}$$

$$\begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 1 \neq 0$$

$\underline{\underline{S}}$

$$T(1,1,1) = 2 + 2t + 2t^2 = -2(1) + 2(1+t) + 2(1+t^2)$$
$$T(1,2,3) = 3 + 5t + 4t^2 = -6(1) + 5(1+t) + 4(1+t^2)$$
$$T(0,-1,-1) = -1 - 2t - t^2 = 2(1) - 2(1+t) - 1(1+t^2)$$

$$\alpha + \beta t + \gamma t^2 = a_0(1) + a_1(1+t) + a_2(1+t^2)$$
$$= (a_0 + a_1 + a_2) + a_1 t + a_2 t^2$$

$a_0 + a_1 + a_2 = 2$

$a_1 = 2$

$a_2 = 2$

$\Rightarrow a_0 = -2$

$a_0 = 3 - 5 - 4 = -6$

$a_0 = -1 + 2 + 1$

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$$[T]_{SS'} = \begin{pmatrix} -2 & -6 & 2 \\ 2 & 5 & -2 \\ 2 & 4 & -1 \end{pmatrix}$$

Theorem:-

Let T be a LT on a finite dimensional v.s V and let S be a basis of V, then for any vector $v \in V$,

$$[T]_S[v]_S = [T(v)]_S$$

Change of basis matrix from S to S':-

Let $S = \{u_1, u_2, \dots, u_n\}$

$S' = \{v_1, v_2, \dots, v_n\}$

$v_1 = a_{11}u_1 + a_{12}u_2 + \dots + a_{1n}u_n$

$v_2 = a_{21}u_1 + a_{22}u_2 + \dots + a_{2n}u_n$

$\vdots$

$v_n = a_{n1}u_1 + a_{n2}u_2 + \dots + a_{nn}u_n$

$[v_1]_S = (a_{11}, a_{12}, \dots, a_{1n})^T$

$\vdots$

$[v_n]_S = (a_{n1}, a_{n2}, \dots, a_{nn})^T$

$$P = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & a_{n2} \\ \vdots & & & \\ a_{1n} & a_{2n} & \dots & a_{nn} \end{bmatrix} \leftarrow \text{Transition matrix}$$

Change of basis matrix from S' to S is $Q = P^{-1}$.

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Theorem

Let $T: V \longrightarrow V$ be a linear operator, and let S be a (finite) basis of V. Then, for any vector $v \in V$,

$$[T]_S [v]_S = [T(v)]_S$$

Proof :- Given $T: V \longrightarrow V$ is a linear operator.

S be a basis of V

Suppose $S = \{u_1, u_2, \dots, u_n\}$

$$T(u_i) = a_{1i}u_1 + a_{2i}u_2 + \dots + a_{ni}u_n$$

$$= \sum_{j=1}^{n} a_{ji} u_j$$

Then $[T]_S$ is the nth square matrix whose jth ~~term~~ row is, $(a_{j1} \quad a_{j2} \quad \dots \quad a_{jn})$

Now, suppose,

$$v = k_1u_1 + k_2u_2 + \dots + k_nu_n = \sum_{i=1}^{n} k_iu_i$$

$$\therefore [v]_S = (k_1 \quad k_2 \quad \dots \quad k_n)^T$$

Now jth entry of $[T]_S[v]_S$ is obtained by multiplying jth row of $[T]_S$ and $[v]_S$

i.e. $a_{j1}k_1 + a_{j2}k_2 + \dots + a_{jn}k_n$

Now, $T(v) = T \sum_{i=1}^{n}(k_iu_i)$

$$= \sum_{i=1}^{n} k_i T(u_i)$$

$$= \sum_{i=1}^{n} k_i\left(\sum_{j=1}^{n} a_{ji}u_j\right)$$

$$= \sum_{j=1}^{n}\left(\sum_{i=1}^{n} a_{ji}k_i\right)u_j$$

$$= \sum_{j=1}^{n}\left(a_{j1}k_1 + a_{j2}k_2 + \dots + a_{jn}k_n\right)u_j$$

Now, $[T(v)]_S$ is a column vector whose jth entry is, $a_{j1}k_1 + a_{j2}k_2 + \dots + a_{jn}k_n$

$\therefore$ jth entry of $[T]_S[v]_S$ = jth entry of $[T(v)]_S$

Hence $[T]_S[v]_S = [T(v)]_S$. □

12.02.2020

$T: \mathbb{R}^2 \to \mathbb{R}^2$

Q:- $T(x,y) = (2x+3y,\ x-y)$

$S = \{(1,1), (1,2)\}$

$v = (2,3)$

$T(1,1) = (5,0) \Rightarrow [(5,0)]_S = (10,-5)^T$

$T(1,2) = (8,-1) \Rightarrow [(8,-1)]_S = (17,-9)^T$

$(a,b) = \alpha(1,1) + \beta(1,2)$

$= (\alpha+\beta,\ \alpha+2\beta)$

$\alpha + \beta = a$

$\alpha + 2\beta = b$

$\beta = b - a$

$\alpha = a - \beta$

$= a - b + a = 2a - b$

$$[T]_S = \begin{pmatrix} 10 & 17 \\ -5 & -9 \end{pmatrix}$$

$[(2,3)]_S = (1,1)^T$

$$[T]_S[v]_S = \begin{pmatrix} 10 & 17 \\ -5 & -9 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 27 \\ -14 \end{pmatrix}$$

$T(2,3) = (13,-1) \Rightarrow [(13,-1)]_S = (27,-14)^T$

$\therefore [T]_S[V]_S = [T(V)]_S$

P Kalika Maths

* Find T^{-1}.

$$T(1,0) = (2,1)$$

$$T(0,1) = (3,-1)$$

$$[T]_S = \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}$$

$$T^{-1} = \frac{-1}{5}\begin{bmatrix} -1 & -3 \\ -1 & 2 \end{bmatrix}$$

$$T^{-1}(x,y) = \left(\frac{1}{5}x + \frac{3}{5}y \,,\ \frac{1}{5}x - \frac{2}{5}y\right)$$

* $[P]_{SS'}[v]_{S'} = [v]_S$ prove this

$$[P^{-1}(v)]_S = [v]_{S'}$$

$$Q_{S'S}[v]_S = [v]_{S'}$$

Result:-

Let P be the change of basis matrix from S to a basis S' in a vectorspace V, then for any linear operator T on V,

$$[T]_{S'} = P^{-1}[T]_S P$$

* Two matrices represent the same linear operator iff matrices are similar.

