P KALIKA NOTES

[For NET/GATE/SET/NBHM/JAM/CUCET/MSc/PhD Exams…etc.]

Special Function

[Handwritten Study Material]

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Your Note/Remarks

Content/Syllabus

Subject: SPECIAL FUNCTIONS

UNIT-I: Beta and Gamma Functions, Euler Reflection Formula, Stirling's Asymptotic Formula, Gauss's Multiplication Formula, Ratio of two gamma functions, Integral Representations for Logarithm of Gamma function and Beta functions. **(10 Lectures)**

UNIT-II. Hypergeometric Differential Equations, Gauss Hypergeometric Function, Elementary Properties, Conditions of convergence, Integral Representation, Gauss Theorem, Vandermonde's theorem, Kummer's theorem, Linear transformation, Generalized Hypergeometric Functions, Elementary Properties, Integral Representation. **(10 Lectures)**

UNIT-III: Legendre polynomials and functions, Solution of Legendre's differential equations, Generating Functions, Rodrigue's Formula, Orthogonality of Legendre polynomials, Recurrence relations. Bessel functions, Bessel differential equation and it's solution, Recurrence relation, Generating functions, Integral representation. **(10 Lectures)**

Recommended Books:

1. G. E. Andrews, R. Askey, Ranjan Roy, Special Functions, Encyclopedia of Mathematics and its Applications, Cambridge University Press, 1999.
2. E. D. Rainville, Special Functions, Macmillan, New York, 1960.

Special f^n (syllabus)

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UNIT-I
61-74

BETA & GAMMA f^n, Euler's Reflection formula, Stirling's Asymptotic formula. Gauss's Multiplication formula, Ratio of two gamma f^{ns}. Integral Representation for Logarithime of Gamma & Beta function.

UNIT-II
127-157

Hypergeometric DE, Gauss hypergeometric f^n, elementary properties conditions of Conguerence. Integral Representation, Gauss theorem, Vandermonde's theorem, Kummer's theorem, Linear transformation, Generalised hypergeometric f^n. Elementary properties. Integral Representation.

UNIT-III
103-127
Bessel 75-102

Legendre Poly. & f^n, sol^n of Legendre DE, Generating f^ns, Orthogonality of Legendre poly. Recurrence Rel^n, Bessel f^n, Bessel DE & its sol^n, Recursive Rel^n, Generating f^n. Integral Representation

1.

Recurrence Relation/

Formula for $P_n(x)$

(1) $(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$

(2) $nP_n(x) = xP_n'(x) - P_{n-1}'(x)$

(3) $(2n+1)P_n(x) = P_{n+1}'(x) - P_{n-1}'(x)$

(4) $P_n'(x) = xP_{n-1}'(x) + nP_{n-1}(x)$

(5) $(1-x^2)P_n'(x) = n\left[P_{n-1}(x) - xP_n(x)\right]$

Proof (1) Using the generating f^n of Legendre's polynomial:

$$(1-2xz+z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(x) \quad \text{--- (1)}$$

Now eq^n (1),

differentiating partially w.r.t. z, then we have

$$(x-z)(1-2xz+z^2)^{-3/2} = \sum_{n=0}^{\infty} nz^{n-1}P_n(x) \quad \text{--- (2)}$$

in eq^n (2), equating the co-efficient of z^n.

$$x\sum_{n=0}^{\infty} z^n P_n(x) - \sum_{n=0}^{\infty} z^{n+1}P_n(x) = \sum_{n=0}^{\infty} nz^{n-1}P_n(x) - 2x\sum_{n=0}^{\infty} nz^n P_n(x) + \sum_{n=0}^{\infty} nz^{n+1}P_n(x).$$

$\Rightarrow$ Equating co-efficient of z^n.

$$xP_n(x) - P_{n-1}(x) = (n+1)P_{n+1}(x) - 2xnP_n(x) + (n-1)P_{(n-1)}(x)$$

$$\Rightarrow (x+2xn)P_n(x) - [1+n-1]P_{n-1}(x) - (n+1)P_{n+1}(x) = 0$$

$$\Rightarrow (2n+1)\cdot xP_n(x) - nP_{n-1}(x) = (n+1)P_{n+1}(x).$$

$$\Rightarrow \boxed{(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)}$$

Proof (2):

Diff^n eq^n (1), partially w.r.t x, we have

$$\left(-\tfrac{1}{2}\right)(1-2xz+z^2)^{-3/2}\cdot(-2z) = \sum_{n=0}^{\infty} z^n P_n'(x)$$

$$\Rightarrow (1-2xz+z^2)^{-1/2}(1-2xz+z^2)^{-1}\cdot z = \sum_{n=0}^{\infty} z^n \cdot P_n'(x)$$

$$\Rightarrow \sum_{n=0}^{\infty} z^n P_n(x) \cdot (1-2xz+z^2)^{-1} z = \sum_{n=0}^{\infty} z^n P_n'(x)$$

$$\Rightarrow (1-2xz+z^2) \sum_{n=0}^{\infty} z^n P_n'(x) = \sum_{n=0}^{\infty} z \cdot z^n P_n(x) \quad — (3)$$

Now, dividing eq^n (2) by (3), we have —

$$\frac{(x-z) \cancel{\sum_{n=0}^{\infty} z^n P_n(x)}}{z \cdot \cancel{\sum_{n=0}^{\infty} z^n P_n(x)}} = \frac{(1-2xz+z^2) \sum_{n=0}^{\infty} n z^{n-1} P_n(x)}{(1-2xz+z^2) \sum_{n=0}^{\infty} z^n P_n'(x)}$$

$$\frac{x-z}{z} = \frac{\sum_{n=0}^{\infty} n P_n(x) z^{n-1}}{\sum_{n=0}^{\infty} P_n'(x) z^n}$$

$$\Rightarrow (x-z) \sum_{n=0}^{\infty} P_n'(x) z^n = z \sum_{n=0}^{\infty} n P_n(x) z^{n-1}$$

$$\Rightarrow x \sum_{n=0}^{\infty} P_n'(x) z^n - \sum_{n=0}^{\infty} P_n'(x) z^{n+1} = \sum_{n=0}^{\infty} n P_n(x) z^n$$

Now equating, co-efficient of z^n both sides —

$$\Rightarrow x P_n'(x) - P_{n-1}'(x) = n P_n(x)$$

$$\boxed{n P_n(x) = x P_n'(x) - P_{n-1}'(x)} \quad \text{Proved}$$

Proof (3) Using the first result

$$(n+1) P_{n+1}(x) = (2n+1) x P_n(x) - n P_{n-1}(x)$$

diff^n w.r.t. x —

$$(n+1) P_{n+1}'(x) = (2n+1) P_n(x) + (2n+1) x P_n'(x) - n P_{n-1}'(x)$$

$$\Rightarrow (2n+1) P_n(x) = (n+1) P_{n+1}'(x) + n P_{n-1}'(x) - (2n+1) x P_n'(x) —$$

putting $x P_n'(x) = n P_n(x) + P_{n-1}'(x)$ (from second result

$$\Rightarrow (n+1) P_{n+1}'(x) = (2n+1) \left[P_n(x) + n P_n(x) + P_{n-1}'(x) \right] - n P_{n-1}'(x)$$

$$\Rightarrow (n+1) P_{n+1}'(x) = (2n+1) \left[(n+1) P_n(x) + P_{n-1}'(x) \right] - n P_{n-1}'(x).$$

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$\Rightarrow (n+1) P'_{n+1}(x) = (2n+1)(n+1) P_n(x)$

$+ (2n+1) P'_{n-1}(x) - n P'_{n-1}(x).$

$\Rightarrow (n+1) P'_{n+1}(x) = (2n+1)(n+1) P_n(x) + (n+1) P'_{n-1}(x).$

dividing by $(n+1)$.

$\Rightarrow \boxed{(2n+1) P_n(x) = P'_{n+1}(x) - P'_{n-1}(x).}$ Proved.

Proof (4) Rewriting eq^n (4) -

$(n+1) P'_{n+1}(x) = (2n+1) P_n(x) + (n+1) x P'_n(x) + x n P'_n(x)$
$- n P'_{n-1}(x).$

$\Rightarrow (n+1) P'_{n+1}(x) = (2n+1) P_n(x) + (n+1) x P'_n(x) + \underbrace{(x P'_n(x) - P'_{n-1}(x)) n}$

$\Rightarrow (n+1) P'_{n+1}(x) = (2n+1) P_n(x) + (n+1) x P'_n(x) + n \cdot [n P_n(x)]$
(from second result)

$\Rightarrow (n+1) P'_{n+1}(x) = (2n+1) P_n(x) + (n+1) x P'_n(x) + n^2 P_n(x).$

$\Rightarrow (n+1) P'_{n+1}(x) = (n^2+2n+1) P_n(x) + (n+1) x P'_n(x)$

$\Rightarrow P'_{n+1}(x) = (n+1) P_n(x) + x P'_n(x)$

Now putting $n \to n-1$, we have

$\boxed{P'_n(x) = n P_{n-1}(x) + x P'_{n-1}(x)}$ Proved

Proof (5): Rewriting the result second and fourth & multiply -

$x P'_n(x) - P'_{n-1}(x) = n P_n(x)$ —— (5) $\times x$

$P'_n(x) - x P'_{n-1}(x) = n P_{n-1}(x)$ —— (6) $\times 1$

Now, Consider (6) - (5) we have

$(1-x^2) P'_n(x) + \cancel{x P'_{n-1}(x)} - \cancel{x P'_{n-1}(x)} = n [P_{n-1}(x) - x P_n(x)]$

$\boxed{(1-x^2) P'_n(x) = n [P_{n-1}(x) - x P_n(x)]}$ Proved

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Que: Prove that

$$(2n+1)(1-x^2)P_n'(x) = n(n+1)\left[P_{n-1}(x) - P_{n+1}(x)\right]$$

Sol^n L.H.S.

$(2n+1)(1-x^2)P_n'(x)$

Using the recurrence relation —

$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$ (first Rel^n)

$\Rightarrow (2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x)$

$\Rightarrow (n+1+n)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x)$

$\Rightarrow (n+1)xP_n(x) + nxP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x)$

$\Rightarrow (n+1)xP_n(x) - (n+1)P_{n+1}(x) = nP_{n-1}(x) - nxP_n(x)$

$\Rightarrow (n+1)\left[xP_n(x) - P_{n+1}(x)\right] = n\left[P_{n-1}(x) - xP_n(x)\right]$

$\because$ We know that

$(1-x^2)P_n'(x) = n\left[P_{n-1}(x) - xP_n(x)\right]$

$\Rightarrow$ putting this 4 above, we have —

$(n+1)\left[xP_n(x) - P_{n+1}(x)\right] = (1-x^2)P_n'(x)$

$\Rightarrow xP_n(x) = P_{n+1}(x) + \dfrac{(1-x^2)P_n'(x)}{(n+1)}$ —— (1)

||^{ly} $xP_n(x) = P_{n-1}(x) - \dfrac{(1-x^2)P_n'(x)}{n}$ —— (2)

(from 5^th relation)

Now eq^n (1) & (2), we have —

$P_{n-1}(x) - \dfrac{(1-x^2)P_n'(x)}{n} = P_{n+1}(x) + \dfrac{(1-x^2)P_n'(x)}{n+1}$

$(1-x^2)P_n'(x)\left[\dfrac{1}{n} + \dfrac{1}{n+1}\right] + P_{n+1}(x) = P_{n-1}(x)$

$\Rightarrow \dfrac{(1-x^2)P_n'(x).(2n+1)}{n(n+1)} + P_{n+1}(x) = P_{n-1}(x)$

$\Rightarrow (2n+1)(1-x^2)P_n'(x) = n(n+1)\left[P_{n-1}(x) - P_{n+1}(x)\right]$

Proved.

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Prove that —

Que: $$\frac{1-z^2}{(1-2xz+z^2)^{3/2}} = \sum_{n=0}^{\infty} (2n+1) z^n P_n(x)$$

Sol^n:- We know that the generating f^n of Legendre poly. —

$$[1-2xz+z^2]^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(x) \quad \text{——— (1)}.$$

diff (1), w.r.t z (partially)

$$-\frac{1}{2}(1-2xz+z^2)^{-3/2}(2z-2x) = \sum_{n=0}^{\infty} n.z^{n-1} P_n(x)$$

$$\Rightarrow \frac{(-z+x)}{(1-2xz+z^2)^{3/2}} = \sum_{n=0}^{\infty} n.z^{n-1} P_n(x) \quad \text{—(2)}$$

multiply eq^n (2) on both sides of eq^n (2) by $2z$, we have—

$$\frac{2z(x-z)}{(1-2xz+z^2)^{3/2}} = 2\cdot\sum_{n=0}^{\infty} n z^n P_n(x) \quad \text{— (3)}$$

Now, on adding eq^n (1) + (3), gives —

$$\frac{1}{(1-2xz+z^2)^{3/2}}\left[2xz-2z^2+(1-2xz+z^2)\right] =$$

$$= \sum_{n=0}^{\infty} z^n P_n(x) + 2\sum_{n=0}^{\infty} n z^n P_n(x)$$

$$\Rightarrow \frac{(1-z^2)}{[1-2xz+z^2]^{3/2}} = \sum_{n=0}^{\infty} (2n+1) z^n P_n(x)$$

Proved

Que: P.T. $$\frac{1+z}{z\sqrt{1-2xz+z^2}} - \frac{1}{z} = \sum_{n=0}^{\infty} z^n \{P_n(x) + P_{n+1}(x)\}$$

$\because$ We know that

$$(1-2xz+z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(x) \quad \text{——— (1)}$$

taking L.H.S

$$\frac{1+z}{z\sqrt{1-2xz+z^2}} - \frac{1}{z} = \frac{1}{z}(1-2xz+z^2)^{-\frac{1}{2}} + (1-2xz+z^2)^{-1/2}$$

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$$= \frac{1}{2}\sum_{n=0}^{\infty} z^n P_n(x) + \sum_{n=0}^{\infty} P_n(x) - \frac{1}{2} \quad \text{--- (2)}$$

$$\sum_{n=0}^{\infty} z^n P_n(x) = P_0 + z\cdot P_1(x) + z^2 P_2(x) + \cdots + z^n P_n(x) + z^{n+1} P_{n+1}(x)$$

$$= 1 + z \sum_{n=0}^{\infty} z^n P_{n+1} \quad \text{--- (3)}$$

Using eqn (3) in (ii), we get —

$$\frac{1}{2}\left[1 + z\sum_{n=0}^{\infty} z^n P_{n+1}(x)\right] + \sum_{n=0}^{\infty} z^n P_n(x) - \frac{1}{2}$$

$$= \sum_{n=0}^{\infty} z^n P_{n+1}(x) + \sum_{n=0}^{\infty} z^n P_n(x)$$

$$= \sum_{n=0}^{\infty} z^n \left(P_n(x) + P_{n+1}(x)\right)$$

$$= R.H.S$$

Proved

Que: Verify that $[1-2xz+z^2]^{-1/2}$ is a solⁿ of the eqⁿ

$$z\frac{\partial^2}{\partial z^2}(zv) + \frac{\partial}{\partial x}\left[(1-x^2)\frac{dv}{dx}\right] = 0$$

Solⁿ: Let $v = [1-2xz+z^2]^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(x)$

$$\therefore zv = z\sum_{n=0}^{\infty} z^n P_n(x)$$

$$\frac{\partial^2}{\partial z^2}(zv) = \frac{\partial^2}{\partial z^2}\left[\sum_{n=0}^{\infty} z^{n+1} P_n(x)\right]$$

$$= \frac{\partial}{\partial z}\left[\sum_{n=0}^{\infty} (n+1) z^n P_n(x)\right]$$

$$= \sum_{n=0}^{\infty} n(n+1) z^{n-1} P_n(x)$$

∵ Legendre eqⁿ
$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$

$$z\frac{\partial^2}{\partial z^2}(zv) = \sum_{n=0}^{\infty} (n)(n+1) z^n P_n(x) \quad \text{--- (I)}$$

Again $\frac{\partial}{\partial x}\left[(1-x^2)\frac{dv}{dx}\right] = (1-x^2)\frac{d^2v}{dx^2} - 2x\frac{dv}{dx}$

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Recurrence Formulae for $P_n(x)$

(i) $(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$

(2) $nP_n(x) = xP_n'(x) - P_{n-1}'(x)$

(3) $(2n+1)P_n(x) = P_{n+1}'(x) - P_{n-1}'(x)$

(4) $P_n'(x) = xP_{n-1}'(x) + nP_{n-1}(x)$

(5) $(1-x^2)P_n'(x) = n\left[P_{n-1}(x) - xP_n(x)\right]$

Proof: Using the generating f^n of Legendre's poly.

$$(1-2xz+z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(x) \quad \text{--- (1)}$$

Now eq^n (1) differentiating partially w.r.t z, then we have

$$(x-z)(1-2xz+z^2)^{-3/2} = \sum_{n=0}^{\infty} nz^{n-1}P_n(x)$$

$$(x-z)\sum_{n=0}^{\infty} z^n P_n(x) = (1-2xz+z^2)\sum_{n=0}^{\infty} nz^{n-1}P_n(x) \quad \text{--- (2)}$$

In eq^n (2), equating the co-eff of z^n,

$$x\sum_{n=0}^{\infty} z^n P_n(x) - \sum_{n=0}^{\infty} z^{n+1}P_n(x) = \sum_{n=0}^{\infty} nz^{n-1}P_n(x) - 2x\sum_{n=0}^{\infty} nz^nP_n(x) + \sum_{n=0}^{\infty} nz^{n+1}P_n(x).$$

Equating Co-efficient of z^n.

$$xP_n(x) - P_{n-1}(x) = (n+1)P_{n+1}(x) - 2xnP_n(x) + (n-1)P_{(n-1)}(x)$$

$$\Rightarrow (x+2xn)P_n(x) - (1+n-1)P_{n-1}(x) - (n+1)P_{n+1}(x) = 0$$

$$\Rightarrow (2n+1)\,xP_n(x) - nP_{n-1}(x) = (n+1)P_{n+1}(x)$$

$$\Rightarrow \boxed{(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)}$$

Proved.

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$$= -n(n+1)\, v$$
$$= -n(n+1)\,(1-2xz+z^2)^{-1/2}$$
$$= -n(n+1)\cdot \sum_{n=0}^{\infty} z^n P_n(x)$$
$$= -\sum_{n=0}^{\infty} n(n+1)\, z^n P_n(x) \quad \text{---} (2)$$

$\Rightarrow$ (1), (2),

$$z\frac{\partial^2}{\partial z^2}(zv) + \frac{\partial}{\partial x}\left[(1-x^2)\frac{dv}{dx}\right] = \sum_{n=0}^{\infty} n(n+1)\, z^n P_n(x) - \sum_{n=0}^{\infty} n(n+1)\, z^n P_n(x)$$
$$= 0$$

— x — — x —

5/8/16

Prove that

(I). $1 + \frac{1}{2}P_1(\cos\theta) + \frac{1}{3}P_2(\cos\theta) + \frac{1}{4}P_3(\cos\theta) + \cdots$
$$\cdots = \log\left[\frac{1+\sin\theta/2}{\sin\theta/2}\right]$$

(II). $$\sum_{n=0}^{\infty} \frac{P_n(\cos\theta)}{n+1} = \log\{1+\operatorname{cosec}\theta/2\}$$

Pf:— Using the generating f^n of legendre poly. —
$$\sum_{n=0}^{\infty} z^n P_n(x) = [1-2xz+z^2]^{-1/2} \quad \text{---} (1)$$

Now putting $x = \cos\theta$
$$\sum_{n=0}^{\infty} z^n P_n(\cos\theta) = [1-2z\cos\theta+z^2]^{-1/2}$$

integrating the above expression w.r.t. z, from 0 to 1.

$$\sum_{n=0}^{\infty} P_n\cos\theta \int_0^1 z^n dz = \int_0^1 [1-2z\cos\theta+z^2]^{-1/2}$$

$$\sum_{n=0}^{\infty} \frac{P_n\cos\theta}{(n+1)} = \int_0^1 \frac{dz}{\sqrt{(z-\cos\theta)^2+\sin^2\theta}}$$

$$\left(\because \int_0 \frac{dx}{\sqrt{x^2+a^2}} = \log\left[x + \sqrt{x^2+a^2}\right] \right)$$

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$$\sum_{n=0}^{\infty} \frac{P_n \cos\theta}{(n+1)} = \int_0^1 \frac{dz}{\sqrt{(z-\cos\theta)^2+\sin^2\theta}}$$

$$= \log\left[(z-\cos\theta) + \sqrt{(z-\cos\theta)^2+\sin^2\theta}\right]_0^1$$

$$= \log\left((1-\cos\theta) + \sqrt{(1-\cos\theta)^2+\sin^2\theta}\right) - \log\left((-\cos\theta) + \sqrt{\cos^2\theta+\sin^2\theta}\right)$$

$$= \log\left[(1-\cos\theta) + \sqrt{2-2\cos\theta}\right] - \log(-\cos\theta+1)$$

$$= \log\left[1 + \frac{\sqrt{2(1-\cos\theta)}}{(1-\cos\theta)}\right]$$

$\cos 2\theta = 1-2\sin^2\theta$

$$= \log\left[1 + \frac{\sqrt{2}\sqrt{\cancel{1-}2\sin^2\theta/2}}{2\sin^2\theta/2}\right]$$

$$= \log\left[1 + \frac{2\sin\theta/2}{2\sin^2\theta/2}\right] = \log\left[1 + \frac{1}{\sin\theta/2}\right]$$

$$\sum_{n=0}^{\infty} \frac{P_n \cos\theta}{(n+1)} = \log\left[\frac{1+\sin\theta/2}{\sin\theta/2}\right]$$

$$\Rightarrow \frac{P_0(\cos\theta)}{1} + \frac{P_1(\cos\theta)}{2} + \frac{P_3(\cos\theta)}{3} + \cdots = \log\left[\frac{1+\sin\theta/2}{\sin\theta/2}\right]$$

$$\Rightarrow 1 + \frac{P_1(\cos\theta)}{2} + \frac{P_3(\cos\theta)}{3} + \cdots = \log\left[\frac{1+\sin\theta/2}{\sin\theta/2}\right]$$

$\because P_0(x) = 1$
$\therefore P_0(\cos x) = 1$

Trignometric Series for $P_n(x)$

To show that

$$P_n(\cos\theta) = \frac{1.3.5.\dots(2n-1)}{2^{(n-1)}.\,n!}\left[\cos n\theta + \frac{1.n}{1.(2n-1)}\cos(n-2)\theta + \frac{1.3.n(n+1)}{1.2.(2n-1)(2n-3)}\cos(n-4)\theta + \cdots\right]$$

Proof :- $\cancel{P_n(\cos\theta) = \frac{1.3.5\dots(2n-1)}{2}}$

$$\because \sum_{n=0}^{\infty} z^n P_n(x) = \left[1-2xz+z^2\right]^{-1/2}$$

put $x = \cos\theta$

$$\sum_{n=0}^{\infty} z^n P_n(\cos\theta) = \left[1 - 2\cos\theta . z + z^2\right]^{-1/2}$$

$\because 2\cos\theta = e^{i\theta} + e^{-i\theta}$

$$\Rightarrow \sum_{n=0}^{\infty} z^n P_n(\cos\theta) = \left[1 - (e^{i\theta} + e^{-i\theta}) . z + z^2\right]^{-1/2}$$

$$= \left[1 - e^{i\theta} z - e^{-i\theta} z + z^2\right]^{-1/2}$$

$$= \left[(1 - z e^{i\theta})(1 - z e^{-i\theta})\right]^{-1/2}$$

$$= \left[1 + \frac{1}{2} z e^{i\theta} + \frac{1.3}{2.4} z^2 e^{2i\theta} + \frac{1.3.5}{2.4.6} z^3 e^{3i\theta} + \dots \frac{1.3.5\dots(2n-1)}{2.4.6\dots(2n)} . z^n e^{ni\theta}\right] \times \left[1 + \frac{z e^{-i\theta}}{2} + \frac{1.3}{2.4} z^2 e^{-2i\theta} + \frac{1.3.5}{2.4.6} z^2 e^{-3i\theta} + \dots + \frac{1.3.5\dots(2n-1)}{2.4.6\dots(2n)} z^n e^{-ni\theta}\right]$$

In above expression equating the co-efficient of z^n, then we have—

$$\sum_{n=0}^{\infty} z^n P_n(\cos\theta) = \frac{1.3.5\dots(2n-1)}{2.4.6\dots(2n)} \left[e^{ni\theta} + e^{-ni\theta}\right] + \frac{1.3.5\dots(2n-3)}{2.4.6\dots(2n-2).2} \left[e^{(n-2)i\theta} + e^{-(n-2)i\theta}\right] + \frac{1.3.5\dots(2n-5).1.3}{2.4.6\dots(2n-4).2.4} \left[e^{(n-4)i\theta} + e^{-(n-4)i\theta}\right]$$

$$\Rightarrow \sum_{n=0}^{\infty} z^n P_n(\cos\theta) = \frac{1.3.5\dots(2n-1)}{2.4.6\dots(2n)} (2\cos n\theta) + \frac{1.3.5\dots(2n-3)}{2.4.6\dots(2n-2).2} . 2\cos(n-2)\theta + \frac{1.3.5\dots(2n-5).1.3}{2.4.6\dots(2n-4)2.4} . 2\cos(n-4)\theta$$

$$= \frac{1.3.5\cdots(2n-1)}{2.4.6\cdots(2n)}\left[2\cos n\theta + \frac{1.2n}{2(2n-1)}\times 2\cos(n-2)\theta + \frac{1.3}{2.4}\cdot\frac{2n(2n-2)}{(2n-1)(2n-3)}\cdot 2\cos(n-4)\theta + \cdots\right]$$

$$= \frac{1.3.5\cdots(2n-1)}{2^n.2^{-1}.1.2\cdots n}\left[\cos n\theta + \frac{2n\cos(n-2)\theta}{2(2n-1)} + \frac{1.3.2n(2n-2)}{2.4.(2n-1)(2n-3)}\cos(n-4)\theta + \cdots\right]$$

$$= \frac{1.3.5\cdots(2n-1)}{2^{n-1}.n!}\left[\cos n\theta + \frac{n.\cos(n-2)\theta}{(2n-1)} + \frac{1.3.n}{1.2.}\cdot\frac{(n-1)}{(2n-1)(2n-3)}\cdot\cos(n-4)\theta + \cdots\right]$$

Theorem: If z is solution of legendre ~~poly~~. Eqn.
$$(1-x^2)y_2 - 2xy_1 + n(n+1)y = 0,$$
then ST, $(1-x^2)^{m/2}\frac{d^m z}{dx^m}$ is a sol^n of the eq^n
$$(1-x^2)y_2 - 2xy_1 + \left[n(n+1) - \frac{m^2}{1-x^2}\right]y = 0$$

Proof: $\because (1-x^2)y_2 - 2xy_1 + (n(n+1))y = 0$ — (1)

$\because z$ is sol^n of (1), then
$$(1-x^2)z_2 - 2xz_1 + n(n+1)z = 0$$
$$\Rightarrow (1-x^2)\frac{d^2z}{dx^2} - 2x\frac{dz}{dx} + n(n+1)z = 0$$

integrating m times w.r.t. x, we have
$$\frac{d^m}{dx^m}\left[(1-x^2)\frac{d^2z}{dx^2}\right] + \frac{d^m}{dx^m}\left[(-2x).\frac{dz}{dx}\right] + n(n+1).\frac{d^m z}{dx^m} = 0$$

Now by using, Lebniz theorem (formula)

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formula -

$$D^n(uv) = uv_n + {}^nC_1 u' v_{n-1} + {}^nC_2 u'' v_{n-2} + \dots$$

$$\Rightarrow (1-x^2)\frac{d^{m+2}z}{dx^{m+2}} + m(-2x)\frac{d^{m+1}z}{dx^{m+1}} + \frac{m(m-1)}{2}(-2)\frac{d^m}{dx^m} + 0 + 0 + \dots$$

$$+ (-2x)\frac{d^{m+1}}{dx^{m+1}}z + (-2)\frac{m}{1}\frac{d^m z}{dx^m} + 0 + \dots$$

$$+ n(n+1)\frac{d^m z}{dx^m} = 0$$

$$\Rightarrow (1-x^2)\frac{d^{m+2}z}{dx^{m+2}} - 2(m+1)x\frac{d^{m+1}z}{dx^{m+1}} + \left[n(n+1) - m\cdot(m+1)\right]\frac{d^m z}{dx^m} = 0 \quad \text{--- (3)}$$

let $u = \dfrac{d^m z}{dx^m}$ --- (4)

then eqn (3) reduces —

$$(1-x^2)\frac{d^2u}{dx^2} - 2(m+1)x\frac{du}{dx} + \left[n(n+1) - m(m+1)\right]u = 0 \quad \text{--- (5)}$$

putting $v = (1-x^2)^{m/2} u$

$$\boxed{u = (1-x^2)^{-m/2} v}$$

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Gamma & Beta Function

Laplace

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt \quad , \quad R(s) > 0$$

$$L\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}} = \int_0^\infty e^{-st}\, t^n\,dt \, , \quad n \in R$$

let $s=1,\ n=x-1$

$$\boxed{\Gamma x = \int_0^\infty e^{-t}\, t^{x-1}\,dt \, , \quad x>0}$$

cts

For laplace transform:

(i) $f(t)$ is piecewise cts in $[0,t]$, $t>0$

(ii). $f(t)$ is of exponential order.

$$|e^{-st} f(t)| < M \quad , \quad M \text{ is any finite No.}$$

When $s=1$, $f(t) = t^{x-1}$, $x>0$, then

$t \to \infty$ $\quad \boxed{e^{-st}\, t^{x-1} \to 0} \quad \Rightarrow$ Convergent

(1)* $$\boxed{\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt \, , \quad x>0,\ y>0}$$

* $\Gamma(1) = \int_0^\infty e^{-t}\,dt = 1 \quad \Rightarrow \boxed{\Gamma(1) = 1}$

*(2) $\Gamma(x+1) = \int_0^\infty \underset{I}{e^{-t}}\, \underset{II}{t^x}\,dt$

Integration by part, we get —

$$= \left[t^x(-e^{-t})\right]_0^\infty - \int_0^\infty -e^{-t}\cdot x\cdot t^{x-1}\,dt$$

$$= 0 + x\int_0^\infty e^{-t}\cdot t^{x-1}\,dt$$

$$= x\,\Gamma x$$

(3) $$\Rightarrow \boxed{\Gamma(x+1) = x\,\Gamma x}$$

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(4) let x is a non. neg. integer
then

$$\Gamma(x+1) = x\,\Gamma x$$
$$= x(x-1)\,\Gamma(x-1)$$
$$= x(x-1)(x-2)\,\Gamma(x-2)$$
$$\cdots\cdots$$
$$= x(x-1)(x-2)\cdots 3.2.1$$

$$\boxed{\Gamma(x+1) = (x)!}$$

(5) $$\boxed{\int_0^{\pi/2} \sin^{2x-1}\theta \cdot \cos^{2y-1}\theta\, d\theta = \frac{\Gamma x\,\Gamma y}{2\,\Gamma(x+y)}}$$

$\because \iint_R \exp(-u^2-t^2)\, u^{2x-1}.\, t^{2y-1}\, du\, dt;$ $R = \{(t,u) \mid t>0, u>0\}$

$$= \int_0^\infty e^{-u^2}.\, u^{2x-1}\left(\int_0^\infty e^{-t^2}.\, t^{2y-1}\, dt\right) du$$
$$= \Gamma(y)/2.$$

Now put $t = u^2$, in the defn of Γx

$$= \frac{\Gamma(y)}{2}\int_0^\infty e^{-u^2}.\, u^{2x-1}\, du$$

$$= \frac{\Gamma x\,\Gamma y}{4} \quad \text{—— (1)}$$

$du = -r\sin\theta\, d\theta + \cos\theta\, dr$
$dt = r\cos\theta\, d\theta + \sin\theta\, dr = r\, d\theta$
when θ = small
$\therefore du\, dt = r\, d\theta\, dr$

Again, let $u = r\cos\theta,\ t = r\sin\theta$
$\Rightarrow du\, dt \Rightarrow r\, d\theta\, dr$ ($\because \theta$ is very small $\theta \to 0$)

