Point Set Topology

(Handwritten Classroom Study Material)

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Your Note/Remarks

POINT SET TOPOLOGY

* Neighbourhood (nbd) of a point :-

Let $a \in \mathbb{R}$

then a set N is said to be a nbd of 'a'.

If $\exists \epsilon > 0$ s.t. $(a-\epsilon, a+\epsilon) \subseteq N$

OR

N is said to be a nbd of a.

If $\exists$ an open interval I s.t. $a \in I \subseteq N$.

So, a nbd of a is $(a-\epsilon, a+\epsilon)$ for some $\epsilon > 0$.

$\Rightarrow$ A nbd of a point in $\mathbb{R}$ is always an uncountable set. (Because, it contain an open interval).

So, a non-empty countable set can never be a nbd of any point.

Eg. $\rightarrow$ finite set, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, A

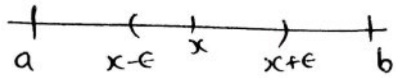
However, If a set is uncountable than it may not be a nbd of any point.

Ex:-

(1) Empty set (ϕ) is a nbd of each of it's points.

(2) $\mathbb{R}$ is a nbd of every real number.

(3) (a, b) is a nbd of each of it's point.



(4) $[a, b]$ is a nbd of each of it's point except 'a' and 'b'.

Ex:- $[1, 2]$

$$(1-\epsilon, 1+\epsilon) = (1-\epsilon, 1) \cup [1, 1+\epsilon)$$

$(1-\epsilon, 1)$ set $[1,2]$ is not Contain.

~~So, this set 1 and 2~~

So, 1 and 2 Can't have a nbd of this set.

(5) $\mathbb{Q}$ Can never be a nbd of any of it's points.

(6) $\mathbb{Q}^c$ Can never be a nbd of any of it's points. (Because open interval not Contain).

(7) A Countable set Can never be a nbd of any of it's point's (Because, open interval uncountable set)

* Some properties of a nbd of a point :-

① Every super set of a nbd is also a nbd.

② Union of two nbd is also a nbd.

③ Intersection of two nbd is also a nbd.

proof:-

Let $N_1 = (a-\epsilon_1, a+\epsilon_1)$ is a nbd of a.

$N_2 = (a-\epsilon_2, a+\epsilon_2)$ is a nbd of a.

If $\epsilon = \min(\epsilon_1, \epsilon_2)$

then

$N_1 \cap N_2 = (a-\epsilon, a+\epsilon)$ is also a nbd of a.

④ Arbitrary union of nbd is a nbd.

⑤ Finite intersection of nbd is a nbd.

⑥ Arbitrary intersection of a nbd need not be a nbd.

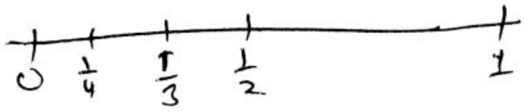
Ex $\left(\frac{-1}{n}, \frac{1}{n}\right)$ is a nbd of 0. $\forall n \in \mathbb{N}$

But $\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$ is not nbd of 0.

Q. $N = (1,2] \cup (3,4) \cup \{5,6,7\}$ is a nbd of.

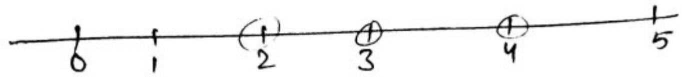
① 2 ② 3 ③ 15 ④ 17 ⑤ 1.5

Q. $N = \left(\frac{1}{2}, 1\right) \cup \left(\frac{1}{3}, \frac{1}{2}\right) \cup \left(\frac{1}{4}, \frac{1}{3}\right)$ is a nbd of:



① $\frac{1}{2}$ ② $\frac{1}{3}$ ③ $\frac{1}{4}$ ④ $\frac{2}{3}$

Q. $N = (1,3) \cup (2,4) \cup (3,5)$ is a nbd of - ?



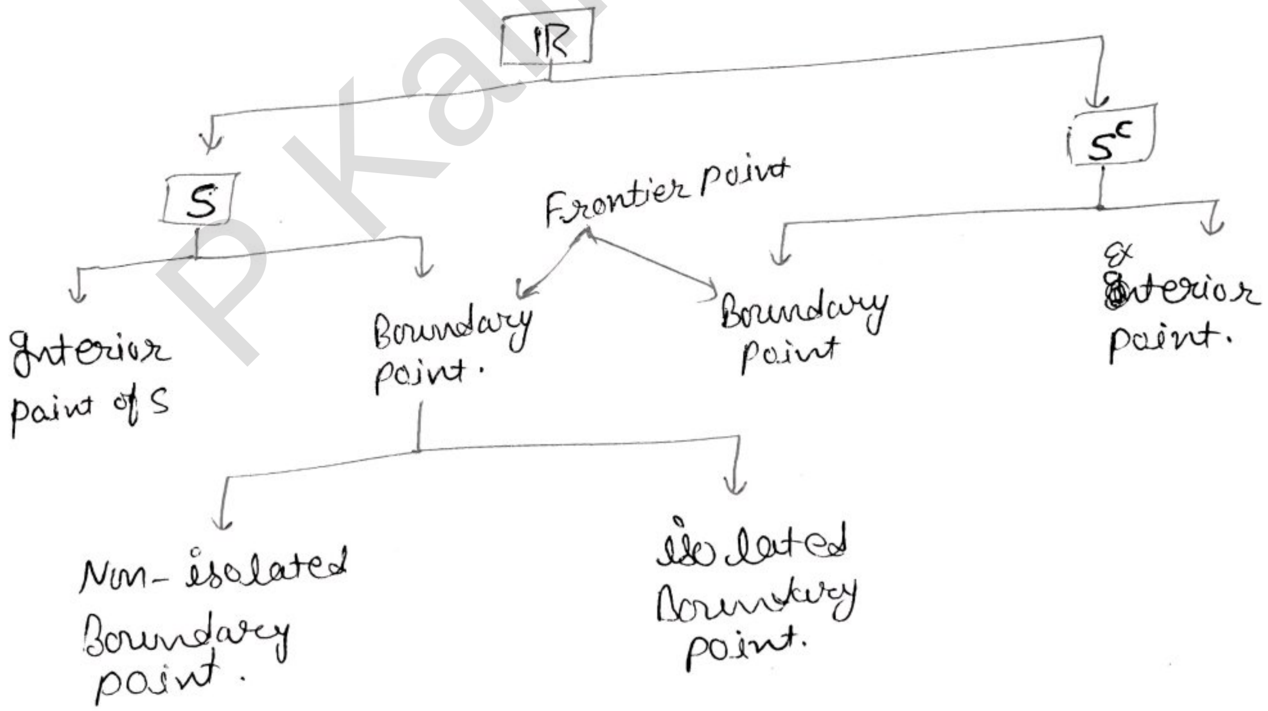
$\boxed{2, 3, 4}$ ✓

☆ Deleted nbd of a point :- A set N is said to be a deleted nbd of a point $a \in \mathbb{R}$.

iff $\exists\, \epsilon > 0$ s.t. $(a-\epsilon, a+\epsilon) \setminus \{a\} \subseteq N$, but $a \notin N$.

ex $N = [5, 10]$, $a = \sqrt{50}$ is nbd.

☆ Let S be any subset of $\mathbb{R}$ then



☆ Interior point :

A point $x \in S$ is said to be an interior point of a set.

If $\exists \epsilon > 0$ s.t. $(x-\epsilon, x+\epsilon) \subseteq S$.

⇒ A point $x \in S$ is said to be an interior point of a set S.

If S is a nbd of x.

