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Partial Differential Equation

(Handwritten Study Material for MSc, GATE, NET...etc)

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YouTube Playlists Link:

PDE (Partial Differential Equations):
https://www.youtube.com/playlist?list=PLDu0JgProGz4SXYf6nxRvIOVOp5aD6596

ODE (Ordinary Differential equations):
https://www.youtube.com/playlist?list=PLDu0JgProGz6plNOouv2oMyGMlq5RweT1

Partial Differential Equation

Definition:- A differential equation which involves partial derivatives of one or more dependent variable with respect to one or more independent variables

Example:- Consider a first order partial differential equation (PDE) as

$$\frac{\partial u}{\partial x} = x^2+y^2$$

in which u is the dependent variable and x and y are independent variables.

Derivation of PDE

① By elimination of arbitary constant :-

Consider the equation,

$$f(x,y,z,a,b)=0 \quad \text{———} (1)$$

where $z=z(x,y)$ and a,b are constants.

Differentiate eq(1) partially w.r.t x and y respectively, we get

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial x}=0 \quad \text{———} (2), \qquad \text{and} \qquad \frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial y}=0 \quad \text{———} (3)$$

eliminating a and b from (1), (2) and (3), we get required pde of the form

$$f(x,y,z,p,q)=0$$

Note:- $p=\frac{\partial z}{\partial x}$, $q=\frac{\partial z}{\partial y}$, $r=\frac{\partial^2 z}{\partial x^2}$, $s=\frac{\partial^2 z}{\partial x\partial y}$, $t=\frac{\partial^2 z}{\partial y^2}$

Q Eliminate arbitrary constants to form the pde

1) $(x-h)^2+(y-k)^2+z^2=c^2$, where h and k are parameter.

2) $z=axe^y+\frac{1}{2}a^2e^{2y}+b$.

Sol:- (1) Given, $(x-h)^2+(y-k)^2+z^2=c^2$ —— (1)

Differentiating eq(1) partially w.r.t x, we get

$$2(x-h)+2z\frac{\partial z}{\partial x}=0$$

$\Rightarrow (x-h)+zp=0$

$\Rightarrow (x-h)^2=p^2z^2$ —— (2)

Again differentiating eq(1) partially w.r.t y, we get

$$2(y-k)+2z\frac{\partial z}{\partial y}=0$$

$\Rightarrow 2(y-k)+2zq=0$

$\Rightarrow (y-k)=-zq$

$\Rightarrow (y-k)^2=q^2z^2$ —— (3)

From equations (1), (2) and (3), we have

$$p^2z^2+q^2z^2+z^2=c^2$$

$\Rightarrow (p^2+q^2+1)z^2=c^2$, which is required pde.

2) Given, $z=ae^yx+\frac{1}{2}a^2e^{2y}+b$ —— (1)

Differentiating eq(1) partially w.r.t x, we get

$$\frac{\partial z}{\partial x}=ae^y+0$$

$\Rightarrow p=ae^y$ —— (2)

Again differentiating eq(1) partially w.r.t y, we get

$$\frac{\partial z}{\partial y}=axe^y+a^2e^{2y}$$

$\Rightarrow q=xp+p^2$ [Using(2)]

$\Rightarrow q=(x+p)p$, which is the required pde

Q:- Form a pde by eliminating a, b, c from

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

Sol:- Given, $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ ——— (1)

Differentiating eq(1) partially w.r.t x we get,

$$\frac{2x}{a^2}+\frac{2z}{c^2}\cdot\frac{\partial z}{\partial x}=0$$

$$\Rightarrow \frac{x}{a^2}+\frac{z}{c^2}\cdot\frac{\partial z}{\partial x}=0 \quad \text{——— (2)}$$

Differentiating eq(1) partially w.r.t y we get,

$$\frac{2y}{b^2}+\frac{2z}{c^2}\frac{\partial z}{\partial y}=0$$

$$\Rightarrow \frac{y}{b^2}+\frac{z}{c^2}\frac{\partial z}{\partial y}=0 \quad \text{——— (3)}$$

$$\Rightarrow \cancel{\frac{y}{b^2}} +$$

Again differentiating eq(2) w.r.t y we get,

$$0+\frac{1}{c^2}\left(\frac{\partial z}{\partial y}\cdot\frac{\partial z}{\partial x}+z\cdot\frac{\partial^2 z}{\partial y\partial x}\right)=0$$

$$\Rightarrow \frac{1}{c^2}(qp+zs)=0$$

$\Rightarrow \boxed{qp+zs=0}$, which is the required pde.

(2) By eliminating arbitary functions :-

Consider the function / equation

$$f(u,v)=0 \quad \text{——— (1)}$$

where u and v are functions of x, y, z and $z = z(x,y)$

Differentiating eq(1) partially w.r.t x, we have

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x}\right)+\frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial x}\right)=0$$

$$\Rightarrow \frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x}\right)=-\frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\frac{\partial z}{\partial x}\right) \quad \text{——— (2)}$$

Again differentiating eq(1) partially w.r.t y, we have

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial y}\right)+\frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial y}\right)=0$$

$$\Rightarrow \frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial y}\right)=-\frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial y}\right) \quad \text{——— (3)}$$

Dividing eq(2) by (3), we have

$$\frac{\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x}}{\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial y}}=\frac{\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial x}}{\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial y}}$$

$$\Rightarrow \frac{\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}}{\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}}=\frac{\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}}{\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}}$$

$\Rightarrow \boxed{Pp+Qq=R}$, which is required pde

Q:- Form the pde from the equation

$$f(x+y+z,\ x^2+y^2+z^2)=0$$

Sol:- Given. $f(x+y+z,\ x^2+y^2+z^2)=0$ ——— (1)

Here $u = x+y+z$, $v = x^2+y^2+z^2$

Differentiating eq(1) partially w.r.t x we get

$$\frac{\partial f}{\partial u}\left(1+1\cdot\frac{\partial z}{\partial x}\right)+\frac{\partial f}{\partial v}\left(2x+2z\frac{\partial z}{\partial x}\right)=0$$

$$\Rightarrow \frac{\partial f}{\partial u}(1+p) = -\frac{\partial f}{\partial v}(2x+2zp) \quad \text{——— (2)}$$

Again differentiating eq(1) partially w.r.t y we get

$$\frac{\partial f}{\partial u}\left(1+1\cdot\frac{\partial z}{\partial y}\right)+\frac{\partial f}{\partial v}\left(2y+2z\frac{\partial z}{\partial y}\right)=0$$

$$\Rightarrow \frac{\partial f}{\partial u}(1+q) = -\frac{\partial f}{\partial v}(2y+2zq) \quad \text{——— (3)}$$

Dividing eq(2) and (3), we have

$$\frac{1+p}{1+q} = \frac{2x+2zp}{2y+2zq}$$

$$\Rightarrow \frac{1+p}{1+q} = \frac{x+zp}{y+zq}$$

$$\Rightarrow (1+p)(y+zq) = (1+q)(x+zp)$$

$$\Rightarrow y+zq+py+\cancel{pzq} = x+zp+qx+\cancel{qzp}$$

$$\Rightarrow \boxed{(y-z)p+(z-x)q = x-y}$$

which is the required pde.

Q:- Form the pde from the equation

$$z = f(x+ay) + \phi(x-ay)$$

Sol:- Given, $z = f(x+ay) + \phi(x-ay)$ —— (1)

Here $u = x+ay$, $v = x-ay$

Differentiating eq(1) partially w.r.t x, we get

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial u}\cdot 1 + \frac{\partial \phi}{\partial v}\cdot 1$$

$$\Rightarrow \frac{\partial z}{\partial x} = \frac{\partial f}{\partial u} + \frac{\partial \phi}{\partial v} \quad \text{—— (2)}$$

Differentiating eq(1) partially w.r.t y, we get

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial u}\cdot a + \frac{\partial \phi}{\partial v}(-a) \quad \text{—— (3)}$$

Again differentiating eq(2) partially w.r.t x, we get

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 f}{\partial u^2}\cdot 1 + \frac{\partial^2 \phi}{\partial v^2}\cdot 1$$

$$\Rightarrow \frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 \phi}{\partial v^2} \quad \text{—— (4)}$$

Differentiating eq(3) partially w.r.t y, we get

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 f}{\partial u^2}\cdot a^2 + \frac{\partial^2 \phi}{\partial v^2}\cdot a^2$$

$$\Rightarrow \boxed{\frac{\partial^2 z}{\partial y^2} = a^2 \cdot \frac{\partial^2 z}{\partial x^2}} \quad [\text{Using } (4)]$$

which is the required pde.

* Given, $z = f(x+ay) + \phi(x-ay)$ —— (1)

Differentiating eq(1) partially w.r.t x we get

$$\frac{\partial z}{\partial x} = f'(x+ay) + \phi'(x-ay) \quad \text{—— (2)}$$

Differentiating eq(1) partially w.r.t y, we get

$$\frac{\partial z}{\partial y} = af'(x+ay) - a\phi'(x-ay) \quad \text{—— (3)}$$

Differentiating eq(2) partially w.r.t x we get

$$\frac{\partial^2 z}{\partial x^2} = f''(x+ay) + \phi''(x-ay) \quad \text{——— (4)}$$

Differentiating eq(2) partially w.r.t y, we get

$$\frac{\partial^2 z}{\partial y^2} = a^2 f''(x+ay) + a^2\phi''(x-ay)$$

$$= a^2\left[f''(x+ay) + \phi''(x-ay)\right]$$

$$= a^2 \frac{\partial^2 z}{\partial x^2}$$

$\Rightarrow \boxed{t = a^2 r}$, which is the required pde.

Complete integral

If the partial differential equation is

$$f(x,y,z,p,q) = 0 \quad \text{——— (1)}$$

then the solution of the form

$$\phi(x,y,z,a,b) = 0 \quad \text{——— (2)}$$

where a, b are arbitary constants known as complete integral of eq(1).

Particular integral

If we give some particular values to the constants a and b occuring in the complete integral is known as particular integral of eq(1).

Singular integral

The equation of the envelope of the surface represented by the complete integral of eq(1) is called its singular integral.

Note:- In order to obtain singular integral we eliminate a and b from the complete integral $\phi = 0$ and $\frac{\partial f}{\partial a} = 0$ and $\frac{\partial f}{\partial b} = 0$, which is part of singular solution.

General integral:-

The solution of the form $f(u,v)=0$ of eq(1) where u and v are functions of x, y and z is known as general integral of eq(1).

(1) $f(u,v)=0$

(2) $u=f(v)$

(3) $v=f(u)$

Lagrange's solution of linear pde:-

The pde of the form $Pp+Qq=R$ ——— (1), where P, Q and R are functions of x,y,z and

$$P=\frac{\partial(u,v)}{\partial(y,z)}, \quad Q=\frac{\partial(u,v)}{\partial(z,x)}, \quad R=\frac{\partial(u,v)}{\partial(x,y)}$$

are obtained by eliminating arbitary function f from $f(u,v)=0$ ——— (2)

then clearly (2) is solution of eq(1)

therefore, we have to find the values of u and v.

$\Rightarrow$ Let $u=a$ and $v=b$ be such that a and b are arbitary constants.

$$\therefore du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz=0 \quad \text{——— (3)}$$

$$\text{and } dv=\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy+\frac{\partial v}{\partial z}dz=0 \quad \text{——— (4)}$$

hence from eq(3) & (4), we get

$$\frac{dx}{\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial y}}=\frac{dy}{\frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial z}}=\frac{dz}{\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x}}$$

$$\Rightarrow \frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R} \quad \text{——— (5)}$$

eq(5) is known as <u>Lagrange auxiliary equation</u>

⇒ Thus solution of above differential equation are $u=a$ & $v=b$

⇒ Solution of given eq(1) is found as

$$f(u,v)=0$$

Q:- Solve $(y^2+z^2-x^2)p - 2xyq + 2xz = 0$

Sol:- Given, $(y^2+z^2-x^2)p - 2xyq + 2xz = 0$

$$\Rightarrow (y^2+z^2-x^2)p - 2xyq = -2xz \quad \text{---(1)}$$

∴ Lagrange's auxiliary equation is,

$$\frac{dx}{y^2+z^2-x^2} = \frac{dy}{-2xy} = \frac{dz}{-2xz}$$

taking 2nd and 3rd ratio, we have

$$\frac{dy}{-2xy} = \frac{dz}{-2xz} \Rightarrow \frac{dy}{y} = \frac{dz}{z}$$

On integrating, we have

$$\log y = \log z + \log c_1$$

$$\Rightarrow y = c_1 z$$

$$\Rightarrow \frac{y}{z} = c_1 \quad \text{---(2)}$$

Again using multipliers x, y, z we have

$$\text{each ratio} = \frac{x\,dx + y\,dy + z\,dz}{xy^2 + xz^2 - x^3 - 2xy^2 - 2xz^2}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{-x(y^2+z^2+x^2)}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{-x(x^2+y^2+z^2)}$$

$\therefore$ taking $\frac{dz}{-2xz} = \frac{x\,dx + y\,dy + z\,dz}{-x(x^2+y^2+z^2)}$

$$\Rightarrow \frac{dz}{z} = \frac{2(x\,dx + y\,dy + z\,dz)}{x^2+y^2+z^2}$$

On integrating, we have

$$\log z = \log(x^2+y^2+z^2) + \log c_2$$

$$\Rightarrow z = c_2(x^2+y^2+z^2)$$

$$\Rightarrow \frac{x^2+y^2+z^2}{z} = \frac{1}{c_2} = c_3$$

$\therefore$ The general integral of (1) is

$$\boxed{f\left(\frac{y}{z}, \frac{x^2+y^2+z^2}{z}\right) = 0}$$

Q:- Solve $(mz - ny)p + (nx - lz)q = (ly - mx)$

Sol:- Given, $(mz - ny)p + (nx - lz)q = ly - mx$ ——— (1)

$\therefore$ Lagrange's auxiliary equation is,

$$\frac{dx}{mz - ny} = \frac{dy}{nx - lz} = \frac{dz}{ly - mx}$$

using multipliers x, y, z, we have

$$\text{each ratio} = \frac{x\,dx + y\,dy + z\,dz}{\cancel{mxz} - \cancel{nxy} + \cancel{nxy} - \cancel{lyz} + \cancel{lyz} - \cancel{mxz}}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{0}$$

$$\Rightarrow x\,dx + y\,dy + z\,dz = 0$$

$$\Rightarrow x^2 + y^2 + z^2 = c_1$$

Again using multipliers as l, m, n, then

each ratio $= \dfrac{l\,dx + m\,dy + n\,dz}{\cancel{lmz} - \cancel{lny} + \cancel{mnx} - \cancel{mlz} + \cancel{nly} - \cancel{nmx}}$

$= \dfrac{l\,dx + m\,dy + n\,dz}{0}$

$\Rightarrow l\,dx + m\,dy + n\,dz = 0$

$\Rightarrow lx + my + nz = c_2$

$\therefore$ general solution of (1) is given by:

$$\boxed{f(x^2+y^2+z^2,\ lx+my+nz) = 0}$$

Q-2 Solve $z(xp - yq) = y^2 - x^2$

Sol:- Given, $zxp - zyq = y^2 - x^2$ ——— (1)

$\therefore$ Lagrange's auxiliary equation is,

$$\frac{dx}{zx} = \frac{dy}{-zy} = \frac{dz}{y^2 - x^2}$$

Taking 1st and 2nd ratio, we have

$$\frac{dx}{zx} = \frac{dy}{-zy}$$

$$\Rightarrow \frac{dx}{x} = -\frac{dy}{y}$$

Integrating we get

$$\log x = -\log y + \log c_1$$

$$\Rightarrow \log x + \log y = \log c_1$$

$$\Rightarrow xy = c_1$$

Using multiplier as x, y, z, then

$$\text{each ratio} = \frac{x\,dx + y\,dy + z\,dz}{z\cancel{x^2} - zy^2 + zy^2 - z\cancel{x^2}}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{0}$$

$\Rightarrow \quad x\,dx + y\,dy + z\,dz = 0$

$\Rightarrow \quad x^2 + y^2 + z^2 = c_2$

$\therefore$ General solution of (1), is given by

$$\boxed{f(xy,\ x^2+y^2+z^2) = 0}$$

Q:-2 Solve $\dfrac{y^2zp}{x} + zxq = y^2$

Sol:- Given $\dfrac{y^2zp}{x} + zxq = y^2$ ——— (1)

$\therefore$ Lagrange's auxiliary equation is given by,

$$\frac{dx}{\frac{y^2z}{x}} = \frac{dy}{zx} = \frac{dz}{y^2}$$

$$\Rightarrow \quad \frac{x\,dx}{y^2z} = \frac{dy}{zx} = \frac{dz}{y^2} \quad \text{——— (2)}$$

Taking 1st and 2nd ratio, we have

$$\frac{x\,dx}{y^2z} = \frac{dy}{zx} \Rightarrow x^2dx = y^2dy$$

On integrating, we get

$$\frac{x^3}{3} = \frac{y^3}{3} + c_1$$

$$\Rightarrow \quad x^3 - y^3 = 3c_1 = c_2 \quad \text{——— (3)}$$

Taking 1st and 3rd ratio, we have

$$\frac{x\,dx}{y^2 z} = \frac{dz}{y^2}$$

$\Rightarrow x\,dx = z\,dz$

Integrating, we have

$$\frac{x^2}{2} = \frac{z^2}{2} + C_3$$

$\Rightarrow x^2 - z^2 = 2C_3$

$\Rightarrow x^2 - z^2 = C_4$ ——— (4)

$\therefore$ General solution of (1) is given by,

$$\boxed{f(x^3 - y^3,\ x^2 - z^2) = 0}$$

Q:-3 Solve $\frac{y-z}{yz}p + \frac{z-x}{xz}q = \frac{x-y}{xy}$

Sol:- Given, $\frac{y-z}{yz}p + \frac{z-x}{xz}q = \frac{x-y}{xy}$ ——— (1)

Lagrange's auxiliary equation is

$$\frac{dx}{\frac{y-z}{yz}} = \frac{dy}{\frac{z-x}{xz}} = \frac{dz}{\frac{x-y}{xy}}$$

$$\Rightarrow \frac{yz\,dx}{y-z} = \frac{xz\,dy}{z-x} = \frac{xy\,dz}{x-y} \quad \text{——— (2)}$$

Using multipliers as x, y, z, we have

$$\text{each ratio} = \frac{xyz\,dx + xyz\,dy + xyz\,dz}{xy - xz + yz - yx + xz - yz}$$

$$= \frac{xyz(dx + dy + dz)}{0}$$

$\Rightarrow xyz(dx + dy + dz) = 0$

$\Rightarrow dx + dy + dz = 0$

Sol:- Given, $\frac{y-z}{yz}p + \frac{z-x}{xz}q = \frac{x-y}{xy}$ ——— (1)

$$\Rightarrow xyz\left(\frac{y-z}{yz}p + \frac{z-x}{xz}q\right) = xyz\left(\frac{x-y}{xy}\right)$$

$$\Rightarrow x(y-z)p + y(z-x)q = z(x-y)$$

Lagrange's auxiliary equation is,

$$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)} \quad \text{——— (2)}$$

Using ~~x, y, z~~ 1, 1, 1 as multipliers, we get

$$\text{each ratio} = \frac{dx+dy+dz}{\cancel{xy}-\cancel{xz}+\cancel{yz}-\cancel{yx}+\cancel{zx}-\cancel{zy}}$$

$$= \frac{dx+dy+dz}{0}$$

$\therefore\ dx+dy+dz = 0$

On integration, we get

$$x+y+z = c_1 \quad \text{——— (3)}$$

Again using $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as multipliers, we get

$$\text{each ratio} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{\cancel{y}-\cancel{z}+\cancel{z}-\cancel{x}+\cancel{x}-\cancel{y}}$$

$$= \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$$

$$\Rightarrow \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

Integrating, we get

$$\log x + \log y + \log z = \log c_2$$

$$\Rightarrow xyz = c_2 \quad \text{———} (4)$$

Hence, general solution of (1) is,

$$\boxed{f(x+y+z,\, xyz) = 0}.$$

Q:- Solve, $p+3q = 5z + \tan(y-3x)$

Sol:- Given, $p+3q = 5z + \tan(y-3x)$ ——— (1)

Lagrange's auxiliary equation is given by,

$$\frac{dx}{1} = \frac{dy}{3} = \frac{dz}{5z+\tan(y-3x)}$$

Taking 1st and 2nd ratio, we get

$$3dx = dy$$

$$\Rightarrow y - 3x = c_1 \quad \text{———} (2)$$

Taking 1st and 3rd ratio, we get

$$dx = \frac{dz}{5z+\tan(y-3x)}$$

$$\Rightarrow dx = \frac{dz}{5z+\tan c_1} \quad [\text{By using (2)}]$$

$$\Rightarrow 5dx = \frac{5dz}{5z+\tan c_1}$$

$$\Rightarrow 5x = \log(5z + \tan c_1) + \log c_2$$

$$\Rightarrow e^{5x} = c_2(5z+\tan c_1)$$

$$\Rightarrow e^{-5x}(5z+\tan(y-3x)) = k_2 \quad \text{———} (3)$$

$\therefore$ general solution of (1) is,

$$\boxed{f\left(y-3x,\, e^{-5x}(5z+\tan(y-3x))\right) = 0}.$$

Q:-1 $x^2(y-z)p+(z-x)y^2q = z^2(x-y)$

Sol:- Given, $x^2(y-z)p+(z-x)y^2q = z^2(x-y)$ —— (1)