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* A linear operator T is said to be diagonalisable if $\exists$ a basis S of a Vector space V such that T is represented by a diagonal matrix.

The basis S is said to diagonalise T.

Theorem:-

Let A be the matrix representation of a linear operator T. then T is diagonalisable iff $\exists$ an invertible matrix P such that $P^{-1}AP = D$

i.e. T is diagonalisable iff its matrix representation can be diagonalised by a similarity transformation.

Q:- Suppose $A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 5 & -4 \\ 1 & -2 & 2 \end{bmatrix}$

Find a matrix B that represents the Linear operator A relative to the basis $S = \{(1,1,0)^T, (0,1,1)^T, (1,2,2)^T\}$

$$\begin{bmatrix} 1 & 3 & 1 \\ 2 & 5 & -4 \\ 1 & -2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 4 & 9 \\ 7 & 1 & 4 \\ -1 & 0 & 1 \end{bmatrix}$$

~~$(a,b,c) = \alpha(1,1,0) + \beta(0,1,1) + \gamma(1,2,2)$~~

$= \cancel{\alpha+\gamma}, \quad \underset{\underset{v_1}{\shortparallel}}{[A(u_1)]_S} \quad \underset{\underset{v_2}{\shortparallel}}{[A(u_2)]_S} \quad \underset{\underset{v_3}{\shortparallel}}{[A(u_3)]_S}.$

Another method

$E = \{e_1, e_2, e_3\}$

Change of basis $P \rightarrow E$ to S

$[B] = P^{-1}AP$.

P Kalika Maths

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Suppose $S = \{u_1, u_2\}$ is a basis of V and $T : V \to V$ is defined by $T(u_1) = 3u_1 - 2u_2$ and $T(u_2) = u_1 + 4u_2$. Suppose $S' = \{w_1, w_2\}$ is a basis of V for which $w_1 = u_1 + u_2$ and $w_2 = 2u_1 + 3u_2$:

(a) Find the matrices A and B representing T relativ to the bases S and S' respectively.

(b) Find the matrix P such that $B = P^{-1}AP$.

Sol^n:-

(a) $A = [T]_S$

$$T(u_1) = 3u_1 - 2u_2$$
$$T(u_2) = u_1 + 4u_2$$

$$\therefore A = [T]_S = \begin{pmatrix} 3 & 1 \\ -2 & 4 \end{pmatrix}$$

$$B = [T]_{S'}$$

Now, $T(w_1) = T(u_1 + u_2) = T(u_1) + T(u_2)$
$$= 3u_1 - 2u_2 + u_1 + 4u_2$$
$$= 4u_1 + 2u_2$$

$$T(w_2) = T(2u_1 + 3u_2)$$
$$= 2T(u_1) + 3T(u_2)$$
$$= 2(3u_1 - 2u_2) + 3(u_1 + 4u_2)$$
$$= 6u_1 - 4u_2 + 3u_1 + 12u_2$$
$$= 9u_1 + 8u_2$$

Let $(a, b) = \alpha w_1 + \beta w_2$
$$= \alpha(u_1 + u_2) + \beta(2u_1 + 3u_2)$$
$$= (\alpha + 2\beta)u_1 + (\alpha + 3\beta)u_2$$

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$\Rightarrow \alpha + 2\beta = a$
$\alpha + 3\beta = b$

$\Rightarrow \beta = b - a$
$\alpha = a - 2b + 2a$
$= 3a - 2b$

$\therefore T(\omega_1) = 4u_1 + 2u_2 = 8\omega_1 - 2\omega_2$
$T(\omega_2) = 9u_1 + 8u_2 = 11\omega_1 - \omega_2$

$\therefore B = [T]_{S'} = \begin{pmatrix} 8 & 11 \\ -2 & -1 \end{pmatrix}$

Now let's do the change of basis matrix from S to S'

$\omega_1 = u_1 + u_2$
$\omega_2 = 2u_1 + 3u_2$

$\therefore P = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$

13.02.2020

or

$\omega_1 = u_1 + u_2$
$\omega_2 = 2u_1 + 3u_2$

$$P^{-1}AP = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$$
$$= \begin{pmatrix} 13 & -5 \\ -5 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$$
$$= \begin{pmatrix} 8 & 11 \\ -2 & -1 \end{pmatrix} = B$$

$\Rightarrow 2\omega_1 = 2u_1 + 2u_2$
$\omega_2 = 2u_1 + 3u_2$

$u_2 = \omega_2 - 2\omega_1$
$u_1 = \omega_1 - \omega_2 + 2\omega_1 = 3\omega_1 - \omega_2$

~~$T(\omega_1) = 3\omega_1 - 2\omega\ 3(3\omega_1 - \omega_2) - 2(\omega_2 - 2\omega_1) = 13\omega_1 - 5\omega_2$~~

$T(\omega_1) = 4u_1 + 2u_2 = 4(3\omega_1 - \omega_2) + 2(\omega_2 - 2\omega_1)$
$= 8\omega_1 - 2\omega_2$

$T(\omega_2) = 9u_1 + 8u_2 = 9(3\omega_1 - \omega_2) + 8(\omega_2 - 2\omega_1)$
$= 11\omega_1 - \omega_2$

P Kalika Maths

Q Check Similarity of matrices is equivalence rel^n.

A:-

$A = I^{-1} A I$

$\Rightarrow A \approx A$ (Reflexive)

Let $A \approx B$

$\Rightarrow B = P^{-1}AP$

$\Rightarrow PBP^{-1} = PP^{-1}APP^{-1}$

$\Rightarrow (P^{-1})^{-1}BP^{-1} = A$

$\Rightarrow B \approx A$ (symmetric)

Let $A \approx B$, $B \approx C$

$\Rightarrow B = P^{-1}AP$, $C = T^{-1}BT$

$C = T^{-1}P^{-1}APT$

$= (PT)^{-1}A(PT)$

$\Rightarrow A \approx C$ (Transitive)

$\therefore$ Similarity of matrices is an equivalence relation.