☆☆ ⇒ A non-empty countable set can never have an interior point.
(Because, a nbd of a point must be uncountable).

S	$\text{int}(S)$	$\text{bd}(S)$
ϕ	×	ϕ
$\{a_1, a_2, ---, a_n\}$	×	$\{a_1, a_2, ---, a_n\}$
$\{a_n : n \in \mathbb{N}\}$	×	$\{a_n : n \in \mathbb{N}\}$
$\mathbb{Z}$	×	$\mathbb{Z}$
$\mathbb{N}$	×	$\mathbb{N}$
$\mathbb{Q}$	×	$\mathbb{Q}$
$\mathbb{Q}^c$	×	$\mathbb{Q}^c$
T	×	T

$T, \mathbb{Q}, \mathbb{Q}^c, \mathbb{N}, \mathbb{Z}$ → Don't have a interior point.
→ Every point is a boundary point.

S	int(S)	bd(S)
$\mathbb{Q}^c$	ϕ	all paint.
$\mathbb{R}$	all paint	ϕ
(a, b)	all point	ϕ
$[a, b)$	(a, b)	$\{a\}$
$[a, b]$	(a, b)	$\{a, b\}$
$(1,2) \cup (2,3] \cup [4,5]$	$(1,2) \cup (2,3) \cup (4,5)$	$\{3,4,5\}$
T	ϕ	T

☆ Boundary paint :-

A paint $x \in S$ is said to be a boundary paint.

If $\forall \epsilon > 0$, $(x-\epsilon, x+\epsilon) \cap S^c \neq \phi$
and $(x-\epsilon, x+\epsilon) \cap S \neq \phi$

⇒ Two Types of boundary paint :-

① Isolated point :-

A paint $x \in S$ is said to be isolated point.

If $\exists\ \epsilon > 0$ s.t. $(x-\epsilon, x+\epsilon) \cap S = \{x\}$.

② Non-isolated point :- A boundary point $x \in S$ is said to be non-isolated. If $\forall \epsilon > 0$, $(x-\epsilon, x+\epsilon) \cap S$ has atleast one point other than x.

↓	Isolated	non-isolated.
$\{a_1, a_2, \ldots, a_n\}$	ϕ	ϕ
$\{a_n : n \in \mathbb{N}\}$	depend on a_n	
$\mathbb{Z}$	$\mathbb{Z}$	ϕ
$\mathbb{N}$	$\mathbb{N}$	ϕ
$\mathbb{Q}$	ϕ	$\mathbb{Q}$
ϕ	ϕ	ϕ

$\Rightarrow \mathbb{N}, \mathbb{Z}$ → Every point is isolated.
→ But $\mathbb{Q}$ → don't have an isolated point
→ Has a non-isolated point.

☆ Frontier point of a set :- A point $x \in \mathbb{R}$ is said to be frontier point of a set S.

If $\forall \epsilon > 0$, $(x-\epsilon, x+\epsilon) \cap S \neq \phi$ and $(x-\epsilon, x+\epsilon) \cap S^c \neq \phi$

$\therefore$ $\boxed{Fr(S) = bd(S) + bd(S^c)}$ $\boxed{S = int(S) + bd(S)}$

★ Exterior point :-

A point $x \in \mathbb{R}$ is said to be exterior point of a set S.

If $\exists\ \epsilon > 0$ s.t. $(x-\epsilon, x+\epsilon) \cap S = \phi$

or $(x-\epsilon, x+\epsilon) \subseteq S^c$

So, x is an exterior point of S iff x is an interior point of S^c.

$\Rightarrow$ ★ $\boxed{Ext(S) = int(S^c)}$

① $S \cup S^c = \mathbb{R}$

② $Int(S) \cup bd(S) \cup Int(S^c) \cup bd(S^c)$

$= int(S) \cup Fr(S) \cup ext(S)$

★ Adherent point :-

A real number $a \in \mathbb{R}$ is said to be an adherent point of a set S in $\mathbb{R}$.

If $\forall\ \epsilon > 0$, $(a-\epsilon, a+\epsilon) \cap S \neq \phi$

So, Every element of S is an adherent point of S.

$\Rightarrow$ Every boundary point of S^c is an adherent point of S.

So, $adh(S) = S \cup bd(S^c)$
$= int(S) \cup Fr(S)$
$= \mathbb{R} - ext(S)$

⇒ ① Every interior point of S is adherent point of S.

② Every non-isolated boundary point of S is an adherent point.

③ Every isolated point of S is an adherent point.

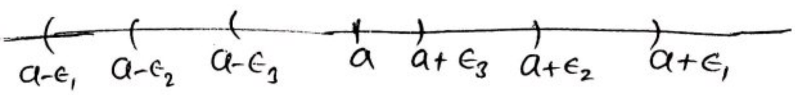
④ Every point in boundary of S^c is adherent point of S.

⑤ ~~Ram~~ Any point of ext(S) is never an adherent point of S.

☆ Limit point of a set :- A real number $a \in \mathbb{R}$ is said to be a limit point of a set S in $\mathbb{R}$.

If $\forall \epsilon > 0$, $(a-\epsilon, a+\epsilon)$ Contains atleast one element of S other than x.

OR $a \in \mathbb{R}$ is a limit point of a set S in $\mathbb{R}$.
If $\forall \epsilon > 0$, $(a-\epsilon, a+\epsilon)$ has infinite elements of S.



⇒ ① Every exterior point of s is a limit of s.

② Every non-isolated boundary point is a limit point.

③ Isolated point is never limit point.

④ Every point in boundary of s^c is a limit point of s.

⑤ Exterior point of a set is never a limit point of s.

☆ Closure set :- The union of a set S and it's derived set S' is called the closure of S.

⇒ It is denoted by $\boxed{\bar{S} = S \cup S'}$.

⇒ $\boxed{\bar{S} = \text{Adh}(S)}$ and here $\bar{S} = [\text{ext}(S)]^c$

⇒ $\boxed{\bar{S} = S + \text{bd}(S^c)}$

☆ Derived set :- The set of all the limit points of S is called the derived set of S.

⇒ It is denoted by S'

$$S' = \text{int}(S) \cup \text{Fr}(S) \setminus \text{iso}(S)$$
$$= \mathbb{R} \setminus (\text{ext}(S) \cup \text{iso}(S))$$

So, Every interior point of S is a limit point of S.

Set of limit point :- $\mathbb{N} \longrightarrow \phi$
$\mathbb{Q} \longrightarrow \mathbb{R}$
$\mathbb{Q}^c \longrightarrow \mathbb{R}$

$\Rightarrow$ $\mathbb{Q}$ and $\mathbb{Q}^c$ $\rightarrow$ Every point is non-isolated boundary point.

Q. Find the derived set of the following sets:-

1. $\phi \longrightarrow \phi$
2. $\{a_1, a_2, \cdots\} \rightarrow \phi$
3. $\mathbb{N} \longrightarrow \phi$
4. $\mathbb{Z} \longrightarrow \phi$
5. $\mathbb{Q}, \mathbb{Q}^c \longrightarrow \mathbb{R}$
6. $\mathbb{R} \longrightarrow \mathbb{R}$
7. $(a, b) \rightarrow [a,b]$
8. $(a, b] \rightarrow [a, b]$
9. $[a, b) \rightarrow [a, b]$
10. $[a, b] \rightarrow [a, b]$
11. $\{\frac{1}{n}, n \in \mathbb{N}\} \longrightarrow \{0\}$
12. $\{n^2 : n \in \mathbb{N}\} \rightarrow \phi$
13. $\{\sin \frac{n\pi}{4} : n \in \mathbb{N}\} \rightarrow \phi$
14. $\{\sin n : n \in \mathbb{N}\} \rightarrow [-1, 1]$

☆ Limit point:- A number $x \in \mathbb{R}$ is said to be a limit point of a set $S \subseteq \mathbb{R}$.