Lagrange's auxiliary equation is,

$$\frac{dx}{x^2(y-z)} = \frac{dy}{(z-x)y^2} = \frac{dz}{z^2(x-y)} \quad \text{—— (2)}$$

Using $\frac{1}{x^2}, \frac{1}{y^2}, \frac{1}{z^2}$ as multipliers, we get

$$\text{each ratio} = \frac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{\cancel{y}-\cancel{z}+\cancel{z}-\cancel{x}+\cancel{x}-\cancel{y}}$$

$$= \frac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{0}$$

$$\Rightarrow \frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz = 0$$

Integrating, we get

$$-\frac{1}{x} - \frac{1}{y} - \frac{1}{z} = C_1$$

$$\Rightarrow \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = k_1 \quad \text{—— (3)}$$

Using $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as multipliers, we get

$$\text{each ratio} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{\cancel{xy}-\cancel{xz}+\cancel{yz}-\cancel{yx}+\cancel{zx}-\cancel{zy}}$$

$$= \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$$

$$\Rightarrow \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

Integrating, we get

$$\log x + \log y + \log z = \log c_2$$

$$\Rightarrow xyz = c_2 \quad \text{—— (4)}$$

Hence, general solution of (1) is,

$$\boxed{f\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z},\ xyz\right)=0}$$

Q:-2 $x(y^2+z)p-y(x^2+z)q=z(x^2-y^2)$

Sol:- Given, $x(y^2+z)p-y(x^2+z)q=z(x^2-y^2)$ —— (1)

Lagrange's auxiliary equation is,

$$\frac{dx}{x(y^2+z)}=\frac{dy}{-y(x^2+z)}=\frac{dz}{z(x^2-y^2)} \quad \text{—— (2)}$$

Using $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as multipliers, we get

$$\text{each ratio}=\frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{\cancel{y^2}+\cancel{z}-\cancel{x^2}-\cancel{z}+\cancel{x^2}-\cancel{y^2}}$$

$$=\frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{0}$$

$\Rightarrow \frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz=0$

Integrating, we get

$\log x+\log y+\log z=\log c_1$

$\Rightarrow xyz=c_1$ —— (3)

Using $x, y, -1$ as multipliers, we get

$$\text{each ratio}=\frac{x\,dx+y\,dy-dz}{\cancel{x^2y^2}+\cancel{x^2z}-\cancel{y^2x^2}-\cancel{y^2z}-\cancel{zx^2}+\cancel{zy^2}}$$

$$=\frac{x\,dx+y\,dy-dz}{0}$$

$\Rightarrow x\,dx+y\,dy-dz=0$

Integrating, we get

$$\frac{x^2}{2}+\frac{y^2}{2}-z=c_2'$$

$$\Rightarrow x^2+y^2-2z=c_3 \quad \text{———} \quad (4)$$

$\therefore$ The general solution of (1) is,

$$\boxed{f(xyz,\ x^2+y^2-2z)=0}$$

non-linear pde

Solution of ~~linear~~ pde of order one (and higher degree)

① Standard form – I :-

If the given pde is of the form

$$f(p,q)=0 \quad \text{———} \quad (1)$$

then its solution is given by,

$$z=ax+by+c \quad \text{———} \quad (2)$$

where a and b are related by

$$f(a,b)=0 \Rightarrow b=\phi(a)$$

$$\therefore eq(2) \Rightarrow z=ax+\phi(a)y+c \quad \text{———} \quad (3)$$

which is required complete integral.

General integral:-

Take $c=\psi(a)$, then

$$eq(3) \Rightarrow z=ax+\phi(a)y+\psi(a) \quad \text{———} \quad (4)$$

differentiating eq(4) w.r.t a, we get

$$0=x+\phi'(a)y+\psi'(a) \quad \text{———} \quad (5)$$

eliminating a from (4) and (5), we will get the required general integral.

singular integral :-

Differentiating eq(2) partially w.r.t a and c respectively, we get

$$0 = x + \phi'(a)y + 0$$

and $0 = 1$ (absurd)

$\Rightarrow$ singular solution doesn't exist.

Q:- Solve $p+q=1$.

Sol:- Given, $p+q=1$ ——— (1)

which is of the form $f(p,q)=0$, then complete integral of (1) is given by

$$z = ax + by + c \quad \text{———} (2)$$

where $a+b=1 \Rightarrow b = 1-a$

$\therefore$ Complete integral is,

$$\boxed{z = ax + (1-a)y + c}$$

Q:- Solve, $x^2p^2 + y^2q^2 = z^2$

Sol:- Given, $x^2p^2 + y^2q^2 = z^2$

$$\Rightarrow \frac{x^2p^2}{z^2} + \frac{y^2q^2}{z^2} = 1$$

$$\Rightarrow \left(\frac{x}{z}\frac{\partial z}{\partial x}\right)^2 + \left(\frac{y}{z}\frac{\partial z}{\partial y}\right)^2 = 1 \quad \text{———} (1)$$

Put $\frac{dx}{x} = dX$, $\frac{dy}{y} = dY$, $\frac{dz}{z} = dZ$

On integrating, we get

$$\log x = X, \quad \log y = Y, \quad \log z = Z$$

$$\therefore \text{eq}(1) \Rightarrow \left(\frac{\partial Z}{\partial X}\right)^2 + \left(\frac{\partial Z}{\partial Y}\right)^2 = 1$$

which is of the form $f(p,q)=0$

then the complete integral is given by,

$$Z = aX + bY + C \quad , \text{ where } a^2+b^2=1$$
$$\Rightarrow b = \sqrt{1-a^2}$$

$$\therefore \; Z = aX + \sqrt{1-a^2}\, Y + C$$

$$\Rightarrow \boxed{\log z = a \log x + \sqrt{1-a^2} \log y + c}$$

Q:- Solve

1) $q = 3p^2$
2) $(x+y)(p+q)^2 + (x-y)(p-q)^2 = 1$
3) $(y-x)(qy-px) = (p-q)^2$

Solutions:-

1) Given, $q = 3p^2$ ——— (1)

which is of the form $f(p,q)=0$, then the complete integral of (1) is given by,

$$z = ax+by+c \text{ ——— (2)}$$

where $b = 3a^2$

$\therefore$ Complete integral is,

$$\boxed{z = ax + 3a^2 y + c}$$

2) Given, $(x+y)(p+q)^2 + (x-y)(p-q)^2 = 1$ ——— (1)

Let us put $x+y = X^2$ and $x-y = Y^2$

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\cdot\frac{\partial X}{\partial x} + \frac{\partial z}{\partial Y}\cdot\frac{\partial Y}{\partial x} = \frac{\partial z}{\partial X}\cdot\frac{1}{2\sqrt{x+y}} + \frac{\partial z}{\partial Y}\cdot\frac{1}{2\sqrt{x-y}}$$
$$= \frac{1}{2X}\frac{\partial z}{\partial X} + \frac{1}{2Y}\frac{\partial z}{\partial Y}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial X}\cdot\frac{\partial X}{\partial y} + \frac{\partial z}{\partial Y}\cdot\frac{\partial Y}{\partial y}$$

$$= \frac{\partial z}{\partial X}\cdot\frac{1}{2\sqrt{x+y}} - \frac{\partial z}{\partial Y}\cdot\frac{1}{2\sqrt{x-y}}$$

$$= \frac{1}{2X}\frac{\partial z}{\partial X} - \frac{1}{2Y}\frac{\partial z}{\partial Y}$$

Now, $p+q = \frac{1}{X}\frac{\partial z}{\partial X}$

and $p-q = \frac{1}{Y}\frac{\partial z}{\partial Y}$

Now, substituting these in eq(1), we get

$$X^2\cdot\frac{1}{X^2}\left(\frac{\partial z}{\partial X}\right)^2 + Y^2\cdot\frac{1}{Y^2}\left(\frac{\partial z}{\partial Y}\right)^2 = 1$$

$$\Rightarrow \left(\frac{\partial z}{\partial X}\right)^2 + \left(\frac{\partial z}{\partial Y}\right)^2 = 1$$

which is of the form $f(P,Q)=0$

then the complete integral is given by,

$$z = aX + bY + c, \quad \text{where } a^2+b^2=1$$

$$\Rightarrow b = \sqrt{1-a^2}$$

$$\therefore\ z = aX + \sqrt{1-a^2}\,Y + c$$

$$\Rightarrow \boxed{z = a\sqrt{x+y} + \sqrt{1-a^2}\sqrt{x-y} + c}$$

3) Given, $(y-x)(qy-px) = (p-q)^2$ ——— (1)

Let X, Y be two new variables such that

$$X = x+y \quad \text{and} \quad Y = xy$$

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\cdot\frac{\partial X}{\partial x} + \frac{\partial z}{\partial Y}\cdot\frac{\partial Y}{\partial x} = \frac{\partial z}{\partial X} + y\frac{\partial z}{\partial Y}$$

$$\text{and } q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial X}\cdot\frac{\partial X}{\partial y} + \frac{\partial z}{\partial Y}\cdot\frac{\partial Y}{\partial y} = \frac{\partial z}{\partial X} + x\frac{\partial z}{\partial Y}$$

Substituting the above values of p and q in (1), we get

$$(y-x)\left\{\left(\frac{\partial z}{\partial X}+x\frac{\partial z}{\partial Y}\right)y-\left(\frac{\partial z}{\partial X}+y\frac{\partial z}{\partial Y}\right)x\right\}$$

$$=\left(\frac{\partial z}{\partial X}+y\frac{\partial z}{\partial Y}-\frac{\partial z}{\partial X}-x\frac{\partial z}{\partial Y}\right)^2$$

$$\Rightarrow (y-x)(y-x)\cdot\frac{\partial z}{\partial X}=(y-x)^2\left(\frac{\partial z}{\partial Y}\right)^2$$

$$\Rightarrow \frac{\partial z}{\partial X}=\left(\frac{\partial z}{\partial Y}\right)^2$$

$$\Rightarrow P=Q^2$$

where $P=\frac{\partial z}{\partial X}$, $Q=\frac{\partial z}{\partial Y}$

which is of the form $f(P,Q)=0$

$\therefore$ The complete integral is given by

$z=aX+bY+c$, where $a=b^2$

$$\Rightarrow z=b^2X+bY+c$$

$$\Rightarrow \boxed{z=b^2(x+y)+bxy+c}$$

(2) Standard form II

If the pde is of the form $f(p,q,z)=0$ —— (1)

then we put $X = x+ay$,

then we have

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\cdot\frac{\partial X}{\partial x} = \frac{\partial z}{\partial X} = \frac{dz}{dX}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial X}\cdot\frac{\partial X}{\partial y} = a\frac{\partial z}{\partial X} = a\frac{dz}{dX}$$

then eq(1) becomes

$$f\left(\frac{dz}{dX}, a\frac{dz}{dX}, z\right) = 0$$

which is ~~pde~~ ODE.

Q:- Find the complete integral of

$$z^2(p^2z^2+q^2) = 1$$

Find also the singular solution if it exists.

Sol:- Given $z^2(p^2z^2+q^2) = 1$ —— (1)

which is of the form $f(p,q,z)=0$

then if we put $X = x+ay$, we get

$$p = \frac{dz}{dX} \text{ and } q = a\frac{dz}{dX}$$

From eq(1), we have

$$z^2\left\{\left(\frac{dz}{dX}\right)^2 z^2 + \left(a\frac{dz}{dX}\right)^2\right\} = 1$$

$$\Rightarrow z^2(z^2+a^2)\left(\frac{dz}{dX}\right)^2 = 1$$

$$\Rightarrow z\sqrt{z^2+a^2}\,\frac{dz}{dX} = 1$$

$$\Rightarrow \int z\sqrt{z^2+a^2}\,dz = \int dX$$

$\Rightarrow \int t^2\,dt = \int dx$

put $z^2+a^2=t^2$

$\Rightarrow 2z\,dz = 2t\,dt$

$\Rightarrow z\,dz = t\,dt$

$\Rightarrow \frac{t^3}{3} = x+b$

$\Rightarrow \frac{(z^2+a^2)^{3/2}}{3} = x+b$

$\Rightarrow \frac{(z^2+a^2)^{3/2}}{3} = x+ay+b$

$\Rightarrow (z^2+a^2)^{3/2} = 3(x+ay+b)$

$\Rightarrow \boxed{(z^2+a^2)^3 = 9(x+ay+b)^2}$ ——— (2)

which is complete integral for (1).

Now, differentiating eq(2) partially w.r.t a, we get

$3(z^2+a^2)\cdot 2a = 18y(x+ay+b)$

$\Rightarrow a(z^2+a^2) = 3y(x+ay+b)$ ——— (3)

Again differentiating eq(2) partially w.r.t b, we get

$0 = 18(x+ay+b)$

$\Rightarrow x+ay+b=0$ ——— (4)

From (2), (3) and (4), we get

$z=0$

Clearly, $z=0$ doesn't satisfy the given pde and hence singular solution doesn't exist.

Q:- Solve.

① $p^2 = z^2(1-pq)$

② $pz = 1+q^2$

③ $q(p^2z+q^2) = 4$

④ $p(1+q^2) = q(z-a)$

⑤ $p^3+q^3 = 27z$

Solutions

1) Given, $p^2 = z^2(1-pq)$ ——— (1)

which is of the form $f(p,q,z)=0$,

then if we put $X = x+ay$, we get

$$p = \frac{dz}{dX} \quad \text{and} \quad q = a\frac{dz}{dX}$$

From (1), we have

$$\left(\frac{dz}{dX}\right)^2 = z^2\left(1-\frac{dz}{dX}\cdot a\frac{dz}{dX}\right)$$

$$\Rightarrow \left(\frac{dz}{dX}\right)^2 = z^2\left(1-a\left(\frac{dz}{dX}\right)^2\right)$$

$$\Rightarrow \left(\frac{dz}{dX}\right)^2 = z^2 - az^2\left(\frac{dz}{dX}\right)^2$$

$$\Rightarrow (1+az^2)\left(\frac{dz}{dX}\right)^2 = z^2$$

$$\Rightarrow \left(\frac{dz}{dX}\right)^2 = \frac{z^2}{1+az^2}$$

$$\Rightarrow \frac{dz}{dX} = \frac{z}{\sqrt{1+az^2}}$$

$$\Rightarrow \frac{\sqrt{1+az^2}}{z}dz = dX$$

$$\Rightarrow \int\frac{\sqrt{1+az^2}}{z}dz = \int dX$$

$$\Rightarrow \int \frac{1+az^2}{z\sqrt{1+az^2}}\,dz = x+c$$

$$\Rightarrow \int \frac{dz}{z\sqrt{1+az^2}} + \int \frac{az\,dz}{\sqrt{1+az^2}} = x+c$$

$$\Rightarrow \int \frac{\cancel{t}\,dt}{(t^2-1)\cancel{t}} + \int \frac{\cancel{t}\,dt}{\cancel{t}} = x+c$$

put $1+az^2 = t^2$

$\Rightarrow 2az\,dz = 2t\,dt$

$\Rightarrow az\,dz = t\,dt$

$\Rightarrow dz = \frac{t\,dt}{az}$

$$\Rightarrow \frac{1}{2}\ln\left(\frac{t-1}{t+1}\right) + t = x+c$$

$$\Rightarrow \frac{1}{2}\ln\left(\frac{\sqrt{1+az^2}-1}{\sqrt{1+az^2}+1}\right) + \sqrt{1+az^2} = x+ay+c$$

2) Given, $pz = 1+q^2$ ——— (1)

which is of the form $f(p,q,z)=0$, then if we put $X = x+ay$, then

$p = \frac{dz}{dX}$, $q = a\frac{dz}{dX}$

$\therefore$ from eq(1), we have

$$\frac{dz}{dX}\cdot z = 1 + a^2\left(\frac{dz}{dX}\right)^2$$

$$\Rightarrow \ln\left(\frac{\sqrt{1+az^2}-1}{\sqrt{1+az^2}+1}\right)^{1/2} + \sqrt{1+az^2} = x+ay+c$$

$$\Rightarrow \ln\left(\frac{(\sqrt{1+az^2}-1)^2}{\cancel{1}+az^2-\cancel{1}}\right)^{1/2} + \sqrt{1+az^2} = x+ay+c$$

$$\Rightarrow \boxed{\ln\left(\sqrt{1+az^2}-1\right) - \ln\sqrt{a}z + \sqrt{1+az^2}} = x+ay+c$$

which is the complete integral for (1).

2) Given, $pz = 1+q^2 \longrightarrow (1)$

which is of the form $f(p,q,z)=0$, then if we put $X = x+ay$, then

$$p = \frac{dz}{dX}, \quad q = a\frac{dz}{dX}$$

$\therefore$ from (1), we have

$$\frac{dz}{dX}\cdot z = 1 + a^2\left(\frac{dz}{dX}\right)^2$$

$$\Rightarrow a^2\left(\frac{dz}{dX}\right)^2 - z\left(\frac{dz}{dX}\right) + 1 = 0$$

$$\Rightarrow \frac{dz}{dX} = \frac{z \pm \sqrt{z^2-4a^2}}{2a^2}$$

$$\Rightarrow \frac{dz}{z\pm\sqrt{z^2-4a^2}} = \frac{dX}{2a^2}$$

$$\Rightarrow \frac{\left(z\mp\sqrt{z^2-4a^2}\right)dz}{\left(z\pm\sqrt{z^2-4a^2}\right)\left(z\mp\sqrt{z^2-4a^2}\right)} = \frac{dX}{2a^2}$$

$$\Rightarrow \frac{\left(z\mp\sqrt{z^2-4a^2}\right)dz}{4a^2} = \frac{dX}{2a^2}$$

$$\Rightarrow \left(z\mp\sqrt{z^2-4a^2}\right)dz = 2dX$$

Integrating we get,

$$\frac{z^2}{2} \mp \left[\frac{z}{2}\sqrt{z^2-4a^2} - \frac{4a^2}{2}\log\left\{z+\sqrt{z^2-4a^2}\right\}\right] = 2X + \frac{b}{2}$$

$$\Rightarrow \boxed{z^2 \mp \left[z\sqrt{z^2-4a^2} - 4a^2\log\left\{z+\sqrt{z^2-4a^2}\right\}\right] = 4(x+ay)+b}$$

which is the required complete integral.

3) Given,

$9(p^2 z + q^2) = 4$ ——— (1)

which is of the form $f(p,q,z) = 0$, then if we put $X = x + ay$, then

$p = \frac{dz}{dX}$, $q = a\frac{dz}{dX}$

$\therefore$ from (1) we have

$$9\left[\left(\frac{dz}{dX}\right)^2 z + a^2\left(\frac{dz}{dX}\right)^2\right] = 4$$

$$\Rightarrow \left(\frac{dz}{dX}\right)^2 \cdot 9(z + a^2) = 4$$

$$\Rightarrow \left(\frac{dz}{dX}\right)^2 = \frac{4}{9(z+a^2)}$$

$$\Rightarrow \frac{dz}{dX} = \pm\frac{2}{3\sqrt{z+a^2}}$$

$$\Rightarrow \pm\left(\frac{3}{2}\right)\sqrt{z+a^2}\, dz = dX$$

$$\Rightarrow \pm\left(\frac{3}{2}\right)\cdot\frac{2}{3}(z+a^2)^{3/2} = X + b$$

$$\Rightarrow \pm(z+a^2)^{3/2} = x + ay + b$$

$$\Rightarrow \boxed{(z+a^2)^3 = (x+ay+b)^2}$$

which is the required complete integral.

4) Given, $p(1+q^2) = q(z-a)$ ——— (1)

which is of the form $f(p,q,z) = 0$,

Let $X = x + ay$, then

$p = \frac{dz}{dX}$ and $q = a\frac{dz}{dX}$

$\therefore$ From (1), we have

$$\frac{dz}{dX}\left\{1+a^2\left(\frac{dz}{dX}\right)^2\right\} = a\frac{dz}{dX}(z-a)$$

$$\Rightarrow 1+a^2\left(\frac{dz}{dX}\right)^2 = a(z-a)$$

$$\Rightarrow a^2\left(\frac{dz}{dX}\right)^2 = a(z-a)-1$$

$$\Rightarrow \left(\frac{dz}{dX}\right)^2 = \frac{a(z-a)-1}{a^2}$$

$$\Rightarrow \frac{dz}{dX} = \frac{\pm\sqrt{a(z-a)-1}}{a}$$

$$\Rightarrow \pm\frac{a\,dz}{\sqrt{a(z-a)-1}} = dX$$

Integrating, we get

$$\Rightarrow \pm 2\sqrt{a(z-a)-1} = X+b$$

$$\Rightarrow \boxed{4\{a(z-a)-1\} = (x+ay+b)^2}$$

which is the required complete integral.