Q:- If $B \approx A$ then show that for any polynomial $f(t)$, $f(B) \approx f(A)$

*Prove that if $A \approx B$, then $B^n = P^{-1}A^nP$

$B = P^{-1}AP$

$\Rightarrow B^2 = (P^{-1}AP)^2$

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$\Rightarrow B = (P^{-1}AP)(P^{-1}AP)$

$= P^{-1}A(PP^{-1})AP$

$= P^{-1}AIAP$

$= P^{-1}A^2P$

Similarly doing in this manner

$B^k = P^{-1}A^kP$ (Assume)

$B^{k+1} = (P^{-1}A^kP)(P^{-1}AP)$

$= P^{-1}A^k(PP^{-1})AP$

$= P^{-1}A^kIAP$

$= P^{-1}A^{k+1}P$

$\therefore B^n = P^{-1}A^nP$

Q-A:- $f(t) = a_nt^n + a_{n-1}t^{n-1} + \dots + a_1t + a_0$

$f(A) = a_nA^n + a_{n-1}A^{n-1} + \dots + a_1A + a_0I$

$f(B) = a_nB^n + a_{n-1}B^{n-1} + \dots + a_1B + a_0I$

$B = P^{-1}AP$

$f(B) = f(P^{-1}AP)$

$= a_n(P^{-1}AP)^n + a_{n-1}(P^{-1}AP)^{n-1} + \dots + a_1(P^{-1}AP) + a_0I$

$= a_n(P^{-1}A^nP) + a_{n-1}(P^{-1}A^{n-1}P) + \dots + a_1(P^{-1}AP) + a_0P^{-1}IP$

$= P^{-1}(a_nA^n)P + P^{-1}(a_{n-1}A^{n-1})P + \dots + P^{-1}(a_1A)P$ $\cancel{+a_0IP}$ $+ P^{-1}a_0IP$

$= P^{-1}(a_nA^n + a_{n-1}A^{n-1} + \dots + a_1A + a_0I)P$

$= P^{-1}f(A)P$

$\therefore f(A) \approx f(B).$

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Theorem:-

Let $T : V \longrightarrow U$ be a linear transformation and $r(T) = r$. Then $\exists$ basis of V and U such that the matrix representation of T has the following form.

$$A = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$$

$\dim(V) = m$, $\dim(U) = n$.

Sol:-

$$n(T) = m - r$$

Suppose $\{v_{m-r+1}, v_{m-r+2}, \ldots, v_m\}$ is the basis of $\text{Ker}(T)$

$$T(v_i) = 0 \quad , \quad i = m-r+1,\ m-r+2, \ldots, m$$

Extend the basis of $\text{Ker}(T)$ as

$\{$ ~~$v_1, v_2, \ldots, v_{m-r}$,~~ $\{v_{m-r+1}, \ldots, v_m\}, v_1, v_2, \ldots, v_{m-}$

$$T(v_1) = u_1$$

Let $\{u_1, u_2, \ldots, u_r\}$ is the basis of $R(T)$

Let's extend the basis of $R(T)$ to basis of U as

$$\{u_1, u_2, \ldots, u_r, u_{r+1}, \ldots, u_n\}$$

$$T(v_1) = u_1 = 1 \cdot u_1 + 0 \cdot u_2 + \cdots + 0 \cdot u_n$$

$$\vdots$$

$$T(v_{-r}) = u_r = 0 \cdot u_1 + \cdots + 1 \cdot u_r + \cdots + 0 \cdot u_n$$

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$$A = \begin{array}{c} \\ \\ \\ r\text{th row} \\ \\ \\ \\ \\ \end{array}\begin{bmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & & & & & \\ 0 & 0 & \cdots & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & & & & & \\ \vdots & & & & & \\ 0 & \cdots & & & & \end{bmatrix}_{n\times m}$$

$$= \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}_{n\times m}$$

* $T : V \longrightarrow U$, $W \subseteq V$ 14.02.2020

The restriction of T to W is the map

$T/W : W \longrightarrow U$ st $T/W(v) = T(v)$, for every $v \in W$

Prove that T/W is linear.

Sol:- (i) let $u_1, u_2 \in W$ st $T/W(u_1) = T(u_1)$

$T/W(u_2) = T(u_2)$

$$T/W\,(u_1+u_2) = T(u_1+u_2)$$
$$= T(u_1) + T(u_2)$$
$$= T/W(u_1) + T/W(u_2)$$

$$T/W\,(\alpha u_1) = T(\alpha u_1)$$
$$= \alpha T(u_1)$$
$$= \alpha T/W(u_1)$$

$\therefore$ T/W is linear.

(ii) * Prove that $\ker(T/W) = \ker(T) \cap W$.

Let $x \in \ker(T/W)$

$\Rightarrow T/W(x) = 0$

$\Rightarrow T(x) = 0$

$\Rightarrow x \in \ker T$ and $x \in W$

$\therefore x \in \ker(T) \cap W$.

$\therefore \ker(T/W) \subseteq \ker(T) \cap W$.

Again, let $y \in \ker(T) \cap W$.

$\Rightarrow y \in \ker T \wedge y \in W$

$\Rightarrow y \in \ker(T/W)$

$\therefore \ker(T) \cap W \subseteq \ker(T/W)$

$\therefore \ker(T/W) = \ker(T) \cap W$.

(iii) $\operatorname{Im}(T/W) = T(W)$

Sol:- let any arbitary, $y \in \operatorname{Im}(T/W)$

$\Rightarrow (T/W)(x) = y$, for some $x \in W$

$\Rightarrow T(x) = y$

$\Rightarrow y \in T(W)$

Since y is arbitarry,

$\operatorname{Im}(T/W) = T(W)$

P Kalika Maths

17.02.2020

INNER PRODUCT

Def^n:- Let V be a real vectorspace. A mapping $\langle \cdot,\cdot \rangle$, $\langle \cdot,\cdot \rangle : V \times V \to \mathbb{R}$ is said to be an inner product on V if it satisfies the following axioms

I1 Linear property

$$\langle \alpha u_1 + \beta u_2, v \rangle = \alpha \langle u_1, v \rangle + \beta \langle u_2, v \rangle$$

I2 Symmetric Property

$$\langle u, v \rangle = \langle v, u \rangle$$

I3 Positive definite property

$$\langle u,u \rangle \geq 0 \text{ and } \langle u,u \rangle = 0 \text{ iff } u=0$$

↳ The vector space V with innerproduct is called innerproduct space.