Then

$$\int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty} \exp(-r^2\cos^2\theta - r^2\sin^2\theta).\, r^{2x-1+2y-1}.\cos^{2x-1}\theta.\, \sin^{2y-1}\theta.\ r\, dr\, d\theta$$

$$= \int_0^\infty e^{-r^2}.\, r^{2(x+y-1)}.\, r.\, dr \int_0^{\pi/2} \cos^{2x-1}\theta.\, \sin^{2y-1}\theta\, d\theta$$

$$= \int_0^\infty e^{-r^2}.\, r^{2(x+y)-1}\, dr \int_0^{\pi/2} \cos^{2x-1}\theta.\, \sin^{2y-1}\theta.\, d\theta$$

$$= \frac{\Gamma(x+y)}{2}.\int_0^{\pi/2} \cos^{2x-1}\theta.\, \sin^{2y-1}\theta\, d\theta$$

$$\Rightarrow \boxed{\int_0^{\pi/2} \sin^{2y-1}\theta.\cos^{2x-1}\theta\, d\theta = \frac{\Gamma x\,\Gamma y}{2\,\Gamma(x+y)}} \quad (\text{By } (1)).$$

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(6) $\sqrt{1/2}$

$x = y = 1/2$, then

$$\int_0^{\pi/2} \sin^0\theta \cdot \cos^0\theta \, d\theta = \frac{\Gamma(1/2)\,\Gamma(1/2)}{2\,\Gamma 1}$$

$$\Rightarrow \int_0^{\pi/2} d\theta = \frac{\left[\Gamma 1/2\right]^2}{2\,\Gamma 1}$$

$(\because \Gamma 1 = 1)$ $\Rightarrow \frac{\pi}{2} = \frac{(\Gamma 1/2)^2}{1\times 2} \Rightarrow \boxed{\Gamma 1/2 = \pm\sqrt{\pi}}$

But gamma f^n for any positive no. is positive —

$$\Rightarrow \boxed{\Gamma(1/2) = \sqrt{\pi}}$$

(7) $\because \boxed{\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt}$

let $t = \cos^2\theta$

$\Rightarrow dt = -2\sin\theta \cdot \cos\theta \, d\theta$

$$\Rightarrow \beta(x,y) = \int_{\pi/2}^{0} \cos^{2x-2}\theta \cdot \sin^{2y-2}\theta \cdot (-2\cos\theta\cdot\sin\theta)\,d\theta$$

$$= 2\int_0^{\pi/2} \cos^{2x-1}\theta \cdot \sin^{2y-1}\theta \, d\theta$$

$$= \frac{\Gamma x \, \Gamma y}{\Gamma(x+y)}$$

$$\Rightarrow \boxed{\beta(x,y) = \frac{\Gamma x\,\Gamma y}{\Gamma x+y}}$$

(8) Duplication Formula $\Rightarrow$

$$\boxed{\Gamma 2x = \frac{2^{2x-1}}{\sqrt{\pi}}\,\Gamma(x)\,\Gamma(x+1/2)}$$

Pf:- $\because \beta(x,y) = \frac{\Gamma x\,\Gamma y}{\Gamma(x+y)}$

let $x = y$, then $\beta(x,x) = \frac{\Gamma x\,\Gamma x}{\Gamma 2x}$

$$\Rightarrow \Gamma 2x = \frac{(\Gamma x)^2}{\beta(x,x)}$$

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$$\beta(x,x) = \frac{\Gamma x \, \Gamma x}{\Gamma 2x}$$

$$\Rightarrow \int_0^1 t^{x-1}(1-t)^{x-1}\,dt = \frac{\Gamma x \, \Gamma x}{\Gamma 2x}$$ (By defⁿ of β)

Let $t \to \frac{1}{2}(s+1)$ $\quad dt = \frac{ds}{2}$

then $$\int_{-1}^{1} \left(\frac{s+1}{2}\right)^{x-1} \cdot \left[1 - \frac{s+1}{2}\right]^{x-1} \cdot \frac{ds}{2} = \frac{\Gamma x \, \Gamma x}{\Gamma 2x} \quad (*)$$

taking L.H.S

$$\Rightarrow \int_{-1}^{1} \frac{1}{2^{x-1}} (1+s)^{x-1} \cdot \frac{1}{2^{x-1}} \cdot (1-s)^{x-1} \frac{ds}{2}$$

$$= \frac{1}{2^{2x-1}} \int_{-1}^{1} (1-s^2)^{x-1}\,ds$$

$(1-s^2)$ is an even f^n

$$= \frac{2}{2^{2x-1}} \int_0^1 (1-s^2)^{x-1}\,ds$$

Let $s^2 = u$

$$\Rightarrow 2s\,ds = du \Rightarrow ds = \frac{du}{2s} = \frac{du}{2\sqrt{u}}$$

$$= \frac{2}{2^{2x-1}} \int_0^1 (1-u)^{x-1} \cdot \frac{1}{2\sqrt{u}} \cdot du$$

$$= \frac{1}{2^{2x-1}} \int_0^1 u^{-1/2} (1-u)^{x-1}\,du$$

$$= \frac{1}{2^{2x-1}} \cdot \beta\left(\tfrac{1}{2}, x\right)$$

$$= \frac{1}{2^{2x-1}} \cdot \frac{\Gamma(1/2)\,\Gamma x}{\Gamma(1/2 + x)}$$

$$\frac{\Gamma x \, \Gamma x}{\Gamma 2x}$$

from (*), we have —

$$\boxed{\Gamma 2x = \frac{2^{2x-1}}{\sqrt{\pi}} \Gamma x \, \Gamma(x + 1/2)}$$

* * *

18/8/16 (9) $\Gamma x = \int_0^\infty e^{-t} \cdot t^{x-1}\,dt \;;\; x > 0$

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* $\Gamma x = \frac{x\Gamma x}{x} = \frac{\Gamma x+1}{x} = \frac{\Gamma x+2}{x(x+1)} = \frac{\Gamma x+n}{x(x+1)\cdots(x+n-1)}$

>0 $\quad x+1>0$ i.e $x>-1$ $\quad x>-2$

Theorem: (10) $\Gamma x = \infty$, if x is zero or a $(-)$ve integer.

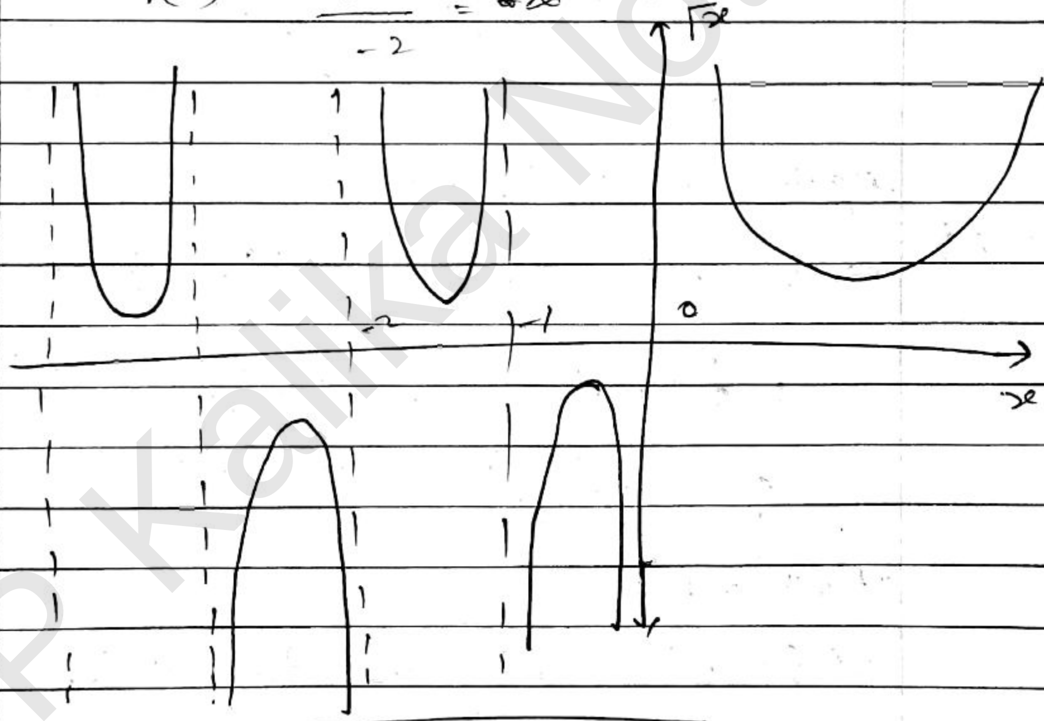
Proof:– Where $x \to 0$, $\Gamma(x+1) \to \Gamma 1 = 1$

Hence, $\Gamma x = \frac{\Gamma(x+1)}{x} = \infty$, when $x \to 0$

$\Rightarrow \Gamma(0) = \infty$

$\Rightarrow \Gamma(-1) = \frac{(-1)\Gamma(-1)}{(-1)} \cdot \frac{\Gamma(0)}{-1} = -\infty$

$\Gamma(-2) = \frac{\Gamma -1}{-2} = \infty$



Theorem (11) $\boxed{\Gamma(x)\,\Gamma(1-x) = \frac{\pi}{\sin \pi x}}$

Pf: In order to set our result, we shall need two priliminary results –

(i) $\sin\theta = \prod_{n=1}^{\infty}\left(1 - \frac{\theta^2}{n^2\pi^2}\right)$

(2) $\frac{1}{\sin\theta} = \sum_{n=-\infty}^{\infty} (-1)^n/(\theta - n\pi)$

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(i) $\sin\theta = 2\sin\frac{\theta}{2}\cdot\cos\frac{\theta}{2}$ — (2)

$= 2\sin\frac{\theta}{2}\sin\left(\frac{\pi}{2}+\frac{\theta}{2}\right)$ — (1)

By (1),

$$\sin\frac{\theta}{2} = 2\sin\frac{\theta}{4}\cdot\sin\left(\pi/2+\theta/4\right)$$

$$\sin\left(\pi/4+\theta/2\right) = 2\sin\left(\pi/4+\frac{\theta}{4}\right)\cdot\sin\left(\pi/2+\pi/4+\frac{\theta}{4}\right)$$

II step $\therefore \sin\theta = 2\left\{2\sin\frac{\theta}{4}\cdot\sin\left(\frac{\pi}{2}+\frac{\theta}{4}\right)\right\}\left\{2\sin\left(\frac{\pi}{4}+\frac{\theta}{4}\right)\cdot\sin\left(\frac{\pi}{2}+\frac{\pi}{4}+\frac{\theta}{4}\right)\right\}$

$$= 2^3\sin\frac{\theta}{4}\cdot\sin\frac{2\pi+\theta}{4}\cdot\sin\frac{\pi+\theta}{4}\cdot\sin\frac{3\pi+\theta}{4}$$

$$= 2^3\sin\frac{\theta}{2^2}\cdot\sin\frac{2\pi+\theta}{2^2}\cdot\sin\frac{\pi+\theta}{2^2}\cdot\sin\frac{3\pi+\theta}{2^2}$$

Again, using (1), we have —

III step $\sin\theta = 2^3\left\{2\sin\frac{\theta}{2^3}\cdot\sin\left(\frac{\pi}{2}+\frac{\theta}{2^3}\right)\right\}\left\{\sin\frac{\pi+\theta}{2^2}\right\}\left\{\sin\frac{2\pi+\theta}{2^2}\right\}\ldots$

$$= 2^7\sin\frac{\theta}{2^3}\cdot\sin\frac{\pi+\theta}{2^3}\cdot\sin\frac{2\pi+\theta}{2^3}\cdot\sin\frac{3\pi+\theta}{2^3}\cdot\sin\frac{4\pi+\theta}{2^3}$$ ← middle term

$$\sin\frac{5\pi+\theta}{2^3}\cdot\sin\frac{6\pi+\theta}{2^3}\cdot\sin\frac{7\pi+\theta}{2^3}$$

at n^{th} step —

$$\sin\theta = 2^{2^n-1}\sin\frac{\theta}{2^n}\sin\frac{\pi+\theta}{2^n}\cdot\sin\frac{2\pi+\theta}{2^n}\cdots\sin\frac{(2^n-1)\pi+\theta}{2^n}$$

$\therefore$ last term $\sin\left[\frac{(2^n-1)\pi+\theta}{2^n}\right] = \sin\left(\pi-\frac{\pi+\theta}{2^n}\right)$

$$= \sin\left(\frac{\pi-\theta}{2^n}\right)$$

||'ly, second last term —

$$\sin\frac{(2^n-2)\pi+\theta}{2^n} = \sin\frac{2\pi-\theta}{2^n}$$

r^{th} terms from last —

$$\sin\frac{(2^n-r)\pi+\theta}{2^n} = \sin\frac{r\pi-\theta}{2^n}$$

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$\sin(A+B)\cdot\sin(A-B) = \frac{1}{2}[\cos(2B) - \cos 2A]$

$= \sin^2 A - \sin^2 B$

$\therefore \sin\theta = 2^{2^n-1} \cdot \sin\frac{\theta}{2^n} \cdot \sin\frac{\pi+\theta}{2^n} \cdot \sin\frac{2\pi+\theta}{2^n} \cdots$

$\cdots \sin\frac{(2^{n-1}-1)\pi+\theta}{2^n} \cdot \sin\frac{(2^{n-1}-1)\pi-\theta}{2^n} \cdot \sin\frac{2\pi-\theta}{2^n} \cdot$

$\sin\frac{\pi-\theta}{2^n} \cdot \sin\left(\frac{2^{n-1}\pi+\theta}{2^n}\right)$

$= 2^{2^n-1} \sin\frac{\theta}{2^n} \left\{\sin\frac{\pi+\theta}{2^n} \cdot \sin\frac{\pi-\theta}{2^n}\right\} \cdot \left\{\sin\frac{2\pi+\theta}{2^n} \cdot \sin\frac{2\pi-\theta}{2^n}\right\}$

$\cdots \left\{\sin\frac{(2^{n-1}-1)\pi+\theta}{2^n} \cdot \sin\frac{(2^{n-1}-1)\pi-\theta}{2^n}\right\} \times$

$\sin\frac{(2^{n-1}\pi+\theta)}{2^n}$ ← (middle term remaing single)

$\left(\sin\frac{(2^n-(2^{n-1}-1))\pi+\theta}{2^n} = \sin\frac{(2^{n-1}-1)\pi-\theta}{2^n}\right)$

$\Rightarrow \sin\theta = 2^{2^n-1} \sin\frac{\theta}{2^n}\left\{\sin^2\frac{\pi}{2^n} - \sin^2\frac{\theta}{2^n}\right\}\left\{\sin^2\frac{2\pi}{2^n} - \sin^2\frac{\theta}{2^n}\right\} \cdots \left\{\sin^2\frac{(2^{n-1}-1)\pi}{2^n} - \sin^2\frac{\theta}{2^n}\right\}\cos\frac{\theta}{2^n}$ —— (3)

$\sin\left(\frac{2^{n-1}\pi+\theta}{2^n}\right) = \sin\left[\frac{\pi}{2}+\frac{\theta}{2^n}\right] = \cos\frac{\theta}{2^n}$

deviding by $\sin\frac{\theta}{2^n}$, both sides

$\lim_{\theta\to 0} \frac{\sin\theta}{\sin\frac{\theta}{2^n}} = 2^{2^n-1} \cdot \sin^2\frac{\pi}{2^n} \cdot \sin^2\frac{2\pi}{2^n} \cdots \sin^2\frac{(2^{n-1}-1)\pi}{2^n}$

$= P$ —— (4)

Dividing (3) by (4), we get —

— (2)

$\frac{\sin\theta}{P} = \sin\frac{\theta}{2^n}\left\{1 - \frac{\sin^2\theta/2^n}{\sin^2\pi/2^n}\right\}\left\{1 - \frac{\sin^2\theta/2^n}{\sin^2 2\pi/2^n}\right\} \cdots$

$\cdots \left\{1 - \frac{\sin^2\theta/2^n}{\sin^2\frac{(2^{n-1}-1)\pi}{2^n}}\right\} \cdot \cos\theta/2^n$ —— (5)

(i) $\lim_{p\to\infty} p\sin\theta/p = \lim_{p\to\infty} \frac{\sin\theta/p}{\theta/p} = \theta$

(ii) $\lim_{p\to\infty} \frac{\sin^2\theta/p}{\sin^2\frac{r\pi}{p}} = \frac{\theta^2}{r^2\pi^2}$

Here $p \neq P$ & $p = 2^n$

$\because$ p $\lim_{\theta \to 0} \frac{\sin\theta}{\sin\theta/2^n}$ then $\lim_{\theta \to 0} \frac{\sin\theta}{\sin\theta/2^n} \cdot \lim_{2^n \to \infty} \sin\theta/2^n = \lim_{n \to \infty} \frac{\sin\theta \cdot (\theta/2^n)}{\theta \cdot \sin(\theta/2^n)}$

$= \frac{\theta}{\theta/2^n} \cdot \lim_{2^n \to \infty} \sin\frac{\theta}{2^n} = \lim_{2^n \to \infty} 2^n \sin\frac{\theta}{2^n}$

$= \lim_{p \to \infty} p \cdot \sin\theta/p = \theta$

(3) $\lim_{p \to \infty} \cos\theta/p = 1$

taking $p \to \infty$ in (5), we get –

$$\sin\theta = \theta\left(1 - \frac{\theta^2}{\pi^2}\right)\left(1 - \frac{\theta^2}{2^2\pi^2}\right)\left(1 - \frac{\theta^2}{3^2\pi^2}\right)$$

$$\Rightarrow \boxed{\sin\theta = \theta \prod_{n=1}^{\infty}\left(1 - \frac{\theta^2}{n^2\pi^2}\right)} \quad \text{Proved}$$

(ii) $\frac{1}{\sin\theta} = \frac{d}{d\theta} \ln\left(\tan\frac{\theta}{2}\right)$

P.34 (ii)

$$= \frac{d}{d\theta} \ln\left[\frac{\sin\theta/2}{\cos\theta/2} \cdot \frac{2\sin\theta/2}{2\sin\theta/2}\right]$$

$$= \frac{d}{d\theta} \ln\left[\frac{2(\sin\theta/2)^2}{\sin\theta}\right]$$

$$= \frac{d}{d\theta}\left\{\ln 2 + 2\ln\sin\theta/2 - \ln\sin\theta\right\} \quad \left[\because \frac{d}{d\theta}(\ln 2) = 0\right]$$

$$= \frac{d}{d\theta}\left\{2\ln\sin\theta/2 - \ln\sin\theta\right\}$$

$$= \frac{d}{d\theta}\left\{2\ln\prod_{n=1}^{\infty}\frac{\theta}{2}\left(1 - \frac{\theta^2}{4\pi^2 n^2}\right) - \ln\prod_{n=1}^{\infty}\theta\left(1 - \frac{\theta^2}{n^2\pi^2}\right)\right\}$$

$$= \frac{d}{d\theta}\left\{2\ln\frac{\theta}{2} + 2\sum_{n=1}^{\infty}\ln\left(1 - \frac{\theta^2}{4n^2\pi^2}\right) - \ln\theta - \sum_{n=1}^{\infty}\ln\left(1 - \frac{\theta^2}{n^2\pi^2}\right)\right\}$$

$$= \frac{d}{d\theta}\left\{2\ln\theta - 2\ln 2 + 2\left[\sum_{n=1}^{\infty}\ln\left(1 - \frac{\theta}{2n\pi}\right) + \sum\ln\left(1 + \frac{\theta}{2n\pi}\right)\right] - \ln\theta - \sum_{n=1}^{\infty}\ln\left(1 + \frac{\theta}{n\pi}\right) - \sum_{n=1}^{\infty}\ln\left(1 - \frac{\theta}{n\pi}\right)\right\}$$

$$= \frac{d}{d\theta}\left\{\ln\theta - \ln 4 + 2\left[\sum_{n=1}^{\infty}\ln(2n\pi - \theta) + \sum\ln(2n\pi + \theta)\right] - \sum_{n=1}^{\infty}\ln(n\pi + \theta) - \sum_{n=1}^{\infty}\ln(n\pi - \theta)\right\}$$

(making $\frac{d}{d\theta}$ of constant to 0)

$$= \frac{1}{\theta} + \left(2\sum_{n=1}^{\infty}\frac{1}{2n\pi - \theta}\right) + 2\sum_{n=1}^{\infty}\frac{1}{2n\pi + \theta} - \sum_{n=1}^{\infty}\frac{1}{n\pi + \theta} - \sum_{n=1}^{\infty}\left(\frac{1}{n\pi - \theta}\right)$$

$$= \frac{1}{\theta} + \left(2\sum_{n=1}^{\infty} \frac{1}{\theta - 2n\pi} - \sum_{n=1}^{\infty}\frac{1}{\theta - n\pi}\right) + \left(2\sum_{n=1}^{\infty}\frac{1}{\theta+2n\pi} - \sum_{n=1}^{\infty}\frac{1}{\theta+n\pi}\right)$$

$$= \frac{1}{\theta} + \sum_{n=1}^{\infty}\left(\frac{(-1)^n}{\theta - n\pi}\right) + \sum_{n=1}^{\infty}\frac{(-1)^n}{\theta+n\pi}$$

$$= \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{\theta - n\pi}$$

P_35 (iii) $\Rightarrow$

$$\therefore \Gamma x\, \Gamma 1-x = B(x, 1-x)\, \Gamma(x+1-x)$$

$$= B(x, 1-x)$$

$$= \int_0^1 t^{x-1} \cdot (1-t)^{1-x-1}\, dt$$

$$= \int_0^1 t^{x-1} \cdot (1-t)^{-x}\, dt$$

for $0 < x < 1$

put $t = 1/(1+v) \Rightarrow dt = \frac{-1}{(1+v)^2}$

$t \to 0 \Rightarrow v \to \infty$ & $t \to 1 \Rightarrow v \to 0$

$$\therefore \Gamma(x)\,\Gamma(1-x) = \int_{\infty}^{0} \frac{1}{(1+v)^{x-1}} \cdot \left(\frac{v}{1+v}\right)^{-x} \cdot \left(\frac{-1}{(1+v)^2}\right) dv$$

$$= \int_0^{\infty} \frac{v^{-x}}{1+v}\, dv$$

$$= \int_0^1 \frac{v^{-x}}{1+v}\, dv + \int_1^{\infty} \frac{v^{-x}}{1+v}\, dv$$

$u = 1/v \Rightarrow dv = -\frac{du}{u^2}$

$$\therefore \int_1^{\infty} \frac{v^{-x}}{1+v}\, dv = \int_0^1 \frac{u^x}{1+1/u} \cdot \frac{du}{(u^2)} \quad \cancel{\int_1^{0} \frac{u^x}{1+\frac{1}{u}}\left(-\frac{1}{u^2}\right) du}$$

$$\therefore \Gamma x\, \Gamma 1-x = \int_0^1 \frac{v^{-x}}{1+v}\, dv + \int_0^1 \frac{u^{x-1}}{(u+1)}\, du$$

$$= \int_0^1 \frac{u^{-x} + u^{x-1}}{(1+u)}\, du$$

$$= \int_0^1 (u^{-x} + u^{x-1}) \left(\sum_{n=0}^{\infty} (-1)^n u^n\right) du$$

$$= \sum_{n=0}^{\infty} (-1)^n \int_0^1 \left(u^{n-x} + u^{n+x-1}\right) du$$

$$= \sum_{n=0}^{\infty} (-1)^n \left[\frac{u^{n-x+1}}{n-x+1} + \frac{u^{n+x}}{n+x}\right]_0^1$$

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$$= \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{n-x+1} + \frac{1}{n+x} \right]$$

$$= \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{x-n} \qquad (0<x<1)$$

$$= \sum_{n=-\infty}^{\infty} \frac{(-1)^n \pi}{x\pi - n\pi} = \frac{\pi}{\sin \pi x}$$

except $0<x<1$, $x = y+N$ N = integer
$0<y<1$

$$\boxed{\Gamma x \, \Gamma 1-x = \Gamma(y+N)\,\Gamma(y+N-1)}$$

$$= (y+N-1)(y+N-2) \cdots y \, \Gamma y \,.$$

$$\frac{1}{(y+N-1)} \cdot \frac{1}{(y+N-2)} \cdots \frac{1}{y} \Gamma 1-y$$

$$= (-1)^N \, \Gamma y \, \Gamma(1-y) = (-1)^N \frac{\pi}{\sin \pi y}$$

$$= \cancel{(-1)^N \, \Gamma y \, \Gamma}$$

$$= \frac{\pi}{\sin(n\pi + y\pi)} = \frac{\pi}{\sin \pi x}$$

Solve * * * 22-8-16

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Que:

$$\boxed{\int_0^1 y^{q-1} \left(\log \frac{1}{y}\right)^{p-1} dy = \int_0^\infty e^{-(q-1)u} \cdot u^{p-1} (e^{-u}) \, du}$$

Let $\log \frac{1}{y} = u \Rightarrow y = e^{-u}$
$\Rightarrow dy = -e^{-u} du$

$$= \int_0^\infty u^{p-1} e^{-qu} du = \int_0^\infty e^{-v} \frac{v^{p-1}}{q^{p-1}} \frac{dv}{q}$$

let $qu = v$

$$= \frac{1}{q^p} \int_0^\infty e^{-v} v^{p-1} dv$$

$$= \frac{\Gamma p}{q^p}$$

Que: Prove that —

$$B(m,n) = B(m+1,n) + B(m, n+1)$$

Pf:

$$\frac{\Gamma m+1 \, \Gamma n}{\Gamma m+n+1} + \frac{\Gamma m \, \Gamma n+1}{\Gamma m+n+1}$$

$$= \frac{\Gamma m \, \Gamma n \,(m+n)}{\Gamma m+n+1}$$

R.H.S. $B(m+1,n) + B(m, n+1)$

$$= \int_0^1 t^{m+1-1} (1-t)^{n-1} dt + \int_0^1 t^{m-1} (1-t)^{n+1-1} dt$$

$= \int_0^1 t^{m-1}(1-t)^{n-1}[t+1-t]\,dt$

$= \int_0^1 t^{m-1}(1-t)^{n-1}\,dt$

$= B(m,n)$ — Hence Proved

Que: ST. $B(p,q) = \int_0^\infty \frac{y^{q-1}}{(1+y)^{p+q}}\,dy = \int_0^1 \frac{x^{p-1}+y^{q-1}}{(1+x)^{p+q}}\,dx$

Sol^n:- $B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,dx$

let $y = \frac{x}{1-x}$

$y - xy = x \Rightarrow \cancel{x+y = xy}$ $x+xy = y$

$\Rightarrow \cancel{x = y(x-1)} \cancel{\Rightarrow y = \frac{x}{x-1}}$

$\Rightarrow x(y+1) = y \Rightarrow x = \frac{y}{y+1}$

$\therefore \frac{dx}{dy} = \frac{(y+1) - y}{(y+1)^2} = \frac{+1}{(y+1)^2}$

$\beta(p,q) = \int_0^\infty \left(\frac{y}{y+1}\right)^{p-1} + \left(\frac{1}{y+1}\right)^{q-1} \cdot \left(\frac{+1}{(y+1)^2}\right) dy$

$\cancel{= \int_0^\infty \frac{y^{p-1} + y^{q-1}(y-1)^{p-1}}{(y+1)^{p-1} \times (y+1+y)^{p+q}} \times \frac{+1}{(y+1)^2}\,dy}$

$= \int_0^\infty \frac{y^{p-1}}{(y+1)^{p-1+q-1+2}}\,dy = \int_0^\infty \frac{y^{p-1}}{(y+1)^{p+q}}\,dy$

$\beta(p,q) = \int_0^\infty \frac{y^{p-1}}{(1+y)^{p+q}}\,dy = \int_0^\infty \frac{y^{q-1}}{(y+1)^{p+q}}\,dy$

$(\because \beta(p,q) = \beta(q,p))$

<u>Que</u>

Show that

$$\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta = \int_0^{\pi/2} (\sin\theta)^{1/2} (\cos\theta)^{-1/2}\, d\theta$$

<u>Soln</u> -
$$\left[\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \cdot d\theta = \frac{\Gamma m\, \Gamma n}{2\,\Gamma(m+n)} \right.$$

$$\& \quad \left. \int_0^{\pi/2} \sin^m\theta \cdot \cos^n\theta\, d\theta = \frac{\Gamma\frac{m+1}{2}\,\Gamma\frac{n+1}{2}}{2\,\Gamma\frac{m+n+2}{2}} \right]$$

<u>R.H.S</u>
$$\int_0^{\pi/2} \sin^{1/2}\theta \cos^{-1/2}\theta\, d\theta \qquad (\text{i.e } m = 3/4,\ n = 1/4)$$

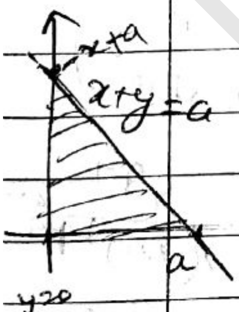
$$= \frac{\Gamma 3/4\ \Gamma 1/4}{2\,\Gamma 1} \qquad \left\{ \because \Gamma x\, \Gamma 1-x = \frac{\pi}{\sin \pi x} \right.$$

$$= \frac{1}{2} \cdot \frac{\pi}{\sin \frac{3}{4}\pi} \quad \text{or} \quad \frac{1}{2} \frac{\pi}{\sin \pi/4} = \frac{\pi}{\sqrt{2}}$$

<u>Que</u>
$$\iint_D X^{l-1} Y^{m-1}\, dX\, dY = \frac{\Gamma l\, \Gamma m}{\Gamma l+m+1}\, a^{l+m}$$

D : domain $x \geq 0,\ y \geq 0,\ x+y \leq a$

$X = ax,\quad Y = ay$



$$= \int_0^1 \int_{y=0}^1 a^{l-1} \cdot a^{m-1}\, x^{l-1} \cdot y^{m-1}\, a^2\, dx\, dy$$

$$= a^{l+m} \int_0^1 x^{l-1} \left[\int_0^{1-x} y^{m-1}\, dy \right] dx$$

$$= a^{l+m} \int_0^1 x^{l-1} \frac{(1-x)^{m+1-1}}{m}\, dx$$

$$= \frac{a^{l+m}}{m} \cdot \frac{\Gamma l\, \Gamma m+1}{\Gamma(m+l+1)}$$

$$= \frac{a^{l+m}}{\cancel{m}} \cdot \frac{\Gamma l\, \Gamma m \cdot \cancel{m}}{\Gamma m+l+1} = \frac{a^{l+m}\, \Gamma m\, \Gamma l}{\Gamma m+l+1}$$

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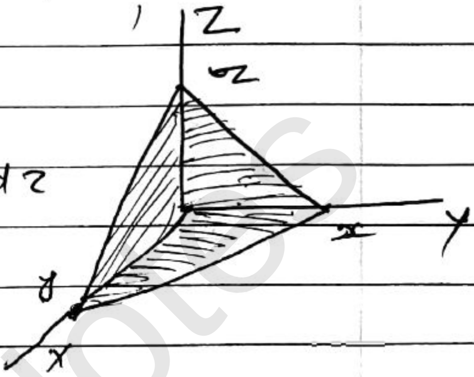
$$\boxed{\iint_D x^{l-1} y^{m-1}\,dx\,dy = a^{l+m}\frac{\Gamma l\,\Gamma m}{\Gamma(l+m+1)}}$$ Proved

* $$\iiint_V x^{l-1} y^{m-1} z^{n-1}\,dx\,dy\,dz = \frac{\Gamma l\,\Gamma m\,\Gamma n}{\Gamma(l+m+n+1)}$$

Where V is region, $x \geq 0, y \geq 0, z \geq 0$ & $x+y+z \leq 1$



Sol^n: L.H.S.

$$\iiint_V x^{l-1} y^{m-1} z^{n-1}\,dx\,dy\,dz$$

let $y+z \leq 1-x = a$

$$\int_0^1\int_0^{1-x}\int_0^{z=1-x-y} x^{l-1} y^{m-1} z^{n-1}\,dx\,dy\,dz$$

$$= \int_0^1\int_0^a\int_0^{a-y} x^{l-1} y^{m-1} z^{n-1}\,dx\,dy\,dz$$

$$= \int_0^1\int_0^a x^{l-1} y^{m-1}\left[\int_0^{a-y} z^{n-1}\,dz\right]dx\,dy$$

$$= \int_0^1\int_0^a x^{l-1} y^{m-1}\cdot\frac{(a-y)^n}{n}\,dy\,dx$$

$$= \frac{1}{n}\left[\int_0^1 x^{l-1}\left\{\int_0^a y^{m-1}(a-y)^{n+1-1}\,dy\right\}dx\right]$$

let $y = au$

$$= \frac{1}{n}\left[\int_0^1 x^{l-1}\left\{\int_0^1 a^{m-1+n+1}\, u^{m-1}(1-u)^{n+1-1}\,du\right\}dx\right]$$

$$= \frac{1}{n}\cdot a^{m+n}\left[\int_0^1 x^{l-1}\left\{\beta[m, n+1]\right\}dx\right]$$

$$= \frac{\Gamma m\,\Gamma n+1}{\Gamma m+n+1}\cdot\frac{1}{n}\left\{\int_0^1 x^{l-1}\, a^{m+n}\,dx\right\}$$

let $a = (1-x)$

$$= \frac{\Gamma m\, n\,\Gamma n}{n\,\Gamma m+n+1}\int_0^1 x^{l-1}(1-x)^{m+n+1-1}\,dx$$

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$$= \frac{\Gamma m \Gamma n}{\cancel{\Gamma m+n+1}} \cdot \frac{\Gamma l \cancel{\Gamma m+n+1}}{\Gamma l+m+n+1}$$

$$= \frac{\Gamma m \Gamma n \Gamma l}{\Gamma l+m+n+1}$$

Proved

— P KALIKA NOTES
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UNIT-III

P. KALIKA

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* Bessel Polynomial :

2^{nd} order DE

$$\boxed{x^2y'' + xy' + (x^2 - \nu^2)y = 0} \quad \text{--- (1)}$$ where ν is non-integer.