5) Given, $p^3+q^3 = 27z$ ——— (1)

which is of the form $f(p,q,z)=0$,

Let $X = x+ay$, then $p=\frac{dz}{dX}$, $q=a\frac{dz}{dX}$

$\therefore$ From (1), we have

$$\left(\frac{dz}{dX}\right)^3 + a^3\left(\frac{dz}{dX}\right)^3 = 27z$$

$$\Rightarrow (1+a^3)\left(\frac{dz}{dX}\right)^3 = 27z$$

$$\Rightarrow \frac{dz}{dX} = \frac{3z^{1/3}}{(1+a^3)^{1/3}}$$

$$\Rightarrow \frac{1}{3}(1+a^3)^{1/3} z^{-1/3}\,dz = dX$$

 Integrating, we get

$$\frac{1}{3}\cdot\frac{3}{2}(1+a^3)^{1/3}\, z^{2/3} = X + b$$

$$\Rightarrow \frac{1}{2}(1+a^3)^{1/3} z^{2/3} = x+ay+b$$

$$\Rightarrow \boxed{z^2(1+a^3) = 8(x+ay+b)^3}$$

which is the required complete integral.

③ Standard form III

If the given pde can be written as

$$f(p,x) = g(q,y)$$

then it can be solved by integrating,

$$dz = p\,dx + q\,dy$$

Note:- $f(p,x) = g(q,y) = a$ (say)

$$\Rightarrow p = f(x,a) \text{ and } q = g(y,a)$$

Q:- Solve $p^2+q^2 = x+y$

Sol:- Given; $p^2+q^2 = x+y$

$$\Rightarrow p^2 - x = y - q^2 = a \text{ (say)}$$

$\therefore p^2 - x = a$ and $y - q^2 = a$

$$\Rightarrow p = (a+x)^{1/2} \text{ and } q = (y-a)^{1/2}$$

Now, we have

$$dz = p\,dx + q\,dy$$

$$\Rightarrow dz = (x+a)^{1/2}\,dx + (y-a)^{1/2}\,dy$$

On integrating we get

$$\boxed{z = \frac{2}{3}(x+a)^{3/2} + \frac{2}{3}(y-a)^{3/2} + b}$$

which is the required complete integral.

Q:- Solve

1) $\sqrt{p} + \sqrt{q} = 2x$

2) $yp = 2yx + \log q$

Solutions:-

1) Given, $\sqrt{p} + \sqrt{q} = 2x$ —— (1)

$$\Rightarrow \sqrt{p} - 2x = -\sqrt{q}$$

Equating each side to an arbitary constant a, we get

$\sqrt{p} - 2x = a$ and $-\sqrt{q} = a$

$\Rightarrow \sqrt{p} = a + 2x$ $\quad\Rightarrow q = a^2$

$\Rightarrow p = (2x+a)^2$

Now, we have

$$dz = p\,dx + q\,dy$$

$$\Rightarrow dz = (2x+a)^2\,dx + a^2\,dy$$

Integrating we get

$$\boxed{z = \frac{(2x+a)^3}{6} + a^2 y + b}$$

which is the required complete integral.

3) Given, $yp = 2yx + \log q$ ——— (1)

$\Rightarrow yp - 2yx = \log q$

$\Rightarrow y(p-2x) = \log q$

$\Rightarrow p - 2x = \frac{\log q}{y}$

Equating each side to an arbitary constant a, we get

$p - 2x = a$ and $\frac{\log q}{y} = a$

$\Rightarrow p = 2x + a$ and $\log q = ay$

$\Rightarrow q = e^{ay}$

Now, we have $dz = p\,dx + q\,dy$

$\Rightarrow dz = (2x+a)\,dx + e^{ay}\,dy$

Integrating we get,

$$\boxed{z = x^2 + ax + \frac{e^{ay}}{a} + b}$$

which is the required complete integral.

④ Standard form IV

If the pde of the form

$z = px + qy + f(p,q)$

then its complete integral is

$z = ax + by + f(a,b)$

Q:- Find the complete integral of

$z = px + qy + p^2 + q^2$

Sol:- Given, $z = px + qy + p^2 + q^2$ ——— (1)

which is of the form $z = px + qy + f(p,q)$

then the complete integral of (1) is,

$$z = ax + by + a^2 + b^2$$

Q:- Find complete and singular solution of

$$z = px + qy + c\sqrt{1+p^2+q^2}$$

Sol:- Given, $z = px + qy + c\sqrt{1+p^2+q^2}$ ——— (1)

which is in the form $z = px + qy + f(p,q)$

then complete integral is,

$$z = ax + by + c\sqrt{1+a^2+b^2} \quad \text{———} (2)$$

In order to find singular solution, we have to differentiate eq(2) partially w.r.t a and b, then we get

$$0 = x + \frac{ca}{\sqrt{1+a^2+b^2}} \quad \text{———} (3)$$

and $$0 = y + \frac{cb}{\sqrt{1+a^2+b^2}} \quad \text{———} (4)$$

From eq(3) & (4), we get

$$x^2 + y^2 = \frac{(a^2+b^2)c^2}{1+a^2+b^2} = \frac{(a^2+b^2+1-1)c^2}{1+a^2+b^2}$$

$$= \frac{(a^2+b^2+1)c^2}{1+a^2+b^2} - \frac{c^2}{1+a^2+b^2}$$

$$\Rightarrow c^2 - x^2 - y^2 = \frac{c^2}{1+a^2+b^2} \quad \text{———} (5)$$

Again from (3), $a = \dfrac{-x\sqrt{1+a^2+b^2}}{c}$

$$= \frac{-x}{c} \cdot \frac{c}{\sqrt{c^2-x^2-y^2}} \quad [\text{Using } (5)]$$

$$\Rightarrow a = \frac{-x}{\sqrt{c^2-x^2-y^2}}$$

Similarly, $b = \dfrac{-y}{\sqrt{c^2-x^2-y^2}}$

Putting the value of a and b in (2), we get

$$z = \frac{-x^2}{\sqrt{c^2-x^2-y^2}} - \frac{y^2}{\sqrt{c^2-x^2-y^2}} + \frac{c^2}{\sqrt{c^2-x^2-y^2}}$$

$$\Rightarrow z = \sqrt{c^2-x^2-y^2}$$

$$\Rightarrow \boxed{x^2+y^2+z^2 = c^2}$$

which is the required singular solution

Charpit's method:-

If the given pde is of the form

$$f(p,q,z,x,y) = 0 \quad \text{——— (1)}$$

then Charpit auxiliary equation is

$$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}$$

then our aim is to find the values of p and q and then substituting these values in

$$dz = p\,dx + q\,dy$$

On integrating we will get the required solution.

Q:- Solve $(p^2+q^2)y = qz$.

Sol:- Given, $(p^2+q^2)y = qz$ ——— (1)

Let $f = (p^2+q^2)y - qz = 0$

The Charpit auxiliary equation is,

$$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}$$

$$\Rightarrow \frac{dp}{0+p(-q)} = \frac{dq}{p^2+q^2+q(-q)} = \frac{dz}{-p(2py)-q(2qy-z)} = \frac{dx}{-2py} = \frac{dy}{-2qy+z}$$

$$\Rightarrow \frac{dp}{-pq} = \frac{dq}{p^2} = \frac{dz}{-2y(p^2+q^2)+qz} = \frac{dx}{-2py} = \frac{dy}{-2qy+z}$$

Taking 1st and 2nd ratio, we have

$$\frac{dp}{-pq} = \frac{dq}{p^2}$$

$$\Rightarrow \frac{-dp}{q} = \frac{dq}{p}$$

$$\Rightarrow pdp + qdq = 0$$

$$\Rightarrow p^2+q^2 = a^2 \quad \text{——— (2)}$$

From (1), $q = \frac{a^2y}{z}$ ——— (3)

and from (2) and (3), we have

$$p^2 = a^2 - \frac{a^4y^2}{z^2} = \frac{a^2z^2 - a^4y^2}{z^2} = \frac{a^2}{z^2}(z^2-a^2y^2)$$

$$\Rightarrow p = \frac{a}{z}\sqrt{z^2-a^2y^2}$$

$\therefore$ We have, $dz = pdx + qdy$

$$\Rightarrow dz = \frac{a}{z}\sqrt{z^2-a^2y^2}\,dx + \frac{a^2y}{z}\,dy$$

$$\Rightarrow z\,dz = a\sqrt{z^2-a^2y^2}\,dx + a^2y\,dy$$

$$\Rightarrow z\,dz - a^2y\,dy = a\sqrt{z^2-a^2y^2}\,dx$$

$$\Rightarrow \frac{z\,dz - a^2y\,dy}{\sqrt{z^2-a^2y^2}} = a\,dx$$

Integrating, we get

$$\int \frac{z\,dz - a^2y\,dy}{\sqrt{z^2-a^2y^2}} = \int a\,dx$$

put $z^2-a^2y^2 = t^2$

$\Rightarrow 2z\,dz - 2a^2y\,dy = 2t\,dt$

$\Rightarrow z\,dz - a^2y\,dy = t\,dt$

$$\Rightarrow \int \frac{t\,dt}{t} = \int a\,dx$$

$$\Rightarrow t = ax+b$$

$$\Rightarrow \sqrt{z^2-a^2y^2} = ax+b$$

$$\Rightarrow z^2-a^2y^2 = (ax+b)^2$$

$$\Rightarrow \boxed{z^2 = a^2y^2+(ax+b)^2} \quad \text{—— (4)}$$

which is required complete integral.

General integral :-

Take $b = \phi(a)$

$$\therefore \quad z^2 = a^2y^2 + (ax+\phi(a))^2 \quad \text{—— (5)}$$

Differentiating eq (5) partially w.r.t 'a', we get

$$0 = 2ay^2 + 2(ax+\phi(a))(x+\phi'(a)) \quad \text{—— (6)}$$

Eliminating a from (5) and (6), we will get the required general integral.

Singular integral :-

Differentiating eq(4) partially w.r.t a, we get

$$0 = 2ay^2 + 2(ax+b)x \quad \text{——— (7)}$$

Again differentiating eq(4) partially w.r.t b, we get

$$0 = 2(ax+b) \quad \text{——— (8)}$$

Eliminating a and b from (4), (7) and (8), we get

$z = 0$,

which satisfies the given pde.

$\therefore z=0$ is singular integral of (1).

Q:- Solve $2zx - px^2 - 2qxy + pq = 0$

Sol:- Given, $2zx - px^2 - 2qxy + pq = 0$ ——— (1)

Let $f = 2zx - px^2 - 2qxy + pq = 0$ ——— (2)

Then charpit auxiliary equation is

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}$$

$$\Rightarrow \frac{dp}{2z - \cancel{2px} - 2qy + \cancel{2px}} = \frac{dq}{-2qx + 2qx} = \frac{dz}{-p(-x^2+q) - q(-2xy+p)}$$

$$= \frac{dx}{x^2 - q} = \frac{dy}{2xy - p}$$

$$\Rightarrow \frac{dp}{2z - 2qy} = \frac{dq}{0} = \frac{dz}{-p(-x^2+q) - q(-2xy+p)} = \frac{dx}{x^2 - q} = \frac{dy}{2xy - p}$$

From 1st and 2nd ratio, we get

$dq = 0 \Rightarrow q = a$ (constant) ——— (3)

Put $q = a$ in eq(1), we get

$$2zx - px^2 - 2axy + pa = 0$$

$$\Rightarrow p(a - x^2) = 2axy - 2xz$$

$$\Rightarrow p = \frac{2axy - 2xz}{a - x^2} = \frac{2xz - 2axy}{x^2 - a}$$

$\therefore$ We have, $dz = p\,dx + q\,dy$

$$\Rightarrow dz = \left(\frac{2xz - 2axy}{x^2 - a}\right) dx + a\,dy$$

$$\Rightarrow dz - a\,dy = \frac{2x(z - ay)}{x^2 - a}\,dx$$

$$\Rightarrow \frac{dz - a\,dy}{z - ay} = \frac{2x\,dx}{x^2 - a}$$

Integrating, we get

$$\log(z - ay) = \log(x^2 - a) + \log b$$

$$\Rightarrow z - ay = (x^2 - a)b$$

$$\Rightarrow \boxed{z = ay + b(x^2 - a)}$$

which is required complete integral

Solve :-

1) $z^2(p^2z^2 + q^2) = 1$

2) $px + qy = z(1 + pq)^{1/2}$

3) $pxy + pq + qy = yz$

4) $p = (qy + z)^2$

Solutions :-

1) Given, $z^2(p^2z^2+q^2)=1$ ——— (1)

Let $f = z^2(p^2z^2+q^2)-1=0$

The Charpit auxilliary equation is,

$$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}$$

$$\Rightarrow \frac{dp}{0+p(4p^2z^3+2zq^2)} = \frac{dq}{0+q(4p^2z^3+2zq^2)} = \frac{dz}{-2p^2z^4-2q^2z^2}$$

$$= \frac{dx}{-2pz^4} = \frac{dy}{-2qz^2}$$

From 1st and 2nd ratio, we get

$$\frac{dp}{p} = \frac{dq}{q}$$

$$\Rightarrow \log p = \log q + \log a$$

$$\Rightarrow p = qa \quad \text{——— (2)}$$

From (1), $p^2z^4+z^2q^2=1$

$$\Rightarrow p^2z^4+z^2\frac{p^2}{a^2}=1 \quad [\text{Using (2)}]$$

$$\Rightarrow a^2p^2z^4+z^2p^2=a^2$$

$$\Rightarrow p^2z^2(a^2z^2+1)=a^2$$

$$\Rightarrow p = \frac{a}{z\sqrt{a^2z^2+1}} \quad \Rightarrow q = \frac{p}{a} = \frac{1}{z\sqrt{a^2z^2+1}}$$

Again from (1), $p^2z^4+z^2q^2=1$

$$\Rightarrow q^2a^2z^4+z^2q^2=1 \quad [\text{Using (2)}]$$

$$\Rightarrow z^2q^2(a^2z^2+1)=1$$

$$\Rightarrow q = \frac{1}{z\sqrt{a^2z^2+1}}$$

$$\therefore dz = p\,dx + q\,dy$$

$$= \frac{a}{z\sqrt{a^2z^2+1}}\,dx + \frac{1}{z\sqrt{a^2z^2+1}}\,dy$$

$$= \frac{a\,dx+dy}{z\sqrt{a^2z^2+1}}$$

$$\Rightarrow a\,dx + dy = z\sqrt{a^2z^2+1}\,dz$$

Integrating, we get

$$\int (a\,dx + dy) = \int z\sqrt{a^2z^2+1}\,dz \qquad \text{put } a^2z^2+1 = t^2$$

$$\Rightarrow 2a^2z\,dz = 2t\,dt$$

$$\Rightarrow z\,dz = \frac{t}{a^2}dt$$

$$\Rightarrow ax+y = \int \frac{t}{a^2}\cdot t\,dt$$

$$\Rightarrow ax+y+b = \frac{t^3}{3a^2}$$

$$\Rightarrow \frac{(a^2z^2+1)^{3/2}}{3a^2} = ax+y+b$$

$$\Rightarrow (a^2z^2+1)^{3/2} = 3a^2(ax+y+b)$$

$$\Rightarrow \boxed{(a^2z^2+1)^3 = 9a^4(ax+y+b)^2}$$

which is required complete integral.

2) Given, $px+qy = z(1+pq)^{1/2}$ ——— (1)

Let $f = px+qy - z(1+pq)^{1/2} = 0$ ——— (2)

Charpit's auxiliary equation is

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}$$

$$\Rightarrow \frac{dp}{p-p(1+pq)^{1/2}} = \frac{dq}{q-q(1+pq)^{1/2}} = \frac{dz}{-p\left[x-\frac{zq}{2\sqrt{1+pq}}\right]-q\left[y-\frac{zp}{2\sqrt{1+pq}}\right]}$$

$$= \frac{dx}{-x+\frac{zq}{2\sqrt{1+pq}}} = \frac{dy}{-y+\frac{zp}{2\sqrt{1+pq}}}$$

Taking first two ratios we get

$$\frac{dp}{p\left[1-\cancel{(1+pq)^{1/2}}\right]} = \frac{dq}{q\left[1-\cancel{(1+pq)^{1/2}}\right]}$$

$$\Rightarrow \frac{dp}{p} = \frac{dq}{q}$$

$$\Rightarrow \log p = \log q + \log a$$

$$\Rightarrow p = qa \quad \text{---------- (3)}$$

Using (3), (1) $\Rightarrow qax + qy = z(1+aq^2)^{1/2}$

$$\Rightarrow q(ax+y) = z(1+aq^2)^{1/2}$$

$$\Rightarrow q^2(ax+y)^2 = z^2(1+aq^2)$$

$$\Rightarrow q^2\{(ax+y)^2 - az^2\} = z^2$$

$$\Rightarrow q = \frac{z}{\sqrt{(ax+y)^2-az^2}}$$

and $p = \dfrac{az}{\sqrt{(ax+y)^2-az^2}}$

Substituting these values in $dz = pdx + qdy$, we have,

$$dz = \frac{az\,dx + z\,dy}{\sqrt{(ax+y)^2-az^2}}$$

$$= \frac{z(adx+dy)}{\sqrt{(ax+y)^2-az^2}}$$

$$\Rightarrow \frac{dz}{z} = \frac{a\,dx + dy}{\sqrt{(ax+y)^2 - az^2}}$$

put $ax + y = \sqrt{a}\,t$

$\Rightarrow a\,dx + dy = \sqrt{a}\,dt$

$$\Rightarrow \frac{dz}{z} = \frac{\sqrt{a}\,dt}{\sqrt{at^2 - az^2}}$$

$$\Rightarrow \frac{dz}{dt} = \frac{z}{\sqrt{t^2 - z^2}}$$

$$\Rightarrow \frac{dt}{dz} = \frac{\sqrt{t^2 - z^2}}{z}$$

$$\Rightarrow \frac{dt}{dz} = \sqrt{\left(\frac{t}{z}\right)^2 - 1} \quad \text{———— (4)}$$

which is linear homogeneous equation. To solve it let us put $\frac{t}{z} = v$ or $t = vz$

$~~eq(4) \Rightarrow~~$ $\frac{dt}{dz} = v + z\frac{dv}{dz}$

$$\therefore eq(4) \Rightarrow v + z\frac{dv}{dz} = \sqrt{v^2 - 1}$$

$$\Rightarrow z\frac{dv}{dz} = \sqrt{v^2 - 1} - v$$

$$\Rightarrow \frac{dz}{z} = \frac{dv}{\sqrt{v^2 - 1} - v}$$

$$\Rightarrow \frac{dz}{z} = \frac{(\sqrt{v^2 - 1} + v)\,dv}{v^2 - 1 - v^2}$$

$$\Rightarrow \frac{dz}{z} = -(\sqrt{v^2 - 1} + v)\,dv$$

$$\Rightarrow \log z = -\left[\frac{v}{2}\sqrt{v^2 - 1} - \frac{1}{2}\log\left(v + \sqrt{v^2 - 1}\right)\right] - \frac{v^2}{2} + b,$$

where $v = \frac{t}{z} = \frac{ax + y}{z\sqrt{a}}$

3) $pxy + pq + qy = yz$ ——— (1)

Let $f = pxy + pq + qy - yz = 0$ ——— (2)

Charpit's auxiliary equation is,

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}$$

$$\Rightarrow \frac{dp}{py - py} = \frac{dq}{px + q - z - qy} = \frac{dz}{-p(xy+q) - q(p+y)}$$

$$= \frac{dx}{-xy-q} = \frac{dy}{-p-y}$$

The first fraction gives, $dp = 0$

$\Rightarrow p = a$ ——— (3)

Putting $p = a$ in eq(1), we get

$axy + aq + qy = yz$

$\Rightarrow q(a+y) = yz - axy = y(z - ax)$

$\Rightarrow q = \dfrac{y(z-ax)}{a+y}$ ——— (4)

Now, $dz = p\,dx + q\,dy$

$$\Rightarrow dz = a\,dx + \frac{y(z-ax)}{a+y}\,dy$$

$$\Rightarrow dz - a\,dx = \frac{y(z-ax)}{a+y}\,dy$$

$$\Rightarrow \frac{dz - a\,dx}{z - ax} = \frac{y\,dy}{a+y} = \frac{(a+y-a)\,dy}{a+y}$$

$$\Rightarrow \frac{dz - a\,dx}{z - ax} = \left(1 - \frac{a}{a+y}\right)dy$$

Integrating, we get

$$\log(z-ax) = y - a\log(a+y) + \log b$$

$$\Rightarrow \log(z-ax) + \log(a+y)^a - \log b = y$$

$$\Rightarrow \boxed{(z-ax)(a+y)^a = be^y}$$

which is the required complete integral.