Ex:- $\mathbb{R}^n \to$ Eucleidian space

$\langle \cdot,\cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ st $\langle u, v \rangle = u \cdot v$

Check it is inner product or not.

Sol^n:- Let $u = (x_1, x_2, \dots, x_n)$

$v = (y_1, y_2, \dots, y_n)$

$$\langle u, v \rangle = u \cdot v = (x_1, x_2, \dots, x_n)(y_1, y_2, \dots, y_n)$$

$$= \sum_{i=1}^{n} x_i y_i$$

I1

$u_1 = (x_1, x_2, \dots, x_n)$

$u_2 = (y_1, y_2, \dots, y_n)$

$v = (z_1, z_2, \dots, z_n)$

$$\langle \alpha u_1 + \beta u_2, v \rangle = \langle \alpha(x_1, \dots, x_n) + \beta(y_1, \dots, y_n), (z_1, z_2, \dots, z_n) \rangle$$

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$$= \langle \alpha x_1+\beta y_1, \alpha x_2+\beta y_2, \dots, \alpha x_n+\beta y_n, z_1, z_2, \dots, z_n \rangle$$

$$= (\alpha x_1+\beta y_1) z_1 + (\alpha x_2+\beta y_2) z_2 + \dots + (\alpha x_n+\beta y_n) z_n$$

$$= \alpha (x_1 z_1 + x_2 z_2 + \dots + x_n z_n) + \beta (y_1 z_1 + y_2 z_2 + \dots + y_n z_n)$$

$$= \alpha \langle u_1, v \rangle + \beta \langle u_2, v \rangle$$

I2

$$\langle u, v \rangle = \sum_{i=1}^{n} x_i y_i$$

$$= \sum_{i=1}^{n} y_i x_i = \langle v, u \rangle$$

I3

$$\langle u, u \rangle = \sum_{i=1}^{n} x_i^2 \geqslant 0$$

$$\langle u, u \rangle = 0$$

$$\Leftrightarrow \sum_{i=1}^{n} x_i^2 = 0$$

$$\Leftrightarrow x_i = 0, \; i = 1, 2, \dots, n$$

$$\Leftrightarrow u = (0, 0, \dots, 0)$$

Since $\langle u, v \rangle : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ st $\langle u, v \rangle = u \cdot v$ satisfy all the proporties, so $\langle u, v \rangle$ is an innerproduct on $\mathbb{R}^n$.

Ex:- $\langle u, v \rangle = u^T v$, $u, v \to$ Row vector

is innerproduct.

$V = \mathbb{R}^n$

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Ex:- $V = C[a,b]$

$$\langle f,g\rangle = \int_a^b f(t)\, g(t)\, dt$$

I1 $\langle \alpha f_1 + \beta f_2, g\rangle = \int_a^b \{\alpha f_1(t) + \beta f_2(t)\} g(t)\, dt$

$$= \int_a^b \alpha f_1(t) g(t)\, dt + \int_a^b \beta f_2(t) g(t)\, dt$$ [∵ Integral operator is linear]

$$= \alpha \int_a^b f_1(t) g(t)\, dt + \beta \int_a^b f_2(t)\, g(t)\, dt$$

$$= \alpha\langle f_1, g\rangle + \beta\langle f_2, g\rangle$$

I2 $\langle f,g\rangle = \int_a^b f(t) g(t)\, dt$

$$= \int_a^b g(t)\, f(t)\, dt$$

$$= \langle g,f\rangle$$

I3 $\langle f,f\rangle = \int_a^b f(t)\, f(t)\, dt$

$$= \int_a^b (f(t))^2\, dt \geqslant 0 .$$

If $\langle f,f\rangle = 0$

$\Leftrightarrow \int_a^b (f(t))^2\, dt = 0$

$\Leftrightarrow f(t) = 0$

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$(A^T)_{n\times m}$ $(A^T)_{n\times m}(B)_{m\times n}$

$(A^TB)_{n\times n}$

B^T

$M = \{ A = [a_{ij}]_{m\times n} \}$

$\langle A, B\rangle = tr\,(B^T A)$

I1 $\langle \alpha A_1 + \beta A_2 , B\rangle = tr\left(B^T(\alpha A_1 + \beta A_2)\right)$

$= tr\left((B^T\alpha A_1) + (B^T\beta A_2)\right)$

$= tr\,(\alpha B^T A_1 + \beta B^T A_2)$

$= tr(\alpha B^T A_1) + tr(\beta B^T A_2)$

$= \alpha\, tr(B^T A_1) + \beta\, tr(B^T A_2)$

$= \alpha\langle A_1, B\rangle + \beta\langle A_2, B\rangle$.

I2) $\langle A, B\rangle = tr\,(B^T A) = tr(B^T A)^T$

$= tr\,(A^T B)$

$= \langle B, A\rangle$

I3) Let $\langle A, A\rangle =$

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$\rightarrow$ $\|u\| = \sqrt{u \cdot u}$

$= \sqrt{\langle u \cdot u \rangle}$

$$\|u\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$
$$= \sqrt{x_1x_1 + x_2x_2 + \cdots + x_nx_n}$$
$$= \sqrt{(x_1, x_2, \cdots, x_n)(x_1, x_2, \cdots, x_n)}$$
$$= \sqrt{u \cdot u}$$
$$= \sqrt{\langle u, u \rangle}$$

$$\|\hat{u}\| = \left\| \frac{u}{\|u\|} \right\| = \frac{\|u\|}{\|u\|} = 1 .$$

$$\|f\| = \sqrt{\langle f, f \rangle}$$
$$= \left(\int_a^b f^2(t)\, dt \right)^{1/2}$$

$$\hat{f} = \frac{f}{\|f\|}$$

18.02.2020

* $u \cdot v = \|u\| \|v\| \cos\theta$

$$\cos\theta = \frac{u \cdot v}{\|u\| \|v\|}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{u \cdot v}{\|u\| \|v\|} \right)$$

If $\frac{u \cdot v}{\|u\| \|v\|} = 0 \Rightarrow u \cdot v = 0$

$\Rightarrow \langle u, v \rangle = 0$

Orthogonal vectors

Let V be an inner product space. The vectors $u, v \in V$ are said to be orthogonal if $\langle u, v \rangle = 0$.