Its solⁿ is of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \quad \text{(for ordinary pt.)}$$

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \quad \text{(for Regular singular pt.)}$$

$$P(x)y'' + Q(x)y' + R(x) = 0$$

(1). $x = x_0$, $P(x_0) \neq 0$ ordinary pt.

$$\boxed{p(x) = \frac{Q(x)}{P(x)}} = \text{finite value in } x$$

$$\boxed{q(x) = \frac{R(x)}{P(x)}} = \text{finite value in } x$$

then x_0 is called ordinary pt.

(i). $p(x)$ & $q(x)$ are called nbd of x_0.

(2) at $x = x_0$, $p(x_0) = 0$

$$\lim_{x \to x_0} (x - x_0)p(x) = A \text{ finite value}$$

$$\lim_{x \to x_0} (x - x_0)^2 q(x) = A \text{ finite value}$$

$x = x_0$ is called Regular singular pt.

$p(x)$ & $q(x)$ are not defined in nbd of x_0.
but defined in deleted nbd of x_0.
and smooth (ctsly differentiable in deleted nbd)

* $P(x) = x^2$, $Q(x) = x$, $R(x) = x^2 - \nu^2$

at $x = 0$ (Regular singular pt.)

$$p(x) = \frac{1}{x}, \quad q(x) = 1 - \frac{\nu^2}{x^2}$$

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$$\lim_{x\to 0} x \cdot p(x) = 1$$

& $$\lim_{x\to 0} x^2 p(x) = -\nu^2$$

So, the soln will be of the form —

$$\boxed{y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}} \quad ; a_0 \neq 0$$

putting this value in equation (1), we have

$$x^2 \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2} + x \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1} + (x^2-\nu^2) \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r} + \sum_{n=0}^{\infty} a_n (n+r) x^{n+r} + (x^2-\nu^2) \cdot \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} \left[(n+r)(n+r-1) + (n+r) - \nu^2\right] a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+2} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} \left[(n+r)^2 - \nu^2\right] a_n x^{n+r} + x^{n+r} \sum_{n=0}^{\infty} a_n x^{n+r+2} = 0$$

Now putting coefficient of lowest power of x equal to zero, we get —

$$(r^2 - \nu^2) a_0 = 0 \quad , \quad a_0 \neq 0$$

$$\Rightarrow \boxed{r = \pm \nu}$$

Again, putting coeff of second lowest power of x (x^{r+1}) equal

$$\left[(1+r)^2 - \nu^2\right] a_1 = 0$$

$$\Rightarrow a_1 = 0 \qquad \left[\text{Bez } \left((1+r)^2 - \nu^2\right) \neq 0\right]$$

Similarly same process —

Again, comparing coefficient of x^{n+r} both sides,

$$\left[(n+r)^2 - \nu^2\right] a_n + a_{n-2} = 0$$

$$\boxed{a_n = \frac{-1}{(n+r)^2 - \nu^2} \cdot a_{n-2}}$$

Case-I, Now taking for $r = \nu$

$$a_n = \frac{-a_{n-2}}{n^2 + 2n\nu} = \frac{-a_{n-2}}{n(n+2\nu)}$$

$(n \geq 2)$

for $n=3$,

$$a_3 = \frac{-a_1}{3(3+2\nu)} = 0 \quad (\because a_1 = 0)$$

for $n=5$, $$a_5 = \frac{-1}{5(5+2\nu)} \cdot a_3 = 0 \quad (\because a_3 = 0)$$

$$\Rightarrow \boxed{a_1 = a_3 = a_5 = \cdots = 0}$$

now,

$n=2$, $$a_2 = \frac{-a_0}{2(2+2\nu)} = \frac{-a_0}{2^2(1+\nu)}$$

$n=4$, $$a_4 = \frac{-a_2}{4(4+2\nu)} = \frac{-a_2}{2^2(2+2\nu)} = \frac{a_0}{2!\,2^4(\nu+1)(\nu+2)}$$

$n=6$, $$a_6 = \frac{-a_4}{6(6+2\nu)} = \frac{-a_0}{3!\,2^6(1+\nu)(2+\nu)(3+\nu)}$$

$n=2m$ $$\boxed{a_{2m} = \frac{-a_{2m-2}}{2m\cdot 2(m+\nu)} = \frac{(-1)^m \cdot a_0}{m!\,2^{2m}(\nu+1)(\nu+2)\cdots(\nu+m)}}$$

now, we want to write the above in β, ν form,

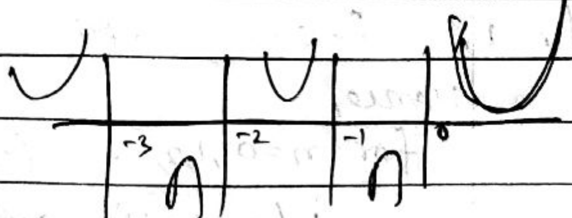
$$\because (\nu+1)(\nu+2)\cdots(\nu+m)\frac{\Gamma(\nu+1)}{\Gamma(\nu+1)} = \frac{\Gamma(\nu+m+1)}{\Gamma(\nu+1)}$$

So, $$\boxed{a_{2m} = \frac{(-1)^m a_0 \,\Gamma(\nu+1)}{m!\,2^{2m}\,\Gamma(\nu+m+1)}} \quad \left(\begin{array}{l}\nu+1 \neq 0,-1,-2\ldots \\ \nu \neq -1,-2,-3\ldots\end{array}\right)$$

to 0.

$\Gamma(x) \rightarrow$ Singular pt's at $x = 0,-1,-2,\ldots$

graph of gamma f^n are



then $$y(x) = \sum_{n=0}^{\infty} a_n \cdot x^{n+\nu}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \,\Gamma(\nu+1)}{n!\,2^{2n}\,\Gamma(\nu+n+1)} \cdot a_0 \cdot x^{2n+\nu}$$

This is the soln of eqn (1).

J_ν has only (+) powers of x

$J_{-\nu}$ has ~~only~~ (−) powers of x also

so J_ν & $J_{-\nu}$ are L.I

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Let $a_0 = \left(1/2^\nu\right)\Gamma(\nu+1)$

then
$$\boxed{J_\nu(x) = y(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(\nu+n+1)}\left(\frac{x}{2}\right)^{2n+\nu}}$$

This special f^n is known as Bessel Polynomial.

Case-II. If we proceed by taking $r=-\nu$ then we get –

$$\boxed{J_{-\nu}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(-\nu+n+1)}\cdot\left(\frac{x}{2}\right)^{2n-\nu}}$$

(Both are sol^n of eqn (1))

These two are sol^n of eq^n (1) if 'ν' is non-integer.

Then $\boxed{y(x) = A\,J_\nu(x) + B\,J_{-\nu}(x)}$ is the complete sol^n of Bessel DE.

(Bcz $J_\nu(x)$ & $J_{-\nu}(x)$ are L.I)

so $y(x)$ is the complete sol^n.

Theorem [4.1] / P-95

Que: Prove that

$J_{-\nu}(x) = (-1)^\nu J_\nu(x)$ if ν is integer

Proof: $$J_{-\nu}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(-\nu+n+1)}\left(\frac{x}{2}\right)^{2n-\nu}$$

if $n=0$, $\Gamma(-\nu+1) \to \infty$ (as ν is integer & $\nu \neq 1$)

$\Rightarrow J_{-\nu}(x) \to 0$

if $n=1$, $\Gamma(-\nu+2) \to \infty \Rightarrow J_{-\nu}(x) \to 0$

Hence,

for $n = 0, 1, 2, \ldots (\nu-1)$

$\Gamma(-\nu+n+1) \to \infty \Rightarrow J_{-\nu}(x) \to 0$

$$\Rightarrow J_{-\nu}(x) = \sum_{n=\nu}^{\infty} \frac{(-1)^n}{n!\,\Gamma(-\nu+n+1)}\left(\frac{x}{2}\right)^{2n-\nu}$$

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Replace n by $m+\nu$

then. $$J_{-\nu}(x) = \sum_{m=0}^{\infty} \frac{(-1)^{m+\nu}}{(m+\nu)!\,\Gamma(m+1)} \left(\frac{x}{2}\right)^{2(m+\nu)-\nu}$$

Since $(m+\nu)!\,\Gamma m+1 = m!\,\Gamma m+\nu+1$

$\therefore$ $(m+\nu)(m+\nu-1)\ldots(m+1)\,m!\,\Gamma m+1$

So, $$J_{-\nu}(x) = (-1)^{\nu}\sum_{m=0}^{\infty} \frac{(-1)^{m}}{m!\,\Gamma m+\nu+1}\left(\frac{x}{2}\right)^{2m+\nu}$$

$$J_{-\nu}(x) = (-1)^{\nu} J_{\nu}(x)$$

* * *

wed (1 hr) 31/8/16

Theorem:

Result: $\boxed{4.2 \mid 96}$ The two independent soln of Bessel DE are given by

$$J_\nu(x)$$

L.I

and

$$Y_\nu(x) = \frac{\cos\nu\pi\, J_\nu(x) - J_{-\nu}(x)}{\sin\nu\pi}$$

for all value of ν.

Proof: Let ν - non integral, $Y_\nu(x)$ is also a soln of Bessel Eqn.

if ν is integral, so,

$$Y_\nu(x) = \frac{(-1)^{\nu}J_\nu(x) - J_{-\nu}(x)}{\sin\nu\pi} \simeq \frac{0}{0}$$

It has no meaning, we may write it as—

$$Y_\nu(x) = \lim_{n\to\nu} \frac{\cos n\pi\, J_n(x) - J_{-n}(x)}{\sin n\pi}$$

$$= \frac{\left\{\frac{\partial}{\partial n}\left[\cos n\pi\, J_n(x) - J_{-n}(x)\right]\right\}_{n=\nu}}{\{\pi\cdot\cos n\pi\}_{n=\nu}}$$

$$= \frac{\left[-\pi\sin n\pi\, J_n(x) + \cos n\pi\left(\frac{\partial}{\partial n}J_n(x)\right) - \left(\frac{\partial}{\partial n}J_{-n}(x)\right)\right]_{n=\nu}}{\pi(-1)^{\nu}}$$

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$$= \frac{\cos\nu\pi\left(\frac{\partial}{\partial n}J_n(x)\right)_{n=\nu} - \left(\frac{\partial}{\partial n}J_{-n}(x)\right)_{n=\nu}}{\pi(-1)^\nu}$$

$$= \frac{1}{\pi}\left[\frac{\partial}{\partial n}J_n(x) - (-1)^n\frac{\partial}{\partial n}J_{-n}(x)\right]_{n=\nu}$$

or $\frac{1}{\pi}\left[\left(\frac{\partial}{\partial n}J_n(x)\right)_{n=\nu} - (-1)^\nu\left(\frac{\partial}{\partial n}J_{-n}(x)\right)_{n=\nu}\right]$

So $Y_\nu(x) = \frac{\cos\nu\pi J_\nu(x) - J_{-\nu}(x)}{\sin\nu\pi} = \frac{1}{\pi}\left[\frac{\partial}{\partial n}J_n(x) - (-1)^n\frac{\partial}{\partial n}J_{-n}(x)\right]_{n=\nu}$

→ Also sol^n of B.D.E

Now since $J_\nu(x)$ is sol^n of (1), so

$$x^2y'' + xy' + (x^2-\nu^2)y = 0 \quad \text{—(1)}$$

so it will satisfy (1), so we write it as

$$x^2\frac{d^2}{dx^2}J_\nu(x) + x\frac{d}{dx}J_\nu(x) + (x^2-\nu^2)J_\nu(x) = 0$$

which on diff^n gives —(w.r.t ν).

$$x^2\frac{d^2}{dx^2}\left(\frac{\partial}{\partial\nu}J_\nu(x)\right) + x\frac{d}{dx}\left(\frac{\partial}{\partial\nu}J_\nu(x)\right) + (x^2-\nu^2)\frac{\partial}{\partial\nu}J_\nu(x)$$
$$- 2\nu J_\nu(x) = 0 \quad \text{—(ii)}$$

Since $J_{-\nu}(x)$ is also sol^n of (1) so also it satisfies (1)

$$\Rightarrow x^2\frac{d^2}{dx^2}\left(\frac{\partial}{\partial\nu}J_{-\nu}(x)\right) + x\frac{d}{dx}\left(\frac{\partial}{\partial\nu}J_{-\nu}(x)\right) + (x^2-\nu^2)$$
$$\cdot\frac{\partial}{\partial\nu}J_{-\nu}(x) - 2\nu J_{-\nu}(x) = 0$$
$$\text{—(iii)}$$

Now multiply (iii) by $(-1)^\nu$ then subtract it from (ii), we get —

$$x^2\frac{d^2}{dx^2}\underbrace{\left[\frac{\partial}{\partial\nu}J_\nu(x) - (-1)^\nu\frac{\partial}{\partial\nu}J_{-\nu}(x)\right]}_{=Y_\nu(x)} + x\frac{d}{dx}\left[\frac{\partial}{\partial\nu}J_\nu(x)\right.$$
$$\left.-(-1)^\nu J_{-\nu}(x)\right] + (x^2-\nu^2)\left[\frac{\partial}{\partial\nu}J_\nu(x) - (-1)^\nu J_\nu(x)\right]$$
$$- 2\nu\left[J_\nu(x) - (-1)^\nu J_{-\nu}(x)\right] = 0$$

NE

Theorem (4
(P-99)

pf: —

So

Case I

of E

$$x^2\frac{d^2}{dx^2}Y_\nu(x) + x\frac{d}{dx}Y_\nu(x) + (x^2-\nu^2)Y_\nu(x) = 0$$

NB ($Y_n(x)$ is L.I with $J_\nu(x)$ when n is integral)

Since when ν is integral then $J_\nu(x)$ & $Y_\nu(x)$ are sol^n of (1) & when ν is non-integral then it also sol^n of (1).

so $\forall$ ν, $J_\nu(x)$ & $J_{-\nu}(x)$ are sol^n of (1).

$\left.J_{-n}(x)\right]_{n=\nu}$

Generating functions for the Bessel f^{n}s

Theorem (4.5) (p-99)

$$\exp\left\{\frac{x}{2}\left(t-\frac{1}{t}\right)\right\} = \sum_{\nu=-\infty}^{\infty} J_\nu(x)\, t^\nu \quad -(1)$$

pf:- If we have to prove (1), the consider L.H.S —

$$\exp\left(\frac{xt}{2}\right)\exp\left(-\frac{x}{2t}\right) = \sum_{r=0}^{\infty}\frac{x^r t^r}{2^r r!}\cdot\sum_{s=0}^{\infty}\frac{(-1)^s x^s}{2^s t^s s!}$$

$$= \sum_{r,s=0}^{\infty}\frac{(-1)^s\left(\frac{1}{2}\right)^{r+s} x^{r+s}\cdot t^{r-s}}{r!\,s!} \quad —(II)$$

so from (1), we have

$$\exp\left(\frac{x}{2}\left(t-\frac{1}{t}\right)\right) = \sum_{\nu=-\infty}^{\infty} J_\nu(x)\,t^\nu = \underbrace{\sum_{\nu=0}^{\infty} J_\nu(x)\,t^\nu}_{(a)} + \underbrace{\sum_{\nu=-1}^{-\infty} J_\nu(x)\,t^\nu}_{(b)} \quad (III)\ (3)$$

from (II), we have two cases —

Case I $r-s=n\geq 0$ (then we have (a)).

co-eff^n of t^n when $n\geq 0$

for fix r, coefficient of t^n are —

$$\frac{(-1)^{r-n}\left(\frac{1}{2}\right)^{2r-n} x^{2r-n}}{r!\,(r-n)!}$$

$r-s=n\geq 0$
$s=r-n\geq 0$
$\Rightarrow r\geq n$

On taking summation, we get coeff^n of t

$$\sum_{r=n}^{\infty}\frac{(-1)^{r-n}\left(\frac{1}{2}\right)^{2r-n} x^{2r-n}}{r!\,(r-n)!}$$

Now if $r \to m+n$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{1}{2}\right)^{2m+n} x^{2m+n}}{(m+n)!\, m!}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+n+1)} \cdot \left(\frac{x}{2}\right)^{2m+n} = J_n(x)$$

If we replace ν by n in eqn (3), then we get.

<u>Case-II</u> $r - s = n < 0$

Co-efficient of t^n when $n<0$

for fix r, co-effn of t^n are

$$\frac{(-1)^{r-n} \left(\frac{1}{2}\right)^{2r-n} \cdot x^{2r-n}}{r!\,(r-n)!}$$

i.e Co-effn of t^n will not be changed $\forall n$

$r - s = n < 0$
$s = r - n$
$\because r \geq 0$ & $n < 0$
so $s \geq 0 \ \forall r$

Now taking summation, we get

$$\sum_{r=0}^{\infty} \frac{(-1)^{r-n} \left(\frac{1}{2}\right)^{2r-n} x^{2r-n}}{r!\,(r-n)!}$$

$$= (-1)^n \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(r-n+1)} \left(\frac{x}{2}\right)^{2r-n}$$

$$= (-1)^n J_{-n}(x)$$

further when n is integral then—

$$= (-1)^n J_{-n}(x) = J_n(x) \quad (\text{if } n<0)$$

On taking ~~all~~ summation of coeffn of ~~part~~ t^n ~~if~~ i.e for neg. value of n & positive value of n.

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we get (3).

* * *

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INTEGRAL REPRESENTATION OF BESSEL F^n

Theorem (4.6)

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\phi - x\sin\phi)\, d\phi, \quad n = \text{integer}$$

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$$= \frac{1}{\pi}\int_0^{\pi} \left[\cos n\phi \cdot \cos(x\sin\phi) + \sin n\phi \sin(x\sin\phi)\right] d\phi \quad \text{—} (2)$$

$\because$ Generating f^n

$$\exp\left\{\frac{x}{2}\left(t - \frac{1}{t}\right)\right\} = \sum_{n=-\infty}^{\infty} J_n(x)\, t^n \quad \text{—} (1)$$

$$= J_0(x) + \sum_{n=1}^{\infty} J_n(x)\, t^n + \sum_{n=1}^{\infty} J_{-n}(x)\, t^{-n}$$

$$= J_0(x) + \sum_{n=1}^{\infty} \left[t^n + (-1)^n t^{-n}\right] J_n(x)$$

If we put $t = e^{i\theta}$ in LHS of (1) then we have

$$\exp\left\{\frac{x}{2}\left(t - \frac{1}{t}\right)\right\} = \exp\{x\sin\theta\}$$

$$\therefore \exp\left\{\frac{x}{2}\left(t - \frac{1}{t}\right)\right\} = J_0(x) + \sum_{n=\text{even}}^{\infty} 2\cos n\phi\, J_n(x) + \sum_{n=\text{odd}}^{\infty} 2i\sin n\phi\, J_n(x)$$

$$\Rightarrow \exp\{x\sin\phi\} = J_0(x) + \sum_{k=1}^{\infty} 2\cos 2k\phi \cdot J_{2k}(x) + \sum_{k=1}^{\infty} 2i\sin(2k-1)\phi\, J_{2k-1}(x)$$

$$\cos(x\sin\phi) = J_0(x) + \sum_{k=1}^{\infty} 2\cos 2k\phi \cdot J_{2k}(x) \quad \text{—} (a)$$

$$\& \quad \sin(x\sin\phi) = \sum_{k=1}^{\infty} 2\sin(2k-1)\phi\, J_{2k-1}(x) \quad \text{—} (b)$$

$\because$ we know that

$$\int_0^{\pi} \cos m\phi \cdot \cos n\phi\, d\phi = \begin{cases} 0 & m \neq n \\ \pi & m = n = 0 \\ \pi/2 & m = n \end{cases}$$

and

$$\int_0^{\pi} \sin m\phi \cdot \sin n\phi\, d\phi = \begin{cases} 0, & m \neq n \\ \pi/2, & m = n \end{cases}$$

Now

(a) $\times \cos n\phi \Rightarrow$

$$\int_0^{\pi} \cos(x\sin\phi) \times \cos n\phi\, d\phi = \begin{cases} 0, & n = \text{odd} \\ \pi J_n(x), & n = \text{even}, n \neq k \end{cases} \quad \text{—} (c)$$

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od (b) $x \sin n\phi \Rightarrow$

$$\int_0^\pi \sin(x\sin\phi)\,d\phi \quad x\sin n\phi = \begin{cases} \pi J_n(x) & n = \text{odd} \\ 0 & n = \text{even} \end{cases} \quad \text{---} (d)$$

Now on adding (c) & (d), we get —
eqn (2) (LHS of (2))

$$\int_0^\pi \cos n\phi\, \cos(x\sin\phi)\,d\phi + \int_0^\pi \sin n\phi.(x\sin\phi)\,d\phi$$

$$= \pi J_n(x)$$

$$\Rightarrow \boxed{\pi J_n(x) = J_n(x)}$$

Theorem (4.7) (p-103)

(2) $$J_n(x) = \frac{(x/2)^n}{\sqrt{\pi}\,\Gamma(n+1/2)} \int_{-1}^{1} (1-t^2)^{n-1/2} e^{ixt}\,dt \quad (n \geq -1/2) \quad \text{---} (1)$$

Duplication formula —
$$\boxed{\Gamma(2z) = \frac{1}{\sqrt{\pi}} 2^{2z-1} \Gamma z\, \Gamma(z+1/2)}$$

~~By duplication formula, we have~~

since —
$$\int_{-1}^{1} (1-t^2)^{n-1/2} \left(\sum_{n=0}^{\infty} \frac{(ixt)^n}{n!}\right) dt = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} \int_{-1}^{1} (1-t^2)^{n-1/2} t^n\,dt$$

Now
$$\int_{-1}^{1} (1-t^2)^{n-1/2} \left(\sum_{k=0}^{\infty} \frac{(ixt)^k}{k!}\right) dt = \sum_{k=0}^{\infty} \frac{(ix)^k}{k!} \int_{-1}^{1} (1-t^2)^{n-1/2} t^k\,dt$$

if k = even

(Bcz if n = odd then integral will be '0')

$$= \sum_{k=0}^{\infty} \frac{(ix)^{2k}}{(2k)!} \int_{-1}^{1} (1-t^2)^{n-1/2} t^{2k}\,dt$$

(we write its in even fun)

Now let $t^2 = u$

$\Rightarrow 2t\,dt = du \Rightarrow dt = du/2\sqrt{u}$

$\therefore$

$$= \sum_{k=0}^{\infty} \frac{(ix)^{2k}}{(2k)!} \, 2\int_0^1 (1-u)^{n-1/2} \frac{1}{2} u^{k-1/2} du$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2k!} \beta(n+1/2, k+1/2)$$

If we want to write above in gamma form

then $$= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2k!} \frac{\Gamma(n+1/2)\Gamma(k+1/2)}{\Gamma(n+k+1)}$$

Now By using Duplication ~~gamma~~ formula, we have

$$= \Gamma(n+1/2) \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{\cancel{2k!}} \frac{\cancel{\Gamma(2k)}\sqrt{\pi}}{\left(2^{2k-1}\Gamma(k)\right) \times \Gamma(n+k+1)} \times \frac{2k}{2k}$$

$$= \sqrt{\pi}\,\Gamma(n+1/2) \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(n+k+1)} \left(\frac{x}{2}\right)^{2k}$$

$$= \left(\frac{x}{2}\right)^{-n} \sqrt{\pi}\,\Gamma(n+1/2) \underbrace{\sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(n+k+1)} \left(\frac{x}{2}\right)^{2k+n}}_{= J_n(x)}$$

$$= \left(\frac{x}{2}\right)^{-n} \sqrt{\pi}\,\Gamma(n+1/2) \cdot J_n(x)$$

Now putting this in eqn (1), we have

Theorem (4.8) (P-104) **Recurrance Relation**

(i) $\frac{d}{dx}\{x^n J_n(x)\} = x^n J_{n-1}(x)$

(ii) $\frac{d}{dx}\{x^{-n} J_n(x)\} = -x^{-n} J_{n+1}(x)$

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(iii) $J_n'(x) = J_{n-1}(x) - \frac{n}{x} J_n(x)$

(iv) $J_n'(x) = \frac{n}{x} J_n(x) - J_{n+1}(x)$

(v) $J_n'(x) = \frac{1}{2}\left\{ J_{n-1}(x) - J_{n+1}(x) \right\}$

(vi) $J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x)$

$\because \quad J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(n+k+1)} \left(\frac{x}{2}\right)^{2k+n} \quad \text{--- (1)}$

Now multiplying (1) ~~w.r.t~~ with x^n, we get

$x^n J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(n+k+1)} \cdot \left(\frac{x}{2}\right)^{2k+2n} \quad \text{--- (2)}$

Now differentiating (2) w.r.t. x, we get

$\frac{d}{dx}\left(x^n J_n(x)\right) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(n+k+1)} \cdot \frac{(2k+2n)}{2} \left(\frac{x}{2}\right)^{2k+2n-1}$

$=$

* Riemann Zeta function

Not in syllabus

$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad R(s) > 0$

$_2F_1(a,b,c,x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (n)!} x^n$

$(a)_n = \begin{cases} 1 \\ a(a+1)\cdots(a+n-1) & , n \neq 0 \end{cases}$

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2-n^2)y = 0$$

Bessel function of 2nd Kind

Then if we replace J_n by Y_n (ie J_{-n} by Y_{-n}) then we get solⁿ after (recurrence relⁿ) for Bessel fⁿ of 2nd kind.

So from prev. Bessel's recurrence Relⁿ we have—

(i) $\frac{d}{dx}\left(x^n Y_n(x)\right) = x^n Y_{n-1}(x)$

(ii) $\frac{d}{dx}\left(x^{-n} Y_n(x)\right) = -x^{-n} Y_{n+1}(x)$

(iii) $Y_n'(x) = Y_{n-1}(x) - \frac{n}{x} Y_n(x)$

(iv) $Y_n'(x) = \frac{n}{x} Y_n(x) - Y_{n+1}(x)$

(v) $Y_n'(x) = \frac{1}{2}\left[Y_{n-1}(x) - Y_{n+1}(x)\right]$

(vi) $\frac{2n}{x} Y_n(x) = Y_{n-1}(x) + Y_{n+1}(x)$

$Y(x) = J_0$

, n = non-integer

we have

$$x^n Y_n(x) = \frac{1}{\sin n\pi}\left[\cos n\pi\, x^n J_n(x) - x^n J_{-n}(x)\right]$$

$$\frac{d}{dx}\left(x^n Y_n(x)\right) = \frac{1}{\sin n\pi}\left[\cos n\pi \frac{d}{dx}\left(x^n J_n(x)\right) - \frac{d}{dx}\left(x^n J_{-n}(x)\right)\right]$$

$$= \frac{1}{\sin n\pi}\left[\cos n\pi\, x^n J_{n-1}(x) + x^n J_{-(n-1)}(x)\right]$$

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As $\sin[(n-1)\pi+\pi] = -\sin(n-1)\pi$

& $\cos[(n-1)\pi+\pi] = -\cos(n-1)\pi$

and $\frac{d}{dx}\left(x^{-n}J_n(x)\right) = -x^{-n}J_{n+1}(x)$

$n \to -m$ $\frac{d}{dx}\left(x^{m}J_{-m}(x)\right) = -x^{m}J_{-(m+1)}(x)$

$$= x^n \cdot \frac{\cos(n-1)\pi\, J_{n-1}(x) - J_{-(n-1)}(x)}{\sin(n-1)\pi}$$

$$= x^n Y_{n-1}(x)$$

For eg. consider the DE—

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (\lambda^2x^2 - n^2)y = 0 \quad — (1)$$

If we are able to reduce λ^2, then done

let $u = \lambda x$

$$\Rightarrow \frac{dy}{dx} = \lambda\frac{dy}{du}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \lambda^2\frac{d^2y}{du^2}$$

∴ (1) becomes —

$$\frac{u^2}{\cancel{\lambda^2}}\cancel{\lambda^2}\frac{d^2y}{du^2} + \frac{u}{\cancel{\lambda}}\cancel{\lambda}\frac{dy}{du} + (u^2 - n^2)y = 0$$

Thus solⁿ of this DE is —

$$\boxed{y(x) = A J_n(\lambda x) + B J_{-n}(\lambda x)} \quad — (2)$$

$\left(\text{as } y(x) = A J_n(u) + B J_{-n}(u)\right)$

It holds only when n is non-integer.

In (1), let $\lambda^2 = -1$, then (1) becomes —

$$\boxed{x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - (x^2+n^2)y = 0} \quad — (3)$$

this form of DE is known as Bessel Modified Diff. Eqn.

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$\therefore \lambda^2 = -1$

$\Rightarrow \lambda = \pm i$

Since we know that ② sol^n of ①

so, sol^n for DE ③ is given as -

$$\boxed{Y(x) = A J_n(ix) + B J_{-n}(ix)} \quad \text{— ⓐ}$$

&

$$= A J_n(-ix) + B J_{-n}(-ix)$$

$$\boxed{I_n(x) = \sum_{r=0}^{\infty} \frac{1}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{2r+n}}$$

which is known as Bessel Modified f^n

$$I_n(x) = i^{-n} J_n(ix)$$

$$= i^{-n} \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)} \left(\frac{ix}{2}\right)^{2r+n}$$

$$= i^{-n} \sum_{r=0}^{\infty} \frac{(-1)^r (i)^{2r+n}}{r!\,\Gamma(n+r+1)} \cdot \left(\frac{x}{2}\right)^{2r+n}$$

further

$$I_{-n}(x) = i^{n} J_{-n}(ix)$$

so we may write ⓐ as

$$\boxed{Y(x) = A_1 I_n(x) + B_1 I_{-n}(x)} \quad \text{— ⓑ}$$

where $A_1 = i^{-n} A$, $B_1 = i^{-n} B$

so if we have given DE of the form ③, so its sol^n is given as - ⓑ.
where n must be non-integer.

And when n = integer, in ③, then

$$I_{-n}(x) = I_n(x)$$

and ⓑ is not complete sol^n of ③

$$I_{-n}(x) = i^{n} J_{-n}(ix)$$

$$= i^{+n} (-1)^n J_n(ix) \quad \text{when } n = \text{integer}$$

1 (B) wed

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$$I_{-n}(x) = \underbrace{i^{-n} i^{2n} (-1)^n}_{=1} J_n(ix)$$
$$= i^{-n} J_n(ix)$$
$$= I_n(x) \quad \text{— (c)}$$

$$\boxed{K_n(x) = \frac{\pi}{2} \frac{I_{-n}(x) - I_n(x)}{\sin n\pi}} \quad , n \neq \text{integer}$$

It is called modified Bessel's function of 2nd kind.
where (c) is the solⁿ of (3).
Here for (c), we have two case
n = integer, n ≠ integer.
When n ≠ integer then (c) is solⁿ of (3).
and when n = integer then (c) is $\frac{0}{0}$ form
i.e indeterminate form so we do following
we take limiting case —

$$K_n(x) = \frac{\pi}{2} \lim_{\nu \to n} \frac{I_{-\nu}(x) - I_\nu(x)}{\sin \nu\pi}$$

$$= \lim_{\nu \to n} \frac{\frac{\pi}{2} \left[\frac{\partial}{\partial\nu} I_{-\nu}(x) - \frac{\partial}{\partial\nu} I_\nu(x)\right]_{\nu=n}}{\pi \left[\cos \nu\pi\right]_{\nu=n}} \quad = (-1)^n$$

$$= \frac{(-1)^n}{2} \left[\frac{\partial}{\partial\nu} I_{-\nu}(x) - \frac{\partial}{\partial\nu} I_\nu(x)\right]_{\nu=n}$$

Examp

As we ~~know~~ know that $I_\nu(x)$ & $I_{-\nu}(x)$ both are solⁿ of (3), so it must satisfy (3).
so by (3) —

$$x^2 \frac{d^2 I_\nu}{dx^2} + x \frac{dI_\nu}{dx} - (x^2+n^2) I_\nu = 0$$

$$\left(\cancel{x^2 \frac{d^2 I_{-\nu}}{dx^2} + x \frac{dI_{-\nu}}{dx} + (x^2+n^2) I_{-\nu} = 0} \right)$$

$$\Rightarrow x^2 \frac{d^2}{dx^2}\left(\frac{\partial}{\partial\nu} I_\nu\right) + x \frac{d}{dx}\left(\frac{\partial}{\partial\nu} I_\nu\right) - (x^2+n^2)\left(\frac{\partial}{\partial\nu} I_\nu\right) = 0 \quad \text{— (4)}$$

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$$\left(x\frac{d^2}{dx^2}\left(\frac{\partial}{\partial \nu} I_{-\nu}\right) + x\frac{d}{dx}\left[\frac{\partial}{\partial \nu} I_{-\nu}\right] - (x^2+n^2)\frac{\partial}{\partial \nu}(-I_{-\nu}) \right) = 0 \quad \text{--- (5)}$$

Now (4)-(5), gives —

$$x^2\frac{d^2}{dx^2}\left[\frac{\partial}{\partial \nu} I_\nu - \frac{\partial}{\partial \nu} I_{-\nu}\right] + x\frac{d}{dx}\left[\frac{\partial}{\partial \nu} I_\nu - \frac{\partial}{\partial \nu} I_{-\nu}\right] - (x^2+n^2)\left[\frac{\partial}{\partial \nu} I_\nu - \frac{\partial}{\partial \nu} I_{-\nu}\right] = 0$$

So (b), becomes —

$$y(x) = A_1 I_n(x) + B_1 K_n(x)$$

NB (Here $I_n(x)$ & $K_n(x)$ are independent, bcz in $I_n(x)$ only +ve power of x while in $K_n(x)$ neg. power of x also available)

i.e when $n =$ integer, then $I_n(x)$ & $K_n(x)$ are independent.