4) Given, $p = (qy+z)^2$ ——— (1)

Let $f = p-(qy+z)^2 = 0$ ——— (2)

Champit's auxiliary equation is,

$$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}$$

$$\Rightarrow \frac{dp}{0+p\{-2(qy+z)\}} = \frac{dq}{-2q(qy+z)+q\{-2(qy+z)\}}$$

$$= \frac{dz}{-p\cdot 1 - q\{-2y(qy+z)\}}$$

$$= \frac{dx}{-1} = \frac{dy}{2y(qy+z)}$$

Taking 1st and 5th ratio, we get

$$\frac{dp}{-2p(qy+z)} = \frac{dy}{2y(qy+z)}$$

$$\Rightarrow \frac{dp}{p} = -\frac{dy}{y}$$

$$\Rightarrow \log p = -\log y + \log a$$

$$\Rightarrow p = \frac{a}{y} \quad \text{——— (3)}$$

$\therefore$ eq (ii) $\Rightarrow \frac{a}{y} = (qy+z)^2$

$$\Rightarrow qy+z = \frac{\sqrt{a}}{\sqrt{y}}$$

$$\Rightarrow qy = \frac{\sqrt{a}}{\sqrt{y}} - z$$

$$\Rightarrow q = \frac{\sqrt{a}}{y\sqrt{y}} - \frac{z}{y} \quad \text{——— (4)}$$

Now, $dz = p\,dx + q\,dy$

$$= \frac{a}{y}dx + \left(\frac{\sqrt{a}}{y\sqrt{y}} - \frac{z}{y}\right)dy$$

$$\Rightarrow y\,dz = a\,dx + \frac{\sqrt{a}}{\sqrt{y}}dy - z\,dy$$

$$\Rightarrow y\,dz + z\,dy = a\,dx + \frac{\sqrt{a}}{\sqrt{y}}dy$$

$$\Rightarrow d(yz) = a\,dx + \sqrt{a}\,y^{-1/2}dy$$

Integrating we get,

$$yz = ax + 2\sqrt{a}\sqrt{y} + b$$

$$\Rightarrow \boxed{yz = ax + 2\sqrt{ay} + b}$$

which is the required complete integral.

* Homogeneous linear pde with constant coefficients :-

Homogeneous linear pde with constant coefficients is written as

$$\frac{\partial^n z}{\partial x^n} + A_1 \frac{\partial^n z}{\partial x^{n-1}\partial y} + A_2 \frac{\partial^n z}{\partial x^{n-2}\partial y^2} + \dots + A_n \frac{\partial^n z}{\partial y^n} = f(x,y) \quad \text{---} (1)$$

where $A_1, A_2, \dots, A_n$ are constants.

Put $\frac{\partial}{\partial x} = D$ and $\frac{\partial}{\partial y} = D'$, then eq(1) reduces to

$$\left(D^n + A_1 D^{n-1} D' + A_2 D^{n-2} D'^2 + \dots + A_n D'^n\right) z = f(x,y)$$

Note:- Auxiliary equation is obtained by putting $D = m$ and $D' = 1$, then

Auxiliary equation of (1) is

$$m^n + A_1 m^{n-1} + A_2 m^{n-2} + \dots + A_n = 0$$

On solving we get n roots, say $\alpha_1, \alpha_2, \dots, \alpha_n$ all are distinct then

$$C.F = f_1(y + \alpha_1 x) + f_2(y + \alpha_2 x) + \dots + f_n(y + \alpha_n x)$$

If α repeated twice then

$$C.F = f_1(y + \alpha x) + x f_2(y + \alpha x)$$

Q:- Solve $2r + 5s + 2t = 0$

A:- Given, $2r + 5s + 2t = 0$

$$\Rightarrow 2\frac{\partial^2 z}{\partial x^2} + 5\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = 0$$

$$\Rightarrow 2D^2 + 5DD' + 2D'^2 = 0$$

Put $D = m$ and $D' = 1$, then auxiliary equation is,

$$2m^2+5m+2=0$$

$$\Rightarrow m = \frac{-5\pm\sqrt{25-16}}{2\cdot 2} = \frac{-5\pm 3}{4}$$

$$\Rightarrow m = \frac{-1}{2}, -2$$

General solution is given by,

$$\boxed{z = f_1\left(y-\tfrac{1}{2}x\right) + f_2(y-2x)}$$

Q:- Solve $(D^3-4D^2D'+4DD'^2)z=0$

Sol:- Given, $(D^3-4D^2D'+4DD'^2)z=0$

Put $D=m$ and $D'=1$, then

Auxiliary equation is given by,

$$m^3-4m^2+4m=0$$

$$\Rightarrow m(m^2-4m+4)=0$$

$$\Rightarrow m(m-2)^2=0$$

$$\Rightarrow m=0,2,2$$

$\therefore$ Complete solution is given by,

$$z = f_1(y)+f_2(y+2x)+xf_3(y+2x)$$

Q:- Solve $(D^4-D'^4)z=0$

Sol:- Given, $(D^4-D'^4)z=0$

Put $D=m$ and $D'=1$ then auxiliary equation is given by,

$$m^4-1=0$$

$$\Rightarrow (m^2-1)(m^2+1)=0$$

$$\Rightarrow m=\pm 1, \pm i$$

$\therefore$ general solution is given by,

$$z=f_1(y+x)+f_2(y-x)+f_3(y+ix)+f_4(y-ix)$$

Particular integral:-

When $f(x,y)$ is in algebraic form then

$$P.I = \frac{1}{F(D,D')} f(x,y)$$

we express $F(D,D')$ by binomial theorem and then find P.I.

Q:- Solve $(D^2 - 2DD' + D'^2) z = 12xy$

Sol:- Given, $(D^2 - 2DD' + D'^2) z = 12xy$ ——— (1)

Put $D = m$ and $D' = 1$ then auxilliary equation is given by,

$$m^2 - 2m + 1 = 0$$
$$\Rightarrow (m-1)^2 = 0$$
$$\Rightarrow m = 1, 1$$

∴ Complementary function (C.F) is

$$C.F = f_1(y+x) + x f_2(y+x)$$

Now, $P.I = \frac{1}{(D^2 - 2DD' + D'^2)}(12xy)$

$$= \frac{1}{(D-D')^2}(12xy)$$

$$= \frac{1}{D^2\left(1 - \frac{D'}{D}\right)^2}(12xy)$$

$$= \frac{1}{D^2}\left(1 - \frac{D'}{D}\right)^{-2}(12xy)$$

$$= \frac{1}{D^2}\left(1 + \frac{2D'}{D} + \frac{3D'^2}{D^2} + \cdots\right)(12xy)$$

$$= \frac{1}{D^2}\left[12xy + 2 \cdot \frac{1}{D}(12x) + 0\right] = \frac{1}{D^2}(12xy + 12x^2)$$

$$= \frac{1}{D}(6x^2y + 4x^3)$$

$$= 2x^3y + x^4$$

$\therefore$ P.I $= 2x^3y + x^4$

Complete integral of (1) is

$z =$ C.F. + P.I.

$$\Rightarrow \boxed{z = f_1(y+x) + x f_2(y+x) + 2x^3y + x^4}$$

Q:- $(D^2 - 6DD' + 9D'^2)z = 12x^2 + 36xy$

Sol:- Given, $(D^2 - 6DD' + 9D'^2)z = 12x^2 + 36xy$ —(1)

Put $D = m$ and $D' = 1$, then auxiliary equation is given by, $m^2 - 6m + 9 = 0$

$$\Rightarrow (m-3)^2 = 0$$

$$\Rightarrow m = 3, 3$$

$\therefore$ Complementary function (C.F) is

C.F $= f_1(y+3x) + x f_2(y+3x)$

Now, P.I $= \dfrac{1}{(D^2 - 6DD' + 9D'^2)}(12x^2 + 36xy)$

$$= \frac{1}{(D - 3D')^2}(12x^2 + 36xy)$$

$$= \frac{1}{D^2\left[1 - \frac{3D'}{D}\right]^2}(12x^2 + 36xy)$$

$$= \frac{1}{D^2}\left(1 - \frac{3D'}{D}\right)^{-2}(12x^2 + 36xy)$$

$$= \frac{1}{D^2}\left[1+\frac{6D'}{D}+\frac{9D'^2}{D^2}+\cdots\right](12x^2+36xy)$$

$$= \frac{1}{D^2}\left[12x^2+36xy+\frac{6}{D}(36x)+\frac{9}{D^2}\cdot 0\right]$$

$$= \frac{1}{D^2}\left[12x^2+36xy+108x^2\right]$$

$$= \frac{1}{D}\left[4x^3+18x^2y+36x^3\right]$$

$$= x^4+6x^3y+9x^4$$

$$= 10x^4+6x^3y$$

$\therefore$ Complete integral of (1) is,

$z = C.F + P.I$

$$\Rightarrow \boxed{z = f_1(y+3x)+xf_2(y+3x)+10x^4+6x^3y}$$

<u>First order and first degree</u>

<u>Q:-</u> Find two families of surfaces that generate the characteristic of the pde

$$(3y-2z)p+(z-3x)q = 2x-y$$

<u>Sol:-</u> Given pde,

$$(3y-2z)p+(z-3x)q = 2x-y \quad \text{--- (1)}$$

Lagrange's auxiliary equation is,

$$\frac{dx}{3y-2z} = \frac{dy}{z-3x} = \frac{dz}{2x-y}$$

Using multipliers 1, 2 and 3, we get

each ratio $= \dfrac{dx+2dy+3dz}{3y-2z+2z-6x+6x-3y}$

$\Rightarrow dx + 2dy + 3dz = 0$

Integrating we get.

$$x + 2y + 3z = c_1 \quad \text{——— (2)}$$

Again using x, y and z as multipliers we get

each ratio $= \dfrac{x\,dx + y\,dy + z\,dz}{3xy - 2xz + yz - 3xy + 2xz - yz}$

$\Rightarrow x\,dx + y\,dy + z\,dz = 0$

$\Rightarrow x^2 + y^2 + z^2 = c_2 \quad \text{——— (3)}$

$\therefore$ $x + 2y + 3z = c_1$ and $x^2 + y^2 + z^2 = c_2$ are the two families of surfaces that generate the characteristic of the given pde.

Q:- Find the general integral of the following differential equation

$$x^2(y-z)p + y^2(z-x)q = z^2(x-y)$$

Sol:- Given pde,

$$x^2(y-z)p + y^2(z-x)q = z^2(x-y) \quad \text{——— (1)}$$

Lagranges auxiliary equation is,

$$\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)}$$

Using $\frac{1}{x^2}, \frac{1}{y^2}$ and $\frac{1}{z^2}$ as multipliers, we get

each ratio $= \dfrac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{y - z + z - x + x - y}$

$\Rightarrow \frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz = 0$

Integrating we get,

$$-\frac{1}{x}-\frac{1}{y}-\frac{1}{z}=c$$

$$\Rightarrow \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=c_1 \quad \text{------} (2)$$

Again using $\frac{1}{x}, \frac{1}{y}$ and $\frac{1}{z}$ as multipliers we get

$$\text{each ratio} = \frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{xy-xz+yz-yx+zx-zy}$$

$$\Rightarrow \frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz=0$$

Integrating, we get

$$\log x+\log y+\log z=\log c_2$$

$$\Rightarrow xyz=c_2 \quad \text{------} (3)$$

$\therefore$ The general integral of (1) is,

$$f\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z},\ xyz\right)=0$$

3) Solve $yz\frac{\partial z}{\partial x}+xz\frac{\partial z}{\partial y}=xy$ and hence find the integral surface passing through $z^2-y^2=1$, $x^2-y^2=y$

Sol:- Given, $yz\frac{\partial z}{\partial x}+xz\frac{\partial z}{\partial y}=xy$ ------ (1)

Lagrange's auxiliary equation is,

$$\frac{dx}{yz}=\frac{dy}{xz}=\frac{dz}{xy}$$

Taking 1st and 2nd ratio, we get

$$\frac{dx}{yz}=\frac{dy}{xz} \Rightarrow \frac{dx}{y}=\frac{dy}{x}$$

$$\Rightarrow x\,dx=y\,dy$$

Integrating we get,

$$x^2 - y^2 = c_1 \quad \text{——— (2)}$$

Taking 2nd and 3rd ratio, we get

$$\frac{dy}{xz} = \frac{dz}{xy}$$

$$\Rightarrow \frac{dy}{z} = \frac{dz}{y}$$

$$\Rightarrow z\,dz = y\,dy$$

Integrating we get,

$$z^2 - y^2 = c_2 \quad \text{——— (3)}$$

Given, the integral surface is passing through $z^2 - y^2 = 1$, $x^2 - y^2 = 4$

$\therefore$ (2) gives $c_1 = 4$ and $c_2 = 1$

Now $c_1 + c_2 = 5$

$$\Rightarrow x^2 - y^2 + z^2 - y^2 = 5$$

$$\Rightarrow \boxed{x^2 - 2y^2 + z^2 = 5},$$

which is the required integral surface.

Q:-4 Find the integral surface of the linear pde.

$$x(y^2+z^2)p + y(x^2+z^2)q = (x^2-y^2)z$$

which contains the straight line $x+y=0$ and $z=1$

Sol:- Given, pde

$$x(y^2+z^2)p + y(x^2+z^2)q = (x^2-y^2)z \quad \text{——— (1)}$$

Lagrange's auxiliary equation is

$$\frac{dx}{x(y^2+z^2)} = \frac{dy}{y(x^2+z^2)} = \frac{dz}{(x^2-y^2)z}$$

Using $\frac{1}{x}$, $\frac{-1}{y}$ and $\frac{1}{z}$ as multipliers, we get

$$\text{each ratio} = \frac{\frac{1}{x}dx - \frac{1}{y}dy + \frac{1}{z}dz}{\cancel{y^2} + \cancel{z^2} - \cancel{x^2} - \cancel{z^2} + \cancel{x^2} - \cancel{y^2}}$$

$$\Rightarrow \frac{1}{x}dx - \frac{1}{y}dy + \frac{1}{z}dz = 0$$

Integrating, we get

$$\log x - \log y + \log z = \log c_1$$

$$\Rightarrow \frac{xz}{y} = c_1 \quad \text{------} (2)$$

Using x, $-y$ and $-z$ as multipliers we get,

$$\text{each ratio} = \frac{x\,dx - y\,dy - z\,dz}{\cancel{x^2y^2} + \cancel{x^2z^2} - \cancel{y^2x^2} - \cancel{y^2z^2} - \cancel{x^2z^2} + \cancel{y^2z^2}}$$

$$\Rightarrow x\,dx - y\,dy - z\,dz = 0$$

Integrating, we get

$$x^2 - y^2 - z^2 = c_2 \quad \text{------} (3)$$

Given the integral surface contains the straight lines $x+y=0$ and $z=1$

$$\Rightarrow x = -y$$

From (2), $\frac{-y \cdot 1}{y} = c_1$

$$\Rightarrow c_1 = -1 \quad \text{------} (4)$$

From (3), $(-y)^2 - y^2 - 1 = c_2$

$$\Rightarrow y^2 - y^2 - 1 = c_2$$

$$\Rightarrow c_2 = -1 \quad \text{------} (5)$$

$\therefore \quad c_2 - c_1 = 0$

$$\Rightarrow \boxed{x^2 - y^2 - z^2 - \frac{xz}{y} = 0}$$

5) Find the integral surface of the linear pde

$$2y(z-3)p + (2x-z)q = y(2x-3)$$

which passes through the circle $x^2+y^2=2x$, $z=0$.

Sol:- Given pde,

$$2y(z-3)p + (2x-z)q = y(2x-3) \quad \text{——— (1)}$$

Lagrange's auxiliary equation is,

$$\frac{dx}{2y(z-3)} = \frac{dy}{(2x-z)} = \frac{dz}{y(2x-3)}$$

Using $x, 3y$ and $-z$ as multipliers, we get

$$\text{each ratio} = \frac{x\,dx + 3y\,dy - z\,dz}{\cancel{2xyz} - \cancel{6xy} + \cancel{6xy} - \cancel{3yz} - \cancel{2xyz} + \cancel{3yz}}$$

$$\Rightarrow x\,dx + 3y\,dy - z\,dz = 0$$

Integrating, we get

$$\frac{x^2}{2} + \frac{3y^2}{2} - \frac{z^2}{2} = c$$

$$\Rightarrow x^2 + 3y^2 - z^2 = c_1 \quad \text{——— (2)}$$

Again using $1, 2y$ and -2 as multipliers we get

$$\text{each ratio} = \frac{dx + 2y\,dy - 2dz}{\cancel{2yz} - \cancel{6y} + \cancel{4xy} - \cancel{2yz} - \cancel{4xy} + \cancel{6y}}$$

$$\Rightarrow dx + 2y\,dy - 2dz = 0$$

Integrating, we get

$$x + y^2 - 2z = c_2 \quad \text{——— (3)}$$

Given the integral surface passes through the circle $x^2+y^2 = 2x$, $z=0$ ——— (4)

$$\Rightarrow y^2 = 2x - x^2$$

Now from (3),

$x+y^2=c_2 \Rightarrow x+2x-x^2=c_2 \Rightarrow 3x-x^2=c_2$

$\Rightarrow x^2=3x-c_2$

eq(2) $\Rightarrow x^2+3y^2=c_1$

$\Rightarrow 3x-c_2+3y^2=c_1$

$\Rightarrow 3(x+y^2)-c_2=c_1$

$\Rightarrow 3c_2-c_2=c_1$

$\Rightarrow 2c_2=c_1$

$\Rightarrow 2c_2-c_1=0$ $\Rightarrow 2(x+y^2-2z)-(x^2+3y^2-z^2)=0$

$\Rightarrow 2(x+y^2)-(x^2+3y^2)=0$

$\Rightarrow \boxed{2x+2y^2-x^2-3y^2=0}$ $\Rightarrow 2x+2y^2-4z-x^2-3y^2+z^2=0$

$\Rightarrow 2x-y^2-4z-x^2+z^2=0$

$\Rightarrow 2x-x^2-y^2=0$

$\Rightarrow \boxed{x^2+y^2-2x=0}$, is the required integral surface passing through the circle $x^2+y^2=2x$ and $z=0$

2) If $f(x,y) = \phi(ax+by)$ then

$$P.I = \frac{1}{F(D,D')}\phi(ax+by)$$

case-i If $F(a,b) \neq 0$, then

$$P.I = \frac{1}{F(a,b)} \underbrace{\iint\dots\int}_{n \text{ times}} \phi(v)\,dv \quad ; \; v = ax+by$$

case-ii If $F(a,b) = 0$, then

$$P.I = \frac{1}{F(D,D')}\phi(ax+by) = \frac{x^n}{b^n n!}\phi(ax+by)$$

Q:- Solve $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 12(x+y)$

Sol:- Given, $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 12(x+y)$ ——— (1)

$$\Rightarrow (D^2 + D'^2)z = 12(x+y)$$

then by putting $D=m$ and $D'=1$, the auxiliary equation is

$$m^2+1=0 \Rightarrow m = \pm i$$

$\therefore$ $C.F = f_1(y+ix) + f_2(y-ix)$

Now, $P.I = \frac{1}{(D^2+D'^2)}\{12(x+y)\} = \frac{12}{D^2+D'^2}(x+y)$

$$= \frac{12}{1+1}\iint v\,dv \quad ; \; v = x+y$$

$$= \cancel{6}\cdot\frac{v^3}{\cancel{6}} = v^3 = (x+y)^3$$

$\therefore$ $P.I = (x+y)^3$

$\therefore$ complete solution is,

$$\boxed{z = f_1(y+ix) + f_2(y-ix) + (x+y)^3}$$

Q:- $(D^2-6DD'+9D'^2)z = 6x+2y$

Sol:- Given, $(D^2-6DD'+9D'^2)z = 6x+2y$ ——— (1)

Put $D=m$ and $D'=1$, then auxiliary equation is

$$m^2-6m+9=0$$

$$\Rightarrow (m-3)^2=0$$

$$\Rightarrow m=3,3$$

$\therefore$ C.F $= f_1(y+3x) + x f_2(y+3x)$

Now, $P.I = \dfrac{1}{(D^2-6DD'+9D'^2)}(6x+2y)$

$$= \frac{2}{(D-3D')^2}(3x+y)$$

$$= \frac{2\cdot x^2}{1^2\cdot 2!}(3x+y) = x^2(3x+y)$$

$$\therefore \boxed{z = f_1(y+3x) + x f_2(y+3x) + x^2(3x+y)}$$

Q:- Find P.I of $(2D-7D')^3 z = \log(14x+4y)$

Sol:- $P.I = \dfrac{1}{(2D-7D')^3}\log(14x+4y)$

$$= \frac{x^3}{2^3\cdot 3!}\log(14x+4y) = \frac{x^3}{48}\log(14x+4y)$$

<u>General method for finding P.I</u>

Let $(D-mD')z = f(x,y)$

$$\Rightarrow p-mq = f(x,y)$$

$\therefore$ Auxiliary equation is given by,

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{f(x,y)}$$

Now, taking 1st and 3rd ratio we have

$$\frac{dx}{1} = \frac{dz}{f(x,y)}$$

$$\Rightarrow dz = f(x,y)\,dx$$

$$\Rightarrow z = \int f(x,y)\,dx \quad ; \quad y = c - mx$$

$$\Rightarrow z = \int f(x, c-mx)\,dx$$

after integrating put the value of c.