Orthonormal vectors

Let V be an inner product space. The vectors $u, v \in V$ are said to be orthonormal if (i) $\langle u, v \rangle = 0$

(ii) $\|u\| = \|v\| = 1$

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* The set $S = \{u_1, u_2, \ldots, u_n\}$ is said to be orthogonal if $\langle u_i, u_j \rangle = 0$ where $i \neq j$.

The set $S = \{u_1, u_2, \ldots, u_n\}$ is said to be orthonormal if (i) $\langle u_i, u_j \rangle = \begin{cases} 0 & \text{for } i \neq j \\ 1 & \text{for } i = j \end{cases}$

* Consider the set

$$\{1, \sin t, \sin 2t, \ldots, \sin nt, \ldots, \cos t, \cos 2t, \ldots, \cos nt, \ldots\}$$

in $[-\pi, \pi]$ is orthogonal set.

$$\langle f(t), g(t) \rangle = \int_{-\pi}^{\pi} f(t)\, g(t)\, dt$$

$$\int_{-\pi}^{\pi} \sin nt \cdot \cos mt\, dt \quad , \text{ let } m \neq n$$

$$= \frac{1}{2} \int_{-\pi}^{\pi} [\sin (n+m)t + \sin (n-m)t]\, dt$$

$$= \frac{1}{2} \left[\frac{\cos (n+m)t}{n+m} + \frac{\cos (n-m)t}{n-m} \right]_{-\pi}^{\pi}$$

$$= \frac{1}{2} \left(\cancel{\frac{(-1)^{n+m}}{n+m}} + \cancel{\frac{(-1)^{n-m}}{n-m}} - \cancel{\frac{(-1)^{n+m}}{n+m}} - \cancel{\frac{(-1)^{n-m}}{n-m}} \right)$$

$$= 0$$

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for m=n

$$\int_{-\pi}^{\pi} \sin^2 nt\, dt$$

$$= \int_{-\pi}^{\pi} \frac{1-\cos 2nt}{2}\, dt$$

$$= \frac{1}{2} t \Big|_{-\pi}^{\pi} - \frac{1}{2} \frac{\sin 2nt}{2n}\Big|_{-\pi}^{\pi}$$

$$= \frac{1}{2}(\pi+\pi) - \frac{1}{4n}(0-0)$$

$$\Rightarrow \pi$$

$$\int_{-\pi}^{\pi} \sin^2 nt\, dt$$

$$= \int_{-\pi}^{\pi} \frac{1-\cos 2nt}{2}\, dt$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} dt - \frac{1}{2}\int_{-\pi}^{\pi}\cos 2nt\, dt$$

$$= \frac{1}{2}(\pi+\pi) - 0$$

$$= \pi$$

For m=n

$$\int_{-\pi}^{\pi} \sin nt \cdot \cos nt\, dt$$

$$= \int_{-\pi}^{\pi} \frac{\sin 2nt}{2}\, dt$$

$$= \frac{1}{2}\, \frac{-\cos 2nt}{2n}\Big|_{-\pi}^{\pi}$$

$$= \frac{-1}{4n}(1-1) = 0$$

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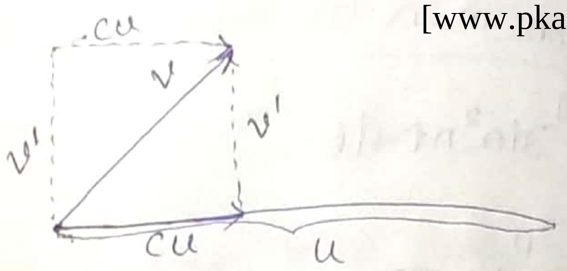
$v = v' + cu$

$v' = v - cu$



$\langle v', u \rangle = \langle v - cu, u \rangle = 0$

$\Rightarrow \langle v, u \rangle - c\langle u, u \rangle = 0$

$\Rightarrow \boxed{c = \dfrac{\langle v, u \rangle}{\langle u, u \rangle}}$ {Projection of v on $u = cu$}

* Let $u \in \mathbb{R}^2$ st $u = (a, b)$

$\langle v, u \rangle = 0$

$(x, y) \cdot (a, b) = 0$

$\Rightarrow ax + by = 0$

Orthogonal Compliment :-

Let S be a subset of an innerproduct space V. The orthogonal compliment of S is denoted by $S^{\perp}$ and defined as $S^{\perp} = \{ v \in V : \langle v, u \rangle = 0 \text{ for every } u \in S \}$

Orthogonal compliment of a single vector

$u^{\perp} = \{ v \in V : \langle v, u \rangle = 0 \}$

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Q:- $S = \{ \underset{u_1}{(1,1)}, \underset{u_2}{(-1,2)} \}$. Find $S^\perp$

A:- Let $v = (x,y)$ is orthogonal to u $\forall$ $u \in S$.

$\langle v, u_1 \rangle = 0$

$\langle v, u_2 \rangle = 0$

$\Rightarrow x + y = 0$ —— (i)

$-x + 2y = 0$ —— (ii)

$x = -y$

$3y = 0$

$\Rightarrow y = 0$

$\therefore (x,y) = (0,0)$

$S^\perp = \{(0,0)\}$

Q:- $S = \{ (1,1), (2,2) \}$

$x + y = 0$

$2x + 2y = 0$

$x = -y$

Let $x = \alpha$

$(x,y) = (\alpha, -\alpha) = \alpha(1,-1)$

$S^\perp = \{\alpha(1,-1), \alpha \in \mathbb{R}\}$

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* $S^{\perp} = \{ v \in V : \langle v, u \rangle = 0 \;\; \forall \; u \in S \}$

let $v_1, v_2 \in S^{\perp}$

$\therefore \langle v_1, u \rangle = 0 \;,\; \langle v_2, u \rangle = 0 \;\; \forall \; u \in S.$

$$\langle v_1 + v_2 , u \rangle = \langle v_1, u \rangle + \langle v_2, u \rangle$$
$$= 0 + 0$$
$$= 0$$