Consider

Example: $$J_n(x) = \frac{2(x/2)^{n-m}}{\Gamma(n-m)}\int_0^1 (1-t^2)^{n-m-1} \times t^{m+1} J_m(xt)\,dt \quad (n>m>-1)$$

let $$I = \int_0^1 (1-t^2)^{n-m-1} t^{m+1} J_m \, dt$$

$$= \int_0^1 (1-t^2)^{n-m-1} t^{m+1}\left(\sum_{r=0}^{\infty}\frac{(-1)^r}{r!\,\Gamma(n+r-1)}\left(\frac{xt}{2}\right)^{2r+m}\right)$$

$$= \sum_{r=0}^{\infty}\frac{(-1)^r}{r!\,\Gamma(r+n-1)}\left(\frac{x}{2}\right)^{2r+m}\int_0^1 (1-t^2)^{n+m-1} \cdot t^{2r+2m+1}\cdot dt$$

$\Big) = 0$ — (4)

$t^2 \to u$

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$$\left(x\frac{d^2}{dx^2}\left(\frac{\partial}{\partial \nu} I_{-\nu}\right) + x\frac{d}{dx}\left[\frac{\partial}{\partial \nu} I_{-\nu}\right] - (x^2+n^2)\left(\frac{\partial}{\partial \nu}(-I_{-\nu})\right) = 0 \right) \quad \text{— (5)}$$

Now (4)-(5), gives—

$$x^2\frac{d^2}{dx^2}\left[\frac{\partial}{\partial \nu} I_\nu - \frac{\partial}{\partial \nu} I_{-\nu}\right] + x\frac{d}{dx}\left[\frac{\partial}{\partial \nu} I_\nu - \frac{\partial}{\partial \nu} I_{-\nu}\right]$$

$$-(x^2+n^2)\left[\frac{\partial}{\partial \nu} I_\nu - \frac{\partial}{\partial \nu} I_{-\nu}\right] = 0$$

$= 0$ — (4)

So (6), becomes—

$$y(x) = A_1 I_n(x) + B_1 K_n(x)$$

NB (Here $I_n(x)$ & $K_n(x)$ are independent, bcz in $I_n(x)$ only +ve power of x while in $K_n(x)$ neg. power of x also available)

i.e. when $n =$ integer, then $I_n(x)$ & $K_n(x)$ are independent.

Consider

Example: $J_n(x) = \dfrac{2(x/2)^{n-m}}{\Gamma(n-m)} \displaystyle\int_0^1 (1-t^2)^{n-m-1} \times t^{m+1} J_m(xt)\, dt \quad (n>m>-1)$

let $I = \displaystyle\int_0^1 (1-t^2)^{n-m-1} t^{m+1} J_m \, dt$

$$= \int_0^1 (1-t^2)^{n-m-1} t^{m+1} \left(\sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r-1)} \left(\frac{xt}{2}\right)^{2r+m}\right)$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(r+n-1)} \left(\frac{x}{2}\right)^{2r+m} \int_0^1 (1-t^2)^{n+m-1} \cdot t^{2r+2m+1} \cdot dt$$

$t^2 \to u$

* $\int_0^\infty e^{-ax} J_0(bx)\,dx = \frac{1}{\sqrt{a^2+b^2}}$ $(a>0)$ —— (1)

Theorem: $\int_0^\infty J_n(bx)\,dx = \frac{1}{b}$ [n = non-neg. integer]

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\lceil(n+r+1)} \left(\frac{x}{2}\right)^{2r+n} \quad \text{—— (2)}$$

Proof: If $a \to 0$ in (1), then (1) becomes

$$\lim_{a\to 0} \Big| \int_0^\infty J_0(bx)\,dx = \frac{1}{b}$$

for $n=0$ result (2) is valid.

$$\int_0^1 J_1(bx)\,dx = 1/b$$

Note: $\because \frac{d}{dx}\left(x^{-n} J_n(x)\right) = -x^{-n} J_{n+1}(x)$

$n=0$, $\quad J_0'(x) = -J_1(x)$

$x \to bx$ $\quad \frac{d}{d(bx)} J_0(bx) = -J_1(bx)$

$$\Rightarrow \Big[J_0(bx)\Big]_0^\infty = -\int_0^\infty b\,J_1(bx)\,dx$$

$J_0(\infty) = 0$

For large value of x

$$J_n(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - (1+\pi/2)\pi/2\right)$$

$x \to \infty$, $\quad \underline{J_n(\infty) = 0}$

$J_0(0) = 1$,
$$-1 = -\int_0^\infty b\,J_1(bx)\,dx$$

OR,

Since $\frac{d}{dx} J_n(x) = \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right]$

Take $x \to bx$

$$\frac{d}{dx} J_n(bx) = \frac{b}{2}\left[J_{n-1}(bx) - J_{n+1}(bx)\right]$$

$$\left[J_n(bx)\right]_0^\infty = \frac{b}{2}\left[\int_0^\infty J_{n-1}(bx)\,dx - \int_0^\infty J_{n+1}(bx)\,dx\right]$$

$J_n(\infty) = 0$

$J_n(0) = 0$ when n = non-neg. integer

$$\int_0^\infty J_{n-1}(bx)\,dx = \int_0^\infty J_{n+1}(bx)\,dx$$

now $\underline{n \to N+1}$

$$\int_0^\infty J_N(bx)\,dx = \int_0^\infty J_{N+2}(bx)\,dx$$

If $\int_0^\infty J_N(bx)\,dx = 1/b$

then $\int_0^\infty J_{N+2}(bx)\,dx = \frac{1}{b}$

from here, we conclude the result (2).

Theorem: (1) $\int_0^\infty J_n(bx)\, x^n e^{-ax}\,dx = \frac{2^n \Gamma(n+\frac{1}{2})}{\sqrt{\pi}} \cdot \frac{b^n}{(a^2+b^2)^{n+1/2}}$

(2) $\int_0^\infty J_n(bx)\, x^{n+1} e^{-ax}\,dx = \frac{2^{n+1}\,\Gamma(n+3/2)}{\sqrt{\pi}} \frac{a b^n}{(a^2+b^2)^{n+3/2}}$ $(a>0)$

If we differentiate (1) w.r.t. a, then we get result (2).

Now for proving (1), we replace $J_n(bx)$

i.e

L.H.S (1), $\int_0^\infty x^n e^{-ax}\left[\sum_{r=0}^\infty \frac{(-1)^r}{r!\,\Gamma(n+r+1)}\left(\frac{bx}{2}\right)^{2r+n}\right]dx$

$$= \sum_{r=0}^\infty \frac{(-1)^r}{r!\,\Gamma(n+r+1)}\left(\frac{b}{2}\right)^{2r+n}\boxed{\int_0^\infty e^{-ax} x^{2r+2n}\,dx}$$

$$\left(\because L\{t^n\} = \int_0^\infty e^{-at} t^n\,dt = \frac{\Gamma(n+1)}{a^{n+1}}\right)$$

$$= \sum_{r=0}^\infty \frac{(-1)^r}{r!\,\Gamma(n+r+1)} \frac{\Gamma(2r+2n+1)}{a^{2r+2n+1}} \times \left(\frac{b}{2}\right)^{2r+n}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \times \frac{2\,\Gamma(2r+2n)}{\Gamma(r+n)} \times \frac{1}{a^{2r+2n+1}} \times \left(\frac{b}{2}\right)^{2r+n}$$

$$\left(\frac{\Gamma 2x}{\Gamma x} = \Gamma(x+1/2) \cdot \frac{2^{2x-1}}{\sqrt{\pi}} \right)$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!}\, 2\,\Gamma(r+n+1/2) \cdot \frac{2^{2r+2n-1}}{\sqrt{\pi}} \cdot \frac{1}{a^{2r+2n+1}} \left(\frac{b}{2}\right)^{2r+n} \quad \text{——— (3)}$$

(∵ Binomial expansion of $\frac{1}{(a^2+b^2)^{n+1/2}}$ is —

$$(1-x)^{-\alpha} = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} x^n$$

$$(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma\alpha} = \begin{cases} \alpha(\alpha+1)\cdots(\alpha+n-1), & n \neq 0 \\ 1 & n=0 \end{cases}$$

$$\text{\&}\quad (1+x)^{-\alpha} = \sum_{n=0}^{\infty} \frac{(-1)^n (\alpha)_n}{n!} x^n$$)

$$\therefore \frac{1}{(a^2+b^2)^{n+1/2}} = \frac{1}{a^{2n+1}}\left(1+\frac{b^2}{a^2}\right)^{-(n+1/2)}$$

$$= \frac{1}{a^{2n+1}} \sum_{r=0}^{\infty} \frac{(-1)^r (n+1/2)_r}{r!} \left(\frac{b^2}{a^2}\right)^r$$

$$= \frac{1}{a^{2n+1}} \sum_{r=0}^{\infty} \frac{(-1)^r \Gamma(n+r+1/2)}{r!\,\Gamma(n+1/2)} \cdot \frac{b^{2r}}{a^{2r}} \quad \text{——— (4)}$$

from (3), we have —

$$= \frac{2^{2n}}{\sqrt{\pi}} \cdot \frac{b^n}{2^n} \cdot \frac{1}{a^{2n+1}} \sum_{r=0}^{\infty} \frac{(-1)^r \Gamma(r+n+1/2)}{r!\, a^{2r}} \cdot \frac{\cancel{2^{2r}}\, b^{2r}}{\cancel{2^{2r}}}$$

$$= \frac{2^{n} \cdot b^n}{\sqrt{\pi} \cdot a^{2n+1}} \sum_{r=0}^{\infty} \frac{(-1)^r \Gamma(n+r+1/2)}{r!} \left(\frac{b^2}{a^2}\right)^r$$

$$= \frac{(2b)^n}{\sqrt{\pi}} \cdot \frac{1}{\cancel{a^{2n+1}}} \cdot \frac{\cancel{a^{2n+1}}\, \Gamma(n+1/2)}{(a^2+b^2)^{n+1/2}}$$

$$= \frac{2^n\, \Gamma(n+1/2)}{\sqrt{\pi}} \cdot \frac{b^n}{(a^2+b^2)^{n+1/2}} = R.H.S$$

Proved

expect for orthonormality result

Wednesday - Exam
↳ β, γ f^n
↳ Bessel f^n

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Theorem: Integral Representation of Bessel function

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Sol^n (b)

Let $P = x^n \int_{1}^{\infty} e^{-xt}(t^2-1)^{n-\frac{1}{2}}\,dt$

Here P satisfies Modified Bessel Eq^n

$$= x^n \int_1^\infty \{ \cancel{n(n+1)} - 2nxt + x^2t^2 + \cancel{n} - xt - x^2 - \cancel{n^2} \} \cdot e^{-xt}(t^2-1)\,dt$$

$$= x^{n+1}\int_1^\infty \{-2nt + xt^2 - t - x\} e^{-xt}(t^2-1)^{n-\frac{1}{2}}\,dt$$

$$= x^{n+1}\int_1^\infty \{x(t^2-1) - 2t(n+\tfrac{1}{2})\} e^{-xt}(t^2-1)^{n-\frac{1}{2}}\,dt$$

$$= -x^{n+1}\int_1^\infty \{(-x)e^{-xt}(t^2-1)^{n+\frac{1}{2}} + (n+\tfrac{1}{2})e^{-xt}(t^2-1)^{n-\frac{1}{2}} \cdot 2t\}\,dt$$

$$= -x^{n+1}\int_1^\infty \frac{d}{dt}\{e^{-xt}(t^2-1)^{n+\frac{1}{2}}\}\,dt$$

$$= -x^{n+1}\left\{e^{-xt}(t^2-1)^{n+\frac{1}{2}}\right\}_1^\infty = 0$$

$$= 0$$

Since P satisfied given DE (1), so

$$y = A I_n(x) + B K_n(x)$$

and $x \to \infty$, $I_n(x) \to \infty$

$$P(x) = A I_n(x) + B K_n(x) \qquad \text{—— (3)}$$

$P(x) > 0$

$x \to \infty$ $P(x) \to 0$

Since $(t^2-1)^{n-\frac{1}{2}} < e^{xt/2}$ for large value of x (let $x > X$)

so $P(x) < x^n \int_1^\infty e^{-xt} e^{xt/2}\,dt$

$$= x^n \int_1^\infty e^{-xt/2}\,dt$$

Now take $x \to \infty$.

then $P(x) \to 0$

As $P(x) \to 0$, $I_n(x) \to \infty$ as $x \to \infty$

$\Rightarrow$ there is no term of $I_n(x)$ in eq^n (3)

Therefore in sol^n terms (1) of $I_n(x)$ will not be there.

80 —

$$\boxed{P(x) = BK_n(x)}$$

Now we have to find B, i.e $B = ??$

where $K_n(x) = \frac{\pi}{2} \frac{I_{-n}(x) - I_n(x)}{\sin n\pi}$

Now lowest term of x in $K_n(x)$ is —

$\therefore K_n(x) = \frac{\pi}{2} \cdot \frac{1}{\Gamma(1-n) \sin n\pi} \left(\frac{x}{2}\right)^{-n}$ for small x.

$\left(\text{i.e lowest term in } I_{-n}(x) = \frac{1}{\Gamma(1-n)} \left(\frac{x}{2}\right)^{-n} \right.$

$\left. \quad \& \quad I_n(x) = \frac{1}{\Gamma(1+n)} \left(\frac{x}{2}\right)^{n} \right)$

$\because \Gamma n \, \Gamma(1-n) = \frac{\pi}{\sin n\pi} \qquad = \frac{\pi}{2} \cdot \frac{\Gamma n}{\pi} \left(\frac{2}{x}\right)^n$

$$= \frac{\Gamma n \cdot 2^{n-1}}{x^n}$$

As $P(x) = x^n \int_1^\infty e^{-xt} (t^2-1)^{n-1/2} \, dt$

put $t = 1 + u/x$

$$= x^n \int_0^\infty e^{-x-u} \left(\frac{2u}{x} + \frac{u^2}{x^2}\right)^{n-1/2} \frac{du}{x}$$

$$= x^n e^{-x} \int_0^\infty e^{-u} \left(1 + \frac{2x}{u}\right)^{n-1/2} \cdot \frac{u^{2n-1}}{x^{2n-1}} \frac{du}{x}$$

$$= \frac{e^{-x}}{x^n} \int_0^\infty e^{-u} \left(1 + \frac{2x}{u}\right)^{n-1/2} u^{2n-1} \, du$$

for small x, $e^{-x} \approx 1 \quad \left(1 + \frac{2x}{u}\right)^{n-1/2} \simeq 1$

for small x, $P(x) = \frac{1}{x^n} \int_0^\infty e^{-u} u^{2n-1} \, du$

$$= \frac{\Gamma 2n}{x^n}$$

For small x, $\frac{\Gamma 2n}{\cancel{x^n}} = B \cdot \frac{\Gamma n \cdot 2^{n-1}}{\cancel{x^n}}$

$$B = \frac{\Gamma 2n}{\Gamma n} \cdot 2^{1-n} = \frac{2^{2n-1} \left(\Gamma(n+1/2)\right)}{\sqrt{\pi}} \cdot 2^{1-n}$$

$$= \frac{2^n \, \Gamma(n+1/2)}{\sqrt{\pi}}$$

Now putting this value in Relⁿ (*),
we get $p(x)$,

$$i\cdot 0 \quad \boxed{P(x) = 2^n \frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}} \cdot k_n(x)}$$

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(details in prob. copy) p-192)

Orthonormality of Bessel function

Theorem (4.23)
(137-8)

$$\int_0^a x J_n(\xi_i x) J_n(\xi_j x)\, dx = \begin{cases} \frac{a^2}{2}\left[J_{n+1}(\xi_i x)\right]^2 \\ 0 & i \neq j \end{cases}$$

where ξ_i & ξ_j are roots of Eqⁿ $J_n(\xi a) = 0$

Solⁿ Pf: If ξ_i & ξ_j are distinct roots of Eqⁿ $J_n(\xi a) = 0$, then $J_n(\xi_i x)$ is Bessel fⁿ

i.e solⁿ of Bessel DE.

$$\Rightarrow x^2 \frac{d^2}{dx^2} J_n(\xi_i x) + x \frac{d}{dx} J_n(\xi_i x) + (\xi_i^2 x^2 - n^2) J_n(\xi_i x) = 0$$

multiply by

$$\frac{1}{x} J_n(\xi_j x) * x \frac{d}{dx}\left\{x \frac{d}{dx} J_n(\xi_i x)\right\} + (\xi_i^2 x^2 - n^2) J_n(\xi_i x) = 0 \quad \text{— (a)}$$

$$\frac{1}{x} J_n(\xi_i x) * J_n(\xi_j x)\, x \frac{d}{dx}\left\{x \frac{d}{dx} J_n(\xi_j x)\right\} + \{\xi_j^2 x^2 - n^2\} J_n(\xi_j x) = 0 \quad \text{— (b)}$$

$$a-b \Rightarrow J_n(\xi_j x) \frac{d}{dx}\left\{x \frac{d}{dx} J_n(\xi_i x)\right\} - J_n(\xi_i x) \frac{d}{dx}\left\{x \frac{d}{dx} J_n(\xi_j x)\right\}$$
$$+ x(\xi_i^2 - \xi_j^2) J_n(\xi_i x) J_n(\xi_j x) = 0$$

first we multiply by $\frac{J_n(\xi_j x)}{x}$ & $\frac{J_n(\xi_i x)}{x}$ resp, then subtracting

$$\Rightarrow \frac{d}{dx}\left\{J_n(\xi_j x)\, x \frac{d}{dx} J_n(\xi_i x)\right\} - \cancel{x \frac{d}{dx} J_n(\xi_j x) \frac{d}{dx} J_n(\xi_j x)}$$

$$- \frac{d}{dx}\left\{J_n(\xi_i x)\, x \frac{d}{dx} J_n(\xi_j x)\right\} + \cancel{x \frac{d}{dx} J_n(\xi_j x) \frac{d}{dx} J_n}$$

$$+ x(\xi_i^2 - \xi_j^2) J_n(\xi_i x) J_n(\xi_j x) = 0$$

Now Integrate w.r.t x between 0 to a, we get—

$$\left[\underbrace{J_n(\xi_j x)\, x \frac{d}{dx} J_n(\xi_i x) - J_n(\xi_i x)\, x \frac{d}{dx} J_n(\xi_j x)}_{=0}\right]_0^a + (\xi_i^2 - \xi_j^2)\int_0^a x J_n(\xi_i x) J_n(\xi_j x)\, dx = 0$$

Because in starting we have $J_n(\xi a) = 0$

$$\therefore \int_0^a x J_n(\xi_i x) J_n(\xi_j x)\, dx = 0$$

If ξ_i is root of eq^n $J_n(\xi a) = 0$ then— $J_n(\xi_i x)$ is Bessel f^n i.e sol^n of Bessel Diff Eq^n.

$$\Rightarrow x^2 \underbrace{\frac{d^2}{dx^2} J_n(\xi_i x)}_{z''} + x \underbrace{\frac{d}{dx} J_n(\xi_i x)}_{z'} + (\xi_i^2 x^2 - n^2) \underbrace{J_n(\xi_i x)}_{z} = 0$$

put $J_n(\xi_i x) = z$, then multiply by $2z'$

$$\Rightarrow 2x^2 z' z'' + 2x z'^2 + 2(\xi_i^2 x^2 - n^2) z z' = 0$$

now on combining—

$$\frac{d}{dx}\left\{x^2 z'^2 - n^2 z^2 + \xi_i^2 x^2 z^2\right\} - 2x \xi_i^2 z^2 = 0$$

On Integrating w.r.t x between 0 to a—

$$\left\{x^2\left[\frac{d}{dx} J_n(\xi_i x)\right]^2 - n^2 [J_n(\xi_i x)]^2 + \xi_i^2 x^2 [J_n(\xi_i x)]^2\right]_0^a$$
$$- 2\xi_i^2 \int_0^a x[J_n(\xi_i x)]^2 dx = 0$$

$$a^2\left[\frac{d}{dx} J_n(\xi_i x)\right]^2_{x=a} - 2\xi_i^2 \int_0^a x [J_n(\xi_i x)]^2 dx = 0$$

(for $i=j$, it gives L.H.S of theorem)

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$$\int_0^a x[J_n(\xi_i x)]^2 dx = \frac{a^2}{2\xi_i^2}\left[\frac{d}{dx} J_n(\xi_i x)\right]^2_{x=a}$$

$$= \frac{a^2}{2\xi_i^2}\left[\frac{n}{x} J_n(\xi_i x) - \xi_i J_{n+1}(\xi_i x)\right]^2_{x=a}$$

$$= \frac{a^2}{2\xi_i^2} \xi_i^2 [J_{n+1}(\xi_i a)]^2$$

(Becaus ~~ξ~~ $J_n(\xi_i a)=0$ so on sq. of first term is '0' also 2 times of both term = 0)

~~Result~~:
Theorem: (4.24) (140) (Copy - 197)

If $f(x)$ is defined in the region $0 \le x \le a$ and can be expanded in the form $\sum_{i=1}^{\infty} C_i J_n(\xi_i x)$, where ξ_i are the roots of the eq^n $J_n(\xi_i a)=0$ then

$$C_i = \frac{2\int_0^a x f(x) J_n(\xi_i x)\, dx}{a^2 \{J_{n+1}(\xi_i a)\}^2}$$

(P-148, Eg-①)

Theorem: $J_n(x+y) = \sum_{r=-\infty}^{\infty} J_r(x) J_{n-r}(y)$

Imp Imt

Prove it by the help of generating f^n

Pf: $\because$ generating f^n of Bessel f^n is –

$$\exp\left\{\frac{x}{2}(t - 1/t)\right\} = \sum_{n=-\infty}^{\infty} J_n(x) t^n$$

Now replace x by $x+y$

$$\underbrace{\exp\left\{\frac{x+y}{2}(t-1/t)\right\}}_{\downarrow} = \sum_{n=-\infty}^{\infty} J_n(x+y) t^n$$

$$\exp\left\{\frac{x}{2}(t-1/t)\right\} \cdot \exp\left\{\frac{y}{2}(t-1/t)\right\}$$
$$= \sum_{r=-\infty}^{\infty} J_r(x) t^r \cdot \sum_{s=-\infty}^{\infty} J_s(y) \cdot t^s$$

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$$= \sum_{r,s=-\infty}^{\infty} J_r(x)\, J_s(y) \cdot t^{r+s}$$

for a fixed value of r.

$s = n - r$ will give co-eff of t^n

that is —

$$\sum_{r=-\infty}^{\infty} J_r(x)\, J_{n-r}(y) = J_n(x+y)$$

(Bessel f^n is finished)

* * *

Legendre polynomial

[3.1 (97)]

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + l(l+1)y = 0 \quad \text{— (I)}$$

(initialy l = real no.)

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$$

Here $\boxed{P(x) = \frac{-2x}{1-x^2}, \quad Q(x) = \frac{l(l+1)}{1-x^2}}$

real analytic f^n $\quad x^2<1$

$\because$ it is ~~regular~~ singular at $x=\pm 1$

so ordinary pt. will be in $-1<x<1$

Now, if want to find sol^n of (1) in $-1<x<1$

then sol^n is given by

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \quad \text{— (ii)}$$

(Similarly sol^n for Bessel eq^n / f^n is —

$$y(x) = \sum_{n=0}^{\infty} a_n x_n^{n+s}\;)$$

So (ii) satisfies (1) —

$\therefore$ (1) becomes —

$$(1-x^2)\left[\sum_{n=0}^{\infty} a_n \cdot n\cdot(n-1)\cdot x^{n-2}\right] - 2x\left[\sum_{n=0}^{\infty} a_n . n . x^{n-1}\right]$$

$$+ l(l+1)\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} a_n n(n-1)x^{n-2} - \sum_{n=0}^{\infty} a_n\underbrace{\left[n(n-1)+2n-l(l+1)\right]}x^n = 0$$

$n \to n+2 = 0$

$= n^2+n-l(l+1)$

$= (n-l)(n+l) + (n-l)$

$= (n-l)(n+l+1)$

$$\Rightarrow \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n + \sum_{n=0}^{\infty} a_n(l-n)(l+n+1)x^n = 0$$

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Now Coeff^n of x^n —

$$a_{n+2}(n+2)(n+1) + a_n(l-n)(l+n+1) = 0$$

$$\Rightarrow a_{n+2} = -\frac{(l-n)(l+n+1)}{(n+1)(n+2)} a_n, \quad n = 0, 1, 2, \ldots \quad (*)$$

for $n=0$ $\quad a_2 = -\frac{l(l+1)}{1.2} a_0$ (even)

$n=1$, $\quad a_3 = \frac{-(l-1)(l+2)}{2 \cdot 3} a_1$

$n=2$ $\quad a_4 = \frac{-(l-2)(l+3)}{3 \cdot 4} a_2$ (even)

$$= \frac{l(l-2)(l+1)(l+3)}{1.2.3.4} a_0$$

$n=3$, $\quad a_5 = -\frac{(l-3)(l+4)}{4.5} a_3$

$$= \frac{(l-3)(l-1)(l+2)(l+4)}{2.3.4.5} a_1$$

— —

NB (Here we have all term of even are written in terms of a_0 & all odd coeff^n can be written in term of a_1)

In general

$$a_{2r} = (-1)^r a_0 \frac{l(l-2)\ldots(l-2r+2)(l+1)(l+3)\ldots(l+2r-1)}{2r!} \quad \text{— (a)}$$

for even Co-efficient.

and for odd Co-eff^n term—

$$a_{2r+1} = (-1)^r a_1 \frac{(l-1)(l-3)\cdots(l-2r+1)(l+2)(l+4)\cdots(l+2r)}{(2r+1)!} \quad \text{—(b)}$$

Now putting these co-eff^n in (ii)—

$$y(x) = a_0 \underbrace{\left\{1 + \sum_{n=1}^{\infty} (-1)^n \frac{l(l-2)\cdots(l-2n+2)(l+1)\cdots(l+2n-1)}{2n!} x^{2n}\right\}}_{=y_1(x)} \quad \text{(iii)}$$

$$+ a_1 \underbrace{\left\{1 + \sum_{n=1}^{\infty} (-1)^n \frac{(l-1)(l-3)\cdots(l-2n+1)(l+2)\cdots(l+2n)}{(2n+1)!} x^{2n+1}\right\}}_{=y_2(x)}$$

$$y(x) = a_0 y_1(x) + a_1 y_2(x)$$

NB (If here both sol^n $y_1(x)$ & $y_2(x)$ are Convergent then $y(x)$ is meaningful)

H.W. Now check the convergent of $y_1(x)$ & $y_2(x)$ only for $-1 < x < 1$ (use Ratio test)

Consider these case l = even integer
l = odd integer

Suppose $l = 2r$
then by (a) $a_{2r} \neq 0$, $a_{2r+2} = 0$
$a_{2r+4} \neq 0$, - - -

If $l = 2r+1$, then by (b), $a_{2r+1} \neq 0$,
$a_{2r+3} = 0$, $a_{2r+5} = 0$, - - -

It means that —
If l is even integer then $y_1(x)$ fun^n reduces to a polynomial. (& hence finite for all)
If l is odd integer then y_2 reduces to a poly. (& hence finite for all finite x)

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Since we have $a_{n+2} = -\frac{(l-n)(l+n+1)}{(n+1)(n+2)} a_n$

Now writting it in reverse order, we have

By (*) —

$$\boxed{a_n = -\frac{(n+1)(n+2)}{(l-n)(l+n+1)} a_{n+2}}$$

it applies to the series for both $y_1(x)$ and $y_2(x)$, we may obtain a single series valid for both even & odd

$n \to l-2$

$$a_{l-2} = -\frac{(l-1)l}{2\cdot(2l-1)} a_l$$

$n \to l-4$,

$$a_{l-4} = \frac{-(l-3)(l-2)}{4\cdot(2l-3)} a_{l-2}$$

$$= \frac{l(l-1)(l-2)(l-3)}{2\cdot4\cdot(2l-1)\cdot(2l-3)} a_l$$

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SNB

∴ general term is

$n \to l-2r$

$$\boxed{a_{l-2r} = (-1)^{r} \frac{l(l-1)(l-2)\dots(l-2r+1)}{2\cdot4\dots(2r)\cdot(2l-1)(2l-3)\dots(2l-2r+1)} \cdot a_l} \quad (4)$$

thus for l = even, $y_1(x)$ reduces to a poly., while for l = odd, reduces to same poly. and we get that poly as

$y(x) = a_l x^l + a_{l-2} x^{l-2} + \dots + \begin{cases} a_0 & l = \text{even} \\ a_1 x & l = \text{odd} \end{cases}$

$$\boxed{y(x) = \sum_{r=0}^{[l/2]} a_{l-2r} x^{l-2r}}$$

when l = odd, $[\cdot]$ = g.i.f.