If $\forall \epsilon > 0$, $(x-\epsilon, x+\epsilon)$ contains infinitely many elements of S.

$\Rightarrow$ So, A finite set has no limit point.

* Different types of sets :-

① Open set :-

A set S is said to be open.

If $\boxed{bd(S) = \phi}$

OR

If every element of S is an interior point of S.

i.e. $\boxed{int(S) = S}$

⇒ A set S is said to be open if S is a nbd of each of it's points.

⇒ Non-empty countable set can never be an open set.

⇒ Empty set is an open set.

Ex $S = (1, 2)$

② Closed set :-

A set S is said to be closed.

If $\boxed{bd(S^c) = \phi}$

OR

If $\boxed{S' \subseteq S}$

→ A set S is said to be closed if every limit point of S is an element of S.

⇒ S is closed if $S = \bar{S}$

☆ ⇒ $\mathbb{R}$ and ϕ are only set which are both open and closed. [∵ bd of ($\mathbb{R}$ & $\mathbb{R}^c$) = ϕ]

⇒ S is open set than bd(S) = ϕ, S is uncountable.

⇒ n^2, $n \in \mathbb{N}$ → open set (No limit point).

Q. which of the following set are open ?

(1) ϕ → Both open & Closed.

(2) $\{a_1, a_2, \dots, a_n\}$ → Not open But closed.

(3) $\mathbb{N}$ → Not open ~~and not~~ But closed.

(4) $\mathbb{Z}$ → Not open and not closed.

(5) $\mathbb{Q}$ → "

(6) $\mathbb{R}$ → open and closed.

(7) $\mathbb{Q}^c$ → Not open and not closed.

(8) (a, b) → opent and not closed.

(9) $[a, b]$ → not open and

(10) $[a, b)$ → not open and not closed.

(11) $\{\frac{1}{n} : n \in \mathbb{N}\}$ → not open and not closed.

☆ Dense in itself :- A set S is said to be dense in itself.

If Every element of S is a limit point of S.

OR

⇒ A set S is dense in itself if $\boxed{S \subseteq S'}$

⇒ A set S is dense in itself if $\boxed{\bar{S} = S'}$

⇒ A set S is dense in itself if $\boxed{iso(S) = \phi}$

Eg → ϕ, $\mathbb{Q}$, $\mathbb{Q}^c$, (a,b), $[a,b)$, $[a,b]$, ~~$\mathbb{R}$~~, ~~$\mathbb{Z}$~~ . → ✓

→ $\mathbb{R}$, $\mathbb{N}$, $\mathbb{Z}$, $\{\frac{1}{n}; n \in \mathbb{N}\}$. → ✗

$\boxed{\mathbb{R} = \bar{S} \cup ext(S)}$

☆ perfect set :-

A set S is said to be perfect. If it is both closed and dense in itself.

OR

⇒ A set S is perfect if $S = S' (= \bar{S})$

Ex $\mathbb{R}$, ϕ, $[a,b]$

⇒ $\boxed{S \text{ is perfect in } \mathbb{R} \text{ if } bd(S^c) = \phi \text{ and } int(S^c) = \phi}$

☆ Dense set or Dense in $\mathbb{R}$ or Everywhere dense set. :-

A set S is dense in $\mathbb{R}$ if $\bar{S} = \mathbb{R}$

OR

S is dense set in $\mathbb{R}$ if $\boxed{ext(S) = \phi}$

OR S is dense in $\mathbb{R}$ if every real number is either an element of S or a limit point of S.

Ex → $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Q}^c$ → Everywhere dense.

⇒ $S = \mathbb{R} - \mathbb{N}$ → dense in $\mathbb{R}$

⇒ $\mathbb{Z}^c$, $\mathbb{N}^c$ → Dense in $\mathbb{R}$

☆ Nowhere dense set ÷ A set S in $\mathbb{R}$ is said to be nowhere dense set.

If $(\bar{S})^c$ is dense in $\mathbb{R}$.

i.e. S is nowhere dense if $\overline{(\bar{S})^c} = \mathbb{R}$.

$\Rightarrow (\bar{S})^c$ is every where dense.

$\boxed{ext(\bar{S})^c = \phi \ , \ int(\bar{S}) = \phi}$

$\Rightarrow$ S is nowhere dense if $ext.[ext(S)] = \phi$

☆ $\Rightarrow$ Every point of $\mathbb{N}, \mathbb{Z}$ is isolated.

Ex $\mathbb{N}$, $\mathbb{Z}$, finite set $\leftarrow$ Nowhere dense.

Ex $S = \mathbb{N}$

$ext(S) = (-\infty, 1) \cup (1,2) \cup (2,3) \cup (3,4) \cup ---$

$\overline{ext(S)} = \mathbb{R}$

☆☆ $\rightarrow$

1. Dense in itself $\rightarrow iso(S) = \phi$
2. perfect set $\rightarrow iso(S) = \phi \ , \ bd(S^c) = \phi$
3. Every where dense $\rightarrow ext(S) = \phi$
4. Nowhere dense $\rightarrow int(\bar{S}) = \phi$

☆ Compact set ÷ A set S in $\mathbb{R}$ is said to be Compact set.

If it is both closed and bounded.

$\Rightarrow$ Every finite set is compact set.

$\Rightarrow [a,b]$ is a compact set.

Heine – Borel Theorem:

A set S in $\mathbb{R}$ is Compact iff it is closed and bounded in $\mathbb{R}$.

Ex
(1) $\phi \to$ Compact set.

(2) $\{a_1, a_2, a_3, \dots, a_n\} \to$ Compact set.

(3) $\mathbb{N} \to$ Don't Compact. [$\because$ Bounded But not closed]

(4) $\mathbb{R} \to$ close But not bounded.

(5) $(a,b) \to$ Bounded But not close.

(6) $[a,b] \to$ close and Bounded.

(7) $\mathbb{Q}, \mathbb{Q}^c, \mathbb{R}, \mathbb{Z} \to$ not Compact set.

(8) $\{\frac{1}{n} : n \in \mathbb{N}\} \to$ not Compact set. [$\because$ Don't closed].

Ex (i) $S = (0,1] \cup [-2,3] \cup \{2,3,4\}$
$= [-2,3] \cup \{4\}$

So, $S \to$ Compact set.

(ii) $S = \{x \in \mathbb{R} : \sin x = \frac{1}{2}\}$

$\sin x = \frac{1}{2}$
$x = \sin^{-1}(\frac{1}{2}) = 2n\pi \pm \frac{\pi}{6}$

$\left(\dots, \frac{11\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}, \dots \right)$

this is not Closed.

So, $S \to$ Don't Compact set.

☆ Connected set :- A set S in $\mathbb{R}$ is said to be Connected.

If it is either empty (ϕ) or a Singleton($\{a\}$) or an interval.

Ex $\mathbb{R} \rightarrow$ Connected set.

$\Rightarrow$ $\mathbb{Q}, \mathbb{Q}^c, \mathbb{N}, \mathbb{Z} \rightarrow$ Don't Connected.

$\Rightarrow$ If a set S can be expressed as a union of two separated sets then S is called disconnected sets.

☆ Disjoint set :- Two sets S and T in $\mathbb{R}$ are said to be disjoint.

If $\boxed{S \cap T = \phi}$

☆ Separated sets :- Two sets S and T in $\mathbb{R}$ are said to be separated.