$\therefore v_1 + v_2 \in S^{\perp}$

Now, $\langle \alpha v_1, u \rangle = \alpha \langle v_1, u \rangle$
$$= \alpha \cdot 0$$
$$= 0$$

$\therefore \alpha v_1 \in S^{\perp}$

$\therefore S^{\perp}$ is a subspace of V. ∎

04.03.2020

$S = \{ u_1, u_2, \ldots, u_n \}$

$v = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n$

$\langle v, u_i \rangle = \langle \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n , u_i \rangle$

$= \alpha_1 \langle u_1, u_i \rangle + \alpha_2 \langle u_2, u_i \rangle + \cdots + \alpha_n \langle u_n, u_i \rangle$

$\langle u_i, u_j \rangle = 0 \;,\; i \neq j$

$= \alpha_i \langle u_i, u_i \rangle$

$\Rightarrow \alpha_i = \dfrac{\langle v, u_i \rangle}{\langle u_i, u_i \rangle} \;,\; i = 1, 2, \ldots, n$, where α_i's are the fourier co-efficients w.r.t u.

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Ex:- $S = \{(1,2,1), (2,1,-4), (3,-2,1)\}$

$v = (7,1,9)$

Sol:- $v = \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3$

$$\alpha_1 = \frac{\langle v, u_1 \rangle}{\langle u_1, u_1 \rangle} = \frac{18}{6} = 3$$

$$\alpha_2 = \frac{\langle v, u_2 \rangle}{\langle u_2, u_2 \rangle} = \frac{-21}{21} = -1$$

$$\alpha_3 = \frac{\langle v, u_3 \rangle}{\langle u_3, u_3 \rangle} = \frac{28}{14} = 2$$

Prove that

→ If S is orthogonal then S is LI.

Theorem:-

Suppose $w_1, w_2, \ldots, w_r$ form an orthogonal set of non-zero vectors in V. Let v be any vector in V and we define $v' = v - (c_1 w_1 + c_2 w_2 + \cdots + c_r w_r)$. Then v' is orthogonal to $w_1, w_2, \ldots, w_r$

Proof:- $\langle v', w_i \rangle$

$= \langle v - c_1 w_1 - c_2 w_2 - \cdots - c_r w_r, w_i \rangle$

$= \langle v, w_i \rangle - c_1 \langle w_1, w_i \rangle - \cdots - c_r \langle w_r, w_i \rangle$

$= \langle v, w_i \rangle - c_i \langle w_i, w_i \rangle \quad , i = 1, 2, \ldots, r$

$= \langle v, w_i \rangle - \frac{\langle v, w_i \rangle}{\langle w_i, w_i \rangle} \langle w_i, w_i \rangle \quad \left[\because c_i = \frac{\langle v, w_i \rangle}{\langle w_i, w_i \rangle} \text{ for } i = 1, 2, \ldots, n \right]$

$= \langle v, w_i \rangle - \langle v, w_i \rangle$

$= 0$

$\Rightarrow v'$ is orthogonal to $w_1, w_2, \ldots, w_r$.

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Projection of V along W:-

If $W = [w_1, w_2, \dots, w_r]$, where w_i's formed an orthogonal set. Then $proj(V, W) = c_1 w_1 + c_2 w_2 + \dots + c_r w_r$ read as projection of V along W, where $c_i = \frac{\langle V, w_i \rangle}{\langle w_i, w_i \rangle}$

Ex:- Find projection of V along S.

$S = \{(1,2,1), (2,1,-4), (3,-2,1)\}$

$V = \{(7,1,9)\}$

$$Proj(V, S) = c_1 u_1 + c_2 u_2 + c_3 u_3$$

$$= 3(1,2,1) + (-1)(2,1,-4) + 2(3,-2,1)$$

$$= (3,6,3) + (-2,-1,4) + (6,-4,2)$$

$$= (7,1,9)$$

Gram-Schmidt Process:-

Let $S = \{u_1, u_2, \dots, u_n\}$ is a basis of an inner product space V. Now we can construct an orthogonal basis $\{w_1, w_2, \dots, w_n\}$ of V as follows.

Step-I $w_1 = u_1$

Step-II $w_2 = u_2 - c_1 w_1$, $c_1 = \frac{\langle u_2, w_1 \rangle}{\langle w_1, w_1 \rangle}$

$$= u_2 - \frac{\langle u_2, w_1 \rangle}{\langle w_1, w_1 \rangle} \cdot w_1$$

$$\langle w_1, w_2 \rangle = \left\langle u_1, u_2 - \frac{\langle u_2, w_1 \rangle}{\langle w_1, w_1 \rangle} \cdot w_1 \right\rangle = 0$$

Step-III : $w_3 = u_3 - \frac{\langle u_3, w_1\rangle}{\langle w_1, w_1\rangle} w_1 - \frac{\langle u_3, w_2\rangle}{\langle w_2, w_2\rangle} \cdot w_2$

$\vdots$

$$w_n = u_n - \sum_{i=1}^{n-1} \frac{\langle u_n, w_i\rangle}{\langle w_i, w_i\rangle} w_i$$

This process is known as Gram-Schmidt ~~Proce~~ Orthogonalization

Ex:- $\{(1,0,1), (1,1,1), (-1,1,0)\}$

$w_1 = (1,0,1)$

$w_2 = u_2 - c_1 w_1 = u_2 - \frac{\langle u_2, w_1\rangle}{\langle w_1, w_1\rangle} w_1$

$= (1,1,1) - \frac{2}{2}(1,0,1) = (0,1,0)$

w_1 and w_2 are orthogonal.

$w_3 = u_3 - c_1 w_1 - c_2 w_2$

$= u_3 - \frac{\langle u_3, w_1\rangle}{\langle w_1, w_1\rangle} w_1 - \frac{\langle u_3, w_2\rangle}{\langle w_2, w_2\rangle} w_2$

$= (-1,1,0) - \frac{(-1)}{2}(1,0,1) - \frac{1}{1}(0,1,0)$

$= (-1,1,0) + \left(\frac{1}{2}, 0, \frac{1}{2}\right) + (0,-1,0)$

$= \left(\frac{-1}{2}, 0, \frac{1}{2}\right)$

$\therefore \{(1,0,1), (0,1,0), (\frac{-1}{2}, 0, \frac{1}{2})\}$ is an orthogonal basis of V.