$n \to l-3$

$$a_{l-3} = \frac{-(l-2)(l-1)}{3\cdot(2l-2)}$$

general term is

$a_{l-2r+1} =$

∴ By using gives ⇒

$\left(\because 2\cdot4\cdot6 \dots 2r = 2^r \cdot 1\cdot2\cdot3\dots r = 2^r \cdot r! \right)$ (5)

$[l/2] \rightarrow$ highest integer less than or equal to $l/2$ ..

$$= a_l \sum_{r=0}^{[l/2]} (-1)^r \frac{l(l-1)\cdots(l-2r+1)}{2\cdot 4 \cdots 2r(2l-1)(2l-3)\cdots(2l-2r+1)} x^{l-2r}$$

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* * *

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$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + l(l+1)y = 0$$

i.e $(1-x^2)y'' - 2xy' + l(l+1)y = 0$

$$y = AY_1(x) + BY_2(x) \qquad -1 < x < 1$$

SNB Here Y_1 & Y_2 are L.I Sol^{n}s and $Y_1(x)$ $Y_2(x)$ will be convergent for $-1 < x < 1$

$$Y(x) = a_l \sum_{r=0}^{[l/2]} (-1)^r \frac{(l)(l-1)\cdots(l-2r+1)}{2\cdot 4 \cdots 2r(2l-1)(2l-3)\cdots(2l-2r+1)}$$

$(l+1)(l-1)(l-3)\ldots$

$\therefore$ $\dfrac{l(l-1)\cdots(l-2r+1)(l-2r)(l-2r-1)\cdots 3.2.1}{(l-2r)(l-2r-1)\cdots 3.2.1} = \dfrac{l!}{(l-2r)!}$

$\cdot a_l$ $+1)$

(4) And also —

$$\Rightarrow \frac{2l(2l-1)(2l-2)(2l-3)\cdots(2l-2r)!}{2l(2l-2)(2l-4)\cdots(2l-2r)} = \frac{(2l)!}{2^r(l-r)!(2l-2r)!}$$

$$= \frac{(2l)!\,(l-r)!}{2^r\, l!\,(2l-2r)!} \quad \text{— (c)}$$

$$\left(\frac{(2l)!\,(l-r)!}{2^r \underbrace{(l)(l-1)(l-2)\cdots(l-r+1) \times (l-r)!}_{l!} \times (2l-2r)!}\right)$$

$l(l-1)\cdots(l-r+1).(2l-2r)! = \dfrac{l!}{(l-r)!}$ — (d)

$\therefore$ By using (b) (c) & (d) in (4) gives $\Rightarrow$

$$\therefore Y(x) = a_l \sum_{r=0}^{[l/2]} (-1)^r \frac{l!\cdot \cancel{2^r}(l!)(2l-2r)!}{(l-2r)!\,\cancel{2^r}\cdot r!(2l)!\,(l-r)!} x^{l-2r}$$

$$= a_l \sum_{r=0}^{[l/2]} (-1)^r \frac{(l!)^2(2l-2r)!}{r!(l-2r)!(l-r)!(2l)!} x^{l-2r}$$

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which is sol^n for any value of a_l.

suppose we choose $a_l = \frac{(2l)!}{(l!)^2 \cdot 2^l}$

then we obtain Legendre poly. of order l

$$\boxed{y(x) = P_l(x) = \sum_{r=0}^{[l/2]} (-1)^r \frac{(2l-2r)! \cdot x^{l-2r}}{2^l r! (l-r)! (l-2r)!}}$$

which is finite for $-1 \le x \le 1$

Legendre's Polynomials :

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$\because [l/2] = \begin{cases} l/2 & : l \text{ is even} \\ \frac{l-1}{2} & : l \text{ is odd} \end{cases}$

$P_0(x) = 1$

$P_1(x) = x$

$$P_2(x) = \sum_{r=0}^{1} (-1)^r \frac{(2l-2r)! \cdot x^{l-2r}}{2^l r! (l-r)! (l-2r)!}$$

$$= \frac{(2l)!\, x^l}{2^l \cdot l!\, l!} - \frac{(2l-2)!\, x^{l-2}}{2^l \cdot (l-1)!\, (l-2)!}$$

$$= \frac{4!\, x^2}{2^2 \cdot 2!\, 2!} - \frac{2!\, x^0}{2^2 \cdot 1!}$$

$$= \frac{4 \cdot 3}{4 \cdot 2} \cdot x^2 - \frac{2}{4} \cdot x^0 = \frac{3}{2} x^2 - \frac{1}{2}$$

$P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$

llly $P_3(x) = \frac{1}{2}(5x^3 - 3x)$

$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$

(3.2/P-46) Theorem: (Generating f^n for the Poly.s)

$$\frac{1}{\sqrt{1-2tx+t^2}} = \sum_{l=0}^{\infty} t^l P_l(x) \quad \text{if } |t|<1 \;\&\; |x|\le 1$$

SNB This means that, when $(1-2xt+t^2)$ is expanded in power of t, the co-eff^n t^l is $P_l(x)$.

$(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma\alpha} = \frac{\lfloor \alpha+n-1}{\lfloor \alpha-1}$

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$(1-2xt+t^2)^{-1/2}$ is called the generating function of the legendre polys.

1. Pf: — Expand $(1-2tx+t^2)^{-1/2}$ by binomial thm.

L.H.S

$$\left[1-t(2x-t)\right]^{-1/2} = \sum_{k=0}^{\infty} (1/2)_k \cdot t^k$$

$$= \sum_{k=0}^{\infty} \frac{(1/2)_k\, t^k}{k!} (2x-t)^k$$

$(\tfrac{1}{2})_k = \frac{\Gamma(k+1/2)}{\Gamma 1/2}$

self

$$(1-t(2x-t))^{-1/2} = 1 + (-1/2)\{-t(2x-t)\} + \frac{(-1/2)(-3/2)}{2!} \cdot \{-t(2x-t)\}^2 + \cdots +$$

$$+ \frac{(-1/2)(-3/2)\cdots\{-(2r-1)/2\}}{r!}\{-t(2x-t)\}^r + \cdots$$

$$= \sum_{r=0}^{\infty} (-1)^r \frac{1.3.5\cdots(2r-1)}{2^r \cdot r!} \cdot (-1)^r \cdot t^r (2x-t)^r.$$

$$= \sum_{r=0}^{\infty} \frac{(2r)!}{2^{2r}(r!)^2} \cdot t^r \cdot (2x-t)^r$$

duplication formula (p-84)

$$= \sum_{k=0}^{\infty} \frac{\Gamma(k+1/2)\, t^k (2x-t)^k}{(k!)\sqrt{\pi}} \qquad \left(\Gamma\tfrac{1}{2} = \sqrt{\pi}\right.$$

$$= \sum_{k=0}^{\infty} \frac{\cancel{\sqrt{\pi}}\, 2\Gamma 2k \cdot 2^k}{2^{2k} k\Gamma k \cdot k!\, \cancel{\sqrt{\pi}}} \cdot t^k (2x-t)^k \cdot \frac{2k}{2k}$$

$$= \sum_{k=0}^{\infty} \frac{\cancel{(2k-1)}(2k)!}{2^{2k}(k!)^2} \cdot t^k \cdot \underbrace{(2x-t)^k}_{\text{expanding by Binomial thm.}}$$

$$= \sum_{k=0}^{\infty} \frac{(2k)!}{2^{2k}(k!)^2} \cdot t^k \left(\sum_{p=0}^{\infty} {}^kC_p (-t)^p (2x)^{k-p}\right)$$

$$= \sum_{k=0}^{\infty} \sum_{p=0}^{k} \frac{k!}{(k-p)!\,p!} \cdot \frac{(2k)!}{2^{2k}(k!)^2} \cdot (-1)^p \cdot t^{p+k} \cdot (2x)^{k-p}$$

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for fix k.

$$p + k = l \Rightarrow p = l - k$$

$$0 \le l - k \le k$$

$$\frac{l}{2} \le k \le l \rightarrow \text{even}$$

$$\frac{l+1}{2} \le k \le l \rightarrow \text{odd}$$

co-eff. of t^l

$$= \sum_{k}^{l} \frac{\cancel{k!}\, 2k!\, (-1)^p \cdot 2 \cdot x^{2k-l}}{(2k-l)!\, 2^{2k}\, (\cancel{k!})\, (l-k)!}$$

$$k = \begin{cases} l/2, & l = \text{even} \\ \frac{l+1}{2} & l = \text{odd} \end{cases}$$

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Generating function (P-46).

Thm 3.1|46

$$P_l(x) = \sum_{r=0}^{[l/2]} (-1)^r \frac{(2l-2r)!}{2^l \cdot r!\, (l-r)!\, (l-2r)!} \cdot x^{l-2r}$$

$$\because \frac{1}{\sqrt{1-2xt+t^2}} = \left[1 - t(2x-t)\right]^{-1/2}$$

$$= \sum_{r=0}^{\infty} \frac{(1/2)_r}{r!} t^r \cdot (2x-t)^r$$

$$(1-x)^{-\alpha} = \sum_{r=0}^{\infty} \frac{(\alpha)_n}{n!} x^n$$

$$\alpha_n = \frac{\Gamma(\alpha+n)}{\Gamma\alpha}$$

$$= \sum_{r=0}^{\infty} \frac{2r!}{2^{2r}(r!)^2} t^r (2x-t)^r \quad \text{(By duplication formula)}$$

$$= \sum_{r=0}^{\infty} \frac{(2r)!}{2^{2r}(r!)^2} \cdot t^r \cdot \sum_{p=0}^{r} {}^rC_p\, (2x)^{r-p} \cdot (-t)^p$$

$$\left(\because (x-y)^t = \sum_{p=0}^{t} {}^tC_p\, x^{t-p} (-y)^p\right)$$

$$= \sum_{r=0}^{\infty} \sum_{p=0}^{r} \frac{(2r)!}{2^{2r}(r!)^2} \times {}^rC_p\, (-1)^p \cdot (2x)^{r-p} \cdot t^{r+p}$$

for fix r, put $r + p = l \Rightarrow p = l - r$, $0 \le p \le r$

$$\Rightarrow l/2 \le r \le l \rightarrow l = \text{even}$$

$$\frac{l+1}{2} \le r \le l \rightarrow l = \text{odd}$$

Total coefficient of $t^l \Rightarrow$

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$$\sum^{l} \frac{2r!}{2^{2r}.(r!)^2} \cdot {}^{r}C_{l-r} (-1)^{l-r} \cdot (2x)^{r-(l-r)}$$

$r = \begin{cases} l/2 & : l = \text{even} \\ \frac{l+1}{2} & : l = \text{odd} \end{cases}$

let $l - r = k$, then $r = l - k$

$$= \sum^{0} \frac{(2l-2k)! \cdot (l-k)! \cdot (-1)^k \cdot (2x)^{2(l-k)-l}}{2^{2l-2k} \cdot [(l-k)!]^2 \cdot (l-k-k)!}$$

$k = \begin{cases} l/2 & : l \text{ even} \\ \frac{l-1}{2} & : l \text{ odd} \end{cases}$

$$P_l(x) = \sum_{k=0}^{[l/2]} \frac{(2l-2k)! \cdot (-1)^k \cdot x^{l-2k}}{2^l \cdot k! \, (l-k)! \, (l-2k)!}$$

FURTHER EXP^{ns} FOR THE LEGENDRE POLY.S

Thm 13.2 (78) Rodrigues' Formula

$$\boxed{P_l(x) = \frac{1}{2^l . l!} \frac{d^l}{dx^l} (x^2-1)^l}$$

~~put $l=1$ in above eq^n~~

Pf:- R.H.S $\frac{1}{2^l . l!} \cdot \frac{d^l}{dx^l} \sum_{r=0}^{l} {}^{l}C_r \, x^{2(l-r)} (-1)^r$

$$= \frac{1}{2^l . l!} \sum_{r=0}^{l} {}^{l}C_r (-1)^r \left(\frac{d^l}{dx^l} . x^{2l-2r} \right)$$

If $2l - 2r < l \Rightarrow r > l/2$ [But the l^{th} derivative of a power of x, less than l is zero, so that ?.]

In this case $\frac{d^l}{dx^l} . x^{2l-2r}$ will be 0

thus we may replace $\sum_{r=0}^{l}$ by $\sum_{r=0}^{[l/2]}$ when l = even, l = odd

$$\frac{1}{2^l . l!} \sum_{r=0}^{\begin{cases} l/2 & : l \text{ even} \\ \frac{l-1}{2} & : l \text{ odd} \end{cases}} {}^{l}C_r (-1)^r \left(\frac{d^l}{dx^l} . x^{2l-2r} \right)$$

Since :- $\boxed{\frac{d^l}{dx^l} x^m = \frac{m!}{(m-l)!} . x^{m-l}}$

$$= \frac{1}{2^l . l!} \sum_{r=0}^{[l/2]} {}^{l}C_r (-1)^r . \frac{(2l-2r)!}{(l-2r)!} . x^{2l-2r-l} = P_l(x)$$

SNB : Expanding $(x^2-1)^l$ by the binomial theorem we have

$$(x^2-1)^l = \sum_{r=0}^{l} {}^{l}C_r \, (-1)^r \cdot x^{2(l-r)}$$

Theorem 3.3 (49) (Laplace's Integral Representation)

$$P_l(x) = \frac{1}{\pi}\int_0^\pi \{x + \sqrt{x^2-1}\cdot\cos\theta\}^l \, d\theta$$

Pf: we know that —

$$\boxed{\int_0^\pi \frac{1}{1+\lambda\cos\theta}\cdot d\theta = \frac{\pi}{\sqrt{(1-\lambda^2)}}} \quad (*)$$

If we now write $\lambda = \dfrac{-u\sqrt{x^2-1}}{1-ux}$ and expand both side of eqn $(*)$, in powers of u & equate co-effs of corresponding power of u, we shall obtain the desired result.

Left term $\dfrac{1}{1+\lambda\cos\theta}$

$$= \frac{1}{1+\left(\dfrac{-u\sqrt{x^2-1}}{1-ux}\right)\cos\theta}$$

$$= \frac{1-ux}{(1-ux) - u\sqrt{x^2-1}\cos\theta}$$

$(1)_n = n!$

$$= (1-ux)\cdot\left(1 - u(x+\sqrt{x^2-1}\cdot\cos\theta)\right)^{-1}$$

$$= (1-ux)\cdot\sum_{l=0}^{\infty} u^l\{x+\sqrt{x^2-1}\cdot\cos\theta\}^l \quad (1)$$

(Since by Binomial thm $(1-a)^{-1} = \sum_{n=0}^{\infty} a^n$)

Now right term $\dfrac{\pi}{\sqrt{(1-\lambda^2)}}$

$$= \frac{\pi}{\sqrt{1-\dfrac{u^2(x^2-1)}{(1-ux)^2}}} = \frac{\pi\cdot(1-ux)}{\sqrt{(1-ux)^2-u^2(x^2-1)}}$$

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$$= \frac{\pi(1-ux)}{\sqrt{\underbrace{1-2ux+u^2}_{\text{generating f}^n\text{ of legendre poly.}}}}$$

$$\text{or } = \frac{\pi(1-ux)}{\sqrt{1-u(2x-u)}} = \pi(1-ux)\left(1-u(2x-u)\right)^{-1}$$

$$= \pi(1-ux) \cdot \sum_{l=0}^{\infty} u^l \cdot P_l(x) \quad -(2)$$

Now put (1) & (2) into (*), then —

$$\int_0^{\pi} \left[\cancel{(1-ux)} \cdot \sum_{l=0}^{\infty} u^l \{x+\sqrt{x^2-1}\cos\theta\}^l \right] d\theta = \pi\cancel{(1-ux)} \sum_{l=0}^{\infty} u^l P_l(x)$$

Now, changing order of sum & integration —

$$\Rightarrow \sum_{l=0}^{\infty} u^l \int_0^{\pi} [x+\sqrt{x^2-1}\cos\theta]^l d\theta = \pi \sum_{l=0}^{\infty} u^l P_l(x)$$

Now equating co-eff of u^l, gives —

$$\pi P_l(x) = \int_0^{\pi} \{x+\sqrt{x^2-1}\cos\theta\}^l d\theta$$

$$\Rightarrow \boxed{P_l(x) = \frac{1}{\pi}\int_0^{\pi} [x+\sqrt{x^2-1}\cos\theta]^l d\theta}$$

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Theorem [3.4] Explicit Exp. for and special value of the Legendre poly.

(1) $P_l(1) = 1$

(2) $P_l(-1) = (-1)^l$

(3) $P_l'(1) = \frac{1}{2}l(l+1)$

(4) $P_l'(-1) = (-1)^{l+1} \cdot \frac{l(l+1)}{2}$

(5) $P_{2l}(0) = (-1)^l \frac{2l!}{2^{2l} \cdot (l!)^2}$

(6) $P_{2l+1}(0) = 0$

Since we have

$$P_l(x) = \sum_{r=0}^{[l/2]} (-1)^r \frac{(2l-2r)!}{2^l r! (l-r)! (l-2r)!} x^{l-2r} \quad \text{—(i)}$$

also generating f^n is —

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{l=0}^{\infty} t^l P_l(x) \quad \text{— (ii)}$$

& Legendre poly. is —

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + l(l+1)y = 0 \quad \text{— (iii)}$$

(1) for (1), putting $x = 1$ in (ii), we get

$$(1-t)^{-1} = \sum_{l=0}^{\infty} t^l P_l(1) = \sum_{l=0}^{\infty} t^l$$

Now comparing co-eff of t^l, we get —

$$\boxed{P_l(1) = 1}$$

(2) put $x = -1$ in (ii), we get —

$$(1+t)^{-1} = \sum_{l=0}^{\infty} (-1)^l t^l = \sum_{l=0}^{\infty} t^l \cdot P_l(-1)$$

Now comparing co-effⁿ of t^l, we get

$$P_l(-1) = (-1)^l$$

(3) put $x = 1$ in (iii), we get — $(\because y = P_l(x))$

$$-2\frac{dP_l(1)}{dx} + l(l+1) P_l(1) = 0$$

$$\Rightarrow \frac{dP_l(1)}{dx} = \frac{l(l+1)P_l(1)}{2}$$

$(\because P_l(1) = 1)$

$$\boxed{P_l'(1) = \frac{1}{2} l(l+1)}$$

$(1)_n = n! = \frac{\Gamma(n+1)}{\Gamma 1}$

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(4) Now put $x = -1$, we have

$$+2\frac{dP_l(-1)}{dx} + l(l+1)P_l(-1) = 0$$

$$\Rightarrow \frac{dP_l(-1)}{dx} = -\frac{l(l+1)P_l(-1)}{2}$$

$$\Rightarrow P_l'(-1) = \frac{l(l+1)}{2}\cdot(-1)\cdot(-1)^l$$

$(\because P_l(-1) = (-1)^l)$

$$\Rightarrow P_l'(-1) = (-1)^{l+1}\cdot\frac{l(l+1)}{2}$$

(5) In eqn (ii), put $x = 0$, then we have

$$(1+t^2)^{-1/2} = \sum_{l=0}^{\infty} t^l \cdot P_l(0)$$

$$\sum_{l=0}^{\infty}\frac{(1/2)_l}{l!}(-1)^l t^{2l} = \sum_{l=0}^{\infty}\frac{\Gamma(l+1/2)}{\sqrt{\pi}\, l!}(-1)^l t^{2l}$$

$\left((\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma\alpha}\right)$

$\sqrt{\pi}\,\Gamma(2x) = 2^{2x-1}\Gamma x\,\Gamma x+1/2$

$l\Gamma l = \underline{l}$

$$= \sum_{l=0}^{\infty}\frac{\cancel{\sqrt{\pi}}\cdot\Gamma 2l \times 2l}{2^{2l-1}\cdot\Gamma l\cdot 2l}\times\frac{1}{\cancel{\sqrt{\pi}}\cdot l!}(-1)^l t^{2l}$$

$$\sum_{l=0}^{\infty} t^l P_l(0) = \sum_{l=0}^{\infty}\frac{\underline{2l}}{2^{2l}\,\underline{l}}\cdot\frac{(-1)^l\cdot t^{2l}}{l!}$$

$$= \sum_{l=0}^{\infty}(-1)^l\cdot\frac{(2l)!}{2^{2l}[(l)!]^2}\cdot t^{2l}$$

Now comparing corresponding co-eff (of even ~~power~~ power of t

$$\boxed{P_{2l}(0) = (-1)^l\cdot\frac{(2l)!}{2^{2l}(l!)^2}}$$ even power

and,

$$\boxed{P_{2l+1}(0) = 0}$$ odd power

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Orthogonality Prop. of the legendre Poly.

Theorem: (3.5/S2)

$$\int_{-1}^{1} P_l(x)\,P_m(x)\,dx = \begin{cases} 0 & \text{if } l \neq m \\ \dfrac{2}{2l+1} & \text{if } l = m \end{cases}$$

Sol^n: ~~If~~ it satisfies legendre poly. Eq^n, then —
Since $P_l(x)$ & $P_m(x)$ satisfy, so ⇒

for $l \neq m$ ⇒ $(1-x^2)\dfrac{d^2P_l(x)}{dx^2} - 2x\dfrac{dP_l}{dx} + l(l+1)P_l = 0$

we rewrite it as —

$$\frac{d}{dx}\left[(1-x^2)\frac{dP_l}{dx}\right] + l(l+1)P_l = 0 \quad \times P_m \quad \text{— (i)}$$

$$\frac{d}{dx}\left[(1-x^2)\frac{dP_m}{dx}\right] + m(m+1)P_m = 0 \quad \times P_l \quad \text{— (ii)}$$

(i) can be rewritten as (after multi. P_m)

$$P_m\frac{d}{dx}\left[(1-x^2)\frac{dP_l}{dx}\right] + l(l+1)P_lP_m = 0$$

$$\Rightarrow \frac{d}{dx}\left[(1-x^2)P_m\frac{dP_l}{dx}\right] - \cancel{(1-x^2)\frac{dP_m}{dx}\frac{dP_l}{dx}} + l(l+1)P_lP_m = 0 \quad \text{— (iii)}$$

$\because u\dfrac{dv}{dx} = \dfrac{d(uv)}{dx} - v\dfrac{du}{dx}$

& (ii) can be re-written as —

$$P_l\frac{d}{dx}\left[(1-x^2)\frac{dP_m}{dx}\right] + l(l+1)P_mP_l = 0$$

$$\Rightarrow \frac{d}{dx}\left[(1-x^2)P_l\frac{dP_m}{dx}\right] - \cancel{(1-x^2)\frac{dP_m}{dx}\frac{dP_l}{dx}} + m(m+1)P_lP_m = 0 \quad \text{— (iv)}$$

Now substitute, ie (iii) − (iv) ⇒

$$\frac{d}{dx}\left[(1-x^2)P_m\frac{dP_l}{dx}\right] - \frac{d}{dx}\left[(1-x^2)P_l\cdot\frac{dP_m}{dx}\right] + \left[l(l+1) - m(m+1)\right]P_lP_m = 0$$

Now integrate w.r.t. x between −1 to 1.

we then

$$\Rightarrow \left[(1-x^2)P_m\cdot\frac{dP_l}{dx} - (1-x^2)P_l\cdot\frac{dP_m}{dx}\right]_{-1}^{1} + (l-m)(l+m+1)$$

$$\int_{-1}^{1} P_l(x)P_m(x)\,dx = 0.$$

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$\because$ first part can't be zero, also $(\because l \neq m)$ $(l-m)(l+m+1) \neq 0$ so, we have

$$\boxed{\int_{-1}^{1} P_l(x) P_m(x)\,dx = 0}$$

for $l = m$

$\because$ Since we have generating f^n

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{l=0}^{\infty} t^l P_l(x)$$

Now, square both side, we have

$$\frac{1}{(1-2xt+t^2)} = \sum_{l=0}^{\infty} t^l \cdot P_l(x) \cdot \sum_{l=0}^{\infty} t^l \cdot P_l(x)$$

$$= \sum_{l,m=0}^{\infty} t^{l+m} \cdot P_l(x)\, P_m(x)$$

Now integrate both side w.r.t. x, (b/w -1 to 1) we have

$$\int_{-1}^{1} \frac{1}{(1-2xt+t^2)}\,dx = \sum_{l,m=0}^{\infty} t^{l+m} \underbrace{\int_{-1}^{1} P_l(x) P_m(x)\,dx}$$

$\because$ by hypothesis if $l \neq m$ then $\int = 0$

so we take $l = m$, then we have,

$$\int_{-1}^{1} \frac{1}{(1-2xt+t^2)}\,dx = \sum_{l,m=0}^{\infty} t^{2l} \int_{-1}^{1} [P_l(x)]^2\,dx$$

Now integrate L.H.S, we have

$$\int_{-1}^{1} \frac{1}{(1-2xt+t^2)}\,dx = \frac{1}{-2t} \int_{-1}^{1} \frac{-2t}{(1-2xt+t^2)}\,dx$$

$$\boxed{\int \frac{1}{a+bx}\,dx = \frac{\log(a+bx)}{b}}$$

$$= \frac{-1}{2t} \left[\log(1-2xt+t^2)\right]_{-1}^{1}$$

$$= \frac{-1}{2t} \left[\log(1-2t+t^2) - \log(1+2t+t^2)\right]$$

$$= \frac{-1}{2t} \left[\log(1-t)^2 - \log(1+t)^2\right]$$

$$= -\frac{1}{t} \left[\log(1-t) - \log(1+t)\right]$$

$$\int_{-1}^{1} \frac{1}{1-2xt+t^2} = \frac{-1}{t} \log\left(\frac{1-t}{1+t}\right) = \frac{1}{t}\left[\log(1+t) - \log(1-t)\right]$$

$$= \cancel{\frac{1}{t}\log\left(\frac{1+t}{1-t}\right)}$$

$$= \frac{1}{t}\left[t - \cancel{\frac{t^2}{2}} + \frac{t^3}{3} + \cdots - \left(-t - \cancel{\frac{t^2}{2}} - \frac{t^3}{3} - \cdots\right)\right]$$

$$= \frac{1}{t}\left[2\cdot\left(t + \frac{t^3}{3} + \frac{t^5}{5} + \cdots\right)\right]$$

$$= 2\left(1 + \frac{t^2}{3} + \frac{t^4}{5} + \cdots\right)$$

$$= 2\cdot\sum_{l=0}^{\infty} \frac{t^{2l}}{2l+1}$$

So, we have

$$2\sum_{l=0}^{\infty} \frac{t^{2l}}{2l+1} = \sum_{l,m=0} t^{2l} \int_{-1}^{1} [P_l(x)]^2 dx$$

Now comparing co-eff of t^{2l}, we have

$$\boxed{\int_{-1}^{1} (P_l[x])^2 dx = \frac{2}{2l+1}}$$ proved

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* * *

[3.6] LEGENDRE SERIES

Theorem: [3.6/55] If $f(x)$ is a poly. of deg. n, then

$$f(x) = \sum_{r=0}^{n} C_r p_r(x) \quad \text{where } C_r = \left(r+\tfrac{1}{2}\right)\int_{-1}^{1} f(x) P_r(x)\, dx$$

Also, if $f(x)$ is even (odd), then only those C_r with even (or odd) suffixes are n-z.

∵ Legendre poly. is given by —

$$P_l(x) = \sum_{r=0}^{[l/2]} (-1)^r \frac{2l-2r!}{2^l \cdot r!\, (l-r)!\, l-2r!} x^{l-2r}$$

(If l = even, then it is even poly,
l = odd then it is odd poly.)

Now, we may write f(x) as -

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$$

$$p_k(x) = \alpha_k x^k + \alpha_{k-2} x^{k-2} + \dots$$

(In Legendre poly. if we take $l = k$ & $r = 0$ then co-eff^n of x^k is denoted by α_k)

$$f(x) - \underbrace{\frac{a_n}{\alpha_n}}_{= C_n} P_n(x) = g_{n-1}(x) \leftarrow \text{poly. of degree } n-1.$$

$$f(x) = C_n p_n(x) + g_{n-1}(x) \quad \text{where } C_n = \frac{a_n}{\alpha_n} \quad \text{--- (1)}$$

Here, we may also write $g_{n-1}(x)$ as

$$g_{n-1}(x) = C_{n-1} p_{n-1}(x) + g_{n-2}(x) \quad (\text{poly. of deg } n-1)$$

Now substituting this value in (1), we get -

$$f(x) = C_n p_n(x) + C_{n-1} P_{n-1}(x) + g_{n-2}(x)$$

Now again we may write $g_{n-2}(x)$ as above, Continuing this process, we get

$$f(x) = C_n p_n(x) + C_{n-1} p_{n-1}(x) + \dots + C_1 P_1(x) + C_0 P_0(x)$$

$$f(x) = \sum_{r=0}^{n} C_r p_r(x)$$

Now for solving (finding C_r), we multiply with $P_s(x)$ and then integrate b/w -1 to 1.

dx

$$\int_{-1}^{1} f(x) P_s(x) = \sum_{r=0}^{n} C_r \int_{-1}^{1} p_r(x) P_s(x)\, dx$$

$$= C_s \int_{-1}^{1} [P_s(x)]^2 dx = C_s \cdot \frac{2}{2s+1}$$

$$\Rightarrow \boxed{C_s = \left(s + \frac{1}{2}\right) \int_{-1}^{1} f(x) P_s(x)\, dx}$$

RMK. Suppose that f(x) is even. Then since $P_r(x)$ is even when r is even & odd when r is odd, so the integrand $f(x) P_r(x)$ is also even (odd) when r is even (odd). But odd f^n integrated over the range -1 to $+1$ will give zero.

Hence C_r is zero when r is add. Similarly $f(x)$ is add, ~~when~~ C_r is zero, when r is zero.

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Theorem: If $f(x)$ is a poly. of degree less than l, then (corollary/56)

$$\int_{-1}^{1} f(x) P_l(x)\, dx = 0$$

Sol^n: from pr. result, we have

$$f(x) = \sum_{r=0}^{l} C_r p_r(x)$$

Now multiply by $P_l(x)$ both side, we have

$$P_l(x) f(x) = \sum_{r=0}^{l} C_r p_r(x) P_l(x)$$

$$\Rightarrow \int_{-1}^{1} P_l(x) f(x)\, dx = \sum_{r=0}^{\infty} C_r \int_{-1}^{1} p_r(x) p_l(x)\, dx \quad \underline{r<l}$$

∴ from thm on p-116

$$\int_{-1}^{1} p_r(x) p_l(x)\, dx = \begin{cases} 0 & \text{if } l \neq m \\ \frac{2}{2r+1} & \text{if } r = m \end{cases}$$

& hence

$$\int_{-1}^{1} P_l(x) f(x)\, dx = 0 \quad \text{when } r < l$$

Theorem: (3.8/58) (1)

$$P_l'(x) = \sum_{r=0}^{\left[\frac{l-1}{2}\right]} (2l-4r-1) P_{l-2r-1}(x)$$

$P_l(x) \rightarrow$ Poly. of deg. l → even (l even), add (l add)

$P_l'(x) \rightarrow$ Poly. of deg. $(l-1)$ → even (l add), add (l even)

$$P_l'(x) = C_{l-1} P_{l-1}(x) + C_{l-3} P_{l-3}(x) + \cdots + C_{l-2r-1} \cdot P_{l-2r-1}(x) + \cdots \begin{cases} a_1 P_1(x) & (l \text{ even}) \\ a_0 p_0(x) & (l \text{ add}) \end{cases}$$

(1)

from prev result, we have C_l

(Since $P_s'(x)$ is a poly. of degree $s-1$, and $s-1$ is always less than l) 1 pr | TWED

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$\therefore C_s(x) = (s+\frac{1}{2}) \int_{-1}^{1} P_l'(x) P_s(x)\, dx$

$\therefore C_s(x) = (s+\frac{1}{2}) \left[P_l(x) \cdot P_s(x) \Big|_{-1}^{1} - \underbrace{\int_{-1}^{1} P_l(x) P_s'(x)}_{=0} \right]$

$= (s+\frac{1}{2}) \left[P_l(1) \cdot P_s(1) - P_l(-1) P_s(-1) + 0 \right]$

($\because \int_{-1}^{1} P_l(x) P_s'(x)\, dx = 0$ from prev. theorem

$\because s \leq l-1 \Rightarrow s < l+1 \Rightarrow s-1 < l$ and

here deg. of $P_s'(x) = s-1$, which is $< l$

so by prev. result, it will be 0)

Now

$C_s(x) = (s+\frac{1}{2}) \cdot \left[P_l(1) \cdot P_s(1) - P_l(-1) \cdot P_s(-1) \right]$

$= (s+\frac{1}{2}) \left[1 - (-1)^{s+l} \right]$

Here,

Since, $s \to l-1, l-3, \ldots$

$s+l \to 2l-1, 2l-3, \ldots$ $\quad (-1)^{s+l} = -1$

$\Rightarrow 1-(-1)^{s+l} = 2$

So

$C_s(x) = (s+\frac{1}{2}) \cdot 2 = 2s+1$

$\therefore C_{l-1}(x) = 2l-1$

$C_{l-3}(x) = 2(l-3)+1 = 2l-5$

$C_{l-5}(x) = 2(l-5)+1 = 2l-9$

$\vdots \quad - \quad - \quad - \quad -$

Now putting in eqn (1), we have

$P_l'(x) = (2l-1) P_{l-1}(x) + (2l-5) P_{l-3}(x) + \cdots +$

$(2l-4r-1) \cdot P_{(l-2r-1)} + \cdots + \begin{cases} 3P_1(x) & \text{if even} \\ P_0(x) & \text{if odd} \end{cases}$

* * *

Rel^n Between the Legendre Poly. & their Derivative

Recurrence Rel^n

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(sallaby) Theorem (3.8) (58)

(a) $$P_l'(x) = \sum_{r=0}^{[\frac{l-1}{2}]} (2l-4r-1) P_{l-2r-1}(x)$$

(b) $$x P_l(x) = \frac{l+1}{2l+1} P_{l+1}(x) + \frac{l}{2l+1} P_{l-1}(x)$$

(c) $$(l+1) P_{l+1}(x) - (2l+1) x P_l(x) + l P_{l-1}(x) = 0$$

(d) $$P'_{l+1}(x) - P'_{l-1}(x) = (2l+1) P_l(x)$$

(e) $$x P_l'(x) - P_{l-1}'(x) = l P_l(x).$$

Pf (d) for part (d)

Consider ~~the~~ (a). put $l = l+1$, then

$$P_{l+1}'(x) = \sum_{r=0}^{[\frac{l}{2}]} (2l-4r+1) P_{l-2r}(x)$$

$$= (2l+1) P_l + (2l-3) P_{l-2} + (2l-7) P_{l-4}$$

$$(2l-11) P_{l-6} + \cdots \quad \text{— (i)}$$

Similarly replacing l by $l-1$, we get

$$P_{l-1}'(x) = \sum_{r=0}^{[\frac{l-2}{2}]} (2l-4r-3) P_{l-2r-2}$$

$$= (2l-3) P_{l-2} + (2l-7) P_{l-4} + \cdots \quad \text{— (ii)}$$

Now (i) − (ii) gives result, i.e

$$P_{l+1}'(x) - P_{l-1}'(x) = (2l+1) P_l(x) \quad \text{Proof}$$

(e) differentiate (c) w.r.t x, we gets

$$(l+1) P_{l+1}'(x) - (2l+1) P_l(x) - (2l+1) x P_l'(x)$$

$$+ l P_{l-1}'(x) = 0$$

Now put value of $P'_{(l+1)}(x)$ by (d) —
we get —

$$\Rightarrow (l+1)\left[(2l+1)P_l(x) + P'_{l-1}(x)\right] - (2l+1)P_l(x) - (2l+1)\,x\,P'_l(x) + l\,P'_{l-1}(x) = 0$$

$$\Rightarrow (l+1)(2l+1)P_l(x) + (l+1)P'_{l-1}(x) - (2l+1)P_l(x) - (2l+1)x \cdot P'_l(x) + l\,P'_{l-1}(x) = 0$$

$$\Rightarrow (2l+1)(l+1-1)P_l(x) + (l+1+l)P'_{l-1}(x) - (2l+1)x\,P'_l(x)$$

$$\Rightarrow \cancel{(2l+1)}\,l\,P_l(x) + \cancel{(2l+1)}P'_{l-1}(x) - \cancel{(2l+1)}x\,P'_l(x) = 0$$

$$\Rightarrow l\,P_l(x) + P'_{l-1}(x) - x\,P'_l(x) = 0$$

$$\Rightarrow \boxed{x P'_l(x) - P'_{l-1}(x) = l\,P_l(x)} \quad \text{——— (e)}$$

(b)

$xP_l(x)$ is a poly. of degree $l+1$, so — even (l odd), odd (l even)

$x P_l(x) \longrightarrow l+1 \longrightarrow$ even / odd l odd / l even

$\because$ $f(x)$ is poly. of deg. n

$$f(x) = \sum_{r=0}^{l} C_r P_r(x), \quad C_s = (s+1/2)\int_{-1}^{1} f(x)P_s(x)\,dx$$

(from prev. Result)

$$xP_l(x) = C_{l+1}P_{l+1}(x) + C_{l-1}P_{l-1}(x) + \cdots + \begin{cases} C_1P_1(x) & (l \text{ even}) \\ C_0P_0(x) & (l \text{ odd}) \end{cases} \quad \text{——— (1)}$$

$n = l$

$$\therefore C_s = (s+1/2)\int_{-1}^{1} xP_l(x)P_s(x)\,dx$$

$$= (s+1/2)\int_{-1}^{1} P_l(x)\{xP_s(x)\}\,dx$$

poly. of deg. $(s+1)$

Since $xP_s(x)$ is a poly. of deg. $< l$

~~$s \le l+1 \Rightarrow s+1 \le l+2$~~

$\Rightarrow s+1 < l \Rightarrow \underline{s < l-1}$

So

$C_s = 0$ when $s < l-1$

So from (i),

$$x P_l(x) = C_{l+1} P_{l+1}(x) + C_{l-1} P_{l-1}(x) \quad \text{--- (ii)}$$

(Bez rest term will be zero, b/z $s < l-1$)

To determine C_{l+1} & C_{l-1}, we set $x=1$

Now if $x=1$ in (ii),

$$P_l(1) = C_{l+1} P_{l+1}(1) + C_{l-1} P_{l-1}(1)$$

$\because$ $P_l(1) = 1$, $P_{l+1}(1) = 1$, $P_{l-1}(1) = 1$

$$\Rightarrow 1 = C_{l+1} + C_{l-1} \quad \text{--- (iii)}$$

Now if diff (ii), w.r.t x, & after that put $x=1$

$$P_l(x) + x P_l'(x) = C_{l+1} P_{l+1}'(x) + C_{l-1} P_{l-1}'(x)$$

$$\left(\because P_l'(1) = \tfrac{1}{2} l(l+1)\right)$$

$x=1$

$$P_l(1) + P_l'(1) = C_{l+1} P_{l+1}'(x) + C_{l-1} P_{l-1}'(1)$$

$$\Rightarrow 1 + \frac{1}{2} l(l+1) = C_{l+1} \cdot \frac{1}{2}(l+1)(l+2) + C_{l-1} \frac{1}{2}(l-1)l \quad \text{--- (iv)}$$

from (iii), & (iv), we may find C_{l+1} & C_{l-1}

$$C_{l+1} = \frac{l+1}{2l+1} \quad \& \quad C_{l-1} = \frac{l}{2l+1}$$

Now putting these values in (ii), we get desired result.