If $S \cap \bar{T} = \phi$ and $\bar{S} \cap T = \phi$

Eg. $S = (1,2)$, $T = [4,5]$ are separated sets.

$S = (1,2)$, $T = (2,3)$ are separated sets.

But $S = (1,2)$, $T = [2,3]$ are not separated set.

☆ $\Rightarrow$ Two separated sets are always disjoint But two disjoint sets need not be separated.

However,

If Both sets are open (or closed) then they are separated iff they are disjoint.

$\Rightarrow$ If S and T both are closed then.

$$S \cap T = \phi$$
$$\Rightarrow \bar{S} \cap T = \phi \quad (\because \bar{S} = S)$$
$$\& \quad S \cap \bar{T} = \phi \quad (\because \bar{T} = T)$$

$\Rightarrow$ If S and T Both are open then

S is open $\Rightarrow bd(S) = \phi$

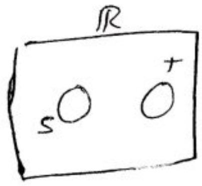
If $S \cap T = \phi \Rightarrow T \subseteq S^c$
$\Rightarrow \bar{T} \subseteq S^c \Rightarrow \bar{T} \cap S = \phi$

T is open $\Rightarrow bd(T) = \phi$

If $S \cap T = \phi \Rightarrow S \subseteq T^c$
$\Rightarrow \bar{S} \subseteq T^c$
$\Rightarrow \bar{S} \cap T = \phi$

$\Leftarrow$ Given:- S and T are open and $S \cap T = \phi$

$\therefore S \cap T = \phi \Rightarrow T \subseteq S^c$



Now, we claim that $\bar{T} \subseteq S^c$.

If possible, Let x is a limit point of T and $x \in S$.

$\Rightarrow x$ is a boundary point of S. But S is open

So, $bd(S) = \phi$

So, if x is a limit point of T then $x \notin S$.

$\therefore \bar{T} \subseteq S^c$

$$S \cap \bar{T} = \phi$$

(In General)

☆ Connected set :- A set S in $\mathbb{R}$ is said to be Connected.

If it can't be expressed as a union of two non-empty separated sets.

Ex → Singlton set → Disconnected.

→ $S = \{x \in \mathbb{R} : x^2 - x - 2 \leq 0\}$ ← Connected set.

→ $S = \{x \in \mathbb{R} : x^2 - x - 2 > 0\}$ ← Disconnected set.

↳ Don't compact set.

→ $S = [-1, 2]$ → Compact.

→ $S = (-\infty, -1] \cup [2, \infty)$ ← Don't Compact.

☆ Properties of different types of sets :-

(1) Union of arbitrary family of open sets is an open set.

Let A be a set of real numbers then

$S = \{S_\alpha : \alpha \in A\}$

$S = \{S_n ; n = 1, 2, 3, \ldots, 1000\}$

$S = \{S_n : n = 1, 2, 3, \ldots\}$

$S = \{S_\alpha : \alpha \in [0,1]\}$

Proof :- Let $S = \bigcup_{\alpha \in A} S_\alpha$ where each S_α is an open set.

Now, Let x be any element of S.

$\because x \in S \Rightarrow x \in \bigcup_{\alpha \in A} S_\alpha$

$\Rightarrow \exists\, \alpha_k \in A$ s.t. $x \in S_{\alpha_k}$

$\because S_{\alpha_k}$ is an open set.

$\Rightarrow S_{\alpha_k}$ is a nbd of x.

$\Rightarrow S$ is a nbd of x. $(\because S_{\alpha_k} \subseteq S)$

$\Rightarrow S$ is ~~an~~ an open set.

(2) Finite intersection of open sets is an open set.

Proof:- Let $S = \bigcap_{\alpha=1}^{n} S_\alpha$, where each S_α is an open set.

Now, Let x be any element of S.

$\because x \in S \Rightarrow x \in \bigcap_{\alpha=1}^{n} S_\alpha$

$\Rightarrow x \in S_\alpha \quad \forall \alpha = 1, 2, 3, \ldots, n$

$\because S_\alpha$ is an open set. $\forall \alpha = 1, 2, 3, \ldots, n$.

$\Rightarrow S_\alpha$ is a nbd of x. $\forall \alpha = 1, 2, \ldots, n$.

$\Rightarrow S$ is a nbd of x. $(\because S = \bigcap_{\alpha=1}^{n} S_\alpha)$

$\Rightarrow S$ is an open set.

(3.) Intersection of arbitrary family of open sets need not be an open set.

Ex:- $S_n = \{(-n, n) : n \in \mathbb{N}\}$

$\bigcap_{n=1}^{\infty} S_n = (-1, 1)$

$S_n = \{(-\frac{1}{n}, \frac{1}{n}) : n \in \mathbb{N}\}$

$\bigcap_{n=1}^{\infty} S_n = \{0\}$ $\qquad \Rightarrow \bigcap_{n=1}^{\infty} \left(\frac{-1}{n}, 1 + \frac{1}{n}\right) = [0, 1]$

(4.) A set S is open iff S^c is closed.

Let S is closed
$\Rightarrow bd(S^c) = \phi$
$\Rightarrow S^c$ is open

Let S is open.
$\Rightarrow bd(S) = \phi$
$\Rightarrow bd((S^c)^c) = \phi$
$\Rightarrow bd(S^c)$ is closed.

(5.) The set of all open disjoint open intervals is always countable.

(6.) Intersection of arbitrary family of closed sets is a closed set.

(7.) Union of finite family of closed sets is a closed set.

(8.) Union of arbitrary family of closed set is need not be a closed set.

$$S = \bigcap_{\alpha \in A} S_\alpha$$

$$\Rightarrow S^c = \left(\bigcap_{\alpha \in A} S_\alpha\right)^c$$

$$= \bigcup_{\alpha \in A} S_\alpha^c$$

$\Rightarrow S^c$ is an open set.
$\Rightarrow S$ is a closed set.

(9.) Intersection of arbitrary family of bounded sets is a bounded set.

(10.) Union of finite family of bounded sets is a bounded set.

(11) Union of arbitrary family of bounded sets is need not be a bounded set.

Ex $S_n = \{[-n,n] : n \in \mathbb{N}\}$ $\bigcup_{n=1}^{\infty} S_n = \mathbb{R}$ is not bounded.

(12) Intersection of arbitrary family of Compact sets is a Compact set.

(13) Union of finite family of Compact sets is a Compact set.

(14) Union of arbitrary family of Compact sets is need not be a Compact set.

(15) Intersection of two Connected sets is also Connected.

(16) Union of two Connected sets need not be Connected.

Ex - $[1,2] \cup [2,3] = [1,3]$

$[1,2] \cup [3,4] = [1,2] \cup [3,4]$

$[1,3] \cup [2,4] = [1,4]$

(17) Union of two Connected sets having non-empty intersection is Connected.

(18) Every subset of a bounded set is bounded.

(19) Every closed subset of a Compact set is Compact.

(20) Continuous image of a Compact set is a Compact set.

(21) Continuous image of a bounded set need not be bounded set.

Ex:- $f:(0,1)\to\mathbb{R}$ s.t. $f(x)=\frac{1}{x}$

(22) Continuous image of a closed set need not be closed set.

Ex $f:[1,\infty)\to\mathbb{R}$ s.t. $f(x)=\frac{1}{x}$

$f([1,\infty)) = (0,1]$

(23) Continuous image of an open set need not be open set.

Ex:- $f:(0,1)\to\mathbb{R}$
s.t. $f(x)=2$

(24) Continuous image of a connected set is always connected set.