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Gram-Schmidt Orthogonalisation Process

$S = \{1, t, t^2\}$ on $[-1, 1]$

$$\langle f, g\rangle = \int_a^b f(t)\,g(t)\,dt$$

$$= \int_{-1}^{1} f(t)\,g(t)\,dt$$

$\omega_1 = f_1 = 1$

$$\Rightarrow \omega_2 = f_2 - \frac{\langle f_2, \omega_1\rangle}{\langle \omega_1, \omega_1\rangle}\omega_1 = t \qquad , \langle f_2, \omega_1\rangle = \int_{-1}^{1} t\,dt = 0$$

$$\omega_3 = f_3 - \frac{\langle f_3, \omega_1\rangle}{\langle \omega_1, \omega_1\rangle}\omega_1 - \frac{\langle f_3, \omega_2\rangle}{\langle \omega_2, \omega_2\rangle}\omega_2$$

$$\langle \omega_1, \omega_1\rangle = \int_{-1}^{1} dt = 2 .$$

$$\langle f_3, \omega_1\rangle = \int_{-1}^{1} t^2\,dt = \left[\frac{t^3}{3}\right]_{-1}^{1} = \frac{2}{3} .$$

$$\langle f_3, \omega_2\rangle = \int_{-1}^{1} t^3\,dt = 0 .$$

Thm:-

Let $\{v_1, v_2, \ldots, v_n\}$ be any basis of an inner product space V. Then $\exists$ an orthonormal basis $\{u_1, u_2, \ldots, u_n\}$ of V such that the change of basis matrix from $\{v_i\}$ to $\{u_i\}$ is triangular.

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↳ Suppose $S = \{w_1, w_2, \dots, w_r\}$ is an orthogonal basis for a subspace W of a vector space V, then one may extended ~~as~~ S to an orthogonal basis for V.

Positive definite matrices :-

Let A be a real symmetric matrix, then A is said to be positive definite if for every non-zero vector $u \in \mathbb{R}^n$ such that

$$\langle u, Au \rangle = (Au)^T u$$
$$= u^T A^T u$$
$$= u^T A u > 0$$

Ex:-

1) $A = \begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix}$, $|A| = -5 < 0$

∴ A is not positive definite

Properties of positive definite matrix :-

(i) $|A| > 0$

(ii) $\lambda_i > 0$

(iii) $\operatorname{tr}(A) > 0$

(iv) determinant of all principal minors > 0.

(2) $\begin{bmatrix} 1 & -2 \\ -2 & \textcircled{3} \end{bmatrix} \rightarrow$ not P.d

(3) $A = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}$ $|A| = 1 > 0$

$|[a_{11}]| = 1 > 0$

$|[a_{22}]| = 5 > 0$

∴ A is positive definite matrix

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$$\begin{vmatrix} 1-\lambda & -2 \\ -2 & 5-\lambda \end{vmatrix} = 0$$

$\Rightarrow (1-\lambda)(5-\lambda)-4=0$

$\Rightarrow 5-6\lambda+\lambda^2-4=0$

$\Rightarrow \lambda^2-6\lambda+1=0$

$\Rightarrow \lambda = \frac{6 \pm \sqrt{36-4}}{2} = \frac{6 \pm \sqrt{32}}{2}$

Thm:-

Let A be a real positive definite matrix, then the function $\langle u, v\rangle = u^T A v$, then the fun^n is an inner product on $\mathbb{R}^n$.

Matrix Representation of an innerproduct

Let V be an innerproduct space with the basis, $S = \{u_1, u_2, \dots, u_n\}$,then the matrix $A = [a_{ij}]$, $i,j = 1,2,\dots,n$ where $a_{ij} = \langle u_i, u_j\rangle$ is called the matrix representation of an innerproduct in V relative to the basis S !

Ex:- $S = \{1, t, t^2\}$, $V = P_2$, $[-1, 1]$

$$\langle f, g\rangle = \int_{-1}^{1} f(t)g(t)\,dt$$

Find the matrix representation of S in V.

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$a_{11}\langle f_1, f_1\rangle = \int_{-1}^{1} dt = 2$

$a_{12} = \langle f_1, f_2\rangle = \int_{-1}^{1} t\, dt = 0 = a_{21}$

$a_{13} = \langle f_1, f_3\rangle = \int_{-1}^{1} t^2 dt = \frac{2}{3} = a_{31}$

$a_{22} = \langle f_2, f_2\rangle = \int_{-1}^{1} t^2 dt = \frac{2}{3}$

$a_{23} = \langle f_2, f_3\rangle = \int_{-1}^{1} t^3 dt = 0 = a_{32}$

$a_{33} = \langle f_3, f_3\rangle = \int_{-1}^{1} t^4 dt = \frac{2}{5}$

$$\therefore \begin{pmatrix} 2 & 0 & 2/3 \\ 0 & 2/3 & 0 \\ 2/3 & 0 & 2/5 \end{pmatrix}$$

→ Let V be a vectorspace over $\mathbb{C}$. Suppose to each pair of vectors $u, v \in V$ there is assigned a complex number denoted by $\langle u, v\rangle$. This function is called a complex inner product on V if it satisfies the following axioms.

I1 $\langle \alpha u_1 + \beta u_2, v\rangle = \alpha\langle u_1, v\rangle + \beta\langle u_2, v\rangle$

I2 $\langle u, v\rangle = \overline{\langle v, u\rangle}$

I3 $\langle u, v\rangle \geq 0$ and $\langle u, u\rangle = 0$ iff $u = 0$.