(c) Eqn (ii) is
$$x P_l(x) = \frac{(l+1)}{(2l+1)} P_{l+1}(x) + \frac{l}{(2l+1)} P_{l-1}(x)$$

~~Now now~~ $\Rightarrow (2l+1) x P_l(x) = (l+1) P_{l+1}(x) + l P_{l-1}(x)$

$$\Rightarrow (l+1) P_{l+1}(x) - (2l+1) x P_l(x) + l P_{l-1}(x) = 0$$

Ex (1) (80)
Ex. Show that

$$\int_{-1}^{1} x^2 P_{l+1}(x) \cdot P_{l-1}(x)\, dx = \frac{2l(l+1)}{(4l^2-1)(2l+3)}$$

Sol^n

$$\int_{-1}^{1} \left(x P_{l+1}(x)\right)\left(x P_{l-1}(x)\right) dx \quad \text{(Rearranging)}$$

∴ from part (b), putting in above integral

$$\int_{-1}^{1} \left[\frac{l+2}{2l+3} P_{l+2}(x) + \frac{l+1}{2l+3} P_l(x)\right]\left[\frac{l}{2l-1} P_l(x) + \frac{l-1}{2l-1} P_{l-2}(x)\right] dx$$

Now, using orthogonality, all three term will be zero, As

$$\int_{-1}^{1} \left(\frac{l+2}{2l+3}\right)\left(\frac{l}{2l-1}\right) \cancel{P_{l+2}(x) P_l(x)}\, dx + \int_{-1}^{1} \left(\frac{l+2}{2l+3}\right)\left(\frac{l-1}{2l-1}\right) \cancel{P_{l+2}(x) P_{l-2}(x)}$$

$$+ \underbrace{\int_{-1}^{1} \left(\frac{l+1}{2l+3}\right)\left(\frac{l}{2l-1}\right) P_l^2(x)} + \int_{-1}^{1} \left(\frac{l+1}{2l+3}\right)\left(\frac{l-1}{2l-1}\right) \cancel{P_l(x) P_{l-2}(x)}\, dx$$

so

$$\int_{-1}^{1} \frac{l+1}{2l+3} \cdot \frac{l}{2l-1} \cdot [P_l(x)]^2\, dx$$

$$= \frac{l+1}{2l+3} \cdot \frac{l}{2l-1} \cdot \int_{-1}^{1} [P_l(x)]^2\, dx$$

$$= \frac{l+1}{2l+3} \cdot \frac{l}{2l-1} \cdot \frac{2}{2l+1} \quad \text{By orthogonality}$$

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$$\Rightarrow \int_0^1 x^2 P_{l+1}(x) P_{l-1}(x)\, dx = \frac{l(l+1)}{(4l^2-1)(2l+3)}$$

Example $\int_0^1 P_l(x)\, dx$, l = odd

$$\int_0^1 \frac{1}{(2l+1)} \cdot \left[P'_{l+1}(x) - P'_{l-1}(x) \right] dx$$

(By using part (d))

$$= \frac{1}{(2l+1)} \left[P_{l+1}(x) - P_{l-1}(x) \right]_0^1$$

$$= \frac{1}{2l+1} \left[\underset{=1}{P_{l+1}(1)} - \underset{=1}{P_{l-1}(1)} - P_{l+1}(0) + P_{l-1}(0) \right]$$

$$= \frac{1}{2l+1} \left[1 - 1 - P_{l+1}(0) + P_{l-1}(0) \right] = \frac{P_{l-1}(0) - P_{l+1}(0)}{(2l+1)}$$

$$\left(\because P_{2l}(0) = (-1)^l \frac{(2l)!}{2^{2l}(l!)^2} \right)$$

$\because l$ is odd $\Rightarrow l+1, l-1$ are even.

$$= \frac{1}{2l+1} \left[\frac{(-1)^{\frac{l+1}{2}+1} \left(2 \cdot \frac{l+1}{2}\right)!}{2^{(l+1)} \cdot \left(\frac{l+1}{2}!\right)^2} + \frac{(-1)^{l-1} \cdot (l-1)!}{2} \right]$$

$$= \frac{1}{2l+1} \left[\frac{(-1)^{\frac{l+3}{2}} (l+1)!}{2^{l+1} \cdot \left(\frac{l+1}{2}!\right)^2} + \frac{(-1)^{\frac{l-1}{2}} (l-1)!}{2^{(l-1)} \cdot \left(\frac{l-1}{2}!\right)^2} \right]$$

$$= \frac{1}{(2l+1)} \left[\frac{(-1)^{\frac{l-1}{2}}}{2^{l-1}} \cdot \frac{(l-1)!}{\left(\frac{l-1}{2}!\right)^2} \left\{ (-1)^2 \right. \right.$$

* * *

AFTER DIWALI (83) BREAK ~~2nd~~ for 3rd Antend

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Hypergometric function

$$_2F_1(a, b, c, x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} x^n \quad \text{--- (1)}$$

when $(a)_n = \dfrac{\Gamma a+n}{\Gamma a}$

if $b=c$ in (1), then

$$= \sum_{n=0}^{\infty} \frac{(a)_n \cancel{(b)_n}}{\cancel{(c)_n}(n!)} x^n$$

$$= (1-x)^{-a}$$

if $a=1$, $\quad = (1-x)^{-1}$

$$= 1 + x + x^2 + x^3 + \cdots$$

Now for convergent of above series, we apply Ratio test

$$= \sum_{n=0}^{\infty} A_n z^n$$

$$\left(\lim_{n\to\infty} \left| \frac{A_{n+1}}{A_n} x \right| = \lim_{n\to\infty} \left| \frac{(a+n)(b+n)}{(c+n)(n+1)} x \right| \right) \quad \text{--- (ii)}$$

and $\because (a)_{n+1} = \dfrac{\Gamma (a+n)+1}{\Gamma n+1} = \dfrac{\Gamma a+n+1}{\Gamma a}$

$$= \frac{(a+n) \Gamma a+n}{\Gamma a} = (a+n)(a)_n$$

$$\& \ \frac{A_{n+1}}{A_n} = \frac{(a+n)(a)_n (b+n)(b)_n}{(c+n)(c)_n (n+1)!} \times \frac{(c)_n n!}{(a)_n (b)_n}$$

then (ii) converges if $|x|<1$

infact absolutely converges

if we consider for complex no. z then $|z|<1$.

$${}_2F_1(a,b,c,x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n\, n!} \cdot x^n \quad \text{--- (1)}$$

is converges a, b are not $0, -1, -2, \ldots$ and $|x| < 1$.

Generalised hypergeometric function is

$$\boxed{{}_pF_q(a_1, a_2, \ldots a_p; b_1, b_2, \ldots b_q; x) = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \ldots (a_p)_n}{(b_1)_n (b_2)_n \ldots (b_q)_n} \frac{x^n}{n!}}$$

If $p=2$, $q=1$, then it becomes hypergeometric function (above f^n)

Now want to convert $\sin x$ in hypergeometric f^n form

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{\Gamma 2n+2} \Rightarrow \frac{x}{2}\left(\frac{x^2}{4}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n+1} \sqrt{\pi}}{2^{2n+2-1}\, \Gamma n+1\, \Gamma n+3/2}$$

(By duplication formula, $\sqrt{\pi}\, \Gamma 2x = \Gamma(x) \cdot 2^{2x-1} \cdot \Gamma x+1/2$)

$\because \Gamma \alpha+n = (\alpha)_n \Gamma \alpha$

$$\therefore \sin x = \frac{x}{2} \sum_{n=0}^{\infty}$$

$$= \frac{x}{\cancel{2}} \sum_{n=0}^{\infty} \frac{\cancel{\sqrt{\pi}}}{(1)_n\, (3/2)_n\, \frac{1}{\cancel{2}} \cancel{\sqrt{\pi}}} \cdot \left(-\frac{x^2}{4}\right)^n$$

$\Gamma 1+n = \frac{(1)_n}{\Gamma 1}$, $(1)_n$ & $\Gamma n+1 = \lfloor n$ $\Rightarrow \boxed{(1)_n = \lfloor n}$

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$$= x \sum_{n=0}^{\infty} \frac{\cancel{(1)_n}}{\cancel{(1)_n} (3/2)_n} \frac{(-x^2/4)^n}{n!}$$

$$= x \, {}_0F_1 \left(-; 3/2; -x^2/4\right)$$

Theorem (9.1) (i)
(204)

r^{th} derivative

* $$P_n(x) = \sum_{r=0}^{n} P_n^{(r)}(1) \frac{(x-1)^r}{r!} \quad \text{---} \quad (1)$$

[i.e every f^n can be written in the form of taylor series)

Taylor series of f(x) about was is

$$f(x) = \sum_{r=0}^{\infty} f_n^{(r)}(a) \cdot \frac{(x-a)^r}{r!}$$

Since generating f^n for legendre poly. is

$$(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x) t^n$$

Now, diff^n w.r.t x, r-times, then —

$$(-2t)^r \left(-\tfrac{1}{2}\right)\left(-\tfrac{1}{2}-1\right) \cdots \left(-\tfrac{1}{2}-r+1\right) (1-2xt+t^2)^{-1/2-r} = \sum_{n=0}^{\infty} P_n^{(r)}(x) t^n$$

$$\Rightarrow \cancel{(-1)^r} \cdot 2^r \cdot t^r \frac{\cancel{(-1)^r}}{2^r} \underbrace{(1 \cdot 3 \cdot 5 \cdots (2r-1))}_{=(2r)!} (1-2xt+t^2)^{-1/2-r} = \sum_{n=0}^{\infty} P_n^{(r)}(x) t^n$$

$$\Rightarrow t^r \cdot \frac{(2r)!}{r! \, 2^r} (1-t)^{-1-2r} = \sum_{n=0}^{\infty} P_n^{(r)}(1) t^n$$

(at x = 1)

L.H.S →

$$\frac{(2r)!}{r! \, 2^r} \sum_{s=0}^{\infty} \frac{(1+2r)_s}{s!} t^{s+r} = \frac{(2r)!}{r! \, 2^r} \sum_{n=r}^{\infty} \frac{(1+2r)_{n-r}}{(n-r)!}$$

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$$= \frac{(2r)!}{r!\,2^r} \sum_{n=r}^{\infty} \frac{\Gamma(1+2r+n-r)}{\Gamma(1+2r)\,(n-r)!} t^n$$

$\Gamma(1+2r) = \lfloor 2r$

$$P_n^{(r)}(1) = \text{Coeff of } t^n \begin{cases} \dfrac{(n+r)!}{r!\,2^r (n-r)!} & n \geq r \\ 0 & n < r \end{cases}$$

In (1), we need $P_n^{(r)}(1)$, which is obtained here

$$\therefore P_n(x) = \sum_{r=0}^{n} \frac{(n+r)!}{r!\,2^r (n-r)!} \frac{(x-1)^r}{r!}$$

(Still we are not able to write in the form of hypergeometric f^n bcz here series is upto n.)

$$\because (-n)_r = (-n)(-n+1)\cdots(-n+r-1)$$
$$= (-1)^r \left(n(n-1)\cdots(n-r+1) \right)$$
$$= (-1)^r \cdot \frac{n!}{(n-r)!}$$

and $(n+1)_r = \dfrac{\Gamma(n+r+1)}{\Gamma(n+1)} = \dfrac{(n+r)!}{n!}$

now put value of $(n+r)!$ & $(n-r)!$ in above formula we get –

$$P_n(x) = \sum_{r=0}^{n} \frac{(n+1)_r\, n!}{(r!)(n-r)!} \times \frac{\left(\frac{x-1}{2}\right)^r}{r!}$$

$$= \sum_{r=0}^{n} \frac{(n+1)_r \cancel{n!}\, (-n)_r}{r!\,(-1)^r \cancel{n!}} \frac{\left(\frac{x-1}{2}\right)^r}{r!}$$

$$= \sum_{r=0}^{\infty} \frac{(n+1)_r (-n)_r}{(1)_r} * \frac{\left(\frac{1-x}{2}\right)^r}{(r!)}$$

$r! = \Gamma(1+r) = (1)_r$

$(-n)_r = 0$ when $r > n$

$\therefore (1)_r = r!$

Bcz when $r > n$ then $(-n)_r$ becomes 0

$$= {}_2F_1\left(-n, n+1, 1, \frac{1-x}{2}\right), \quad \left|\frac{1-x}{2}\right| < 1$$

is (gt when ↗

H.W.

$$\boxed{\therefore P_n(x) = {}_2F_1\left(-n, n+1; 1; \frac{1-x}{2}\right)}$$

* * *

From Notes

Theorem (9.2) (2017) The following f^n in variable of z.

$${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n \, n!} z^n \quad \text{--- (1)}$$

is called hypergeometric f^n, where $(\alpha)_r$ is Pochhammer symbol defined by

$$(\alpha)_r = \alpha(\alpha+1)\ldots(\alpha+r-1)$$

$$= \frac{\Gamma(\alpha+r)}{\Gamma\alpha}, \quad (\alpha)_0 = 1$$

Note that (1) is symmetric to exchange $a \leftrightarrow b$. For convenience, we define the general form —

$$A_n = \frac{(a)_n (b)_n}{(c)_n \, n!}$$

so that hypergeometric f^n represented as

$$\boxed{{}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} A_n z^n}$$

From the std. ratio test for series convergence,

$$\lim_{n\to\infty} \left|\frac{A_{n+1} z^{n+1}}{A_n z^n}\right| = \lim_{n\to\infty} \left|\frac{(n+a)(n+b)}{(n+c)(n+1)} z\right|$$

$$= \left[\lim_{n\to\infty} \frac{(n+a)(n+b)}{(n+c)(n+1)}\right] |z|$$

$$= |z|$$

It follows that hypergeometric f^n is absolutely cgt. inside the unit (circle) disk $|z|<1$, where it is defined as of analytic function.

All elementary functions and special f^ns including all orthogonal polynomials can be obtained as special case of hypergeometric functions.

(i) When $b=c$, hypergeometric series gives

Problem ① 2016 (1)

$$_2F_1(a,b;b;z) = \sum_{n=0}^{\infty} \frac{(a)_n}{n!} z^n = (1-z)^{-a}$$

This expansion converges in $|z|<1$ but can be extended to the whole complex plane. In particular, for $a=1$, one obtain elementary geometric series —

$$_2F_1(a,b;b;z) = (1-z)^{-1}$$

(ii) For $b = a+\frac{1}{2}$ and $c = \frac{3}{2}$, we have

$$_2F_1\left(a, a+\tfrac{1}{2}; \tfrac{3}{2}; z^2\right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\,\Gamma(a+\frac{1}{2}+n)\,\Gamma\frac{3}{2}}{\Gamma a\,\Gamma(\frac{3}{2}+n)\,\Gamma(a+\frac{1}{2})} \frac{z^{2n}}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{\cancel{\Gamma(a+n)}\,\Gamma(2a+2n)\,\cancel{\sqrt{\pi}}\cdot\frac{1}{2}\cancel{\sqrt{\pi}}\; 2^{2a-1}\,\cancel{\Gamma a}\; 2^{2n+2-1}}{2^{2(a+n)-1}\,\cancel{\Gamma(a+n)}\,\cancel{\Gamma a}\,\Gamma 2a\,\cancel{\sqrt{\pi}}\,\Gamma(2n+2)\,\cancel{\sqrt{\pi}}\; n!} z^{2n}$$

By duplication formula

$$\Gamma 2x = \frac{\Gamma x\,\Gamma(x+\frac{1}{2})\cdot 2^{2x-1}}{\sqrt{\pi}} \Rightarrow \Gamma\left(x+\tfrac{1}{2}\right) = \frac{\Gamma 2x}{\Gamma x}\cdot\frac{\sqrt{\pi}}{2^{2x-1}}$$

$$\Rightarrow \Gamma\left(a+n+\tfrac{1}{2}\right) = \frac{\Gamma 2(a+n)}{\Gamma(a+n)}\cdot\frac{\sqrt{\pi}}{2^{2(a+n)-1}} \quad \& \quad \Gamma\left(a+\tfrac{1}{2}\right) = \frac{\Gamma 2a\,\sqrt{\pi}}{\Gamma a\; 2^{2a-1}}$$

$\Gamma\frac{3}{2} = \frac{1}{2}\Gamma\frac{1}{2} = \frac{1}{2}\sqrt{\pi}$

$\cancel{4}\ \Gamma\left(\frac{3}{2}+n\right) = \cancel{\frac{1}{2}\Gamma\left(n+\frac{1}{2}\right) = \frac{1}{2}\left[\frac{\Gamma 2n\ \sqrt{\pi}}{\Gamma n\ 2^{2n-1}}\right]}$

$= \Gamma\left((n+1)+\frac{1}{2}\right) = \frac{\Gamma 2(n+1)\ \sqrt{\pi}}{\Gamma(n+1)\cdot 2^{2(n+1)-1}}$

$= \sum_{n=0}^{\infty} \frac{\Gamma 2(a+n)\cdot 2^{-1+2a-1+2n+2-1}}{2^{2a+2n-1}\cdot\Gamma 2a\ \Gamma 2n+2\ n!} z^{2n}$

$= \sum_{n=0}^{\infty} \frac{\Gamma 2(a+n)}{\Gamma 2a\ (2n+1)!} \frac{z^{2n}}{n!}$

$= \frac{1}{z(2a-1)} \sum_{n=0}^{\infty} \frac{\Gamma(2a-1+2n+1)}{\Gamma(2a-1)\ (2n+1)!} z^{2n+1}$

$\Gamma n = (n-1)\Gamma n-1$

We observe now that only odd powers of z Contribute to the sum over n. Adding and substracting sums of even powers of z, We obtain,

$${}_2F_1\left(a, a+\tfrac{1}{2}; \tfrac{3}{2}; z^2\right) = \frac{1}{2z(2a-1)}\left[\sum_{n=0}^{\infty}\frac{\Gamma(2a-1+2n+1)}{\Gamma(2a-1)}\frac{1}{(2n+1)!}z^{2n+1} + \sum_{n=0}^{\infty}\frac{\Gamma(2a-1+2n)}{\Gamma(2a-1)\ (2n)!}z^{2n}\right] +$$

$$\frac{1}{2z(2a-1)}\left[\sum_{n=0}^{\infty}\frac{\Gamma(2a-1+2n+1)}{\Gamma(2a-1)\ (2n+1)!}z^{2n+1} - \sum_{n=0}^{\infty}\frac{\Gamma(2a-1+2n)}{\Gamma(2a-1)\ 2n!}z^{2n}\right]$$

Which in turn, Can be written as,

$${}_2F_1\left(a, a+\tfrac{1}{2}; \tfrac{3}{2}; z^2\right) = \frac{1}{2z(2a-1)}\left[\sum_{k=0}^{\infty}\frac{\Gamma(2a-1+k)}{\Gamma(2a-1)\ k!}z^k - \sum_{k=0}^{\infty}\frac{\Gamma(2a-1+k)}{\Gamma(2a-1)\ k!}(-z)^k\right]$$

$$= \frac{1}{2z(2a-1)}\left[(1-z)^{-2a+1} - (1+z)^{-2a+1}\right]$$

$\therefore \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} \cdots$

(iii). Problem (1) (ii) (216)

$${}_2F_1(1,1;2;z) = \sum_{n=0}^{\infty} \frac{(1)_n (1)_n}{(2)_n} \frac{z^n}{n!} = \sum_{n=0}^{\infty} \frac{\Gamma(1+n)\,\Gamma(1+n)\,\Gamma 2}{\Gamma(2+n)} \frac{z^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{z^n}{n+1} = \sum_{n=1}^{\infty} \frac{z^{n-1}}{n}$$

$$= -\frac{1}{z} \ln(1-z)$$

(iv). $${}_2F_1(1/2, 1; 3/2; z^2) = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n+1)}$$

$$= \frac{1}{2z}\left\{ 2 \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)} \right\}$$

$$= \frac{1}{2z} \ln\left(\frac{1+z}{1-z}\right)$$

(v). $P_n(x) = {}_2F_1\left(-n, n+1; 1; \frac{1-x}{2}\right)$

Thm 9.1 (ii) (204) From definition

$${}_2F_1\left(-n, n+1; 1; \frac{1-x}{2}\right) = \sum_{r=0}^{\infty} \frac{(-n)_r\,(n+1)_r}{(1)_r\, r!} \left(\frac{1-x}{2}\right)^r$$

We may take n to be non-negative integers, since it is ~~only~~ for only these values of n that legendre's poly. are defined. Then we use

$$(-n)_r = (-n)(-n+1)\ldots(-n+r-1)$$

$$= (-1)^r\, n(n-1)(n-2)\ldots(n-r+1)$$

$$= (-1)^r \cdot \frac{n!}{(n-r)!} \quad \text{if } r \le n$$

if $r > n$

then $(-n)_r = 0$, for it will a zero factor

Also

$$(n+1)_r = (n+1)(n+2)\ldots\overset{n+r}{(n+1+(r-1))}$$

$$= \frac{(n+r)!}{n!}$$

and $(1)_r = r!$, so we ~~take~~ have

$$ {}_2F_1\left(-n, n+1; 1; \frac{1-x}{2}\right) = \sum_{r=0}^{n} \frac{(-1)^r \cancel{n!}\,(n+r)!\,(1-x)^r}{(n-r)!\,\cancel{n!}\,r!\,2^r \cdot r!} $$

$$ = \sum_{r=0}^{n} \frac{(n+r)!}{2^r (n-r)!\,(r!)^2} (x-1)^r $$

To show this, it is easiest to write $P_n(x)$ as power series in $(x-1)$. For this, We use Taylor's series

$$ P_n(x) = \sum_{r=0}^{n} P_n^{(r)}(1) \frac{(x-1)^r}{r!} \quad \text{--- (2)} $$

Where by $P_n^{(r)}(1)$ is r^{th} derivative of $P_n(x)$ evaluated at $x=1$, To calculate $P_n^{(r)}(1)$, We use the generating f^n,

$$ (1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x) \cdot t^n $$

now differentiating r times w.r.t x, we have

$$ \sum_{n=0}^{\infty} P_n^{(r)}(x)\, t^n = \frac{d^r}{dx^r} (1-2xt+t^2)^{-1/2} $$

$$ = (-2t)^r \left(-\tfrac{1}{2}\right)\left(-\tfrac{3}{2}\right) \cdots \left(-\tfrac{1}{2}-(r-1)\right) (1-2xt+t^2)^{-\frac{1}{2}-r} $$

$$ = (-1)^r \cancel{2^r} t^r \cdot \frac{1}{\cancel{2^r}} (-1)^r 1\cdot 3\cdot 5 \cdots (2r-1)\, (1-2xt+t^2)^{-\frac{1}{2}-r} $$

$$ = (-1)^{2r} t^r \frac{(2r)!}{2^r \cdot r!} (1-2xt+t^2)^{-\frac{1}{2}-r} $$

Hence setting $x=1$, We have

$$ \sum_{n=0}^{\infty} P_n^{(r)}(1)\, t^n = t^r \frac{(2r)!}{2^r \cdot r!} \left[(1-t)^2\right]^{\frac{-1-2r}{2}} $$

$$ = t^r \frac{(2r)!}{2^r \cdot r!} (1-t)^{-(1+2r)} $$

$$= \frac{t^r \cdot 2r!}{2^r \cdot r!} \sum_{s=0}^{\infty} \frac{(1+2r)_s}{s!} t^s$$

$$= \frac{(2r)!}{2^r \cdot r!} \sum_{s=0}^{\infty} \frac{(1+2r)_s \; t^{r+s}}{s!}$$

$$= \frac{(2r)!}{2^r r!} \sum_{s=0}^{\infty} \frac{\Gamma(1+2r+s)}{\Gamma(1+2r)\,\Gamma(s+1)} \cdot t^{r+s}$$

$\because \Gamma(1+2r) = 2r!$

$$= \frac{1}{2^r r!} \sum_{n=r}^{\infty} \frac{\Gamma(n+r+1)}{\Gamma(n-r+1)} \cdot t^n \qquad \because r+s=n$$

$$= \frac{1}{2^r r!} \sum_{n=r}^{\infty} \frac{(n+r)!}{(n-r)!} t^n \qquad (n=s+r)$$

Equating Co-efficients of t^n, gives—

$$P_n^{(r)}(1) = \begin{cases} \dfrac{1}{2^r r!} \dfrac{(r+n)!}{(n-r)!} & \text{for } n \geq r \\ 0 & n < r \end{cases}$$

Hence, from (1), We have

$$P_n(x) = \sum_{r=0}^{n} \frac{1}{2^r r!} \frac{(r+n)!}{(n-r)!} \frac{(x-1)^r}{r!}$$

$$= \sum_{r=0}^{n} \frac{(n+r)!}{(n-r)!\,(r!)^2} \left(\frac{x-1}{2}\right)^r$$

$$= {}_2F_1\left(-n, n+1; 1; \frac{1-x}{2}\right)$$

$$\left((-n)_r = \frac{(-1)^r n!}{(n-r)!} \;\; \& \;\; (n+1)_r = \frac{(n+r)!}{n!} \right)$$

INTEGRAL REPRESENTATION

Theorem (9.4)

$${}_2F_1(\alpha, \beta; \gamma; x) = \frac{\Gamma(\gamma)}{\Gamma\beta\,\Gamma\gamma-\beta} \int_0^1 t^{\beta-1} (1-t)^{\gamma-\beta-1} (1-xt)^{-\alpha} dt$$

If $\gamma - \beta > 0$, $\alpha, \beta, \gamma \in \mathbb{R}$.