(25) A function $f: D\to\mathbb{R}$ is continuous on D iff for every open set U in $\mathbb{R}$, $f^{-1}(U)$ is open in D.

$\Rightarrow$ If $f:[0,1]\to\mathbb{R}$ is onto.
then f is not a continuous function.

Ex $f:(0,1)\to\mathbb{R}$ s.t. $f(x)=2$

$f:(0,1)\cup(2,3)\to\mathbb{R}$ s.t. $f(x)=2$. ←

$\Rightarrow$ i.e. Continuous pre-image of an open set is an open set.

(26.) Sum of two open sets is an open set.

(27.) Sum of two bounded set is a bounded set.

(28.) Sum of two Compact set is a Compact set.

(29.) Sum of two Connected set is a Connected set.

(30) Sum of a Closed set and a Compact set a Closed set.

(31) Sum of two closed sets need not be a closed set.

Ex $S = \mathbb{N}$, $S' = \phi$

$T = \{-n + \frac{1}{n} : n \in \mathbb{N}\}$, $T' = \phi$

$S + T = \{m - n + \frac{1}{n} : m, n \in \mathbb{N}\}$

$\Rightarrow$ S is ~~not~~ not open than S^c is not closed. and S is not closed than S^c is not open.

However, If S is not open then S may be Closed or may not be closed

Ex → $\mathbb{Q}$, $\mathbb{N}$.

and if S is not closed than S may be open or may not be open

Ex (a, b)

(32) Continuous pre-image of a close set need not be close in $\mathbb{R}$.

☆ properties of different types of sets :-

(●) Derived set and limit point :-

(1) (i) Every infinite bounded set has [at least] a limit point.

if a infinite bounded set then $S' \neq \phi$.

(Bolzano - weierstress Theorem)

(2) (ii) $A \subseteq B$ than $A' \subseteq B'$.

Let $x \in A'$ $\Rightarrow$ x is a limit point of A.

$\Rightarrow \forall \epsilon > 0$ $[(x-\epsilon), (x+\epsilon)]$ has infinite element of A $\forall$ $x \in B'$.

(iii) $\boxed{(A \cup B)' = A' \cup B'}$

$$A \subseteq (A \cup B) \Rightarrow A' \subseteq (A \cup B)'$$
$$B \subseteq (A \cup B) \Rightarrow B' \subseteq (A \cup B)'$$
$$A' \cup B' \subseteq (A \cup B)'$$

Now, Let $x \in (A \cup B)'$

x is a limit point of $A \cup B$.

$\forall \epsilon > 0$. $(x-\epsilon, x+\epsilon)$ has infinite elements of $A \cup B$.

$\forall \epsilon > 0$, $(x-\epsilon, x+\epsilon)$ has

$\forall \epsilon > 0$ $(x-\epsilon, x+\epsilon)$

$\Rightarrow$ x is a limit point of A or x is a limit point of B.

(iv) $\boxed{(A\cap B)' \subseteq A'\cap B'}$

$A\cap B \subseteq A \Rightarrow (A\cap B)' \subseteq A'$

$A\cap B \subseteq B \Rightarrow (A\cap B)' \subseteq B'$

$(A\cap B)' \subseteq A'\cap B'$

But, $A = \mathbb{Q}$ $\quad B = \mathbb{Q}^c$

$A' = \mathbb{R}$ $\quad B' = \mathbb{R}$

$A\cap B = \phi$ $\quad (A\cap B)' = \phi$

$(A\cap B) \subseteq A'\cap B'$

$\phi \subset \mathbb{R}\cap\mathbb{R}$

$\boxed{\phi \subseteq \mathbb{R}}$

(v) If a set is compact then it has the maximum and minimum element.

(a) If the supremum of a bounded above sets is not in S then it is a limit point of S.

M is the supremum of a set S. then

(i) $\forall x \in S$, $x \leq M$

and (ii) $\forall \epsilon > 0$, $\exists x \in S$, $x > M - \epsilon$

Now let M is the supremum of S and $M \notin S$

than $\forall \epsilon > 0$, $\exists x \in S$ s.t. $M - \epsilon < x \leq M < M + \epsilon$

$\forall \epsilon > 0$, $\exists x \in S$ s.t. $x \in (M-\epsilon, M+\epsilon)$ and $x \neq M$

M is a limit point of S. $\quad$ ($\because x \in S$ and $M \notin S$)

(b) If the infimum of bounded below set is not in S than it is a limit point of S.
m is the infimum of a set S then

(i) $\forall x \in S$, $x > m$

and $\forall \epsilon > 0$, $x < m + \epsilon$

than $\forall \epsilon > 0$, $x \in S$ $\quad m - \epsilon < m \leq x \leq m - \epsilon$

$\forall \epsilon > 0$, $\exists x \in S$ s.t. $x \in (m - \epsilon, m + \epsilon)$ and $x \neq m$.

[$x \in S$ But $m \notin S$]

m is the limit point of S.

(c) If the infimum of a bounded below set S. is not in S then it is a limit point of S.

$\therefore$ If $Sup(S) \notin S$ then greatest limit point $= Sup(S)$.
If $Inf.(S) \notin S$ —— least —— $= Inf.(S)$.

(d) Sup.(S) is an upper bounded of S' and infimum S is a lower bounded of S'

If $Sup(S) \in S$ then $Sup(S) = Sup(S')$
and if $Inf(S) \notin S$ then $inf(S) = Inf(S')$

(e) If a sets is a close/bounded (Compact) $Sup(S) \in S$ and $inf(S) \in S$

But If $Sup(S) \in S$ and $inf(S) \in S$ then S need not be closed.

(vi) For every set S in $\mathbb{R}$ S' is a closed set.

$x \in S'' \Rightarrow x$ is a limit point of S'.

$\Rightarrow \forall \epsilon > 0$, $(x-\epsilon, x+\epsilon)$ has infinite element of S'.

(vii) If $S'' = \phi \Rightarrow S'$ is closed and if $S'' \neq \phi$ then let $x \in S''$.

$\Rightarrow x$ is a limit point of S'.

$\Rightarrow \forall \epsilon > 0$, $(x-\epsilon, x+\epsilon)$ has at least one element of S' other than x. Let $y \in S'$ and

$y \in (x-\epsilon, x+\epsilon)$, $y \neq x$

Let $\delta = \min\{x+\epsilon-y,\ y-x-\epsilon\}$

then $(y-\delta, y+\delta) \subseteq (x-\epsilon, x+\epsilon)$

Since $y \in S'$ so, $(y-\delta, y+\delta)$ has infinite element of S. So, $(x-\epsilon, x+\epsilon)$ has infinite elements of S.

x is a limit point of S.

$x \in S'$

S' is closed.

Close $\rightarrow$ sup./inf. $\in S$

Sup./inf. $\not\Rightarrow$ close not necessary.

always $S' \rightarrow$ is a closed set.

☆ Properties of different types of points :-

① Closure of a set :-

(i) $\bar{A}$ is a closed set.

(ii) If $A \subseteq B$ then $\bar{A} \subseteq \bar{B}$.

(iii) $\bar{A}$ is the smallest closed set which contains A.

(iv) A is closed iff $\bar{A} = A$.

(v) $\overline{A \cup B} = \bar{A} \cup \bar{B}$.