I4 $\langle u, \alpha v\rangle = \overline{\langle \alpha v, u\rangle}$

$= \bar{\alpha}\overline{\langle v, u\rangle} = \bar{\alpha}\langle u, v\rangle$

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* Pythagoras Theorem

let $S = \{u_1, u_2, \dots, u_n\}$ is an orthogonal set then prove that

$$\|u_1 + u_2 + \dots + u_n\|^2 = \|u_1\|^2 + \|u_2\|^2 + \dots + \|u_n\|^2$$

Proof:- $\|u_1 + u_2 + \dots + u_n\|^2 = \langle u_1 + u_2 + \dots + u_n, u_1 + u_2 + \dots + u_n \rangle$

$$= \langle u_1, u_1\rangle + \langle u_1, u_2\rangle + \dots + \langle u_1, u_n\rangle$$
$$+ \langle u_2, u_1\rangle + \langle u_2, u_2\rangle + \dots + \langle u_2, u_n\rangle$$
$$+ \dots$$
$$+ \langle u_n, u_1\rangle + \langle u_n, u_2\rangle + \dots + \langle u_n, u_n\rangle$$

$$= \langle u_1, u_1\rangle + \langle u_2, u_2\rangle + \dots + \langle u_n, u_n\rangle$$

$[\because \langle u_i, u_j\rangle = 0$ for $i \neq j]$

$$= \|u_1\|^2 + \|u_2\|^2 + \dots + \|u_n\|^2 .$$

12.03.2020

Cauchy-Schwartz Inequality :-

For u and v in a real ~~vector~~ inner product space V,

$$|\langle u, v\rangle| \leq \|u\| \, \|v\| .$$

Proof:- $\langle tu + v, tu + v\rangle$; $t \in \mathbb{R}$

$$= \langle tu, tu\rangle + \langle tu, v\rangle + \langle v, tu\rangle + \langle v, v\rangle$$
$$= \langle u, u\rangle t^2 + t\langle u, v\rangle + t\langle v, u\rangle + \langle v, v\rangle$$
$$= \|u\|^2 t^2 + 2t\langle u, v\rangle + \|v\|^2$$

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$\|u\|^2 t^2 + 2t\langle u,v\rangle + \|v\|^2 \geq 0$ $\quad [\because \|tu+v\|^2 \geq 0]$

$at^2 + bt + c \geq 0 \quad \forall\, t$, $a = \|u\|^2$, $b = 2\langle u,v\rangle$, $c = \|v\|^2$

$\therefore\ b^2 - 4ac \leq 0$

$(2\langle u,v\rangle)^2 - 4\|u\|^2\|v\|^2 \leq 0$

$\Rightarrow 4\langle u,v\rangle^2 \leq 4\|u\|^2\|v\|^2$

$\Rightarrow |\langle u,v\rangle| \leq \|u\|^2\|v\|^2$

□

* $(a_1, a_2, \cdots, a_n) \in \mathbb{R}^n$ or $\mathbb{C}^n$

$\|(a_1, a_2, \cdots, a_n)\|_\infty = \max(|a_i|)$

$\|1, 2, \cdots, 10\|_\infty = \max(|1|, |2|, \cdots, |10|) = 10$

$\|(a_1, a_2, \cdots, a_n)\|_1 = \sum_{i=1}^{n} |a_i|$

$\|(a_1, a_2, \cdots, a_n)\|_2 = \sqrt{|a_1|^2 + |a_2|^2 + \cdots + |a_n|^2}$

Ex:- $u = (1, 2, 3, -1)$

Find $\|u\|_\infty$, $\|u\|_1$, $\|u\|_2$

A:- $\|u\|_\infty = \max\{|1|, |2|, |3|, |-1|\}$

$= \max\{1, 2, 3, 1\} = 3$

$\|u\|_1 = |1| + |2| + |3| + |-1| = 7$

$\|u\|_2 = \sqrt{|1|^2 + |2|^2 + |3|^2 + |-1|^2} = \sqrt{15}$

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* For function, f

$$\|f\|_1 = \int_a^b |f|\,dt$$

$$\|f\|_2 = \left(\int_a^b |f|^2\,dt\right)^{1/2}$$

$$\|f\|_\infty = \max\{|f(t)|\}$$

Q) $W \subseteq V$, $W^\perp \subseteq V$

$V = W \oplus W^\perp$

Let V is an n-dimensional vector space.

Proof:- $\dim(V) = n$

Let us assume that $\dim(W) = r \leq n$.

Let $\{w_1, w_2, \ldots, w_r\}$ is an orthogonal basis for W

i.e. $W = [\{w_1, w_2, \ldots, w_r\}]$

$\{w_1, w_2, \ldots, w_r, w_{r+1}, \ldots, w_n\}$ is a basis for V.

$W^\perp = [\{w_{r+1}, \ldots, w_n\}]$

* Prove that if $S \subset V$ then $S \subseteq S^{\perp\perp}$

Proof:- Let $u \in S$ and $v \in S^\perp$

$\langle u, v\rangle = 0$

$\Rightarrow \langle v, u\rangle = 0$

$\Rightarrow u \in (S^\perp)^\perp$

$\therefore S \subseteq (S^\perp)^\perp$

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* If $S_1 \subseteq S_2$ then prove that $S_2^{\perp} \subseteq S_1^{\perp}$

Proof:- Let $u \in S_2^{\perp}$

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Some Useful Links:

1. Free Maths Study Materials (https://pkalika.in/2020/04/06/free-maths-study-materials/**)**

2. BSc/MSc Free Study Materials (https://pkalika.in/2019/10/14/study-material/**)**

3. PhD/MSc Entrance Exam Que. Paper: (https://pkalika.in/que-papers-collection/**)**
[CSIR-NET, GATE(MA), BHU, CUCET,IIT, JAM(MA), NBHM, ...etc]

4. CSIR-NET Maths Que. Paper: (https://pkalika.in/2020/03/30/csir-net-previous-yr-papers/**)**
[Upto Lastest CSIR NET Exams]

5. PhD/JRF Position Interview Asked Questions:
(https://pkalika.in/phd-interview-asked-questions/)

6. List of Maths Suggested Books (https://pkalika.in/suggested-books-for-mathematics/)

7. CSIR-NET Mathematics Details Syllabus (https://wp.me/p6gYUB-Fc)

8. CSIR-NET, GATE, PhD Exams, ...etc Study Materials & Solutions
https://pkalika.in/kalika-notes-centre/

9. CSIR-NET, GATE, ... Solutions (https://wp.me/P6gYUB-1eP)

10. Topic-wise Video Lectures (Free Crash Course)
https://www.youtube.com/pkalika/playlists

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