Q:- $(D^2 - DD' - 2D'^2)z = (y-1)e^x$

Sol:- Given $(D^2 - DD' - 2D'^2)z = (y-1)e^x$

Put $D = m$ and $D' = 1$, then auxiliary equation is,

$$m^2 - m - 2 = 0$$

$$\Rightarrow m = -1, 2$$

$\therefore$ Complementary function is,

$$C.F = f_1(y-x) + f_2(y+2x)$$

Now, $$P.I = \frac{1}{(D^2 - DD' - 2D'^2)}(y-1)e^x$$

$$= \frac{1}{(D+D')(D-2D')}(y-1)e^x$$

$$= \frac{1}{(D+D')}\int (a-2x-1)e^x\,dx \quad ; \quad y+2x = a$$

$$= \frac{1}{(D+D')}\left(ae^x - 2(x-1)e^x - e^x\right)$$

$$= \frac{1}{(D+D')}(a-2x+2-1)e^x$$

$$= \frac{1}{(D+D')}(y+1)e^x$$

$= \int (x+b+1)e^x\,dx \quad ; \quad y-x=b$

$= xe^x - e^x + be^x + e^x$

$= (x+b)e^x$

$= ye^x$

$\therefore$ Complete solution is,

$$\boxed{z = f_1(y-x) + f_2(y+2x) + ye^x}.$$

Q:- $r+s-6t = y\cos(x)$

Sol:- Given $r+s-6t = y\cos(x)$

$\Rightarrow (D^2 + DD' - 6D'^2)z = y\cos(x)$ —— (1)

Put $D=m$ and $D'=1$, then the auxiliary equation is,

$m^2+m-6=0$

$\Rightarrow m^2+3m-2m-6=0$

$\Rightarrow m(m+3)-2(m+3)=0$

$\Rightarrow (m+3)(m-2)=0$

$\Rightarrow m=-3, 2$

$\therefore$ C.F $= f_1(y-3x) + f_2(y+2x)$

$$P.I = \frac{1}{(D^2+DD'-6D'^2)}\, y\cos(x)$$

$$= \frac{1}{(D+3D')(D-2D')}\, y\cos(x) \quad ; \quad y+2x=a$$

$$= \frac{1}{D+3D'} \int (a-2x)\cos x\,dx$$

$$= \frac{1}{(D+3D')}\left\{\int a\cos x\,dx - 2\int x\cos x\,dx\right\}$$

$$= \frac{1}{(D+3D')}\left\{a\sin x - 2\int x\cos x\,dx\right\}$$

Let $I = \int x\cos x\,dx$

$$= x\sin x + \cos x$$

$$= \frac{1}{(D+3D')}(a\sin x - 2x\sin x - 2\cos x)$$

$$= \frac{1}{(D+3D')}(y\sin x - 2\cos x) \qquad ; \; y - 3x = b$$

$$= \int\{(3x+b)\sin x - 2\cos x\}\,dx$$

$$= -(3x+b)\cos x + 3\sin x - 2\sin x$$

$$= -y\cos x + \sin x$$

$\therefore$ Complete solution is,

$$\boxed{z = f_1(y-3x) + f_2(y+2x) - y\cos x + \sin x}$$

Q:- Solve $(2D^2 - DD' - 3D'^2)z = 5e^{x-y}$

Sol:- Given $(2D^2 - DD' - 3D'^2)z = 5e^{x-y}$ ——— (1)

Put $D = m$ and $D' = 1$

The auxiliary equation is given by

$$2m^2 - m - 3 = 0$$

$$\Rightarrow m = \frac{1 \pm \sqrt{1+24}}{4} = \frac{1 \pm 5}{4} = \frac{3}{2} \text{ or } -1$$

$$\therefore \text{C.F} = f_1\left(y + \frac{3}{2}x\right) + f_2(y - x)$$

Now, $P.I = \frac{1}{(2D^2 - DD' - 3D'^2)}(5e^{x-y})$

$$= \frac{1}{(D+D')(2D-3D')} 5e^{x-y}$$

$$= \frac{1}{(D+D')} \cdot \frac{5}{(2+3)} \int e^v dv \quad , v = x-y$$

$$= \frac{1}{D+D'} e^v$$

$$= \frac{1}{D+D'} \cdot e^{x-y}$$

$$= \frac{x}{(1)^1 1!} e^{x-y}$$

$$= x e^{x-y}$$

$\therefore$ Complete integral is

$$\boxed{z = f_1\left(y + \frac{3}{2}x\right) + f_2(y-x) + x e^{x-y}}$$

Q:- 1) $(D^2 + 2DD' + 15D'^2) z = 12xy$

2) $(D^2 + 2DD' + D'^2) z = 2\cos y - x\sin y$

Sol:- 1) Given, $(D^2 - 2DD' - 15D'^2) z = 12xy$ —(1)

Put $D = m$ and $D' = 1$, then the auxiliary equation is, $m^2 - 2m - 15 = 0$

$$\Rightarrow m^2 - 5m + 3m - 15 = 0$$

$$\Rightarrow m(m-5) + 3(m-5) = 0$$

$$\Rightarrow (m-5)(m+3) = 0$$

$$\Rightarrow m = 5, -3$$

$\therefore$ C.F $= f_1(y+5x) + f_2(y-3x)$

Now, $\displaystyle P.I = \frac{1}{(D^2 - 2DD' - 15D'^2)}(12xy)$

$$= \frac{1}{D^2\left[1 - \left(\frac{2D'}{D} + \frac{15D'^2}{D^2}\right)\right]}(12xy)$$

$$= \frac{12}{D^2}\left[1 - \left(\frac{2D'}{D} + \frac{15D'^2}{D^2}\right)\right]^{-1}(12xy)$$

$$= \frac{12}{D^2}\left(1 + \frac{2D'}{D} + \frac{15D'^2}{D^2} + \dots\right)(xy)$$

$$= \frac{12}{D^2}\left[xy + \frac{2}{D}(x) + \frac{15}{D^2}(0) + \dots\right]$$

$$= \frac{12}{D^2}\left[xy + x^2\right]$$

$$= \frac{12}{D}\left[\frac{x^2y}{2} + \frac{x^3}{3}\right] = 12\left(\frac{x^3y}{6} + \frac{x^4}{12}\right)$$

$$= 2x^3y + x^4$$

Hence, the required general solution is,

$$\boxed{z = f_1(y+5x) + f_2(y-3x) + 2x^3y + x^4}.$$

2) Given, $(D^2 + 2DD' + D'^2)z = 2\cos y - x\sin y$ ——(1)

Put $D = m$ and $D' = 1$, then the auxiliary equation is,

$$m^2 + 2m + 1 = 0$$

$$\Rightarrow (m+1)^2 = 0$$

$$\Rightarrow m = -1, -1$$

$\therefore$ C.F. $= f_1(y-x) + xf_2(y-x)$

Now, $\displaystyle P.I = \frac{1}{(D^2 + 2DD' + D'^2)}(2\cos y - x\sin y)$

$$= \frac{1}{(D+D')^2}(2\cos y - x\sin y)$$

$$= \frac{1}{(D+D')(D+D')}(2\cos y - x\sin y) \quad , \quad y-x=a$$

$$= \frac{1}{D+D'}\int[2\cos(x+a) - x\sin(x+a)]\,dx$$

$$= \frac{1}{D+D'}\left[2\sin(x+a) + x\cos(x+a) - \sin(x+a)\right]$$

$$= \frac{1}{D+D'}(\sin y + x\cos y) \quad , \quad \text{take } y-x=b$$

$$= \int[\sin(x+b) + x\cos(x+b)]\,dx$$

$$= -\cos(x+b) + x\sin(x+b) + \cos(x+b)$$

$$= \cancel{x\cos}\; x\sin y$$

$\therefore$ The general solution is,

$$\boxed{z = f_1(y-x) + x f_2(y-x) + x\sin y}$$

* Non-Homogeneous linear pde

Let us consider simplest case

$(D - mD' - a)\,z = 0$

$\Rightarrow p - mq = az$

The Lagrange's auxiliary equation is,

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{az}$$

Then from 1st and 2nd ratio, we get

$dy + m\,dx = 0$

$\Rightarrow y + mx = c_1$ (constant)

Now from 1st and 3rd ratio, we get

$$dx = \frac{dz}{az}$$

$$\Rightarrow \log z = ax + \log c_2$$

$$\Rightarrow z = e^{ax} \cdot c_2$$

$$\Rightarrow \frac{z}{e^{ax}} = c_2$$

$\therefore$ Complete solution is, $\frac{z}{e^{ax}} = f(y+mx)$

$$\Rightarrow z = e^{ax} f(y+mx)$$

Similarly, the complete integral of

$(D-m_1D'-a_1)(D-m_2D'-a_2)\cdots(D-m_nD'-a_n)z = 0$ is

$$\boxed{z = e^{a_1x} f_1(y+m_1x) + e^{a_2x} f_2(y+m_2x) + \cdots + e^{a_nx} f_n(y+m_nx)}$$

In case of repeated factors

$$(D-mD'-a)^3 z = 0$$

The integral is

$$z = e^{ax} f_1(y+mx) + x e^{ax} f_2(y+mx) + x^2 e^{ax} f_3(y+mx)$$

<u>Note:-</u> In case $f(D, D')$ can not be resolved into linear factor of D and D' then we can not be integrated by above methods, then for finding C.F we assume

$$z = \sum A e^{hx+ky}$$

be the C.F corresponding to $f(D, D')$ and then find relation in h and k and putting the value of h and k, we get required C.F.

Q:- Solve $(D^2 - D'^2 + D - D')z = 0$

Sol:- Given equation is,

$$(D^2 - D'^2 + D - D')z = 0$$

$$\Rightarrow \{(D-D')(D+D') + (D-D')\}z = 0$$

$$\Rightarrow (D-D')(D+D'+1)z = 0$$

$\therefore$ Complete integral is,

$$\boxed{z = f_1(y+x) + e^{-x} f_2(y-x)}$$

Q:- $DD'(D - 2D' - 3)z = 0$

Sol:- Given $DD'(D-2D'-3)z = 0$

$\Rightarrow Dz = 0$ or $D'z = 0$ or $(D-2D'-3)z = 0$

$\Rightarrow \frac{\partial z}{\partial x} = 0$ or $\frac{\partial z}{\partial y} = 0$ or $(D-2D'-3)z = 0$

$\Rightarrow z = f_1(y)$ $\Rightarrow z = f_2(x)$ $\Rightarrow z = e^{3x} f_3(y+2x)$

$\therefore$ Complete integral is,

$$\boxed{z = f_1(y) + f_2(x) + e^{3x} f_3(y+2x)}$$

Q:- Solve $(2D^4 - 3D^2D' + D'^2)z = 0$

Sol:- Given, $(2D^4 - 3D^2D' + D'^2)z = 0$ ——(1)

$$\Rightarrow (2D^4 - 2D^2D' - D^2D' + D'^2)z = 0$$

$$\Rightarrow \{2D^2(D^2 - D') - D'(D^2 - D')\}z = 0$$

$$\Rightarrow (2D^2 - D')(D^2 - D')z = 0 \quad \text{——(2)}$$

which is not in linear factors of D and D'.

Let $z = \sum Ae^{hx+ky}$ be the C.F corresponding to $(D^2 - D')z = 0$

$\therefore (D^2 - D')z = 0$

$\Rightarrow \sum Ah^2 e^{hx+ky} - \sum Ak e^{hx+ky} = 0$

$\Rightarrow \sum A(h^2 - k) e^{hx+ky} = 0$

$\Rightarrow h^2 - k = 0$

$\Rightarrow k = h^2$

$\therefore$ C.F corresponding to $(D^2 - D')z$ is

$$z = \sum Ae^{hx+h^2y}$$

Now, let $z = \sum Be^{h'x+k'y}$ be the C.F corresponding to $(2D^2 - D')z = 0$

$\Rightarrow 2\sum Bh'^2 e^{h'x+k'y} - \sum Bk' e^{h'x+k'y} = 0$

$\Rightarrow \sum B(2h'^2 - k') e^{h'x+k'y} = 0$

$\Rightarrow 2h'^2 - k' = 0$

$\Rightarrow k' = 2h'^2$

$\therefore$ C.F corresponding to $(2D^2 - D')z$ is

$$z = \sum Be^{h'x+2h'^2y}$$

Hence, complete solution of (1) is given by

$$\boxed{z = \sum Ae^{hx+h^2y} + \sum Be^{h'x+2h'^2y}}$$

Q:- Solve, $(D-2D'-1)(D-2D'^2-1)z=0$

Sol:- Given, $(D-2D'-1)(D-2D'^2-1)z=0$ ——— (1)

$\Rightarrow (D-2D'-1)z=0$ or $(D-2D'^2-1)z=0$

C.F corresponding $(D-2D'-1)z$ is

$$z = e^{x} f_1(y+2x)$$

Let $z = \sum A e^{hx+ky}$ be the C.F corresponding to $(D-2D'^2-1)z=0$

$\Rightarrow \sum A h e^{hx+ky} - 2\sum A k^2 e^{hx+ky} - \sum A e^{hx+ky} = 0$

$\Rightarrow \sum A(h-2k^2-1)e^{hx+ky} = 0$

$\Rightarrow h-2k^2-1=0$

$\Rightarrow h = 2k^2+1$

$\therefore$ C.F corresponding to $(D-2D'^2-1)z$ is

$$\sum A e^{(2k^2+1)x+ky}$$

$\therefore$ The required solution is,

$$z = e^{x} f_1(y+2x) + \sum A e^{(2k^2+1)x+ky}$$

Particular integral

1) If $f(x,y) = e^{ax+by}$, then

$$P.I = \frac{1}{f(D,D')} e^{ax+by} = \frac{1}{f(a,b)} e^{ax+by}$$, provided $f(a,b) \neq 0$.

2) If $f(x,y) = \sin(ax+by)$ or $\cos(ax+by)$ then

$P.I = \frac{1}{f(D,D')} \sin(ax+by)$ is obtained by putting

$D^2 = -a^2$ and $D'^2 = -b^2$, $DD' = -ab$

3) If $f(x,y) = x^m y^n$ then

$$P.I = \frac{1}{f(D,D')} x^m y^n = [f(D,D')]^{-1} x^m y^n$$

4) If $f(x,y) = e^{ax+by} V$, then

$$P.I = \frac{1}{f(D,D')} e^{ax+by} V = e^{ax+by}\frac{1}{f(D+a, D'+b)} V$$

Q.1 Solve $(D-D'-1)(D-D'-2)z = e^{2x-y} + x$

Sol:- Given $(D-D'-1)(D-D'-2)z = e^{2x-y} + x$ —— (1)

$C.F = e^{x} f_1(y+x) + e^{2x} f_2(y+x)$

and P.I corresponding to e^{2x-y} is

$$\frac{1}{(D-D'-1)(D-D'-2)} e^{2x-y} = \frac{1}{12} e^{2x-y}$$

P.I corresponding to $x = \frac{1}{(D-D'-1)(D-D'-2)} x$

$$= \frac{1}{2}[1-(D-D')]^{-1}\left[1-\left(\frac{D}{2}-\frac{D'}{2}\right)\right]^{-1} x$$

$$= \frac{1}{2}[1+D-D'+(D-D')^2+\dots]\left[1+\frac{D}{2}-\frac{D'}{2}+\left(\frac{D}{2}-\frac{D'}{2}\right)^2+\dots\right]x$$

$$= \frac{1}{2}\left[1+\frac{D}{2}+D+\dots\right]x$$

$$= \frac{1}{2}\left(x+\frac{3}{2}\right)$$

$\therefore$ Complete integral is

$$\boxed{z = e^{x} f_1(y+x) + e^{2x} f_2(y+x) + \frac{x}{2} + \frac{3}{4} + \frac{1}{2}e^{2x-y}}$$

Q.2 $(D^3 - 3DD' + D' + 1)z = e^{2x+3y}$

Sol:- Given $(D^3 - 3DD' + D' + 1)z = e^{2x+3y}$ ——— (1)

Here $(D^3 - 3DD' + D' + 1)$ can not be resolved into linear factors in D and D'. Hence for finding C.F. consider the equation

$$(D^3 - 3DD' + D' + 1)z = 0 \quad \text{——— (2)}$$

Let a trial solution of (2) will be

$$z = \sum A e^{hx+ky}$$

$\therefore$ eq(2) $\Rightarrow \sum Ah^3 e^{hx+ky} - 3\sum Ahk e^{hx+ky} + \sum Ak e^{hx+ky} + \sum A e^{hx+ky} = 0$

$$\Rightarrow \sum A(h^3 - 3hk + k + 1) e^{hx+ky} = 0$$

$$\Rightarrow h^3 - 3hk + k + 1 = 0$$

$$\Rightarrow h^3 + 1 = 3hk - k = (3h-1)k$$

$$\Rightarrow k = \frac{h^3+1}{3h-1}$$

$$\therefore \text{C.F} = \sum A e^{hx + \left(\frac{h^3+1}{3h-1}\right)y}$$

$$P.I = \frac{1}{(D^3 - 3DD' + D' + 1)} e^{2x+3y}$$

$$= \frac{1}{(2)^3 - 3\times 2\times 3 + 3 + 1} e^{2x+3y}$$

$$= \frac{-1}{6} e^{2x+3y}$$

Hence the required general solution is

$$Z = C.F + P.I$$

$$\Rightarrow \boxed{z = \sum A e^{hx + \left(\frac{h^3+1}{3h-1}\right) y} - \frac{1}{6} e^{2x+3y}}$$

Q:-3 $\quad s + p - q = z + xy$

Sol:- Given, $s + p - q = z + xy$

$$\Rightarrow \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = z + xy$$

$$\Rightarrow (DD' + D - D') z = z + xy$$

$$\Rightarrow (DD' + D - D' - 1) z = xy$$

For finding C.F. consider the equation

$$(DD' + D - D' - 1) z = 0$$

$$\Rightarrow \{D(D'+1) - (D'+1)\} z = 0$$

$$\Rightarrow (D-1)(D'+1) z = 0$$

$$\therefore \; C.F = e^{x} f_1(y) + e^{-y} f_2(x)$$

$$P.I = \frac{1}{(D-1)(D'+1)} xy = -\frac{1}{(1-D)(1+D')} xy$$

$$= -(1-D)^{-1}(1+D')^{-1} xy$$

$$= -(1 + D + D^2 + \dots)(1 - D' + D'^2 - \dots) xy$$

$$= -(1 + D + D^2 + \dots)(xy - x) = -(xy - x + y - 1)$$

$\therefore$ P.I $= -xy + x - y + 1$

Hence the required solution is,

$$\boxed{z = e^{x} f_1(y) + e^{-y} f_2(x) - xy + x - y + 1}$$

Q.4 $(D^2 - D'^2 - 3D + 3D')z = xy + e^{x+2y}$

Sol:- Given, $(D^2 - D'^2 - 3D + 3D')z = xy + e^{x+2y}$ ——— (1)

For finding C.F consider the equation,

$$(D^2 - D'^2 - 3D + 3D')z = 0$$

$$\Rightarrow \{(D - D')(D + D') - 3(D - D')\} z = 0$$

$$\Rightarrow (D - D')(D + D' - 3)z = 0$$

$\therefore$ C.F $= f_1(y + x) + e^{3x} f_2(y - x)$

P.I corresponding to $xy = \dfrac{1}{(D - D')(D + D' - 3)} xy$

$$= \frac{1}{D\left(1 - \frac{D'}{D}\right)(-3)\left(1 - \frac{D + D'}{3}\right)} xy$$

$$= \frac{-1}{3D}\left(1 - \frac{D'}{D}\right)^{-1}\left(1 - \frac{D + D'}{3}\right)^{-1} xy$$

$$= \frac{-1}{3D}\left(1 + \frac{D'}{D} + \frac{D'^2}{D^2} + \cdots\right)\left[1 + \frac{D + D'}{3} + \left(\frac{D + D'}{3}\right)^2 + \cdots\right] xy$$

$$= \frac{-1}{3D}\left(1 + \frac{D'}{D} + \frac{D'^2}{D^2} + \cdots\right)\left(1 + \frac{D}{3} + \frac{D'}{3} + \frac{2DD'}{9} + \cdots\right)(xy)$$

$$= \frac{-1}{3D}\left(1 + \frac{D'}{D} + \frac{D'^2}{D^2} + \cdots\right)\left(xy + \frac{y}{3} + \frac{x}{3} + \frac{2}{9}\right)$$

$$= \frac{-1}{3D}\left[xy + \frac{y}{3} + \frac{x}{3} + \frac{2}{9} + \frac{1}{D}\left(x + \frac{1}{3}\right)\right]$$

$$= \frac{-1}{3D}\left[xy + \frac{y}{3} + \frac{x}{3} + \frac{2}{9} + \frac{x^2}{2} + \frac{x}{3}\right]$$

$$= \frac{-1}{3}\left(\frac{x^2y}{2} + \frac{xy}{3} + \frac{x^2}{6} + \frac{2x}{9} + \frac{x^3}{6} + \frac{x^2}{6}\right)$$

$$= \frac{-1}{3}\left(\frac{x^2y}{2} + \frac{xy}{3} + \frac{x^2}{3} + \frac{2x}{9} + \frac{x^3}{6}\right)$$

$$= -\frac{x^2y}{6} - \frac{xy}{9} - \frac{x^2}{9} - \frac{2x}{27} - \frac{x^3}{18}$$

P.I corresponding to e^{x+2y} is

$$= \frac{1}{(D-D')(D+D'-3)} e^{x+2y}$$

$$= \frac{1}{(D+D'-3)} \frac{1}{(D-D')} e^{x+2y}$$

$$= \frac{1}{(D+D'-3)} \frac{1}{(1-2)} e^{x+2y}$$

$$= \frac{-1}{D+D'-3} e^{x+2y}$$

$$= \frac{-e^{x+2y}}{(D+1)+(D'+2)-3}(1)$$

$$= -e^{x+2y} \frac{1}{D+D'}(1)$$

$$= -e^{x+2y} \frac{1}{D\left[1+\frac{D'}{D}\right]}(1)$$

$$= -e^{x+2y} \frac{1}{D}\left(1+\frac{D'}{D}\right)^{-1}(1)$$

$$= -e^{x+2y} \cdot \frac{1}{D}\left(1-\frac{D'}{D}+\dots\right)(1)$$

$$= -e^{x+2y} \cdot x$$

$\therefore$ The required solution is,

$$z = f_1(y+x) + e^{3x} f_2(y-x) - \frac{x^2y}{6} - \frac{xy}{9} - \frac{x^2}{9} - \frac{2x}{27} - \frac{x^3}{18} - xe^{x+2y}$$

Q.5 $(D^2 - DD' + D' - 1)z = \cos(x+2y) + e^y$

Sol:- Given $(D^2 - DD' + D' - 1)z = \cos(x+2y) + e^y$ ——(1)

For finding C.F consider the equation,

$(D^2 - DD' + D' - 1)z = 0$

$\Rightarrow$ ~~D(D-D'~~ $(D^2 - 1 - DD' + D')z = 0$

$\Rightarrow \{(D-1)(D+1) - D'(D-1)\}z = 0$

$\Rightarrow (D-1)(D-D'+1)z = 0$

$\therefore$ C.F $= e^{x} f_1(y) + e^{-x} f_2(y+x)$

P.I corresponding to $\cos(x+2y)$ is,

$$= \frac{1}{(D^2 - DD' + D' - 1)} \cos(x+2y)$$

$$= \frac{1}{-1^2 + (1\times 2) + D' - 1} \cos(x+2y)$$

$$= \frac{1}{D'} \cos(x+2y) = \frac{1}{2}\sin(x+2y)$$

P.I corresponding to e^y i.e. $e^{0.x+1.y}$

$$= \frac{1}{D^2 - DD' + D' - 1} e^{0.x+1.y}$$

$$= e^{0.x+1.y} \frac{1}{(D+0)^2 - (D+0)(D'+1) + (D'+1) - 1} (1)$$

$$= e^y \frac{1}{D^2 - DD' - D + D' + \cancel{1} - \cancel{1}} (1)$$

$$= e^y \frac{1}{D^2 - DD' - D + D'} (1)$$

$$= e^y \frac{1}{D(D-D') - 1(D-D')} (1)$$

$$= e^{y} \frac{1}{(D-1)(D-D')}(1)$$

$$= -e^{y} \frac{1}{(1-D)\,D\left(1-\frac{D'}{D}\right)}(1)$$

$$= -e^{y} \frac{1}{D}(1-D)^{-1}\left(1-\frac{D'}{D}\right)^{-1}(1)$$

$$= -e^{y} \frac{1}{D}(1+D+\dots)\left(1+\frac{D'}{D}+\dots\right)(1)$$

$$= -e^{y} \frac{1}{D}(1) = -xe^{y}$$

$\therefore$ The required general solution is,

$$\boxed{z = e^{x} f_1(y) + e^{-x} f_2(y+x) + \frac{1}{2}\sin(x+2y) - xe^{y}}$$

Equation reducible to homogeneous linear form

An equation in which the coefficient of derivative of any order is multiple of the variables of the same degree, may be transformed into the pde with constant coefficients.