(vi) $\overline{A \cap B} \subseteq \bar{A} \cap \bar{B}$ But $\bar{A} \cap \bar{B}$ need not be subset of $\overline{A \cap B}$.

proof :- (v) $A \subseteq A \cup B$

$\Rightarrow \bar{A} \subseteq \overline{A \cup B}$

& $B \subseteq A \cup B$

$\bar{B} \subseteq \overline{A \cup B}$

$\Rightarrow \bar{A} \cup \bar{B} \subseteq \overline{A \cup B}$

$\overline{A \cup B} \subseteq \bar{A} \cup \bar{B}$

Let $x \in \overline{A \cup B} \Rightarrow x \in A \cup B$ or x is a limit point of $A \cup B$.

If $x \in A \cup B \Rightarrow x \in A$ or $x \in B$

If $x \in A \Rightarrow x \in \bar{A}$ $(\because A \subseteq \bar{A})$

$\Rightarrow x \in \bar{A} \cup \bar{B}$ $(\because \bar{A} \subseteq \bar{A} \cup \bar{B})$

If $x \in B \Rightarrow x \in \bar{A} \cup \bar{B}$.

If x is a limit point of $A \cup B$.

$\Rightarrow$ Every nbd of x Contains infinite elements of $A \cup B$.

$\Rightarrow$ Every nbd of x Contains infinite elements of A.

$\Rightarrow$ Every nbd of x Contains infinite elements of B.

$\Rightarrow$ x is a limit point of A or x is a limit point of B.

$\Rightarrow$ $x \in \overline{A}$ or $x \in \overline{B}$ $\Rightarrow x \in \overline{A} \cup \overline{B}$.

proof:- (vi) $A \cap B \subseteq A$

$\Rightarrow \overline{A \cap B} \subseteq \overline{A}$

& $A \cap B \subseteq B$

$\Rightarrow \overline{A \cap B} \subseteq \overline{B}$

So, $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$.

Ex If $A = \mathbb{Q}$, $B = \mathbb{Q}^c \Rightarrow A \cap B = \phi \Rightarrow \overline{A \cap B} = \phi$

$\overline{A} = \mathbb{R}$, $\overline{B} = \mathbb{R} \Rightarrow \overline{A} \cap \overline{B} = \mathbb{R}$

proof:- Let $x \in \overline{A}$

$\Rightarrow x \in A$ or x is a limit point of A.

If $x \in A \Rightarrow x \in B$ $(\because A \subseteq B)$

$\Rightarrow x \in \overline{B}$ $(\because B \subseteq \overline{B})$

If x is a limit point of A.

$\Rightarrow$ Every nbd of x Contains infinite elements of A.

$\Rightarrow$ Every nbd of x Contains infinite elements of B.

$\Rightarrow$ x is a limit point of B. $(\because A \subseteq B)$

$\Rightarrow x \in \overline{B}$.

② Interior of a set :- ($int(A) = A^{\circ}$)

(i) int(A) is an open set.

(ii) If $A \subseteq B$ then $int(A) \subseteq int(B)$.

proof:- $x \in int(A)$
$\Rightarrow A$ is a nbd of x.
$\Rightarrow B$ is a nbd of x. ($\because A \subseteq B$)
$\Rightarrow x \in int(B)$.

(iii) int(A) is the largest open subset of A.

(iv) A is open iff $int(A) = A$.

(v) $int(A \cap B) = int(A) \cap int(B)$.

proof:- $int(A \cap B) \subseteq int(A) \cap int(B)$
$x \in int(A) \cap int(B)$
$\Rightarrow x \in int(A)$ and $x \in int(B)$
$\Rightarrow A$ is a nbd of x and
B is a nbd of x.
$\Rightarrow A \cap B$ is a nbd of x.
$\Rightarrow x \in int(A \cap B)$.

(vi) $int(A \cup B) \supseteq int(A) \cup int(B)$.
But $int(A \cup B)$ need not be equal to $int(A) \cup int(B)$.

proof:- $A \subseteq A \cup B$
$\Rightarrow int(A) \subseteq int(A \cup B)$
& $B \subseteq A \cup B$
$int(B) \subseteq int(A \cup B)$.

(3) Exterior of a Set :-

(i) $ext(A)$ is an open set.

(ii) If $A \subseteq B$ then $ext(A) \supseteq ext(B)$.

(iii) $ext(A)$ is the largest open subset of A^c.

(iv) A is closed iff $ext(A) = A^c$.

(v) $ext(A \cup B) = ext(A) \cap ext(B)$.

proof:- $int(A \cap B) = int(A) \cap int(B)$

$\Rightarrow int(A^c \cap B^c) = int(A^c) \cap int(B^c)$

$\Rightarrow int((A \cup B)^c) = ext(A) \cap ext(B)$

$\Rightarrow ext(A \cup B) = ext(A) \cap ext(B)$.

(vi) $ext(A \cap B) \supseteq ext(A) \cup ext(B)$ But it's Converse is not true.

proof:- $int(A \cup B) \supseteq int(A) \cup int(B)$.

$\Rightarrow int(A^c \cup B^c) \supseteq int(A^c) \cup int(B^c)$.

$\Rightarrow int[(A \cap B)^c] \supseteq ext(A) \cup ext(B)$

$\Rightarrow ext(A \cap B) \supseteq ext(A) \cup ext(B)$

Ex $A = \mathbb{Q}$, $B = \mathbb{Q}^c$

(4) Derived set of a set :-

(i) A' is always a closed set.

(ii) If $A \subseteq B$ then $A' \subseteq B'$.

(iii) $(A \cup B)' = A' \cup B'$.

(iv) $(A \cap B)' \subseteq A' \cap B'$.

(9.) Frontier of a set :

(i) $Fr(A) = \bar{A} - int(A)$.

(ii) $Fr(A) = \phi$ iff A is both open and closed.

(iii) A is open iff $Fr(A) \subseteq A^c$.

(iv) A is closed iff $Fr(A) \subseteq A$.

Q. Set $S = \left\{ \frac{x^2}{1+x^2} , x \in \mathbb{R} \right\}$

Solⁿ: $S = [0, 1)$ $\Rightarrow$ S is Connected set.

Q. $S = \{x \in \mathbb{R} : x^6 - x^5 \leq 100\}$

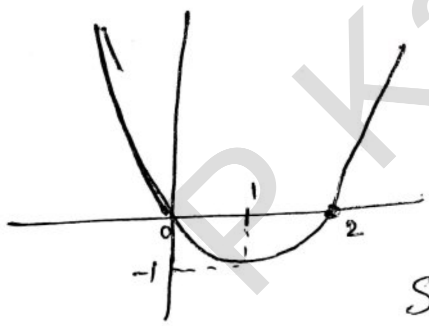
$T = \{x^2 - 2x : x \in (0, \infty)\}$

$S \cap T = ?$

Solⁿ: $x^6 - x^5 \leq 100$

$S = [\alpha, \beta]$ $\qquad 2 < \beta < 3$

$-3 < \alpha < -2$



$T = [-1, \infty)$

$S \cap T = [-1, \beta]$

$S \cap T \rightarrow$ Both Closed and Bounded.
So, this is Compact set.

Q. $E = \left\{ \frac{n}{n+1} : n \in \mathbb{N} \right\}$, $F = \left\{ \frac{1}{1-x} : 0 \leq x < 1 \right\}$

Solⁿ: $E = \left\{ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \ldots \right\} = [\frac{1}{2}, 1)$

E = does not Closed set.

$F = [1, \infty)$ So, F is a Closed set.

Q. Let S be an infinite subset of $\mathbb{R}$ s.t. $S \cap \mathbb{Q} = \phi$ then.

× ① S must have a limit point which belongs to $\mathbb{Q}$.

× ② S must have a limit point which belongs to $\mathbb{Q}^c$.

③ S^c must have a limit point which belongs to S.

× ④ S can't be a closed set in $\mathbb{R}$.