Proof: — From definition, we have

$$_2F_1(\alpha, \beta; \gamma; x) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n} \frac{x^n}{n!}$$

$$\therefore \; _2F_1(\alpha, \beta; \gamma; x) = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n)\,\Gamma(\beta+n)\,\Gamma\gamma}{\Gamma\alpha\,\Gamma\beta\,\Gamma(\gamma+n)} \cdot \frac{x^n}{n!}$$

$$= \frac{\Gamma\gamma}{\Gamma\alpha\,\Gamma\beta\,\Gamma(\gamma-\beta)} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n)\,\Gamma(\beta+n)\,\Gamma(\gamma-\beta)}{\Gamma(\gamma+n)} \frac{x^n}{n!}$$

$$= \frac{\Gamma\gamma}{\Gamma\alpha\,\Gamma\beta\,\Gamma(\gamma-\beta)} \sum_{n=0}^{\infty} \Gamma(\alpha+n)\, B(\gamma-\beta, \beta+n) \cdot \frac{x^n}{n!}$$

$$= \frac{\Gamma\gamma}{\Gamma\alpha\,\Gamma\beta\,\Gamma(\gamma-\beta)} \sum_{n=0}^{\infty} \Gamma(\alpha+n) \int_0^1 (1-t)^{\gamma-\beta-1}\, t^{\beta+n-1}\, dt \cdot \frac{x^n}{n!}$$

(By def^n of Beta f^n, which is valid for $\gamma-\beta>0$ & $\beta+n>0$)

$$= \frac{\Gamma\gamma}{\Gamma\beta\,\Gamma(\gamma-\beta)} \int_0^1 (1-t)^{\gamma-\beta-1}\, t^{\beta-1} \left\{\sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n)}{\Gamma\alpha} \frac{(xt)^n}{n!}\right\} dt$$

$$= \frac{\Gamma\gamma}{\Gamma\beta\,\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1} (1-t)^{\gamma-\beta-1} \cdot \left\{(1-xt)^{-\alpha}\right\} dt$$

= R.H.S. Proved if $\gamma-\beta>0$

9.4 Examples P-212

eg(1) Show that (Gauss thm)

$$_2F_1(\alpha, \beta; \gamma; 1) = \frac{\Gamma\gamma\,\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)}$$

Pf: — From integral Representation of $_2F_1$, we have

$$_2F_1(\alpha, \beta; \gamma; 1) = \frac{\Gamma\gamma}{\Gamma\beta\,\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1} (1-t)^{\gamma-\beta-1} (1-t)^{-\alpha}\, dt$$

$$= \frac{\Gamma\gamma}{\Gamma\beta\,\Gamma\gamma-\beta}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\alpha-\beta-1}\,dt$$

$$= \frac{\Gamma\gamma}{\Gamma\beta\,\Gamma\gamma-\beta}\cdot B(\beta,\gamma-\alpha-\beta)$$

$$= \frac{\Gamma\gamma}{\Gamma\beta\,\Gamma\gamma-\beta}\cdot\frac{\Gamma\beta\,\Gamma\gamma-\alpha-\beta}{\Gamma\gamma-\alpha}$$

$$= \frac{\Gamma\gamma\,\Gamma\gamma-\alpha-\beta}{\Gamma\gamma-\alpha\,\Gamma\gamma-\beta} \quad \text{Proved}$$

Eg. 2 p-213

S.T.

$${}_2F_1(\alpha,\beta;\gamma;x) = (1-x)^{-\alpha}\,{}_2F_1\left(\alpha,\gamma-\beta;\gamma;\frac{x}{x-1}\right)$$

Pf. putting $\tau = 1-t$ in integral representation of ${}_2F_1$, we have

$${}_2F_1(\alpha,\beta;\gamma;x) = \frac{\Gamma\gamma}{\Gamma\beta\,\Gamma(\gamma-\beta)}\int_0^1 \tau^{\gamma-\beta-1}(1-\tau)^{\beta-1}\left(1-x(1-\tau)\right)^{-\alpha}d\tau$$

$$= \frac{\Gamma\gamma}{\Gamma\beta\,\Gamma\gamma-\beta}\int_0^1 (1-\tau)^{\beta-1}\tau^{\gamma-\beta-1}\cdot(1-x)^{-\alpha}\left(1-\frac{x\tau}{1-x}\right)^{-\alpha}d\tau$$

$$= \frac{\Gamma\gamma\,(1-x)^{-\alpha}}{\Gamma\beta\,\Gamma(\gamma-\beta)}\int_0^1 (1-\tau)^{\beta-1}\tau^{\gamma-\beta-1}\left(1-\frac{x\tau}{1-x}\right)^{-\alpha}d\tau$$

$$= \frac{\Gamma\gamma\,(1-x)^{-\alpha}}{\Gamma\gamma-\beta\,\Gamma\gamma-(\gamma-\beta)}\int_0^1 \tau^{\gamma-\beta-1}(1-\tau)^{\beta-1}\left(1-\tau\left(\frac{x}{x-1}\right)\right)^{-\alpha}d\tau$$

$$= (1-x)^{-\alpha}\,{}_2F_1\left(\alpha,\gamma-\beta;\gamma;\frac{x}{x-1}\right)$$

Eg. 3 Prove the Relation of Contiguity

$$\gamma\{(\gamma-1)-(2\gamma-1-\alpha-\beta)x\}\,{}_2F_1(\alpha,\beta;\gamma;x) + (\gamma-\alpha)(\gamma-\beta)x\,{}_2F_1(\alpha,\beta;\gamma+1;x) - \gamma(\gamma-1)(1-x)\cdot{}_2F_1(\alpha,\beta;\gamma-1;x) = 0$$

Sol^n. We prove this result by expanding the hypergeometric f^{n}s in power series, since —

$$_2F_1(\alpha,\beta;\gamma;x) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n\, n!} x^n$$

We see that the coefficients of x^n on the LHS above must be,

$$\frac{\gamma(\gamma-1)(\alpha)_n(\beta)_n}{(\gamma)_n n!} - \frac{\gamma(2\gamma-1-\alpha-\beta)(\alpha)_{n-1}(\beta)_{n-1}}{(\gamma)_{n-1}(n-1)!} + \frac{(\gamma-\alpha)(\gamma-\beta)(\alpha)_{n-1}(\beta)_{n-1}}{(\gamma+1)_{n-1}(n-1)!} - \frac{\gamma(\gamma-1)(\alpha)_n(\beta)_n}{(\gamma-1)_n\, n!}$$

$$+ \frac{\gamma(\gamma-1)(\alpha)_{n-1}(\beta)_{n-1}}{(\gamma-1)_{n-1}(n-1)!}$$

$$\Rightarrow \frac{\gamma(\gamma-1)\Gamma(\alpha+n)\Gamma(\beta+n)\Gamma\gamma}{\Gamma\alpha\Gamma\beta\Gamma(\gamma+n)\, n!} - \gamma(2\gamma-1-\alpha-\beta)\frac{\Gamma(\alpha+n-1)\Gamma(\beta+n-1)\Gamma\gamma}{\Gamma\alpha\Gamma\beta\Gamma(\gamma+n-1)(n-1)!} + (\gamma-\alpha)(\gamma-\beta)\Gamma(\alpha+n-1)$$

$$\frac{\Gamma(\beta+n-1)\Gamma(\gamma+1)}{\Gamma\alpha\Gamma\beta\Gamma(\gamma+n)(n-1)!} - \frac{\gamma(\gamma-1)\Gamma\alpha+n\,\Gamma\beta+n\,\Gamma\gamma-1}{\Gamma\alpha\Gamma\beta\Gamma(\gamma+n-1)\, n!}$$

$$+ \frac{\gamma(\gamma-1)\Gamma(\alpha+n-1)\Gamma(\beta+n-1)\Gamma\gamma-1}{\Gamma\alpha\Gamma\beta\Gamma(\gamma+n-2)(n-1)!}$$

$$= \frac{\Gamma(\alpha+n-1)\Gamma(\beta+n-1)\Gamma(\gamma-1)}{\Gamma\alpha\Gamma\beta\Gamma(\gamma+n-2)(n-1)!}\left\{\frac{\gamma(\gamma-1)(\alpha+n-1)(\beta+n-1)}{(\gamma+n-1)(\gamma+n-2)n}\right.$$

$$- \frac{\gamma(\gamma-1)(2\gamma-1-\alpha-\beta)}{(\gamma+n-2)} + \frac{\gamma(\gamma-\alpha)(\gamma-\beta)\gamma(\gamma-1)}{(\gamma+n-2)(\gamma+n-1)}$$

$$\left.- \frac{\gamma(\gamma-1)(\alpha+n-1)(\beta+n-1)}{(\gamma+n-2)\cdot n} + \frac{\gamma(\gamma-1)}{(\ldots)}\right\}$$

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$$= \underbrace{\frac{\Gamma(\alpha+n+1)\,\Gamma(\beta+n-1)\,\Gamma\gamma-1.\gamma.(\gamma-1)}{\Gamma\alpha\,\Gamma\beta\,\Gamma(\gamma+n-2)\,(n-1)!}\cdot\frac{1}{n.(\gamma+n-2)(\gamma+n-1)}}_{=A}$$

$$\{(\alpha+n-1)(\beta+n-1).\gamma - n(2\gamma-1-\alpha-\beta)(\gamma+n-1)$$

$$+ n(\gamma-\alpha)(\gamma-\beta) - (\alpha+n-1)(\beta+n-1)(\gamma+n-1)$$

$$+ (\gamma+n-2)(\gamma+n-1)\,n\}$$

$$= A\{(\alpha+n-1)(\beta+n-1)(\cancel{\gamma}-\cancel{\gamma}-n+1)$$
$$-n(\gamma+n-1)(2\gamma-1-\alpha-\beta-\cancel{\gamma}-n+2)$$
$$+n(\gamma-\alpha)(\gamma-\beta)\}$$

$$= A\{(\alpha+(n-1))(\beta+(n-1))(1-n) - n(\gamma+n-1)$$
$$(\gamma-\alpha-\beta-n+1) + n(\gamma-\alpha)(\gamma-\beta)\}$$

$$= A\{[\alpha\beta+\alpha(n-1)+(n-1)\beta+(n-1)^2](1-n)$$

$$= A\{-2n\gamma^2 - 2n^2\gamma + 2n\gamma + n\gamma\alpha + n^2\alpha - n\alpha$$
$$+ n\gamma\beta + n^2\beta - n\beta + n\gamma + n^2 - n + n\gamma^2 - n\gamma\beta - n\gamma\alpha$$
$$+ n\alpha\beta - (n-1)(\beta\gamma + \beta n - \beta + \gamma n + n^2 - n - \gamma - n + 1)$$
$$+ n(\gamma^2 + n\gamma - 2\gamma + n\gamma + n^2 - 2n - \gamma - n + 2)]$$

$= 0$ Hence proved

Eg. 4 (vw) Show that

$$\frac{d}{dx}\,{}_2F_1(a,b;c;x) = \frac{ab}{c}\,{}_2F_1(a+1,b+1;c+1;x)$$

Sol^n From series representation of ${}_2F_1$, we have

$$\frac{d}{dx}\left({}_2F_1(a,b;c;x)\right) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{n\,(x)^{n-1}}{n!} \quad (n-1)! \quad (1)$$

$$= \sum_{n=0}^{\infty} \frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}} \cdot \frac{x^n}{n!}$$

$$= \frac{ab}{c} \sum_{n=0}^{\infty} \frac{(a+1)_n (b+1)_n}{(c+1)_n} \frac{x^n}{n!}$$

$$= \frac{ab}{c} \cdot {}_2F_1(a+1, b+1; c+1; x)$$

$(a)_n = \frac{\Gamma(a+n)}{\Gamma a}$

$\Rightarrow (a)_{n+1} = \frac{\Gamma(a+n+1)}{\Gamma a}$

$\cancel{(a+1)_n = \frac{(a)_{n+1}}{a}}$

$(a+1)_n = \frac{\Gamma(a+n+1)}{\Gamma(a+1)}$

$= \frac{1}{a}(a)_{n+1}$

$\Rightarrow (a)_{n+1} = a(a+1)_n$

Hence deduce that

$$\frac{d^n}{dx^n}\,{}_2F_1(a,b;c;x) = \frac{(a)_n(b)_n}{(c)_n}\,{}_2F_1(a+n, b+n; c+n; x)$$

Now replacing n by n+2 in (1), we have

$$= \sum_{n=0}^{\infty} \frac{(a)_{n+2}(b)_{n+2}}{(c)_{n+2}} \frac{x^{n+1}}{(n+2)!}$$

$$=$$

$\cancel{(a)_{n+2} = \frac{\Gamma(a+n+2)}{\Gamma a}}$

$\cancel{(a+2)_n = \frac{\Gamma(a+n+2)}{\Gamma(a+2)}}$

$\cancel{= \frac{\Gamma(a+n+2)}{(a+1)a\,\Gamma a}}$

Eg. 5 Show that (Vandermonde's theorem)

$$ {}_2F_1(-n, b; c; 1) = \frac{(c-b)_n}{(c)_n} $$

$\therefore$ ~~${}_2F_1(-n, b; c; 1) =$~~ Γc

Theorem: ① Gauss theorem

$$ {}_2F_1(\alpha; \beta; \gamma; 1) = \frac{\Gamma\gamma\, \Gamma\gamma-\alpha-\beta}{\Gamma\gamma-\alpha\, \Gamma\gamma-\beta} $$

pf: (P-137)

Now using above theorem replace $\alpha = -n$, $\beta = b$, $\gamma = c$, we have

$$ {}_2F_1(-n, b; c; 1) = \frac{\Gamma c\, \Gamma c-b+n}{\Gamma c+n\, \Gamma c-b} $$

$=$

$$ \because (c)_n = \frac{\Gamma c+n}{\Gamma c} \Rightarrow \frac{\Gamma c}{\Gamma c+n} = \frac{1}{(c)_n} $$

and $(c-b)_n = \dfrac{\Gamma c-b+n}{\Gamma c-b}$

$$ \therefore {}_2F_1(-n; b; c; 1) = \frac{(c-b)_n}{(c)_n} $$

Eg-6 ST, ${}_2F_1(-n, a+n; c; 1) = \dfrac{(-1)^n (1+a-c)_n}{(c)_n}$

As from Gauss theorem, we have

$$ {}_2F_1(-n, a+n; c; 1) = \frac{\Gamma c\, \Gamma c+\cancel{n}-a-\cancel{n}}{\Gamma c+n\, \Gamma c-a-n} $$

if $c-a$ is non-integer then ①

$$ \frac{\Gamma(1-a-n)}{\Gamma 1-a} = \frac{\Gamma(1-a-n)}{(-a)\Gamma(1-a)} = \frac{\Gamma(1-a-n)}{(-a)(-a-1)\Gamma(-a-1)} $$

$$ = \frac{\Gamma(1-a-n)}{(-a)(-a-1)(-a-2)\cdots(-a-(n-1))\Gamma(1-a-} $$

$$ = \frac{1}{(-1)a} = \frac{(-1)^n}{(a)_n} $$

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$$= \frac{1}{(-1)\left[a(a+1)(a+2)\cdots(a+(n-1))\right]}$$

$$= \frac{(a-1)!}{(-1)^n (n+a-1)!} = \frac{\Gamma a}{(-1)^n \Gamma n+a} = \frac{(-1)^n}{(a)_n}$$

From here (1) $\Rightarrow$ $${}_2F_1(-n, a+n; c; 1) = \frac{\Gamma c\, \Gamma c-a}{\Gamma c+n\, \Gamma c-a-n} \quad -(*)$$

$$\frac{\Gamma c-a+n}{\Gamma c-a} = \frac{\Gamma(c-a-n)}{(c-a-1)\,\Gamma c-a-1}$$

$$\frac{\Gamma(c-a+n)}{\Gamma(c-a)} = \frac{\Gamma(c-a-n)}{(c-a-1)(c-a-2)\,\Gamma c-a-2}$$

$$= \frac{\Gamma(c-a-n)}{(c-a-1)(c-a-2)\cdots(c-a-n)\,\Gamma c-a-n}$$

$$= \frac{1}{(-1)^n\left[(a-c+1)(a-c+2)\cdots((a-c+n))\right]}$$

$$= \frac{1}{(-1)^n (a-c+1)!}$$

$$= \frac{1}{(-1)^n\{((a-c)+1)\,((a-c)+2)\cdots((a-c)+n)\}}$$

$$= \frac{1\,(a-c)!}{(-1)^n\,((a-c)+n)!} = \frac{\Gamma a-c+1}{(-1)^n\,\Gamma a-c+n+1}$$

$$= \frac{1}{(-1)^n (1+a-c)_n}$$

Now putting in $(*)$, we get —

$${}_2F_1(-n, a+n; c; 1) = \frac{1}{(c)_n} \cdot (-1)^n (1+a-c)_n$$

$$= \text{RHS} \quad \text{Proved}$$

Hypergeometric DE

Consider a DE of the form —

$$y'' + p(x)y' + q(x)y = 0 \quad \text{—} (1)$$

Where $p(x)$ and $q(x)$ are cts. f^ns in some nbd of x_0 and analytic at x_0. meaning that each has power series expansion valid in some nbd of this pt., the pt. x_0 is called the Ordinary pt. of (1).

It is known that every sol^n of eq^n (1) at the ordinary pt. x_0 is also analytic

On the other hand, if $p(x_0) = 0$, then $p(x)$ and or $q(x)$ become unbounded as $x \to x_0$ such a pt.s are called a singular pt. of eq^n (1).

For Eg., $x_0 = 0$ is a singular pt. of the Bessel Eq^n and all other pt.s are ordinary pt.s.

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0$$

The singular pt.s of legendre's DE

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

are $x = \pm 1$ and its all other pt.s are Ordinary pt.s.

If $x = x_0$ is a singular pt. and

$$\lim_{x \to x_0} (x - x_0)p(x) \;\&\; \lim_{x \to x_0} (x-x_0)^2 q(x)$$

are finite, then $x = x_0$ is known as regular singular pt. of (1). As $x = 1$ is singular pt. of Legendre DE and

$$\lim_{x \to 1} (x-1)\left(\frac{-2x}{1-x^2}\right) = 2 \quad \text{and}$$

$$\lim_{x \to 1} (x-1)^2 \frac{n(n+1)}{1-x^2} = 0$$

are finite, hence $x=1$ is a regular singular pt. of legendre DE.

If x_0 is an ordinary pt. of Eq^n (1), then

$$y(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n = a_0 y_1(x) + a_1 y_2(x)$$

where a_0 & a_1 are constant in sol^n of eq^n (1).

Here y_1 & y_2 are two LI sol^{n}s which are analytic at x_0. The co-efficients a_i's are determined by substituting this series in Eq^n(1).

If x_0 is a regular singular pt. of (1) then sol^n of (1) near $x=x_0$ is in the form

$$y(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^{n+r}, \quad a_0 \neq 0$$

To obtain value of r, substitute y, y' and $y''\left(=\frac{d^2y}{dx^2}\right)$ in (1), We get a quadratic Eq^n in r by equating zero the co-efficient of lowest power of $(x-x_0)$. This Eq^n is known as the indicial Eq^n.

The roots of indicial Eq^n may be equal, different & differing by an integer.

Case-I If roots of indicial Eq^n are different and differ by a quantity not an integer the sol^n of the eq^n may be taken as

$$y(x) = \sum_{n=0}^{\infty} c_n (x-x_0)^{n+r}$$

for different values of r, we get two independent sol^n $y_1(x)$ & $y_2(x)$

For Eg.

$x=0$, is regular singular pt. of Bessel DE.

$$x^2y'' + xy' + (x^2 - n^2)y = 0 \quad \text{—— (2)}$$

We assume that $y = \sum_{n=0}^{\infty} a_n x^{n+r}$, $a_0 \neq 0$

is sol^n of (2), putting this in Eq^n (2) and equating co-efficients of lowest power of x equal to zero, We get indicial Eq^n.

$$a_0(r^2 - m^2) = 0 \quad \Rightarrow r = \pm m$$

If $m \neq$ an integer, then both the roots of indicial Eq^{n}s differ by integer. Hence two independent sol^{n}s are given by

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+m} \quad \& \quad y_2(x) = \sum_{n=0}^{\infty} a_n x^{n-m}$$

Here we need to find values of co-efficients a_is.

Case-II If indicial Eq^n has two equal roots $r = r_1$ (say) then the other independent sol^n will be obtained by putting $r = r_1$ in $\frac{\partial y}{\partial r}$. Here the second sol^n will always Consists the product of first sol^n (or a numerical multiple of it) and $\log x$, added to another series.

Case-III, Many time it is desired to obtain solⁿ of (1) for large value of x. In such case the solⁿ can be obtained by

$$\boxed{y(x) = \sum_{n=0}^{\infty} a_n x^{r-n}}$$

* The second order DE

$$x(1-x)\frac{d^2y}{dx^2} + \{\gamma - (1+\alpha+\beta)x\}\frac{dy}{dx} - \alpha\beta y = 0 \quad \text{--- (3)}$$

in which α, β and γ are constants is called hypergeometric Eqⁿ / or Gauss hypergeometric Eqⁿ.

Note that $x = 0, 1$ and ∞ are three regular singular points of this Eqⁿ. The solⁿ in the nb.d of $x=0$ is of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad a_0 \neq 0 \quad \text{--- (4)}$$

Case-I, Solⁿ near $x=0$

Substituting values of y, y' and y'' in eqⁿ (3)

We get -

$$(x - x^2)\sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2} + \{\gamma - (1+\alpha+\beta)x\} \cdot \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1} - \alpha\beta \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

or $$\sum_{n=0}^{\infty} a_n [(n+r)(n+r-1) + \gamma(n+r)] x^{n+r-1} - \sum_{n=0}^{\infty} a_n [(n+r) \cdot (n+r-1) + (1+\alpha+\beta)(n+r) + \alpha\beta] x^{n+r} = 0$$

or $$\sum_{n=0}^{\infty} a_n (n+r)(n+r-1+\gamma) x^{n+r-1} - \sum_{n=0}^{\infty} a_n (n+r+\alpha) \cdot (n+r+\beta) x^{n+r} = 0 \quad \text{--- (5)}$$

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Equating to zero, the co-efficient of the lowest power of x (i.e x^{r-1}), we get the indicial Eqn, as

$$a_0 r(r-1+\gamma) = 0 \quad , \quad a_0 \neq 0$$

This gives $r=0,\ 1-\gamma$. Hence two independent soln of Eqn (3) are -

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^n \quad \& \quad y_2(x) = \sum_{n=0}^{\infty} a_n x^{n+1-\gamma}$$

Equating to zero the co-efficient of general term x^{n+r}, we get—

$$a_{n+1}(n+r+1)(n+r+\gamma) - (n+r+\alpha)(n+r+\beta)a_n = 0 \quad \text{—(6)}$$

$$\Rightarrow \quad a_{n+1} = \frac{(n+r+\alpha)(n+r+\beta)}{(n+r+1)(n+r+\gamma)} a_n$$

the soln for $r=0$, reduces to

$$\boxed{a_{n+1} = \frac{(n+\alpha)(n+\beta)}{(n+1)(n+\gamma)} a_n}$$

For which

$$a_1 = \frac{\alpha \cdot \beta}{1 \cdot \gamma} a_0$$

$$a_2 = \frac{(\alpha+1)(\beta+1)}{2(1+\gamma)} a_1 = \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(1+\gamma)\, 1\cdot 2} a_0$$

$$= \frac{(\alpha)_2 (\beta)_2}{(\gamma)_2 \cdot 2!} a_0$$

and so general term is

$$\boxed{a_n = \frac{(\alpha)_n (\beta)_n}{(\gamma)_n \cdot n!} a_0} \quad \text{— (7)}$$

Substituting (7) in (4) for $\gamma=0$ and replacing a_0 by A, we, find one of

Sol^n of GHE (Gauss hypergeometric Eq^n) (3) near $x =$ as

$$\boxed{Y_1(x) = A \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n \, n!} x^n = A\,{}_2F_1(\alpha, \beta; \gamma; x)}$$

For sol^n at $r = 1-\gamma$, when $1-\gamma$ is neither zero nor an integer, the recurrence rel^n (6) reduces to

$$\boxed{a_{n+1} = \frac{(n+\alpha+1-\gamma)(n+\beta+1-\gamma)}{(n+1)(n+2-\gamma)} a_n}$$

From which it follows that –

$$a_1 = \frac{(\alpha+1-\gamma)(\beta+1-\gamma)}{1\cdot(2-\gamma)} a_0$$

$$a_2 = \frac{(2+\alpha-\gamma)(\beta+2-\gamma)}{2\cdot(3-\gamma)} a_1$$

$$= \frac{(\alpha+1-\gamma)(\alpha+2-\gamma)(\beta+1-\gamma)(\beta+2-\gamma)}{1\cdot 2\cdot(2-\gamma)(3-\gamma)} a_0$$

$$a_2 = \frac{(\alpha+1-\gamma)_2 (\beta+1-\gamma)_2}{n!\,(2-\gamma)_2} a_0$$

In general

$$a_n = \frac{(\alpha+1-\gamma)_n (\beta+1-\gamma)_n}{n!\,(2-\gamma)_n} a_0$$

Substituting a_n in (4) for $r = 1-\gamma$, we get

$a_0 = B$

$$Y_2(x) = B \sum_{n=0}^{\infty} \frac{(\alpha+1-\gamma)_n (\beta+1-\gamma)_n}{n!\,(2-\gamma)_n} x^{n+1-\gamma}$$

$$\boxed{Y_2(x) = B x^{1-\gamma} {}_2F_1(\alpha-\gamma+1, \beta-\gamma+1; 2-\gamma; x)}$$

Hence complete sol^n of GHE (3) near $x = 0$ when γ is not a positive integer is given by

$$\boxed{y = A\,{}_2F_1(\alpha, \beta; \gamma; x) + B x^{1-\gamma} {}_2F_1(\alpha-\gamma+1, \beta-\gamma+1; 2-\gamma; x)} \quad (8)$$

Case-II Sol^n in the nbd of $x=1$. Let us substitute $\xi = 1-x$ in eq^n (3), we get

$$\xi(1-\xi)\frac{d^2y}{d\xi^2} + \{\gamma-(1+\alpha+\beta)(1-\xi)\}\left(-\frac{dy}{d\xi}\right) - \alpha\beta y = 0$$

$$\Rightarrow \xi(1-\xi)\frac{d^2y}{d\xi^2} + \{(\alpha+\beta+1-\gamma)-(\alpha+\beta+1)\xi\}\frac{dy}{d\xi} - \alpha\beta y = 0 \quad (9)$$

On comparing (3) and (9), we find that eq^n (8) is same as (3) except that γ is replaced by $\alpha+\beta+1-\gamma$ and x by ξ. Hence sol^n (8) of (3) near $x=0$ will be valid for (9) near $\xi=0$ i.e. near $x=1$. Hence the required sol^n

$$y = A\ {}_2F_1(\alpha,\beta;\alpha+\beta-\gamma+1;1-x) + B\,(1-x)^{\gamma-\beta-\alpha}\ {}_2F_1(\gamma-\alpha,\gamma-\beta;\gamma-\alpha+1;1-x)$$

when $\gamma-\alpha-\beta$ is neither 0 nor an integer.

Eg. Show that functions
$$w = (1-z)^{-a}\ {}_2F_1\left(a, c-b; c; -\frac{z}{1-z}\right)$$
is sol^n of DE
$$x(1-x)\frac{d^2u}{dx^2} + [c-(a+c-b+1)x]\frac{du}{dx} - a(c-b)u = 0$$
where $x = -\dfrac{z}{1-z}$

Sol^n We know that $w = {}_2F_1(a,b;c;z)$ is sol^n of Eq^n
$$z(1-z)w'' + [c-(a+b+1)z]w' - abw = 0 \quad (1)$$

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put $w = (1-z)^{-a} u$ in (1), then

$w' = (1-z)^{-a} u' + a(1-z)^{-a-1} u$,

$w'' = (1-z)^{-a} u'' + 2a(1-z)^{-a-1} u' + a(a+1)(1-z)^{-a-2} u$

Hence the new eqn is

$z(1-z)\left[(1-z)^{-a}u'' + 2a(1-z)^{-a-1}u' + a(a+1)(1-z)^{-a-2}u\right] + \left[c-(a+b+1)z\right]\left[(1-z)^{-a}u' + a(1-z)^{-a-1}u\right] - ab(1-z)^{-a}u = 0$

cancel $(1-z)^{a}$ both side

$z(1-z)u'' + \left[2az + c - az - bz - z\right]u' + \left[z(a^2+a) + ac - (a^2+ab+a)z - ab(1-z)\right]u = 0$

$\Rightarrow z(1-z)u'' + \left[c + (a-b-1)z\right]u' + a(c-b)u = 0$ —— (2)

Now put $x = -\frac{z}{1-z}$ then $z = \frac{-x}{1-x}$

$\Rightarrow 1-z = \frac{1}{1-x}$, and

$\frac{dx}{dz} = \frac{-1}{(1-z)^2} = -(1-x)^2$

$\frac{d^2x}{dz^2} = \frac{-2}{(1-z)^3} = -2(1-x)^3$

Hence (2), may be written as ——

$\frac{d^2y}{dz^2} + \left[\frac{c}{z(1-z)} + \frac{a-b-1}{1-z}\right]\frac{dy}{dz} + \frac{a(c-b)}{z(1-z)}u = 0$

which then leads to the new Eqⁿ —

$$x(1-x)^4 \frac{d^2u}{dx^2} + \left[-2(1-x)^3 - (1-x)^2\left\{\frac{c(1-x)^2}{-x} + (a-b-1)(1-x)\right\}\right]\frac{du}{dx} - \frac{a(c-b)}{x}$$

$$\Rightarrow x(1-x)\frac{d^2u}{dx^2} + \left[-2x - \{-c(1-x) + (a-b-1)x\}\right]\frac{du}{dx} - a(c-b)u = 0$$

$$\Rightarrow x(1-x)\frac{d^2u}{dx^2} + \left[c - (a-b+c+1)x\right]\frac{du}{dx} - a(c-b)u = 0 \quad -(3)$$

This is hypergeometric Eqⁿ with parameters $\gamma = c$, $\alpha+\beta+1 = a-b+c+1$, $\alpha\beta = a(c-b)$

Hence $\alpha = a$, $\beta = c-b$, $\gamma = c$ Hence

$$u = {}_2F_1(a, c-b; c; x)$$

is one solⁿ of eqⁿ (3), so one solⁿ of eqⁿ (1) is —

$$w = (1-z)^{-a}\, {}_2F_1(a, c-b; c; -z/1-z)$$

(1) Show that

(a) $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cdot \cos x$ (b) $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$

Hence prove that

$$\left[J_{1/2}(x)\right]^2 + \left[J_{-1/2}(x)\right]^2 = \frac{2}{\pi x}$$

Sol^n:- We know that—

$$J_n(x) = \frac{x^n}{2^n \Gamma(n+1)}\left[1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2\cdot 4\cdot(2n+2)(2n+4)} + \cdots\right]$$

$$\left(J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(r+n+1)}\left(\frac{x}{2}\right)^{2r+n} = \sum_{r=0}^{\infty} \frac{(-1)^r x^{2r}}{\Gamma(n+1)\, 2^{2r}}\left(\frac{x}{2}\right)^{n} \right.$$

As $J_n(x) = \left(\frac{x}{2}\right)^n \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{2r}$

$$= \left(\frac{x}{2}\right)^n \left[\frac{1}{\Gamma(n+1)} - \frac{x^2}{2^2\cdot\Gamma(n+2)} + \frac{x^4}{2^4\cdot 2!\,\Gamma(n+3)} + \cdots + \frac{x^{2r}}{\cdots}\right] \quad \text{--- (1)}$$

$$= \left(\frac{x}{2}\right)^n \frac{1}{\Gamma(n+1)}\left[1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2\cdot 4\,(2n+4)(2n+2)}\right]$$

(a) putting $n = -1/2$, we have in (1)

$$J_{-1/2}(x) = \left(\frac{x}{2}\right)^{-1/2} \frac{1}{\Gamma(-\frac{1}{2}+1)}\left[1 - \frac{x^2}{2(-1+2)} + \frac{x^4}{2\cdot 4(-1+4)(-1+2)} + \cdots\right]$$

$\Gamma\frac{1}{2} = \sqrt{\pi}$

$$= \sqrt{\frac{2}{x}} \cdot \frac{1}{\sqrt{\pi}}\left[1 - \frac{x^2}{2} + \frac{x^4}{2\cdot 4\cdot 3} - \cdots\right]$$

$$= \sqrt{\frac{2}{\pi x}} \cdot \cos x$$

(b) put $n = 1/2$, in (1), we have—

$$J_{1/2}(x) = \left(\frac{x}{2}\right)^{1/2} \frac{1}{\Gamma(\frac{1}{2}+1)}\left[1 - \frac{x^2}{2(1+2)} + \frac{x^4}{2\cdot 4(1+4)(1+2)} + \cdots\right]$$

$$= \sqrt{\frac{x}{2}} \frac{1}{\Gamma\frac{3}{2}}\left[1 - \frac{x^2}{2\cdot 3} + \frac{x^4}{2\cdot 4\cdot 3\cdot 5} - \cdots\right]$$

Multiply & divide by x, we get

$$J_{1/2}(x) = \frac{\sqrt{x}\,2}{\sqrt{2}\sqrt{\pi x}}\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right]$$

$$= \sqrt{\frac{2}{\pi x}}\sin x$$

On squaring & adding, we get,

$$\left[J_{1/2}(x)\right]^2 + \left[J_{-1/2}(x)\right]^2 = \frac{2}{\pi x} \quad \text{Proved}$$

Eg. 2 P.T,

$$\sqrt{\frac{\pi x}{2}}\, J_{3/2}(x) = \frac{\sin x}{x} - \cos x$$

Sol^n We know that

$$J_n(x) = \frac{x^n}{2^n \Gamma(n+1)}\left\{1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2.4(2n+2)(2n+4)} - \cdots\right]$$

now substitute $n = 3/2$, we get—

$$J_{3/2}(x) = \frac{x^{3/2}}{2^{3/2}\Gamma(3/2+1)}\left\{1 - \frac{x^2}{2(3+2)} + \frac{x^4}{2.4(3+2)(3+4)} + \cdots\right\}$$

$$J_{3/2}(x) = \left(\frac{x}{2}\right)^{3/2}\frac{1}{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}\left\{1 - \frac{x^2}{2.5} + \frac{x^4}{2.4.5.7} - \frac{x^6}{2.4.6.5.7.9}\right\}$$

$$\sqrt{\frac{\pi x}{2}}\, J_{3/2}(x) = \left(\frac{x}{2}\right)^{1/2}\cancel{\sqrt{\pi}}\left(\frac{x}{2}\right)^{3/2}\frac{2.2}{3\cancel{\sqrt{\pi}}}\left\{1 - \frac{x^2}{2.5} + \frac{x^4}{2.4.5.7} - \right\}$$

$$= \left(\frac{x}{\cancel{2}}\right)^2 \frac{4}{\cancel{3}}\left\{\frac{1}{3} - \frac{x^2}{2.3.5} + \frac{x^4}{2.3.4.5.7} - \frac{x^6}{2.4.6.5.7.9} + \right\}$$

$$= \left\{\frac{x^2}{2!} - \frac{x^4}{3!} + \frac{x^6}{4!} \ldots - \frac{x^4}{5!} + \frac{x^6}{6!} - \frac{x^6}{7!}\right\}$$

$$= \frac{1}{x}\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right] - \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right]$$

$$= \frac{1}{x}.\sin x - \cos x$$

Eg. 3 Prove that

$$J_{n-1} = \frac{2}{x}\left[n J_n - (n+2) J_{n+2} + (n+4) J_{n+4} - \cdots\right]$$