For that let $x = e^{X}$ and $y = e^{Y}$

$$\Rightarrow X = \log x \text{ and } Y = \log y$$

$$\therefore \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\cdot\frac{\partial X}{\partial x} = \frac{1}{x}\frac{\partial z}{\partial X} \Rightarrow x\frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} = Dz \text{ (say)}$$

Now, $x\frac{\partial}{\partial x}\left(x^{n-1}\frac{\partial^{n-1} z}{\partial x^{n-1}}\right) = x^{n}\frac{\partial^{n} z}{\partial x^{n}} + (n-1)x^{n-1}\frac{\partial^{n-1} z}{\partial x^{n-1}}$

$$\Rightarrow x^{n}\frac{\partial^{n} z}{\partial x^{n}} = x\frac{\partial}{\partial x}\left(x^{n-1}\frac{\partial^{n-1} z}{\partial x^{n-1}}\right) - (n-1)x^{n-1}\frac{\partial^{n-1} z}{\partial x^{n-1}}$$

$$\Rightarrow x^{n}\frac{\partial^{n} z}{\partial x^{n}} = \left(x\frac{\partial}{\partial x} - n + 1\right)x^{n-1}\frac{\partial^{n-1} z}{\partial x^{n-1}}$$

$$= (D-n+1)\,x^{n-1}\frac{\partial^{n-1} z}{\partial x^{n-1}}$$

Putting $n = 2, 3, 4, \ldots$ we have

$$x^2 \frac{\partial^2 z}{\partial x^2} = (D-1)\, x \frac{\partial z}{\partial x} = D(D-1)z$$

$$x^3 \frac{\partial^3 z}{\partial x^3} = (D-2)\, x^2 \frac{\partial^2 z}{\partial x^2} = D(D-1)(D-2)z$$

Similarly, $y \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y} = D'z$

$\therefore \; y^2 \frac{\partial^2 z}{\partial y^2} = D'(D'-1)z$ etc.

and $xy \frac{\partial^2 z}{\partial x \partial y} = DD'z$ etc.

Substituting these values in $f(D, D')$ then it reduces to pde with constant coefficients.

1) Solve $x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = 0$ ——— (1)

Sol:- Put $x = e^X$ and $y = e^Y$

i.e. $X = \log x$ and $Y = \log y$

and denote $\frac{\partial}{\partial X} = D$ and $\frac{\partial}{\partial Y} = D'$ then eq(1) reduces to

$$\{D(D-1) + 2DD' + D'(D'-1)\} z = 0$$

$$\Rightarrow (D^2 - D + 2DD' + D'^2 - D')z = 0$$

$$\Rightarrow (D^2 + DD' + DD' + D'^2 - D - D')z = 0$$

$$\Rightarrow \{D(D+D') + D'(D+D') - (D+D')\} z = 0$$

$$\Rightarrow (D+D')(D+D'-1)z = 0$$

$\therefore$ Complete integral is,

$$z = f_1(Y - X) + e^X f_2(Y - X)$$

$$= f_1\left(\log \frac{y}{x}\right) + x\, f_2\left(\log \frac{y}{x}\right)$$

$$\Rightarrow \boxed{z = \phi_1(y/x) + x\,\phi_2(y/x)}$$

2) $x^2 \dfrac{\partial^2 z}{\partial x^2} + 2xy \dfrac{\partial^2 z}{\partial x \partial y} + y^2 \dfrac{\partial^2 z}{\partial y^2} - nx \dfrac{\partial z}{\partial x} - ny \dfrac{\partial z}{\partial y} + nz = x^2 + y^2$

Put $x = e^X$ and $y = e^Y$ i.e. $X = \log x$ and $Y = \log y$, $\frac{\partial}{\partial X} = D$ and $\frac{\partial}{\partial Y} = D'$

Sol:- $\{D(D-1) + 2DD' + D'(D'-1) - nD - nD' + n\} z = e^{2X} + e^{2Y}$

$$\Rightarrow \{D^2 - D + 2DD' + D'^2 - D' - nD - nD' + n\} z = e^{2X} + e^{2Y}$$

$$\Rightarrow (D + D' - 1)(D + D' - n) z = e^{2X} + e^{2Y}$$

$$\therefore C.F = e^X f_1(Y - X) + e^{nX} f_2(Y - X)$$

$$= x\,\phi_1(y/x) + x^n \phi_2(y/x)$$

$$P.I = \frac{1}{(D + D' - 1)(D + D' - n)} (e^{2X} + e^{2Y})$$

$$= \frac{1}{(D + D' - 1)(D + D' - n)} e^{2X} + \frac{1}{(D + D' - 1)(D + D' - n)} e^{2Y}$$

$$= \frac{1}{(2 + 0 - 1)(2 + 0 - n)} e^{2X} + \frac{1}{(0 + 2 - 1)(0 + 2 - n)} e^{2Y}$$

$$= \frac{e^{2X} + e^{2Y}}{2 - n} = \frac{x^2 + y^2}{2 - n}$$

$\therefore$ Complete solution is,

$$\boxed{z = x\,\phi_1(y/x) + x^n \phi_2(y/x) + \frac{x^2 + y^2}{2 - n}}$$

3) $x^2\dfrac{\partial^2 z}{\partial x^2} - 4xy\dfrac{\partial^2 z}{\partial x \partial y} + 4y^2\dfrac{\partial^2 z}{\partial y^2} + 6y\dfrac{\partial z}{\partial y} = x^3y^4$

Sol:- $\left[D(D-1) - 4DD' + 4D'(D'-1) + 6D'\right]z = e^{3X}\cdot e^{4Y}$

$\Rightarrow (D^2 - D - 4DD' + 4D'^2 - 4D' + 6D')z = e^{3X+4Y}$.

$\Rightarrow (D^2 - 4DD' + 4D'^2 - D + 2D')z = e^{3X+4Y}$

$\Rightarrow \{(D-2D')^2 - (D-2D')\}z = e^{3X+4Y}$

$\Rightarrow (D-2D')(D-2D'-1)z = e^{3X+4Y}$

$$C.F = f_1(Y+2X) + e^X f_2(Y+2X)$$
$$= f_1(\log y + \log x^2) + x f_2(\log y + \log x^2)$$
$$= f_1(\log(yx^2)) + x f_2(\log(yx^2))$$
$$= \phi_1(yx^2) + x\phi_2(yx^2)$$

$$P.I = \frac{1}{(D-2D')(D-2D'-1)}e^{3X+4Y}$$
$$= \frac{1}{(3-(2\times4))(3-(2\times4)-1)}e^{3X+4Y}$$
$$= \frac{1}{30}x^3y^4$$

$\therefore$ The required general solution is,

$$\boxed{z = \phi_1(yx^2) + x\phi_2(yx^2) + \frac{1}{30}x^3y^4}$$

Introduction to Cauchy's problem :-

Let $Pp + Qq = R$ ——— (1)

be the given equation of pde

and let $u(x,y,z) = c_1$ and $v(x,y,z) = c_2$ ——— (2)

be two independent solutions of (1)

then we wish to obtain the Integral surface which passes through the curve

$$x = x(t) \ , \ y = y(t) \ \text{and} \ z = z(t) \quad \text{——— (3)}$$

where t is a parameter.

then eq(2) becomes,

$u[x(t), y(t), z(t)] = c_1$ and

$v[x(t), y(t), z(t)] = c_2$ ——— (4)

After eliminating t from (4), we get a relation in c_1 and c_2.

Finally by replacing c_1 and c_2 with the help of (2), we obtained required integral surface.

1) Find integral surface of the linear pde

$$x(y^2+z)p - y(x^2+z)q = (x^2-y^2)z$$

which contains the straight line $x+y=0$, $z=1$.

Sol:- Given linear pde,

$$x(y^2+z)p - y(x^2+z)q = (x^2-y^2)z \quad \text{——— (1)}$$

Lagrange's auxiliary equation is,

$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{(x^2-y^2)z}$$

Using $\frac{1}{x}, \frac{1}{y}$ and $\frac{1}{z}$ as multipliers, we get each ratio = $\dfrac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{\cancel{y^2}+\cancel{z}-\cancel{x^2}-\cancel{z}+\cancel{x^2}-\cancel{y^2}}$

$\Rightarrow \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$

Integrating, we get

$\log x + \log y + \log z = \log c_1$

$\Rightarrow xyz = c_1$ ———— (2)

Again, using x, y and -1 as multipliers, we get

each ratio = $\dfrac{x\,dx + y\,dy - dz}{\cancel{x^2y^2}+\cancel{x^2z}-\cancel{y^2x^2}-\cancel{y^2z}-\cancel{x^2z}+\cancel{y^2z}}$

$\Rightarrow x\,dx + y\,dy - dz = 0$

Integrating, we get

$\frac{x^2}{2} + \frac{y^2}{2} - z = c_2$

$\Rightarrow x^2+y^2-2z = c_3$ ———— (3)

Taking t as parameter then the straight line $x+y=0$ and $z=1$ can be put in parametric form

$x=t,\ y=-t,\ z=1$

$\therefore$ eq (2) $\Rightarrow -t^2 = c_1$

eq (3) $\Rightarrow 2t^2 - 2 = c_3$

$\Rightarrow 2(-c_1) - 2 = c_3$

$\Rightarrow 2c_1 + c_3 + 2 = 0$

By putting the value of c_1 and c_3 from (2) & (3) respectively we get the required surface i.e. $\boxed{2xyz + x^2 + y^2 - 2z + 2 = 0}$.

2) Integral surface of pde $x^2p + y^2q = -z^2$ which passes through hyperbola $xy = x+y$, $z = 1$ is given by,

(a) $xy + 2yz + xz = 3xyz$ (b) $yz + 2xy + xz = 3xyz$

(c) $xz + 2yz + y^2 = 3xyz$ (d) $xy + 2xz + yz = 3xyz$

Sol:- Given pde, $x^2p + y^2q = -z^2$ ——— (1)

Lagrange's auxiliary equation is,

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{-z^2}$$

Taking 1st two ratios, we get

$$\frac{dx}{x^2} = \frac{dy}{y^2} \Rightarrow \frac{dx}{x^2} - \frac{dy}{y^2} = 0$$

Integrating, we get

$$-\frac{1}{x} + \frac{1}{y} = c_1 \quad \text{——— (2)}$$

Taking 2nd and 3rd ratios, we get

$$\frac{dy}{y^2} + \frac{dz}{z^2} = 0$$

Integrating we get,

$$\frac{-1}{y} - \frac{1}{z} = c_2$$

$$\Rightarrow \frac{1}{y} + \frac{1}{z} = c_3 \quad \text{——— (3)}$$

Taking t as parameter, the hyperbola $xy = x+y$ and $z = 1$ can be put in parametric form

$$x = t, \quad y = \frac{t}{t-1}, \quad z = 1$$

$\therefore$ eq(2) $\Rightarrow \frac{-1}{t} + \frac{t-1}{t} = c_1$

$\Rightarrow \frac{t-2}{t} = c_1 \Rightarrow 1 - \frac{2}{t} = c_1 \Rightarrow \frac{2}{t} = 1 - c_1 \Rightarrow t = \frac{2}{1-c_1}$

eq(3) $\Rightarrow \frac{t-1}{t} + 1 = c_3$

$\Rightarrow \frac{t-1+t}{t} = c_3$

$\Rightarrow \frac{2t-1}{t} = c_3$

$\Rightarrow 2 - \frac{1}{t} = c_3$

$\Rightarrow 2 - \frac{1-c_1}{2} = c_3$

$\Rightarrow 4 - 1 + c_1 = 2c_3$

$\Rightarrow 3 + c_1 = 2c_3$

By putting the value of c_1 and c_3 from (2) & (3) respectively, we get

$$3 + \left(\frac{-1}{x} + \frac{1}{y}\right) = 2\left(\frac{1}{y} + \frac{1}{z}\right)$$

$$\Rightarrow \frac{3xy - y + x}{xy} = \frac{2z + 2y}{yz}$$

$$\Rightarrow \frac{3xy - y + x}{xy} - \frac{2z + 2y}{yz} = 0$$

$$\Rightarrow \frac{3xyz - yz + xz - 2zx - 2xy}{xyz} = 0$$

$$\Rightarrow 3xyz - yz - xz - 2xy = 0$$

$$\Rightarrow \boxed{yz + 2xy + xz = 3xyz}$$

Existence and Uniqueness of integral surface passing through given curve

If given pde is $Pp + Qq = R$ with two initial condition $x_0(t)$, $y_0(t)$ and $z_0(t)$ then

(i) pde has unique solution if

$$P(x_0, y_0, z_0)\frac{dy_0}{dt} - Q(x_0, y_0, z_0)\frac{dx_0}{dt} \neq 0$$

(ii) No solution if

$$\frac{P(x_0, y_0, z_0)}{\frac{dx_0}{dt}} = \frac{Q(x_0, y_0, z_0)}{\frac{dy_0}{dt}} \neq \frac{R(x_0, y_0, z_0)}{\frac{dz_0}{dt}}$$

(iii) infinite solution if

$$\frac{P(x_0, y_0, z_0)}{\frac{dx_0}{dt}} = \frac{Q(x_0, y_0, z_0)}{\frac{dy_0}{dt}} = \frac{R(x_0, y_0, z_0)}{\frac{dz_0}{dt}}$$

Q:- Consider the Cauchy problem $p - q = 2$, check whether equation has unique solution passing through the curve $(2t, t, 2t)$ and also find the solution.

Sol:- Given pde, $p - q = 2$ ——— (1)

Here $P = 1$, $Q = -1$ and $R = 2$

$x_0(t) = 2t$, $y_0(t) = t$, $z_0(t) = 2t$

Since, $P\frac{dy_0}{dt} - Q\frac{dx_0}{dt} = 1 \cdot 1 - (-1)\cdot 2 = 3 \neq 0$

$\therefore$ given pde has unique solution with given condition

Now, the Lagrange's auxiliary equation is given by,

$$\frac{dx}{1} = \frac{dy}{-1} = \frac{dz}{2}$$

then $x + y = c_1$ and $2y + z = c_2$ are two independent solutions corresponding to given pde ——— (2)

$\because x=2t,\ y=t,\ z=2t$

$\therefore$ eq(2) becomes,

$$2t+t=c_1 \quad \text{and} \quad 2t+2t=c_2$$

$$\Rightarrow 3t=c_1 \quad \text{and} \quad 4t=c_2$$

$$\Rightarrow 4c_1=3c_2$$

$\therefore$ By putting the values of c_1 and c_2, we get required integral surface

$$4(x+y)=3(2y+z)$$

$$\Rightarrow \boxed{4x-2y-3z=0}$$

Q:- Consider the cauchy's problem $p+q=1$ which passes through $z(x,x)=x$ then find solution

Sol:- Given, $p+q=1$ ——— (1)

Here $P=1$, $Q=1$ and $R=1$

$x_0=t$, $y_0=t$, $z_0=t$

Since $P\dfrac{dy_0}{dt}-Q\dfrac{dx_0}{dt}=1\cdot1-1\cdot1=0$

So uniqueness theorem fails.

Now, Lagrange's auxiliary equation is given by

$$\frac{dx}{1}=\frac{dy}{1}=\frac{dz}{1}$$

then $x-y=c_1$ and $x-z=c_2$ ——— (2)

are two independent solution of (1)

$\Rightarrow$ general solution is given by;

$$x-z=\phi(x-y) \quad \text{——— (3)}$$

Since, $x_0(t)=t$, $y_0(t)=t$, $z_0(t)=t$

$\therefore$ equation (3) $\Rightarrow$ $t-t=\phi(t-t)$

$\Rightarrow \phi(0)=0$

infinite number of such functions exists.

i.e. $\phi(x)=x, x^2, x^3, x^2-x$ etc.

$\therefore$ General solution passes through $z(x,x)=x$ is

$$\boxed{x-z=\phi(x-y) \text{ with } \phi(0)=0}$$

Q:- $z_x+z_y=1$ with initial condition $z(x,x)=1$

Sol:- Given $z_x+z_y=1$ ——— (1)

Here $P=1$, $Q=1$ and $R=1$

$x_0(t)=t$, $y_0(t)=t$ and $z_0(t)=1$

Since, $P\dfrac{dy_0}{dt}-Q\dfrac{dx_0}{dt}=1\cdot1-1\cdot1=0$

i.e. uniqueness theorem fails.

General solution corresponding to (1) is,

$$x-z=\phi(x-y)$$

$\because x_0(t)=t$, $y_0(t)=t$ and $z_0(t)=1$

$\therefore \quad t-1=\phi(t-t)$

$\Rightarrow \phi(0)=t-1$

There doesn't exist any function such that $\phi(0)=t-1$, so pde (1) has no solution.

Q.1 $\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0$ such that $u(0,y) = 4e^{-2y}$ then the value of $u(1,1)$ is ?

Sol:- Given, $\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0$ ——— (1)

Here $P = 1$, $Q = 2$, $R = 0$

$x_0(t) = 0$, $y_0(t) = t$, $u_0(t) = 4e^{-2t}$

$P\frac{dy_0}{dt} - Q\frac{dx_0}{dt} = 1 \neq 0$

∴ unique solution exists

Lagrange's auxiliary equation is given by,

$$\frac{dx}{1} = \frac{dy}{2} = \frac{du}{0}$$

then $2x - y = c_1$ and $u = c_2$

Solution is ; $u(x,y) = \phi(2x-y)$

Now $u(0,y) = \phi(-y)$

$\Rightarrow \phi(-y) = 4e^{-2y}$

$\Rightarrow \phi(y) = 4e^{2y}$

Using this function, $\phi(2x-y) = 4e^{2(2x-y)}$

$\Rightarrow u = 4e^{4x-2y}$

$\Rightarrow u(1,1) = 4e^{2}$

or $x = 0$, $y = t$, $u = 4e^{-2t}$

$\Rightarrow c_1 = -t$, $c_2 = 4e^{-2t} = 4e^{2c_1}$

$\Rightarrow u = 4e^{2(2x-y)}$

$\Rightarrow u(1,1) = 4e^{2}$

Q:- $2u_x + 3u_y = 5$ such that $u = 1$ and $3x - 2y = 0$ then the IVP has

(i) Exactly one solution
(ii) Exactly two solution
(iii) infinite number of solutions
(iv) No solution

Sol:- Given, $2u_x + 3u_y = 5$

Here $P = 2$, $Q = 3$, $R = 5$

$x_0(t) = 2t$, $y_0(t) = 3t$, $u_0(t) = 1$

Now $P\frac{dy_0}{dt} - Q\frac{dx_0}{dt} = 6 - 6 = 0$

So uniqueness theorem fails.