Sol:- ① $S = \{\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{6}, \sqrt{7}, \sqrt{8}, \sqrt{10}, \sqrt{11}, \sqrt{12}, \sqrt{13}, \ldots\}$

So, limit point does not exist.

③ $S = \left((-\infty, \sqrt{2}) \cup (\sqrt{2}, \sqrt{3}) \cup (\sqrt{3}, \sqrt{5}) \cup \ldots \right) \cap \mathbb{Q}^c$

$S^c = \{\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{6}, \ldots\} \cup \mathbb{Q}$

Q. Let S is closed subset of $\mathbb{R}$, T is compact s.t. $S \cap T \neq \phi$ then $S \cap T$ is

① closed but not compact.

② not closed.

③ compact.

④ Neither closed nor compact.

☆ Decimal representation of a real number:

Every real number can be expressed in decimal form (in base 10 form).
The decimal ~~form~~ representation of a rational number is either terminating or non-terminating but repeating.

The decimal representation of an irrational number is non-terminating and non-repeating.

☆ Place value of digits in decimal representation:

$x = 252.735 = 2\times10^2 + 5\times10^1 + 2\times10^0 + 7\times10^{-1} + 3\times10^{-2} + 5\times10^{-3}$

$\forall x \in [0,1]$ $\boxed{x = \sum_{r=1}^{\infty} \frac{a_r}{10^r}}$

where a_r is a decimal digits.

i.e. $a_r \in \{0,1,2,3,4,5,6,7,8,9\}$

$\frac{2}{5} = 0.4 = \sum_{r=1}^{\infty} \frac{a_r}{10^r}$, $a_1 = 4$, $a_r = 0 \quad \forall r \geq 2$

$\frac{1}{3} = 0.3333\ldots = \sum_{r=1}^{\infty} \frac{a_r}{10^r} = \sum_{r=1}^{\infty} \frac{3}{10^r} = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \ldots$

$1 = \sum_{r=1}^{\infty} \frac{9}{10^r} = 0.9999\ldots = \frac{9}{10} + \frac{9}{10^2} + \frac{9}{10^3} + \ldots$

$= \frac{9/10}{1 - 1/10} = \frac{9/10}{9/10} = 1$

$\forall\ x \in [0,10]$, $\boxed{x = \sum_{r=0}^{\infty} \frac{a_r}{10^r}} = a_0 10^0 + \frac{a_1}{10^1} + \frac{a_2}{10^2} + \cdots$

Q. $S = \left\{ \sum_{r=1}^{\infty} \frac{a_r}{10^r} : a_r \in \{0,1,2,\cdots,9\} \right\}$

$S = [0,1]$

So, S is an uncountable set.

Ex $\sqrt{2} = 1.414259\cdots$

$= 1 + \frac{4}{10} + \frac{1}{100} + \frac{4}{10^3} + \frac{2}{10^4} + \frac{5}{10^5} + \frac{9}{10^6} + \cdots$

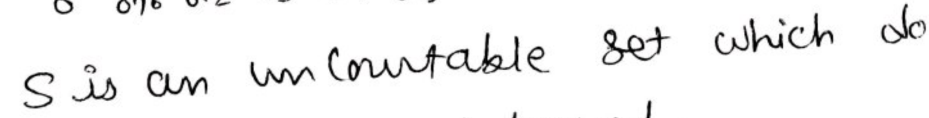
Ex ① $S = \left\{ \sum_{r=2}^{\infty} \frac{a_r}{10^r} : a_r \in \{0,1,2,3,4,5,6,7,8,9\} \right\}$

$S = [0, 0.1] \rightarrow$ Compact & Connected.

$\left(\because \frac{a_2}{10^2} = 0.09999\cdots = 0.1 \right)$

S is an uncountable set which contains an interval.

② $S = \left\{ \sum_{r=1}^{\infty} \frac{a_r}{10^r} : a_r \in \{0,1,2,3,4,6,7,8,9\} \right\}$



So, S is an uncountable set which does not contain any interval.
(If atleast one digit missing in a_r).

$\rightarrow$ S is Compact But not Connected.

☆ Representation of a real no. in base n System :

$\mathcal{D}$ = Digits : $\{0,1,2,3,\cdots,n-1\}$

$\forall x \in \mathbb{R}$

$$x = \sum_{k=-\infty}^{\infty} \frac{a_k}{n^k}, \quad a_k \in \mathcal{D}$$

and $\forall x \in [0,1]$

$$x = \sum_{k=1}^{\infty} \frac{a_k}{n^k}, \quad a_k \in \mathcal{D}.$$

Eg. Express $\frac{1}{4}$ in ternary system?

$a_k \in \{0,1,2\}$

$\boxed{(10)_5 = 1\times 5^1 + 0\cdot 5^0 = 5}$

$$\frac{1}{4} = 0\cdot\frac{1}{3} + 2\cdot\frac{1}{3^2} + 0\cdot\frac{1}{3^3} + 2\cdot\frac{1}{3^4} + \cdots$$

$$\Rightarrow \frac{1}{4} = (0.020202\cdots)_3 \qquad \left[\frac{1}{4} = \sum_{k=1}^{\infty} \frac{a_k}{3^k}\right.$$

☆ $S = \left\{ \sum_{n=1}^{\infty} \frac{a_n}{d^n} : d \in \mathbb{N},\ d > 1,\ a_n \in \{0,1,2,\cdots,d-1\} \right\}$

then $S = [0,1]$

☆ $S = \left\{ \sum_{n=0}^{\infty} \frac{a_n}{d^n} : d \in \mathbb{N},\ d > 1,\ a_n \in \{0,1,2,\cdots,d-1\} \right\}$

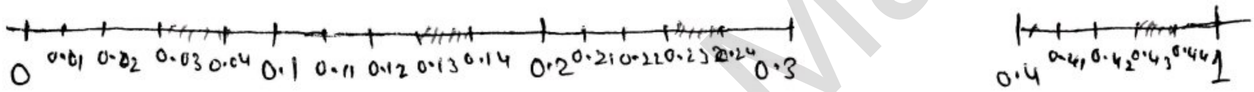
then $S = [0,d]$

Ex $S = \left\{ \sum_{n=1}^{\infty} \frac{a_n}{5^n} : a_n \in \{0,1,2,3,4\} \right\} = [0,1]$

$S = \left\{ \sum_{n=0}^{\infty} \frac{a_n}{5^n} : a_n \in \{0,1,2,3,4\} \right\} = [0,5]$

* ⇒ If $S = \left\{ \sum_{n=1}^{\infty} \frac{a_n}{5^n} : a_n \in \{0,1,2,4\} \right\}$

then S is an uncountable set which does not contain any interval.



* $S = \left\{ \sum_{n=1}^{\infty} \frac{a_n}{2^n} : a_n = 1 \right\}$ then S = Singleton set.

① Closed ② open ③ Compact ④ Connected.

$\Rightarrow 4.4444\ldots = 5 = 4 + \frac{4}{5} + \frac{4}{5^2} + \frac{4}{5^3} + \frac{4}{5^4} + \cdots$

$= \frac{4}{1-\frac{1}{5}} = \frac{4}{4/5} = 5$

$\Rightarrow C_0 \supseteq C_1 \supseteq C_2 \supseteq C_3 \supseteq \cdots$

☆ Cantor set :-

A set $S = \left\{ \sum_{n=1}^{\infty} \frac{a_n}{3^n} : a_n = 0 \text{ or } 2 \right\}$

is called Cantor set which is an uncountable set that does not contain any interval.

So, the ternary representation of Cantor set contains 0 and 2 only.