Hence deduce that

$$\frac{1}{2} x J_n = (n+1) J_{n+1} - (n+3) J_{n+3} + (n+5) J_{n+5} + \cdots$$

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solⁿ :-

From the recurrence relation, we have

$$\frac{2n}{x} J_n = J_{n-1} + J_{n+1} \qquad (1)$$

now replace n by $n+2$ in (1), we get

$$\frac{2(n+2)}{x} J_{n+2} = J_{n+1} + J_{n+3}$$

Changing sign, we have

$$-J_{n+1} - J_{n+3} = -\frac{2}{x}(n+2) J_{n+2} \quad (2)$$

Replacing n by $(n+4)$ in (1), we obtain

$$+J_{n+3} + J_{n+5} = \frac{2}{x}(n+4) J_{n+4} \quad (3)$$

Again replacing n by $n+6$ in (1) & then change the sign, we have

$$-J_{n+5} - J_{n+7} = -\frac{2}{x}(n+6) J_{n+6} \quad (4)$$

and so on

Now adding (1), (2), (3), (4)... etc we have

$$J_{n-1} = \frac{2}{x}\left[nJ_n - (n+2) J_{n+2} + (n+4) J_{n+4} - \cdots \right]$$

Replacing n by $(n+1)$ in above mentioned results, we get

$$\frac{x}{2} J_n = (n+1) J_{n+1} - (n+3) J_{n+3} + (n+5) J_{n+5} - - -$$

Eg. 4

P.T $\frac{d}{dx}\left(J_n^2 + J_{n+1}^2\right) = 2\left(\frac{n}{x} J_n^2 - \frac{n+1}{x} J_{n+1}^2\right)$

solⁿ: We know the recurrence relations

$$J_n' = \frac{n}{x} J_n - J_{n+1} \quad (i)$$

$$J_n' = -\frac{n}{x} J_n + J_{n-1} \quad (ii)$$

Replacing n by $(n+1)$ in (ii), we obtain

$$J_{n+1}' = -\frac{n+1}{x} J_{n+1} + J_n \quad \text{--- (iii)}$$

$$\frac{d}{dx}\left(J_n^2 + J_{n+1}^2\right) = 2J_n J_n' + 2J_{n+1} J_{n+1}' \quad \text{--- (iv)}$$

Now putting value from (i) & (iii) in (iv), we get

$$\frac{d}{dx}\left(J_n^2 + J_{n+1}^2\right) = 2J_n\left[\frac{n}{x}J_n - J_{n+1}\right] - 2J_{n+1}\left[-\frac{n+1}{x}J_{n+1} + J_n\right]$$

$$= \frac{2n}{x}J_n^2 - \cancel{2J_nJ_{n+1}} - \frac{2(n+1)}{x}J_{n+1}^2 - \cancel{2J_nJ_{n+1}}$$

$$= \frac{2n}{x}J_n^2 - \frac{2(n+1)}{x}J_{n+1}^2$$

$$= 2\left(\frac{n}{x}J_n^2 - \frac{(n+1)}{x}J_{n+1}^2\right)$$

Problem: 5. Prove that $\frac{d}{dx}\left(\frac{J_{-n}}{J_n}\right) = -\frac{2\sin n\pi}{\pi x J_n^2}$

or $\left[J_n J_{-n}' - J_n' J_{-n}\right] = -\frac{2\sin n\pi}{\pi x}$

Pf: Since J_n and J_{-n} are sol^{n}s of the Bessel DE's, hence

$$J_n'' + \frac{1}{x}J_n' + \left(1 - \frac{n^2}{x^2}\right)J_n = 0 \quad \text{--- (i)}$$

$$\& \quad J_{-n}'' + \frac{1}{x}J_{-n}' + \left(1 - n^2/x^2\right)J_{-n} = 0 \quad \text{--- (ii)}$$

Now (i) $\times J_{-n}$ & (ii) $\times J_n$ and then substracting, we get —

$$J_{-n}J_n'' - J_nJ_{-n}'' + \frac{1}{x}\left[J_{-n}J_n' - J_nJ_{-n}'\right] = 0$$

$$\therefore \frac{J_{-n}J_n'' - J_nJ_{-n}''}{J_{-n}J_n' - J_nJ_{-n}'} = -\frac{1}{x}$$

which on integration w.r.t. x, yields

$$\log\left(J_{-n}J_n' - J_nJ_{-n}'\right) = -\log x + \log c,$$

where c = arb. constants. Thus

$$J_{-n}J_n' - J_nJ_{-n}' = c/x \quad \text{— (iii)}$$

Here, in order to obtain the value of c, we write down series expansions for the terms in left hand side —

$$\left[\frac{1}{\Gamma(1-n)}\left(\frac{x}{2}\right)^{-n} - \frac{1}{\Gamma(2-n)}\left(\frac{x}{2}\right)^{2-n} + \cdots\right]\left[\frac{n}{\Gamma(1+n)}\left(\frac{x}{2}\right)^{n-1} - \frac{n+2}{\Gamma(2+n)}\cdot\left(\frac{x}{2}\right)^{n+1}\right] - \left[\frac{1}{\Gamma(1+n)}\cdot\left(\frac{x}{2}\right)^{n} - \frac{1}{\Gamma(2+n)}\left(\frac{x}{2}\right)^{n+2} + \cdots\right]$$

$$\left[\frac{-n}{\Gamma(1-n)}\left(\frac{x}{2}\right)^{-n-1} - \frac{(2-n)}{\Gamma(2-n)}\left(\frac{x}{2}\right)^{-n+1} + \cdots\right] = \frac{c}{x}$$

and compare the coefficients of $\frac{1}{x}$ from both sides we get —

$$\frac{2}{\Gamma(1-n)\Gamma(1+n)}\left[n-(-n)\right] = c$$

By the formula — $\Gamma x\, \Gamma 1-x = \frac{\pi}{\sin \pi x}$

$$\Rightarrow c = \frac{4n}{n\Gamma n\cdot\Gamma 1-n} = \frac{4\cdot \sin \pi n}{2\pi} = \frac{2\sin \pi n}{\pi}$$

∴ By (iii)

$$J_{-n}J_n' - J_nJ_{-n}' = \frac{2}{\pi}\sin \pi n \cdot \frac{1}{x}$$

$$\Rightarrow \boxed{J_nJ_{-n}' - J_n'J_{-n} = -\frac{2\sin \pi n}{\pi x}}$$

Problem (6) By collecting powers of x in the summation on the left show that —

$$\sum_{n=0}^{\infty} J_{2n+1}(x) = \frac{1}{2}\int_0^x J_0(y)\,dy$$

Sol^n -

$$\sum_{k=0}^{\infty} J_{2k+1}(x) = \sum_{k,n=0}^{\infty} \frac{(-1)^n}{n!\,(n+2k+1)!}\left(\frac{x}{2}\right)^{2n+2k+1}$$

$n \to n-k$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(-1)^{n-k}}{(n-k)!\,(n+k+1)!}\left(\frac{x}{2}\right)^{2n+1} \quad (1)$$

If $\sum_{i=0}^{\infty} a_i$ and $\sum_{j=0}^{\infty} b_j$ be two infinite series with terms then their product (Cauchy product) is defined by, —

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$$\sum_{i=0}^{\infty} a_i \sum_{j=0}^{\infty} b_j = \sum_{k=0}^{\infty} C_k \quad \text{with} \quad C_k = \sum_{l=0}^{k} a_l b_{k-l}$$

Note that $(-n)_k = (-n)(-n+1)(-n+2)\cdots(-n+k-1)$
$= (-1)^k \, n.(n-1)(n-2)\cdots(n-(k-1))$
$= (-1)^k . \dfrac{n!}{(n-k)!}$ if $k \le n$ — (2)

Using (2), in (1), we get—

$$\sum_{k=0}^{\infty} J_{2k+1}(x) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-n)_k \, (-1)^n}{(n-k)! \, n!} \frac{\left(\frac{x}{2}\right)^{2n+1}}{(n+1)!}$$

$$= \sum_{n=0}^{\infty} {}_2F_1\left[-n, 1; n+2; 1\right] \frac{(-1)^n \, (x/2)^{2n+1}}{n! \, (n+1)!}$$

$$= \sum_{n=0}^{\infty} \frac{\Gamma(n+2) \, \Gamma(2n+1) \, (-1)^n}{\Gamma(2n+2)\,\Gamma(n+1) \; n! \, (n+1)!} \left(x/2\right)^{2n+1}$$

(We have used formulas
$$\sum_{k=0}^{n} \frac{(-n)_k}{(n+1)_k} = {}_2F_1\left[-n, 1; n+2; 1\right]$$
and ${}_2F_1[\alpha, \beta; \gamma;] = \dfrac{\Gamma\gamma \; \Gamma\gamma-\alpha-\beta}{\Gamma\gamma-\alpha \; \Gamma\gamma-\beta}$)

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \left(x/2\right)^{2n+1}}{(2n+1)\, n! \, n!}$$

$$\frac{\Gamma(n+2) \, \Gamma 2n+1}{\Gamma 2n+2 \; \Gamma n+1 \; n! \, (n+1)!} = \frac{(n+1)}{(2n+1) \, n! \, (n+1)!} = \frac{1}{(2n+1)\, (n!)^2}$$

$$= \frac{1}{2} \int_0^x \sum_{n=0}^{\infty} \frac{(-1)^n \left(y/2\right)^{2n}}{n! \, n!} dy$$

$$= \frac{1}{2} \int_0^x J_0(y)\, dy$$

Problem (7) Prove that—

$$J_n(z) = \frac{2 (z/2)^n}{\sqrt{\pi} \, \Gamma(n+1/2)} \int_0^{\pi/2} \sin^{2n} \cos(z \cos\phi) \, d\phi$$

for $(n) > -1/2$

Solⁿ. We know that

$$J_n(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z)^{2k+n}}{2^{2k+n}\, k!\, \Gamma(k+n+1)}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k (1/2)_k}{2^n (2k)!\, \Gamma(k+n+1)} \cdot z^{2k+n}$$

$\left[\because \text{By duplication formula} \right.$

$$\Gamma 2x = \frac{\Gamma x\, \Gamma(x+1/2)\, 2^{2x-1}}{\sqrt{\pi}} \Rightarrow 2^{2x} = \frac{\Gamma 2x\, \sqrt{\pi} \cdot 2}{\Gamma x\, \Gamma(x+1/2)}$$

$$\Rightarrow 2^{2x} = \frac{\Gamma 2x \cdot 2}{\Gamma x\, (1/2)_x} \cdot \frac{2x}{2x} = \frac{(2x)!}{(x)!\, (1/2)_x} \left. \right]$$

$$\therefore \Rightarrow \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+n}}{2^{2k+n}\, k!\, \Gamma(k+n+1)} = \sum_{k=0}^{\infty} \frac{(-1)^k k!\, (1/2)_k\, z^{2k+n}}{2^n (2k)!\, (k!)\, \Gamma(k+n+1)}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k (1/2)_k\, z^{2k+n}}{2^n (2k)!\, \Gamma(k+n+1)}$$

Also,

$$\frac{(1/2)_k}{\Gamma(k+n+1)} = \frac{\Gamma(k+1/2)\, \Gamma(n+1/2)}{\Gamma 1/2\, \Gamma(k+n+1)\, \Gamma(n+1/2)} = \frac{B(k+1/2, n+1/2)}{\Gamma 1/2\, \Gamma(n+1/2)}$$

$$= \frac{2}{\sqrt{\pi}\, \Gamma(n+1/2)} \int_0^{\pi/2} \cos^{2k}\phi\, \sin^{2n}\phi\, d\phi$$

$$\left(\because \frac{\Gamma x\, \Gamma y}{2\Gamma(x+y)} = \int_0^{\pi/2} \sin^{2x-1}\phi \cos^{2y-1}\phi\, d\phi \right)$$

Therefore

$$J_n(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+n}}{2^n (2k)!} \cdot \frac{2}{\sqrt{\pi}\, \Gamma(n+1/2)} \int_0^{\pi/2} \cos^{2k}\phi\, \sin^{2n}\phi\, d\phi$$

$$= \frac{2 (z/2)^n}{\sqrt{\pi}\, \Gamma(n+1/2)} \int_0^{\pi/2} \sin^{2n}\phi \left(\sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!} \cos^{2k}\phi \right) d\phi$$

$$= \frac{2}{\sqrt{\pi}} \frac{(z/2)^n}{\Gamma(n+1/2)} \int_0^{\pi/2} \sin^{2n}\phi \cdot \cos(z\cos\phi)\, d\phi$$

Eg. ST. $|J_n(x)| \leq 1$ for real x and integer

Solⁿ. Bessel integral is

$$J_n(x) = \frac{1}{\pi} \int_0^{\pi} \cos(n\theta - x\sin\theta)\, d\theta$$

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$\because$ for all real x and n, $|\cos(n\theta - x\sin\theta)| \le 1$

Hence

$$|J_n(x)| \le \frac{1}{\pi}\int_0^{\pi} d\theta = \frac{1}{\pi}[\pi - 0] = 1$$

Problem [9] ST (a) $[1+(-1)^n]J_n(z) = \frac{2}{\pi}\int_0^{\pi} \cos(n\theta)\cos(z\sin\theta)\,d\theta$

(b) $[1-(-1)^n]J_n(z) = \frac{2}{\pi}\int_0^{\pi} \sin(n\theta)\sin(z\sin\theta)\,d\theta$

Also for integral k,

$$J_{2k}(z) = \frac{1}{\pi}\int_0^{\pi} \cos(2k\theta)\cos(z\sin\theta)\,d\theta$$

$$J_{2k+1}(z) = \frac{1}{\pi}\int_0^{\pi} \sin(2k+1)\theta\cdot\sin(z\sin\theta)\,d\theta$$

Sol^n We know that

$$J_n(z) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - z\sin\theta)\,d\theta$$

then

$$J_n(z) = \frac{1}{\pi}\int_0^{\pi} \cos n\theta\cdot\cos(z\sin\theta)\,d\theta + \frac{1}{\pi}\int_0^{\pi} \sin n\theta\,\sin(z\sin\theta)\,d\theta \quad \text{——(a)}$$

Now change n to $-n$, we get

$$(-1)^n J_n(z) = \frac{1}{\pi}\int_0^{\pi} \cos n\theta\cos(z\sin\theta) - \frac{1}{\pi}\int_0^{\pi} \sin n\theta\,\sin(z\sin\theta)\,d\theta \quad \text{——(b)}$$

on adding (a) & (b), we get

$$[1+(-1)^n]J_n(z) = \frac{2}{\pi}\int_0^{\pi} \cos n\theta\cos(z\sin\theta)\,d\theta$$

Proved (a)

Similarly (a) − (b) gives Result [b].

Further with $n=2k$ & with $n=2k+1$ in eqn (a) & (b), we obtain another results.

(10) ST (a) $J_{5/2}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{3-x^2}{x^2}\sin x - \frac{3}{x}\cos x\right)$

(b) $J_4(x) = \left(\frac{48}{x^3} - \frac{8}{x}\right) J_1(x) + \left(1 - \frac{24}{x^2}\right) J_0(x).$

(a) We know that

$$J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x) \quad \text{—— (1)}$$

putting $n = 3/2$, we get –

$$J_{5/2}(x) = \frac{3}{x} J_{3/2}(x) - J_{1/2}(x)$$

and on putting $n = 1/2$, we get —

$$J_{3/2}(x) = \frac{1}{x} J_{1/2}(x) - J_{-1/2}(x)$$

$$\therefore J_{5/2}(x) = \frac{3}{x}\left[\frac{1}{x} J_{1/2}(x) - J_{-1/2}(x)\right] - J_{1/2}(x)$$

$$= \left(\frac{3}{x^2} - 1\right) J_{1/2}(x) - \frac{3}{x} J_{-1/2}(x)$$

$$= \left(\frac{3-x^2}{x^2}\right)\sqrt{\frac{2}{\pi x}} \sin x - \frac{3}{x}\sqrt{\frac{2}{\pi x}} \cos x$$

$$= \sqrt{\frac{2}{\pi x}}\left[\left(\frac{3-x^2}{x^2}\right)\sin x - \frac{3}{x}\cos x\right]$$

(b) putting $n = 1, 2$ in (1), we get —

$$J_2(x) = \frac{2}{x} J_1(x) - J_0(x)$$

& $$J_3(x) = \frac{4}{x} J_2(x) - J_1(x)$$

$$J_4(x) = \frac{6}{x} J_3(x) - J_2(x) \quad (\text{By putting } n = 3)$$

$$= \frac{6}{x}\left[\frac{4}{x} J_2(x) - J_1(x)\right] - \left[\frac{2}{x} J_1(x) - J_0(x)\right]$$

$$= \frac{24}{x^2} J_2(x) - \left(\frac{6}{x} + \frac{2}{x}\right) J_1(x) + J_0(x)$$

$$= \frac{24}{x^2}\left[\frac{2}{x} J_1(x) - J_0(x)\right] - \frac{8}{x} J_1(x) + J_0(x)$$

$$= \left(\frac{48}{x^3} - \frac{8}{x}\right) J_1(x) + \left(1 - \frac{24}{x^2}\right) J_0(x) \quad \text{Pr}$$

Problem (11) PT

$$4J_n''(x) = J_{n-2}(x) - 2J_n(x) + J_{n+2}(x)$$

We know that

$$2J_n'(x) = J_{n-1}(x) - J_{n+1}(x) \quad \text{--- (1)}$$

Diffⁿ (1) w.r.t x, we get —

$$2J_n''(x) = J_{n-1}'(x) - J_{n+1}'(x) \quad \text{--- (ii)}$$

Changing n to (n−1) & (n+1) we get — (in (1))

$$2J_{n-1}'(x) = J_{n-2}(x) - J_n(x)$$

$$\& \; 2J_{n+1}'(x) = J_n(x) - J_{n+2}(x)$$

Now putting above eqns in eqn (2) —

$$2J_n''(x) = \frac{1}{2}\left[J_{n-2}(x) - J_n(x)\right] - \frac{1}{2}\left[J_n(x) - J_{n+2}(x)\right]$$

$$= \frac{1}{2}\left[\frac{1}{2}J_{n-2}(x) - 2J_n(x) + \frac{1}{2}J_{n+2}(x)\right]$$

$$\Rightarrow 4J_n''(x) = J_{n-2}(x) - 2J_n(x) + J_{n+2}(x)$$

(12) ST

$$4J_0'''(x) + 3J_0'(x) + J_3(x) = 0$$

$$\frac{d}{dx}x^{-n}J_n(x) = -x^{-n}J_{n+1}(x)$$

Solⁿ: ∵ We know that $J_0'(x) = -J_1(x)$

Diffⁿ w.r.t x, we get

$$J_0''(x) = -J_1'(x) = -\frac{1}{2}\left[J_0(x) - J_2(x)\right]$$

Diffⁿ again w.r.t. x, we get —

$$J_0'''(x) = -\frac{1}{2}\left[J_0'(x) - J_2'(x)\right]$$

$$\left(\because 2J_2'(x) = J_{n-1}(x) - J_{n+1}(x) \text{ with } n=1\right.$$

$$= -\frac{1}{2}J_0'(x) + \frac{1}{2}\left[\frac{1}{2}J_1(x) - \frac{1}{2}J_3(x)\right]$$

$$J_0'''(x) = -\frac{1}{4}\left[2J_0'(x) - J_1(x) + J_3(x)\right]$$

$$\Rightarrow J_0'''(x) = -\frac{1}{4}\left[2J_0'(x) + J_0'(x) + J_3(x)\right]$$

$\Rightarrow 4J_0'''(x) + 3J_0'(x) + J_3(x) = 0$

Prove that -

(13) (i) $\int J_3(x)\,dx = -J_2(x) - \frac{2}{x} J_2(x)$

(ii) $\int x J_0^2(x)\,dx = \frac{1}{2} x^2 \left[J_0^2(x) + J_1^2(x) \right] + C$

where C is a constant.

Pf:
(i) We know that

$$\frac{d}{dx}\left(x^{-n} J_n(x) \right) = -x^{-n} J_{n+1}(x)$$

$$\therefore \int x^{-n} J_{n+1}(x)\,dx = -x^{-n} J_n(x) \quad \text{—(1)}$$

Now

$$\int J_3(x)\,dx = \int x^2 \left[x^{-2} J_3(x) \right] dx$$

$$= x^2 \left[-x^{-2} J_2(x) \right] - \int 2x \left[-x^{-2} J_2(x) \right] dx$$

(By putting $n=2$ in (1))

$$= -J_2(x) + 2\int x^{-1} J_2(x)\,dx$$

$$= -J_2(x) + 2\left[-x^{-1} J_1(x) \right] \quad \left(\text{By putting } n=1 \text{ in (1)}\right)$$

$$= -J_2(x) - \frac{2}{x} J_1(x)$$

(ii)
$$\int x J_0^2(x)\,dx = \int J_0^2(x)\, x\,dx$$

$$= J_0^2(x) \cdot \frac{x^2}{2} - \int 2 J_0 J_0'(x) \cdot \frac{x^2}{2} + C$$

$$= \frac{x^2}{2} J_0^2(x) - \int x^2 J_0(x) \left(-J_1(x) \right) dx + C$$

$$\left[J_0'(x) = -J_1(x) \right]$$

$$= \frac{1}{2} x^2 J_0^2(x) + \int x J_0(x)\, x J_1(x)\,dx + C$$

$$= \frac{1}{2} x^2 J_0^2(x) + \int x J_1(x) \frac{d}{dx}\left(x J_1(x) \right) dx + C$$

$$= \frac{1}{2} x^2 J_0^2(x) + \frac{1}{2}\left[x J_1(x) \right]^2$$

$$= \frac{x^2}{2} J_0^2(x) + \frac{1}{2}\left[x J_1(x) \right]^2$$

$$= \frac{x^2}{2}\left[J_0^2(x) + J_1^2(x) \right] + C$$

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Problem (14) Jacobi Series (Show that)

ⓐ $\cos(x\sin\theta) = J_0(x) + 2[J_2(x)\cos 2\theta + J_4(x).\cos 4\theta + \cdots]$

ⓑ $\sin(x\sin\theta) = 2(J_1(x)\sin\theta + J_3(x)\sin 3\theta + \cdots)$

Sol^n We know that

$$\exp\left\{\frac{x}{2}\left(t - \frac{1}{t}\right)\right\} = \sum_{n=-\infty}^{\infty} t^n J_n(x)$$

Hence, $\exp\left\{\frac{x}{2}\left(t-\frac{1}{t}\right)\right\} = J_0(x) + tJ_1(x) + t^2 J_2(x) + t^3 J_3(x) + \cdots + t^{-1}J_{-1}(x) + t^{-2}J_{-2}(x) + t^{-3}J_{-3}(x) + \cdots$

$\because J_{-n}(x) = (-1)^n J_n(x)$

$= J_0(x) + tJ_1(x) + t^2 J_2(x) + t^3 J_3(x) + \cdots - t^{-1}J_1(x) + t^{-2}J_2(x) - t^{-3}J_3(x) + \cdots$

$= J_0(x) + (t - 1/t)J_1(x) + (t^2 + 1/t^2)J_2(x) + (t^3 - 1/t^3)J_3(x) \longrightarrow (1)$

put $t = \cos\theta + i\sin\theta$, then

$t^n + \frac{1}{t^n} = 2\cos n\theta$ & $t^n - \frac{1}{t^n} = 2i\sin n\theta$

putting these values in (1), we get

$e^{ix\sin\theta} = J_0(x) + 2i\sin\theta J_1(x) + 2\cos 2\theta J_2(x) + 3i\sin 3\theta J_3(x) + \cdots$

equating real & imaginary parts of (2), we get

$\cos(x\sin\theta) = J_0(x) + 2[J_2(x)\cos 2\theta + J_4(x)\cos 4\theta + \cdots]$

$\sin(x\sin\theta) = 2[J_1(x)\sin\theta + J_3(x)\sin 3\theta + \cdots]$

(15) Prove that —

ⓐ $\cos x = J_0 - 2J_2 + 2J_4 -$

ⓑ $\sin x = 2J_1 - 2J_3 + 2J_5 - \cdots$

since as we have Jacobi Series is given as above — put $\theta = \pi/2$ in above Jacobi series we have

$\cos(x) = J_0(x) + 2J_2(x) + 2J_4(x) - \cdots$

$\sin(x) = 2J_1(x) + 2J_3(x) + 2J_5(x) - \cdots$

(16) prove that

(i) $\cos(x\cos\theta) = J_0 - 2J_2\cos 2\theta + 2J_4\cos\theta - \cdots$

(ii) $\sin(x\cos\theta) = 2J_1\cos\theta - 2J_3\cos 3\theta + 2J_5\cos 5\theta - \cdots$

Now replace θ by $\pi/2 - \theta$ in Jacobi series in eq. (14).

$\cos(x\cos\theta) = J_0(x) - 2J_2\cos 2\theta + 2J_4\cos\theta - \cdots$

$\sin(x\cos\theta) = 2J_1\cos\theta - 2J_3\cos 3\theta + 2J_5\cos 5\theta - \cdots$

(17) If $\alpha_1, \alpha_2, \ldots \alpha_n$ are positive roots of $J_0(x) = 0$, then show that —

$$\frac{1}{2} = \sum_{n=1}^{\infty} \frac{J_0(\alpha_n x)}{\alpha_n J_1(\alpha_n)} \quad \text{—— (7)}$$

Sol^n We know that if

$$f(x) = \sum_{i=1}^{\infty} C_i J_n(\alpha_i x), \quad \text{—— (1)}$$

then

$$C_i = \frac{2}{a^2 J_{n+1}^2(\alpha_i a)} \int_0^a x f(x) J_n(\alpha_i x)\, dx$$

Taking $f(x) = 1$, $a = 1$ and $n = 0$ in C_i we get

$$C_i = \frac{2}{J_1^2(\alpha_i)} \int_0^1 x J_0(\alpha_i x)\, dx = \frac{2}{J_1^2(\alpha_i)} \left[\frac{x J_1(\alpha_i x)}{\alpha_i}\right]_0^1$$

$$= \frac{2}{\alpha_i J_1(\alpha_i)}$$

from (1), we have

$$\frac{1}{2} = \sum_{n=1}^{\infty} \frac{J_0(\alpha_n x)}{\alpha_n J_1(\alpha_n)}$$

(18) If $f(x) = x^2$ $(0 < x < a)$, where $(\lambda_n a)$ are positive roots of $J_2(x) = 0$, then show that —

$$x^2 = 2a^2 \sum_{n=1}^{\infty} \frac{J_2(\lambda_n x)}{a\lambda_n J_3(\lambda_n a)}$$

Sol^n:- Let the series representing $f(x) = x^2$ is given by

$$x^2 = \sum_{n=1}^{\infty} C_n J_2(\lambda_n x)$$

Multiply both sides by $x J_2(\lambda_n x)$ and integrating w.r.t. x, between 0 to a, we get —

$$\int_0^a x^3 J_2(\lambda_n x)\,dx = C_n \int_0^a x J_2^2(\lambda_n x)\,dx$$

or $$\left[\frac{x^3 J_2(\lambda_n x)}{\lambda_n}\right]_0^a = C_n \frac{a^2}{2} \cdot J_3^2(\lambda_n a)$$

$$\therefore C_n = \frac{2a^2}{a\lambda_n} \cdot \frac{1}{J_n(\lambda_n a)}$$

Hence,
$$x^2 = 2a^2 \sum_{n=1}^{\infty} \frac{J_2(\lambda_n x)}{a \lambda_n J_3(\lambda_n a)}$$

(19) Show that
$$\frac{d}{dx}\left[x J_n(x) J_{n+1}(x)\right] = x\left[J_n^2(x) - J_{n+1}^2(x)\right]$$

Sol $$\frac{d}{dx}\left[x J_n(x) J_{n+1}(x)\right] = \frac{d}{dx}\left[x^{-n} J_n(x) \, (x^{n+1} J_{n+1}(x))\right]$$

$$= \{x^{-n} J_n(x)\} \frac{d}{dx}\{x^{n+1} J_{n+1}(x)\} + x^{n+1} J_{n+1}(x) \frac{d}{dx}\{x^{-n} J_n(x)\} \quad \text{—(1)}$$

Using recurrence relations
$$\frac{d}{dx}\{x^n J_n(x)\} = x^n J_{n-1}(x)$$

and $$\frac{d}{dx}\{x^{-n} J_n(x)\} = -x^{-n} J_{n+1}(x)$$

in (1), We get—

LHS $x^{-n} J_n(x) \cdot x^{n+1} J_n(x) - x^{n+1} J_{n+1}(x)\, x^{-n} J_{n+1}(x)$

$$= x\left(J_n^2(x) - J_{n+1}^2(x)\right)$$

(20) $\int x^{-1} J_4(x)\,dx$. Evaluate it using the recurrence relation —

$$\frac{d}{dx}\{x^{-n}J_n(x)\} = -x^{-n}J_{n+1}(x) \quad \text{—— (1)}$$

for $n=3$, we have –

$$\frac{d}{dx}\{x^{-3}J_3(x)\} = -x^{-3}J_4(x)$$

$$\therefore \int x^{-3}J_4(x)\,dx = \cancel{\int x^{2}\{x^{-3}J_4(x)\}}\,dx$$

$$= -x^{-3}J_3(x)$$

$$\therefore \int x^{-1}J_4(x)\,dx = \int x^2\{x^{-3}J_4(x)\}\,dx$$

$$= x^2\{-x^{-3}J_3(x)\} - \int 2x\{-x^{-3}J_3(x)\}\,dx$$

$$= -x^{-1}J_3(x) + 2\int x^{-2}J_3(x)\,dx$$

Hlp again from (1) gives

$$\int x^{-1}J_4(x)\,dx = -x^{-1}J_3(x) - 2x^{-2}J_2(x)$$

(21) Express $\int x^{-2}J_2(x)\,dx$ in terms of Bessel f^n.

Sol^n–

$$\int x^{-2}J_2(x)\,dx = \int x^{-4}\{x^2J_2(x)\}\,dx$$

$$= x^2J_2(x)\frac{x^{-3}}{(-3)} - \int \frac{d}{dx}\left(x^2J_2(x)\right)\cdot\frac{x^{-3}}{(-3)}\,dx$$

$$= \frac{-1}{3x}J_2(x) + \frac{1}{3}\int x^{-3}\{x^2J_1(x)\}\,dx$$

$$\left(\text{by } \frac{d}{dx}\{x^nJ_n(x)\} = x^nJ_{n-1}(x)\right)$$

$$= \frac{-1}{3x}J_2(x) + \frac{1}{3}\int x^{-1}J_1(x)\,dx$$

Next

$$\int x^{-1}J_1(x)\,dx = \int x^{-2}(xJ_1(x))\,dx$$

$$= -xJ_1(x)\cdot\frac{1}{x} - \int\frac{x^{-1}}{(-1)}\cdot\frac{d}{dx}\left(xJ_1(x)\right)dx$$

$$= -J_1(x) + \int x^{-1}xJ_0(x)\,dx$$

$$= -J_1(x) + \int J_0(x)\,dx$$

$$\therefore \int x^{-2}J_2(x)\,dx = -\frac{1}{3x}J_2(x) - \frac{1}{3}J_1(x) + \frac{1}{3}\int J_0(x)\,dx$$

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[Special function]

[END]

P. Kalika Notes

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1. **Free Maths Study Materials** (https://pkalika.in/2020/04/06/free-maths-study-materials/)

2. **BSc/MSc Free Study Materials** (https://pkalika.in/2019/10/14/study-material/)

3. **MSc Entrance Exam Que. Paper:** (https://pkalika.in/2020/04/03/msc-entrance-exam-paper/)
   [JAM(MA), JAM(MS), BHU, CUCET, ...etc]

4. **PhD Entrance Exam Que. Paper:** (https://pkalika.in/que-papers-collection/)
   [CSIR-NET, GATE(MA), BHU, CUCET,IIT, NBHM, ...etc]

5. **CSIR-NET Maths Que. Paper:** (https://pkalika.in/2020/03/30/csir-net-previous-yr-papers/)
   [Upto 2019 Dec]

6. **Practice Que. Paper:** (https://pkalika.in/2019/02/10/practice-set-for-net-gate-set-jam/)
   [Topic-wise/Subject-wise]

7. **CSIR-NET Mathematics Details Syllabus** (https://wp.me/p6gYUB-Fc)

8. **CSIR-NET, GATE, PhD Exams, ...etc Study Materials & Solutions**
   https://pkalika.in/kalika-notes-centre/

9. **CSIR-NET, GATE, ... Solutions** (https://wp.me/P6gYUB-1eP)

10. **Topic-wise Video Lectures (Free Crash Course)**
    https://www.youtube.com/pkalika/playlists