* Lagrange's auxilliary equation is given by,

$$\frac{dx}{2} = \frac{dy}{3} = \frac{du}{5}$$

Taking 1st two ratios, we get

$$3x - 2y = c_1$$

Taking 2nd and 3rd ratios, we get

$$3u - 5y = c_2$$

General solution is given by

$$3u - 5y = \phi(3x - 2y)$$

$$\Rightarrow 3 \times 1 - 5 \times 3t = \phi(3 \times 2t - 2 \times 3t)$$

$$\Rightarrow 3 - 15t = \phi(0)$$

There doesn't exist any function such that $\phi(0) = 3 - 15t$, so given pde has no solution.

Classification of second order pde

Let z be a function of two independent variables x and y.

Consider a general pde of second order in the form

$$R\frac{\partial^2 z}{\partial x^2}+S\frac{\partial^2 z}{\partial x\partial y}+T\frac{\partial^2 z}{\partial y^2}+f(x,y,z,p,q)=0 \quad -(1)$$

where R, S, T are continuous function of x and y and possessing partial derivative defined in some domain D on the xy-plane.

Then eq(1) is

(i) Hyperbolic at (x,y) in D if $S^2-4RT>0$

(ii) Parabolic at (x,y) in D if $S^2-4RT=0$

(iii) Elliptic at (x,y) in D if $S^2-4RT<0$

Q:- Classify the following pdes

1) $3r+4s+5t=x+y+p+q$

2) $(x+y)u_{xx}+xyu_{xy}+(x+y)u_{yy}=x+y$ in $0\le x,y\le 1$

3) $\frac{\partial^2 z}{\partial x^2}+(x+y)\frac{\partial^2 z}{\partial x\partial y}+xy\frac{\partial^2 z}{\partial y^2}=5$

Solⁿ:- 1) Here $R=3$, $S=4$, $T=5$

$S^2-4RT=4^2-4\times3\times5=16-60=-44<0$

$\therefore$ given pde is elliptic.

2) Here $R = x+y$, $S = xy$, $T = (x+y)$

$$\therefore S^2 - 4RT = x^2y^2 - 4(x+y)^2$$
$$= \{xy - 2(x+y)\}\{xy + 2(x+y)\}$$
$$< 0 \text{ in } x,y \in (0,1]$$

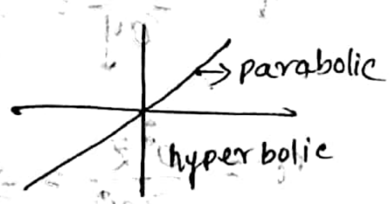
and $S^2 - 4RT = 0$ when $x = y = 0$

$\therefore$ Given pde is elliptic in $(0,1]$ and parabolic at $x = y = 0$.

3) Here $R = 1$, $S = x+y$, $T = xy$

$$S^2 - 4RT = (x+y)^2 - 4xy = (x-y)^2 > 0 \text{ if } x \neq y$$
$$= 0 \text{ if } x = y$$

$\Rightarrow S^2 - 4RT > 0$ if $x \neq y$
and $S^2 - 4RT = 0$ if $x = y$



$\Rightarrow$ given pde is hyperbolic if $x \neq y$ and parabolic if $x = y$.

4) $$xy\frac{\partial^2 z}{\partial x^2} - (x^2 - y^2)\frac{\partial^2 z}{\partial x \partial y} - xy\frac{\partial^2 z}{\partial y^2} + py - qx = 2(x^2 - y^2)$$

Sol:- $R = xy$, $S = -(x^2 - y^2)$, $T = -xy$

$$\therefore S^2 - 4RT = (x^2 - y^2)^2 + 4x^2y^2$$
$$= (x^2 + y^2)^2 > 0, \ \forall x,y \in R$$

except at $x = y = 0$.

When $x = y = 0$, $S^2 - 4RT = 0$

$\therefore$ given pde is hyperbolic for $x,y \in R \setminus \{0\}$ and parabolic when $x = y = 0$.

43) The second order partial differential equation

$$\frac{(x-y)^2}{4}\frac{\partial^2 u}{\partial x^2} + (x-y)\sin(x^2+y^2)\frac{\partial^2 u}{\partial x \partial y} + \cos^2(x^2+y^2)\frac{\partial^2 u}{\partial y^2}$$
$$+ (x-y)\frac{\partial u}{\partial x} + \sin^2(x^2+y^2)\frac{\partial u}{\partial y} + u = 0 \text{ is}$$

(1) ✓ Elliptic in the region $\{(x,y) : x \neq y,\ x^2+y^2 < \pi/6\}$

(2) ✓ Hyperbolic in the region $\{(x,y) : x \neq y;\ \frac{\pi}{4} < x^2+y^2 < \frac{3\pi}{4}\}$

(3) Elliptic in the region $\{(x,y) : x \neq y,\ \frac{\pi}{4} < x^2+y^2 < \frac{3\pi}{4}\}$

(4) Hyperbolic in the region $\{(x,y) : x \neq y,\ x^2+y^2 < \frac{\pi}{4}\}$

Sol:- Here $R = \frac{(x-y)^2}{4}$, $S = (x-y)\sin(x^2+y^2)$, $T = \cos^2(x^2+y^2)$

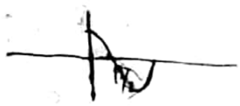
Now, $S^2 - 4RT = (x-y)^2\sin^2(x^2+y^2) - 4\cdot\frac{(x-y)^2}{4}\cdot\cos^2(x^2+y^2)$

$= (x-y)^2\{\sin^2(x^2+y^2) - \cos^2(x^2+y^2)\}$

$= -(x-y)^2 \cos\{2(x^2+y^2)\}$

(1) If $x^2+y^2 < \frac{\pi}{6} \Rightarrow 0 < x^2+y^2 < \frac{\pi}{6}$

$\Rightarrow 2(x^2+y^2) < 2\cdot\frac{\pi}{6} = \frac{\pi}{3}$



$\Rightarrow \cos\{2(x^2+y^2)\} > \cos\frac{\pi}{3} = \frac{1}{2}$

$\therefore S^2 - 4RT < 0$ for $x^2+y^2 < \frac{\pi}{6}$

$= 0$ when $(x,y) = (0,0) \rightarrow$ parabolic

$\therefore$ Elliptic in the region $\{(x,y) : x \neq y,\ x^2+y^2 < \frac{\pi}{6}\}$

~~and para~~

(2) If $\frac{\pi}{4} < x^2+y^2 < \frac{3\pi}{4}$

$\Rightarrow \frac{\pi}{2} < 2(x^2+y^2) < \frac{3\pi}{2}$

$\cos\{2(x^2+y^2)\} < 0$

$\therefore S^2 - 4RT > 0$

$\therefore$ pde is hyperbolic

$\therefore$ (3) is wrong.

(4) $x^2+y^2 < \frac{\pi}{4}$

$\Rightarrow 2(x^2+y^2) < \frac{\pi}{2}$

$\Rightarrow \cos\{2(x^2+y^2)\} > 0$

$\therefore S^2-4RT < 0$, pde is Elliptic.

48) The complete integral of the PDE

$$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = xe^{x+y}$$

involving arbitarry functions ϕ_1 and ϕ_2 is,

(1) $\phi_1(y+x) + \phi_2(y+x) + \frac{1}{4}e^{x+y}$

(2) $\phi_1(y+x) + x\phi_2(y+x) + \left(\frac{x-1}{4}\right)e^{x+y}$

(3) $\phi_1(y-x) + \phi_2(y-x) + \frac{1}{4}e^{x+y}$

~~(4)~~ $\phi_1(y-x) + x\phi_2(y-x) + \frac{(x-1)}{4}e^{x+y}$

Sol:- $(D^2+2DD'+D'^2)u=0$

$m^2+2m+1=0$

$\Rightarrow (m+1)^2=0$

$\Rightarrow m=-1,-1$

$\therefore C.F = \phi_1(y-x) + x\phi_2(y-x)$

$$P.I = \frac{1}{(D+D')^2}xe^{x+y}$$

$$= e^{x+y}\frac{1}{(D+1+D'+1)^2}x$$

$$= e^{x+y}\frac{1}{(D+D'+2)^2}x$$

$$= e^{x+y}\frac{1}{4\left[1+\frac{D+D'}{2}\right]^2}(x)$$

$$= \frac{e^{x+y}}{4}\left[1-\frac{2(D+D')}{2}+\cdots\right](x)$$

$$= \frac{e^{x+y}}{4}(x-1)$$

$$\therefore \boxed{y = \phi_1(y-x)+x\phi_2(y-x)+\frac{(x-1)}{4}e^{x+y}}$$

44) The second order PDE $u_{yy} - y u_{xx} + x^3 u = 0$ is

(1) Elliptic for all $x \in \mathbb{R}$, $y \in \mathbb{R}$

(2) Parabolic for all $x \in \mathbb{R}$, $y \in \mathbb{R}$

✓(3) Elliptic for all $x \in \mathbb{R}$, $y < 0$

(4) Hyperbolic for all $x \in \mathbb{R}$, $y < 0$

Sol:- $R = -y$, $S = 0$, $T = 1$

$S^2 - 4RT = 0 - 4(-y)\cdot 1 = 4y < 0$, when $y < 0$

$\therefore$ Elliptic for all $x \in \mathbb{R}$, $y < 0$

97) The second order pde $u_{xx} + x u_{yy} = 0$ is

✓(1) elliptic for $x > 0$ (2) hyperbolic for $x > 0$

(3) elliptic for $x < 0$ ✓(4) hyperbolic for $x < 0$

Sol:- $R = 1$, $S = 0$, $T = x$

$S^2 - 4RT = 0 - 4x = -4x$

$x > 0$, $S^2 - 4RT < 0 \rightarrow$ elliptic

$x < 0$, $S^2 - 4RT > 0 \rightarrow$ hyperbolic

98) For an arbitarry continuously differentiable function f, which of the following is a general solution of $z(px-qy) = y^2-x^2$

(1) $x^2+y^2+z^2 = f(xy)$ (2) $(x+y)^2+z^2 = f(xy)$

(3) $x^2+y^2+z^2 = f(y-x)$ (4) $x^2+y^2+z^2 = f((x+y)^2+z^2)$

Sol:- Given, pde $z(px-qy) = y^2-x^2$

$$\Rightarrow zxp - zyq = y^2-x^2$$

Lagrange's auxiliary equation gives

$$\frac{dx}{zx} = \frac{dy}{-zy} = \frac{dz}{y^2-x^2}$$

Taking 1st two ratios

$$\frac{dx}{zx} = \frac{dy}{-zy}$$

$$\Rightarrow \ln x = -\ln y + \ln c_1$$

$$\Rightarrow xy = c_1$$

using x, y and z as multipliers,

$$\text{each ratio} = \frac{x\,dx + y\,dy + z\,dz}{zx^2 - zy^2 + zy^2 - zx^2}$$

$$\Rightarrow x^2+y^2+z^2 = c_2$$

$$\therefore \boxed{x^2+y^2+z^2 = f(xy)}$$

Reduction into Canonical and/or Normal form

Let us consider the pde,

$$\underbrace{Rr + Ss + Tt}_{\text{Principal part}} + f(x, y, z, p, q) = 0 \quad \text{--- (1)}$$

where R, S, T are continuous functions of x and y. Then we have to transform eq(1) into one of three canonical forms which can be integrated easily by change of independent variables into u and v.

i.e. $u = u(x,y)$ and $v = v(x,y)$ ——— (2)

$$\therefore \; p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial u}\cdot\frac{\partial^2 u}{\partial x^2} + \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)\cdot\frac{\partial v}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial^2 v}{\partial x^2}$$

$$= \frac{\partial^2 z}{\partial u^2}\left(\frac{\partial u}{\partial x}\right)^2 + \frac{\partial^2 z}{\partial u\partial v}\cdot\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x} + \frac{\partial z}{\partial u}\cdot\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 z}{\partial u\partial v}\cdot\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x} + \frac{\partial^2 z}{\partial v^2}\left(\frac{\partial v}{\partial x}\right)^2 + \frac{\partial z}{\partial v}\cdot\frac{\partial^2 v}{\partial x^2}$$

$$= \frac{\partial^2 z}{\partial u^2}\left(\frac{\partial u}{\partial x}\right)^2 + 2\frac{\partial^2 z}{\partial u\partial v}\cdot\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial^2 z}{\partial v^2}\left(\frac{\partial v}{\partial x}\right)^2 + \frac{\partial z}{\partial u}\cdot\frac{\partial^2 u}{\partial x^2} + \frac{\partial z}{\partial v}\cdot\frac{\partial^2 v}{\partial x^2}$$

Similarly,

$$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial u}\cdot\frac{\partial^2 u}{\partial x \partial y}$$

$$+ \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)\cdot\frac{\partial v}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial^2 v}{\partial x \partial y}$$

$$\cancel{\frac{\partial^2 z}{\partial u^2}\cdot\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial u}\cdot\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 z}{\partial u \partial v}\cdot\frac{\partial u}{}}$$

$$= \frac{\partial^2 z}{\partial u^2}\cdot\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial y} + \frac{\partial^2 z}{\partial v \partial u}\cdot\frac{\partial v}{\partial x}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial u}\cdot\frac{\partial^2 u}{\partial x \partial y}$$

$$+ \frac{\partial^2 z}{\partial u \partial v}\cdot\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y} + \frac{\partial^2 z}{\partial v^2}\cdot\frac{\partial v}{\partial x}\cdot\frac{\partial v}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial^2 v}{\partial x \partial y}$$

$$= \frac{\partial^2 z}{\partial u^2}\cdot\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial y} + \frac{\partial^2 z}{\partial u \partial v}\left(\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y} + \frac{\partial v}{\partial x}\cdot\frac{\partial u}{\partial y}\right)$$

$$+ \frac{\partial^2 z}{\partial v^2}\cdot\frac{\partial v}{\partial x}\cdot\frac{\partial v}{\partial y} + \frac{\partial z}{\partial u}\cdot\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial^2 v}{\partial x \partial y}$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}\right)$$

$$= \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right)\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial u}\cdot\frac{\partial^2 u}{\partial y^2} + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial v}\right)\cdot\frac{\partial v}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial^2 v}{\partial y^2}$$

$$= \frac{\partial^2 z}{\partial u^2}\left(\frac{\partial u}{\partial y}\right)^2 + \frac{\partial^2 z}{\partial v \partial u}\cdot\frac{\partial v}{\partial y}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial u}\cdot\frac{\partial^2 u}{\partial y^2}$$

$$+ \frac{\partial^2 z}{\partial u \partial v}\cdot\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial y} + \frac{\partial^2 z}{\partial v^2}\left(\frac{\partial v}{\partial y}\right)^2 + \frac{\partial z}{\partial v}\cdot\frac{\partial^2 v}{\partial y^2}$$

$$= \frac{\partial^2 z}{\partial u^2}\left(\frac{\partial u}{\partial y}\right)^2 + 2\frac{\partial^2 z}{\partial u \partial v}\cdot\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial y} + \frac{\partial^2 z}{\partial v^2}\cdot\left(\frac{\partial v}{\partial y}\right)^2 + \frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial y^2} + \frac{\partial z}{\partial u}\cdot\frac{\partial^2 u}{\partial y^2}$$

Putting these values of p, q, r, s, t in (1), and simplifying, we get

$$\left\{R\left(\frac{\partial u}{\partial x}\right)^2 + S\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial y} + T\left(\frac{\partial u}{\partial y}\right)^2\right\}\frac{\partial^2 z}{\partial u^2}$$

$$+\left\{2R\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x} + S\left(\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y} + \frac{\partial v}{\partial x}\cdot\frac{\partial u}{\partial y}\right) + 2T\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial y}\right\}\frac{\partial^2 z}{\partial u\partial v}$$

$$+\left\{R\left(\frac{\partial v}{\partial x}\right)^2 + S\left(\frac{\partial v}{\partial x}\right)\left(\frac{\partial v}{\partial y}\right) + T\left(\frac{\partial v}{\partial y}\right)^2\right\}\frac{\partial^2 z}{\partial v^2} + F\left(u,v,z,\frac{\partial z}{\partial u},\frac{\partial z}{\partial v}\right) = 0$$

$$\Rightarrow A\frac{\partial^2 z}{\partial u^2} + B\frac{\partial^2 z}{\partial u\partial v} + C\frac{\partial^2 z}{\partial v^2} + F\left(u,v,z,\frac{\partial z}{\partial u},\frac{\partial z}{\partial v}\right) = 0 \quad \text{---(3)}$$

where $A = R\left(\frac{\partial u}{\partial x}\right)^2 + S\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial y} + T\left(\frac{\partial u}{\partial y}\right)^2$ ---(4)

$B = 2R\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + S\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial u}{\partial y}\right) + 2T\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial y}$ ---(5)

$C = R\left(\frac{\partial v}{\partial x}\right)^2 + S\left(\frac{\partial v}{\partial x}\right)\left(\frac{\partial v}{\partial y}\right) + T\left(\frac{\partial v}{\partial y}\right)^2$ ---(6)

Now, we determine the transformation u and v so that the equation (3) takes the simplest possible form. when discriminant S^2-4RT of $R\lambda^2+S\lambda+T=0$ is <0, >0, $=0$.

<u>Note:-</u> ① Let $\frac{\partial u}{\partial x} = a$, $\frac{\partial u}{\partial y} = b$, $\frac{\partial v}{\partial x} = c$, $\frac{\partial v}{\partial y} = d$, then by the transformation $u=u(x,y)$ and $v=v(x,y)$. Our pde reduces to

$$A\frac{\partial^2 z}{\partial u^2} + B\frac{\partial^2 z}{\partial u\partial v} + C\frac{\partial^2 z}{\partial v^2} + F = 0$$

where, $A = Ra^2 + Sab + Tb^2$

$B = 2Rac + S(ad+bc) + 2Tbd$

$C = Rc^2 + Scd + Td^2$

where the Jacobian of the transformation is non-zero

(2) $B^2 - 4AC = (S^2 - 4RT)(ad - bc)^2$

Proof :- $B^2 - 4AC = [a\ b]\begin{bmatrix}2R & S\\ S & 2T\end{bmatrix}\begin{bmatrix}c\\ d\end{bmatrix}[a\ b]\begin{bmatrix}2R & S\\ S & 2T\end{bmatrix}\begin{bmatrix}c\\ d\end{bmatrix}$

$$-4[a\ b]\begin{bmatrix}R & S/2\\ S/2 & T\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix}[c\ d]\begin{bmatrix}R & S/2\\ S/2 & T\end{bmatrix}\begin{bmatrix}c\\ d\end{bmatrix}$$

$$= [a\ b]\begin{bmatrix}2R & S\\ S & 2T\end{bmatrix}\left\{\begin{bmatrix}c\\ d\end{bmatrix}[a\ b] - \begin{bmatrix}a\\ b\end{bmatrix}[c\ d]\right\}\begin{bmatrix}2R & S\\ S & 2T\end{bmatrix}\begin{bmatrix}c\\ d\end{bmatrix}$$

$$= [a\ b]\begin{bmatrix}2R & S\\ S & 2T\end{bmatrix}\begin{bmatrix}0 & bc-ad\\ ad-bc & 0\end{bmatrix}\begin{bmatrix}2R & S\\ S & 2T\end{bmatrix}\begin{bmatrix}c\\ d\end{bmatrix}$$

$$= (ad-bc)[a\ b]\begin{bmatrix}2R & S\\ S & 2T\end{bmatrix}\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}2R & S\\ S & 2T\end{bmatrix}\begin{bmatrix}c\\ d\end{bmatrix}$$

$$= (ad-bc)[a\ b]\begin{bmatrix}0 & S^2-4RT\\ -S^2+4RT & 0\end{bmatrix}\begin{bmatrix}c\\ d\end{bmatrix}$$

$$= (S^2-4RT)(ad-bc)[a\ b]\begin{bmatrix}0 & +1\\ -1 & 0\end{bmatrix}\begin{bmatrix}c\\ d\end{bmatrix}$$

$$= (S^2-4RT)(ad-bc)[-b\ \ +a]\begin{bmatrix}c\\ d\end{bmatrix}$$

$$= (S^2-4RT)(ad-bc)(-bc+ad)$$

$$= (S^2-4RT)(ad-bc)^2$$

So our given pde and transform pde have same nature.

(3) **Equation** — **Expression**

(a) $xy = c$ $\Rightarrow \dfrac{\partial^2 z}{\partial x \partial y} = f(x, y, z, z_x, z_y)$ (Hyperbolic)

(b) $y^2 = 4ax$ $\Rightarrow \dfrac{\partial^2 z}{\partial y^2} = 1$ (parabolic)

(c) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ $\Rightarrow \dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 1$ (elliptic)

* Explain how to find the co-ordinate transformation $(x,y) \to (u,v)$ which transform the pde $R\frac{\partial^2 z}{\partial x^2} + S\frac{\partial^2 z}{\partial x \partial y} + T\frac{\partial^2 z}{\partial y^2} = f$ into canonical form.