☆ Geometric reprentation of Cantor set :-

Let $C_0 = [0, 1]$

$C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$

$C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$

In this case C_n has 2^n sub-intervals of size $\frac{1}{3^n}$.

Now, $\boxed{\text{Let } C = \bigcap_{n=1}^{\infty} C_n = \lim_{n\to\infty} C_n}$

is called the Cantor set.

⇒ The number of Sub-intervals in $C_n = 2^n$

⇒ Length of each Sub-interval in $C_n = \frac{1}{3^n}$

☆ Properties of Cantor Set :-

① Cantor set is an uncountable set.

② Cantor set does not contain any interval. So, it can never be a nbd of any of it's points. So every element of Cantor set is a boundary of the set.

③ Cantor set is a closed set.

④ Cantor set is a Compact set.

⑤ Cantor set is not a Connected set.

⑥ Cantor set is dense in itself.

i.e. Every element of Cantor set is a limit point of the Cantor set.

⑦ Cantor set is a perfect set.

⑧ Cantor set is nowhere dense set.

$int(\bar{S}^c) = \phi$

~~⑨ Cantor~~

⑨ $int(C) = \phi$, $iso(C) = \phi$, $bd(C^c) = \phi$,

$non\text{-}iso(C) = C$.

(10) Cantor set is an uncountable set But length of Cantor set is zero.

proof:-

$$C = \bigcap_{n=1}^{\infty} C_n$$

$$\Rightarrow C^c = \left(\bigcap_{n=1}^{\infty} C_n\right)^c$$

$$= \bigcup_{n=1}^{\infty} C_n^c$$

$$\therefore m(C^c) = m\left(\bigcup_{n=1}^{\infty} C_n^c\right)$$

$$= \sum_{n=1}^{\infty} m(C_n^c)$$

$$= \frac{1}{3} + \frac{2}{3^2} + \frac{2^2}{3^3} + \cdots$$ [m(c) = major of Cantor]

$$\therefore m(C) = 1 - 1 = \underline{\underline{0}}.$$

$\Rightarrow$ Open sets in $\mathbb{R}$ are either empty set or an open interval or a countable union of disjoint open intervals.

$\Rightarrow$ Closed sets in $\mathbb{R}$ Either Contains and interval or nowhere dense.

Q. Let A be a non-empty subset of $\mathbb{R}$. Let $I(A)$ denote the set of int(A) then no. of elements in $|I(A)|$ can be ?

(1) Empty.
(2) Singleton
(3) A finite set containg more than one element.
(4) Countably infinite set.

Q. Let G and H be non-empty subsets of $\mathbb{R}$. Where G is connected and $G \cup H$ is not connected. which of the following is necessarily true.

(1) If $G \cap H = \phi$ then H is connected.
(2) If $G \cap H = \phi$ then H is not connected.
(3) If $G \cap H \neq \phi$ then H is connected.
(4) If $G \cap H \neq \phi$ then H is not connected.

Sol:- $\to G = [1,4] \to$ Connected, $H = [5,7]$ $\xrightarrow{\text{Conne}}$ or $[5,7] \cup [8,9]$

$G \cap H = \phi$

$G \cup H \to$ Dis connected.

$\to G = [1,3]$, $H = (2,4) \cup \{5\}$

$G \cup H =$ Dis connected

$G \cap H \neq \phi$

Q. Let $S \subseteq \mathbb{R}$ and $\partial S :=$ The set of those points $x \in$ in $\mathbb{R}$.

Such that every nbd of x contains some points of S as well as some points of S^c.

Sol:- $\partial S = Fr(S)$

$\bar{S} :=$ closure of S. (Fauls).

① $\partial \mathbb{Q} = \mathbb{R}$

② $\partial\left(\mathbb{R}/T\right) = \partial T$, $T \subseteq \mathbb{R}$

③ $\partial(T \cup V) = \partial T \cup \partial V$, $T, V \subseteq \mathbb{R}$, $T \cap V \neq \phi$

④ $\partial T = \bar{T} \cap (\overline{\mathbb{R} \setminus T})$

Sol:- ④ $\bar{T} = int(T) \cup Fr(T)$

$(\overline{T^c}) = int(T^c) \cup Fr(T^c)$

③ $T = (0,1)$, $V = (-2, 2)$

$\partial T = \{0, 1\}$ $\quad T \cup V = (-2, 2) = V$

$\partial V = \{-2, 2\}$ $\quad \partial(T \cup V) = \{-2, 2\}$

Q. If P and Q are two non-empty disjoint subsets of $\mathbb{R}$. Then which of the following is/are False?

① If P and Q are compact then $P \cup Q$ is also compact.

② If P and Q are not connected then $P \cup Q$ is also not connected.

③ If $P \cup Q$ and P are closed then Q is closed.

④ If $P \cup Q$ and P are open then Q is open.

Soln: ② $P = \mathbb{Q}$, $Q = \mathbb{Q}^c$

$P \cup Q = \mathbb{R}$ ← Closed & Bounded.

③ $P = \{1, 2\}$

$Q = (1, 2)$

$P \cup Q = [1, 2]$

④ $P = (1, 2)$

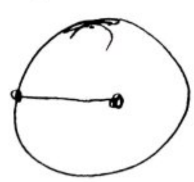
$Q = [2, 3)$

$P \cup Q = (1, 3)$

⇒ In open interval all point interior point.

☆⇒ A set S is said to be convex, if the line segment joining any two point is completly in curve.

Ex



☆ Charatestic function: $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$$

$\Rightarrow$ S is an interval :- $\forall a, b \in S$, $x \in S$
If $a < x < b$.

$\Rightarrow \{(m, n\sqrt{2}) : m, n \in \mathbb{Z}\} \rightarrow$ Set of isolated point.

$\Rightarrow X = \{m + n\sqrt{2} : m, n \in \mathbb{Z}\} \rightarrow$ is dense in $\mathbb{R}$.
i.e. $X' = \mathbb{R}$

Some Useful Links:

1. **Free Maths Study Materials (**https://pkalika.in/2020/04/06/free-maths-study-materials/**)**

2. **BSc/MSc Free Study Materials (**https://pkalika.in/2019/10/14/study-material/**)**

3. **MSc Entrance Exam Que. Paper: (**https://pkalika.in/2020/04/03/msc-entrance-exam-paper/**)**
   [JAM(MA), JAM(MS), BHU, CUCET, ...etc]

4. **PhD Entrance Exam Que. Paper: (**https://pkalika.in/que-papers-collection/**)**
   [CSIR-NET, GATE(MA), BHU, CUCET,IIT, NBHM, ...etc]

5. **CSIR-NET Maths Que. Paper: (**https://pkalika.in/2020/03/30/csir-net-previous-yr-papers/**)**
   [Upto Lastest CSIR NET Exams]

6. **Practice Que. Paper: (**https://pkalika.in/2019/02/10/practice-set-for-net-gate-set-jam/**)**
   [Topic-wise/Subject-wise]

7. **List of Maths Suggested Books** (https://pkalika.in/suggested-books-for-mathematics/)

8. **CSIR-NET Mathematics Details Syllabus (**https://wp.me/p6gYUB-Fc)

9. **CSIR-NET, GATE, PhD Exams, ...etc Study Materials & Solutions**
   https://pkalika.in/kalika-notes-centre/

10. **ONE SHOT Revision(Last Minute Preparation) for NET, GATE, SET, ..etc**
    https://www.youtube.com/playlist?list=PLDu0JgProGz5bU90lRgp2ksdfLe2Hay8I

11. **Topic-wise Video Lectures(Crash Course)**
    https://www.youtube.com/pkalika/playlists