Sol:- We consider the co-ordinate transformation

$$u = u(x,y) \text{ and } v = v(x,y) \quad \text{——} (1)$$

such that $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \neq 0 \quad \text{——} (2)$

Let $\frac{\partial u}{\partial x} = a$; $\frac{\partial u}{\partial y} = b$, $\frac{\partial v}{\partial x} = c$, $\frac{\partial v}{\partial y} = d$

then the principal part of the transformed pde is, $A\frac{\partial^2 z}{\partial u^2} + B\frac{\partial^2 z}{\partial u \partial v} + C\frac{\partial^2 z}{\partial v^2} \quad \text{——} (3)$

where
$$\left.\begin{aligned} A &= Ra^2 + Sab + Tb^2 \\ B &= 2Rac + S(ad+bc) + 2Tbd \\ C &= Rc^2 + Scd + Td^2 \end{aligned}\right\} \text{——} (4)$$

and $B^2 - 4AC = (S^2 - 4RT)(bc-ad)^2 \quad \text{——} (5)$

Now, we consider the following cases –

Case-I :- If $S^2 - 4RT > 0$

and consider the quadratic equation $R\lambda^2 + S\lambda + T = 0 \quad \text{——} (6)$

Let $\lambda = \lambda_1(x,y)$ and $\lambda = \lambda_2(x,y)$ be the roots of eq (6), then

$$\left.\begin{aligned} R\lambda_1^2 + S\lambda_1 + T = 0 \\ R\lambda_2^2 + S\lambda_2 + T = 0 \end{aligned}\right\} \text{——} (7)$$

where λ_1 and λ_2 are real and distinct.

we choose

(i) $\frac{a}{b} = \lambda_1$ and $\frac{c}{d} = \lambda_2$

so that
$$\left.\begin{aligned} A &= b^2\left(R\frac{a^2}{b^2} + S\frac{a}{b} + T\right) \\ &= b^2(R\lambda_1^2 + S\lambda_1 + T) = 0 \\ C &= d^2\left(R\frac{c^2}{d^2} + S\frac{c}{d} + T\right) \\ &= d^2(R\lambda_2^2 + S\lambda_2 + T) = 0 \end{aligned}\right\} \quad \text{---} (8)$$

The choice (i) is equivalent to

$$\frac{\partial u}{\partial x} = \lambda_1 \frac{\partial u}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = \lambda_2 \frac{\partial v}{\partial y}$$

$$\Rightarrow \frac{\partial u}{\partial x} - \lambda_1 \frac{\partial u}{\partial y} = 0 \quad \text{and} \quad \frac{\partial v}{\partial x} - \lambda_2 \frac{\partial v}{\partial y} = 0$$

$\therefore$ By Lagrange's auxiliary equation corresponding to first pde is

$$\frac{dx}{1} = \frac{dy}{-\lambda_1} = \frac{du}{0}$$

$$\Rightarrow \frac{dy}{dx} + \lambda_1 = 0 \text{ and } du = 0$$

$$\Rightarrow f(x,y) = c_1 \quad \text{and} \quad u = c_2$$

$\therefore$ Solution is, $u = f_1(x,y)$ ——— (9)

Similarly from the second pde, we get,

$$v = f_2(x,y) \quad \text{———} (10)$$

* The curve $f_1(x,y) = c_1$ and $f_2(x,y) = c_2$ are called characteristic curves.

Now, it is clear that by the coordinate transformation obtained in (9) and (10), the given pde reduces to

$$B \frac{\partial^2 z}{\partial u \partial v} = f(u, v, z, z_u, z_v)$$

This is the canonical form for hyperbolic pde.

<u>case-ii</u> <u>If $S^2 - 4RT = 0$</u>

This is the case of parabolic pde.

In this case the equation $R\alpha^2 + S\alpha + T = 0$ has two identical roots

Let the roots be λ and we choose

$\frac{a}{b} = \lambda$ ——— (11), then

$$A = b^2\left(R\frac{a^2}{b^2} + S\frac{a}{b} + T\right) = b^2(R\lambda^2 + S\lambda + T) = 0$$

$\therefore$ From (5), $B^2 = 0 \Rightarrow B = 0$

i.e. in this case $A = 0$ and $B = 0$ ——— (12)

As in case (i) we can show that $u = f_1(x, y)$ ——(13)

where $f_1(x, y) = c_1$ is a solution of $\frac{dy}{dx} + \lambda = 0$

Now, we take $v = f_2(x, y)$ ——— (14)

such that $\frac{\partial(u, v)}{\partial(x, y)} \neq 0$

$\therefore$ By the co-ordinate transformation (13) and (14) the principal part of the given pde reduces to

$$C\frac{\partial^2 z}{\partial v^2} + g(u, v, z, z_u, z_v) = 0$$

which is canonical form for parabolic pde.

case-(iii) If $S^2 - 4RT < 0$

In this case both roots λ_1 and λ_2 of the quadratic equation $R\alpha^2 + S\alpha + T = 0$ are imaginary and conjugate to each other.

Hence, we get two solution of $\frac{dy}{dx} + \lambda_1 = 0$ and $\frac{dy}{dx} + \lambda_2 = 0$ as $f_1(x,y) = c_1$ and $f_2(x,y) = c_2$ such that, $f_1(x,y)$ and $f_2(x,y)$ are complex function conjugate to each other.

Hence, if we take $u = f_1(x,y)$ and $v = f_2(x,y)$ then as in case (i) the principal part of the reduced pde is

$$B\frac{\partial^2 z}{\partial u \partial v} = f(u, v, z, z_u, z_v)$$

$$\Rightarrow \frac{\partial^2 z}{\partial u \partial v} = g(u, v, z, z_u, z_v) \quad \text{——— (15)}$$

Again, assume that

$$\left.\begin{array}{l} \xi = u + v \\ \eta = \frac{1}{i}(u - v) \end{array}\right\} \quad \text{——— (16)}$$

then $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial \xi}\cdot\frac{\partial \xi}{\partial u} + \frac{\partial z}{\partial \eta}\cdot\frac{\partial \eta}{\partial u} = \frac{\partial z}{\partial \xi} + \frac{1}{i}\frac{\partial z}{\partial \eta}$

$$\Rightarrow \frac{\partial^2 z}{\partial u \partial v} = \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial \xi}\right) + \frac{1}{i}\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial \eta}\right)$$

$$= \frac{\partial^2 z}{\partial \xi^2}\cdot\frac{\partial \xi}{\partial v} + \frac{\partial^2 z}{\partial \xi \partial \eta}\cdot\frac{\partial \eta}{\partial v} + \frac{1}{i}\left(\frac{\partial^2 z}{\partial \xi \partial \eta}\cdot\frac{\partial \xi}{\partial v} + \frac{\partial^2 z}{\partial \eta^2}\cdot\frac{\partial \eta}{\partial v}\right)$$

$$= \frac{\partial^2 z}{\partial \xi^2} + \frac{\partial^2 z}{\partial \xi \partial \eta}\left(\frac{-1}{i}\right) + \frac{1}{i}\left[\frac{\partial^2 z}{\partial \xi \partial \eta} + \frac{\partial^2 z}{\partial \eta^2}\left(\frac{-1}{i}\right)\right]$$

$$= \frac{\partial^2 z}{\partial \xi^2} + \frac{\partial^2 z}{\partial \eta^2}$$

Hence the transformed pde is of the form

$$\frac{\partial^2 z}{\partial \xi^2} + \frac{\partial^2 z}{\partial \eta^2} + f(\xi, \eta, z, z_\xi, z_\eta) = 0$$

which is the canonical form for elliptic pde.

Method :- $R z_{xx} + S z_{xy} + T z_{yy} = f(x, y, z, z_x, z_y)$

Step-I :- Solve $R\lambda^2 + S\lambda + T = 0$ and find roots.

Step-II :- Solve $\frac{dy}{dx} + \lambda_1 = 0$ and $\frac{dy}{dx} + \lambda_2 = 0$

$\Rightarrow f_1(x,y) = c_1$ and $f_2(x,y) = c_2$

$\Rightarrow$ the transformation is $u = f_1(x,y)$, $v = f_2(x,y)$

Step-(III) :- The reduced pde is,

$$A z_{uu} + B z_{uv} + C z_{vv} + (R u_{xx} + S u_{xy} + T u_{yy})\frac{\partial z}{\partial u} + (R v_{xx} + S v_{xy} + T v_{yy})\frac{\partial z}{\partial v} = f(x, y, z, z_u, z_v)$$

1) Q:- Reduce the equation

$$y^2 z_{xx} - 2xy\, z_{xy} + x^2 z_{yy} = \frac{y^2}{x} z_x + \frac{x^2}{y} z_y \quad \text{——— (1)}$$

Sol:- Here $R = y^2$, $S = -2xy$, $T = x^2$

$\therefore S^2 - 4RT = 4x^2y^2 - 4x^2y^2 = 0$

$\Rightarrow$ The pde is parabolic

Now, the roots of $R\lambda^2 + S\lambda + T = 0$ is

i.e. $y^2\lambda^2 - 2xy\lambda + x^2 = 0$

$\Rightarrow (y\lambda - x)^2 = 0$

$\Rightarrow y\lambda = x$

$\Rightarrow \lambda = \frac{x}{y}$

We solve, $\frac{dy}{dx} + \frac{x}{y} = 0$

$\Rightarrow x\,dx + y\,dy = 0$

$\Rightarrow x^2 + y^2 = c_1$

$\therefore$ We take co-ordinate transformation as,

$$u = x^2 + y^2, \quad v = x^2 - y^2 \quad \text{——— (2)}$$

By this transformation the given pde transformed into

$$c z_{vv} + (R u_{xx} + S u_{xy} + T u_{yy}) \frac{\partial z}{\partial u} + (R v_{xx} + S v_{xy} + T v_{yy}) \frac{\partial z}{\partial v}$$

$$= \frac{y^2}{x} z_x + \frac{x^2}{y} z_y \quad \text{——— (3)}$$

Since, $a = \frac{\partial u}{\partial x} = 2x$, $b = \frac{\partial u}{\partial y} = 2y$, $c = \frac{\partial v}{\partial x} = 2x$,

$d = \frac{\partial v}{\partial y} = -2y$

$u_{xx} = 2$, $u_{xy} = 0$, $u_{yy} = 2$, $v_{xx} = 2$, $v_{xy} = 0$, $v_{yy} = -2$

also, $C = Rc^2 + Scd + Td^2$

$$= y^2(2x)^2 + (-2xy)(2x)(-2y) + x^2(-2y)^2$$

$$= 4x^2y^2 + 8x^2y^2 + 4x^2y^2$$

$$= 16x^2y^2$$

$\therefore$ eq(3) $\Rightarrow$

$$16x^2y^2 z_{vv} + (2y^2 - 2xy\cdot 0 + 2x^2)\frac{\partial z}{\partial u} + (2y^2 - 2x^2)\frac{\partial z}{\partial v} = \frac{y^2}{x} z_x + \frac{x^2}{y} z_y$$

$$\Rightarrow 16x^2y^2 z_{vv} + 2(x^2+y^2)\frac{\partial z}{\partial u} + 2(y^2-x^2)\frac{\partial z}{\partial v} = \frac{y^2}{x} z_x + \frac{x^2}{y} z_y$$

Also, since

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} = 2x\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$$

and $\dfrac{\partial z}{\partial y} = \dfrac{\partial z}{\partial u}\cdot\dfrac{\partial u}{\partial y} + \dfrac{\partial z}{\partial v}\cdot\dfrac{\partial v}{\partial y} = 2y\left(\dfrac{\partial z}{\partial u} - \dfrac{\partial z}{\partial v}\right)$

$$\therefore 16x^2y^2 z_{vv} + 2(x^2+y^2)\frac{\partial z}{\partial u} + 2(y^2-x^2)\frac{\partial z}{\partial v} = \frac{y^2}{x}\cdot 2x\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) + \frac{x^2}{y}\cdot 2y\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)$$

$$\Rightarrow 16x^2y^2 z_{vv} + 2(x^2+y^2)\frac{\partial z}{\partial u} + 2(y^2-x^2)\frac{\partial z}{\partial v} = 2(x^2+y^2)\frac{\partial z}{\partial u} + 2(y^2-x^2)\frac{\partial z}{\partial v}$$

$$\Rightarrow 16x^2y^2 z_{vv} = 0$$

$\Rightarrow \boxed{\dfrac{\partial^2 z}{\partial v^2} = 0}$, which is required canonical form.

$$\Rightarrow \frac{\partial z}{\partial v} = f(u)$$

$$\Rightarrow z = f(u)\cdot v + g(u)$$

$\Rightarrow z = f(x^2+y^2)\cdot(x^2-y^2) + g(x^2+y^2)$

$$\Rightarrow \boxed{z = (x^2-y^2)f(x^2+y^2) + g(x^2+y^2)}$$

2) Reduce the equation $z_{xx} + x^2 z_{yy} = 0$ ——— (1) to a Canonical form.

Sol:- Here $R = 1$, $S = 0$, $T = x^2$

$\therefore S^2 - 4RT = -4x^2$

i.e. given pde is elliptic form

Now the roots of equation $R\lambda^2 + S\lambda + T = 0$ is

i.e. $\lambda^2 + x^2 = 0$

$\Rightarrow \lambda = \pm xi$

Consider the equation $\frac{dy}{dx} + ix = 0$ and $\frac{dy}{dx} - ix = 0$

$\Rightarrow y + \frac{ix^2}{2} = c_1$ and $y - \frac{ix^2}{2} = c_2$

$\therefore$ We take the co-ordinate transformation as,

$$u = y + \frac{ix^2}{2} \text{ and } v = y - \frac{ix^2}{2} \quad \text{——— (2)}$$

By this transformation, the given pde is transformed into,

$$B\frac{\partial^2 z}{\partial u \partial v} + (Ru_{xx} + Su_{xy} + Tu_{yy})\frac{\partial z}{\partial u} + (Rv_{xx} + Sv_{xy} + Tv_{yy})\frac{\partial z}{\partial v} = 0 \quad \text{——— (3)}$$

Since $a = \frac{\partial u}{\partial x} = ix$, $b = \frac{\partial u}{\partial y} = 1$, $c = \frac{\partial v}{\partial x} = -ix$

$d = \frac{\partial v}{\partial y} = 1$

$B = 2acR + S(ad+bc) + 2Tbd$

$= 2x^2 + 2x^2 = 4x^2$

$u_{xx} = i$, $u_{xy} = 0$, $u_{yy} = 0$, $v_{xx} = -i$, $v_{xy} = 0$, $v_{yy} = 0$

$\therefore$ eq(3) $\Rightarrow$

$$4x^2 \frac{\partial^2 z}{\partial u \partial v} + i \frac{\partial z}{\partial u} + (-i) \frac{\partial z}{\partial v} = 0$$

$$\Rightarrow 4x^2 \frac{\partial^2 z}{\partial u \partial v} + i \left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} \right) = 0 \quad \text{——— (4)}$$

We take second transformation as

$$\xi = u + v = 2y \quad \text{and} \quad \eta = \frac{1}{i}(u - v) = x^2 \quad \text{——— (5)}$$

then; $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial \xi} \cdot \frac{\partial \xi}{\partial u} + \frac{\partial z}{\partial \eta} \cdot \frac{\partial \eta}{\partial u} = \frac{\partial z}{\partial \xi} + \frac{1}{i} \frac{\partial z}{\partial \eta}$

and $\frac{\partial z}{\partial v} = \frac{\partial z}{\partial \xi} \cdot \frac{\partial \xi}{\partial v} + \frac{\partial z}{\partial \eta} \cdot \frac{\partial \eta}{\partial v} = \frac{\partial z}{\partial \xi} - \frac{1}{i} \frac{\partial z}{\partial \eta}$

$$\Rightarrow \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = \frac{2}{i} \frac{\partial z}{\partial \eta}$$

Also, $\frac{\partial^2 z}{\partial u \partial v} = \frac{\partial^2 z}{\partial \xi^2} + \frac{\partial^2 z}{\partial \eta^2}$

$\therefore$ eq(4) $\Rightarrow$

$$4\eta \left(\frac{\partial^2 z}{\partial \xi^2} + \frac{\partial^2 z}{\partial \eta^2} \right) + i \cdot \frac{2}{i} \frac{\partial z}{\partial \eta} = 0$$

$$\Rightarrow \boxed{\frac{\partial^2 z}{\partial \xi^2} + \frac{\partial^2 z}{\partial \eta^2} + \frac{1}{2\eta} \frac{\partial z}{\partial \eta} = 0}$$

which is required canonical form.

3) Reduce the equation $\frac{\partial^2 z}{\partial x^2} = x^2 \frac{\partial^2 z}{\partial y^2}$ to canonical form.

Sol:- Here $R = 1$, $S = 0$, $T = -x^2$

$\therefore S^2 - 4RT = 4x^2$

i.e. given pde is hyperbolic.

Now the roots of $R\lambda^2 + S\lambda + T = 0$ is,

i.e. $\lambda^2 - x^2 = 0$

$\Rightarrow \lambda = \pm x$

consider the equation,

$\frac{dy}{dx} + x = 0$ and $\frac{dy}{dx} - x = 0$

$\Rightarrow y + \frac{x^2}{2} = c_1$ and $y - \frac{x^2}{2} = c_2$

We take the coordinate transformation as,

$u = y + \frac{x^2}{2}$, $v = y - \frac{x^2}{2}$ ——— (1)

By this transformation, the given pde is transformed into

$B\frac{\partial^2 z}{\partial u \partial v} + (Ru_{xx} + Su_{xy} + Tu_{yy})\frac{\partial z}{\partial u} + (Rv_{xx} + Sv_{xy} + Tv_{yy})\frac{\partial z}{\partial v} = 0$ ——— (2)

Since, $a = \frac{\partial u}{\partial x} = x$, $b = \frac{\partial u}{\partial y} = 1$, $c = \frac{\partial v}{\partial x} = -x$, $d = \frac{\partial v}{\partial y} = 1$

$\Rightarrow u_{xx} = 1$, $u_{xy} = 0$, $u_{yy} = 0$, $v_{xx} = -1$, $v_{xy} = 0$, $v_{yy} = 0$

and $B = 2Rac + S(ad + bc) + 2Tbd$

$= -2x^2 - 2x^2 = -4x^2$

$\therefore$ eq(2) $\Rightarrow$

$-4x^2 \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = 0$

$\Rightarrow \frac{\partial^2 z}{\partial u \partial v} = \frac{1}{4x^2}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)$

$\Rightarrow \boxed{\frac{\partial^2 z}{\partial u \partial v} = \frac{1}{4(u-v)}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)}$

which is required canonical form.

4) Reduce the equation $\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0$ to canonical form and hence solve it. —— (1)

Sol:- Here $R = 1, \ S = 2, \ T = 1$

$\therefore S^2 - 4RT = 4 - 4 = 0$

$\therefore$ the given pde is parabolic.

Now the roots of $R\lambda^2 + S\lambda + T = 0$ is,

i.e. $\lambda^2 + 2\lambda + 1 = 0$

$\Rightarrow (\lambda+1)^2 = 0$

$\Rightarrow \lambda = -1$

We solve $\frac{dy}{dx} + 1 = 0$

$\Rightarrow y - x = c_1$

$\therefore$ We take co-ordinate transformation as,

$u = y - x$ and $v = y + x$ —— (2)

By this transformation, the given pde transformed into

$$C z_{vv} + (Ru_{xx} + Su_{xy} + Tu_{yy})\frac{\partial z}{\partial u} + (Rv_{xx} + Sv_{xy} + Tv_{yy})\frac{\partial z}{\partial v} = 0 \quad \text{—— (3)}$$

Since $a = \frac{\partial u}{\partial x} = -1, \ b = \frac{\partial u}{\partial y} = 1, \ c = \frac{\partial v}{\partial x} = 1, \ d = \frac{\partial v}{\partial y} = 1$

$u_{xx} = 0, \ u_{xy} = 0, \ u_{yy} = 0, \ v_{xx} = v_{xy} = v_{yy} = 0$.

$C = Rc^2 + Scd + Td^2$

$= 1 + 2 + 1 = 4$

$\therefore$ eq(3) $\Rightarrow$

$$4\frac{\partial^2 z}{\partial v^2} = 0$$

$$\Rightarrow \boxed{\frac{\partial^2 z}{\partial v^2} = 0}$$ which is the required canonical form.

$$\Rightarrow \frac{\partial z}{\partial v} = f(u)$$

$$\Rightarrow z = v f(u) + g(u)$$

$$\Rightarrow \boxed{z = (y+x) f(y-x) + g(y-x)}$$

which is the required solution.

Monge's Method

Let us consider a second order pde as

$$Rr + Ss + Tt = V \quad \text{———} (1)$$

where r, s, t have their usual meaning and R, S, T, V are of functions of x, y, z, p and q.

We have, $dz = p\,dx + q\,dy$ ——— (2)

Now, since p is a function of x and y

$$\therefore dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy$$

$$\Rightarrow dp = r\,dx + s\,dy$$

$$\Rightarrow r = \frac{dp - s\,dy}{dx} \quad \text{———} (3)$$

Again since q is a function of x and y

$$\therefore dq = \frac{\partial q}{\partial x}dx + \frac{\partial q}{\partial y}dy$$

$$\Rightarrow dq = s\,dx + t\,dy$$

$$\Rightarrow t = \frac{dq - s\,dx}{dy} \quad \text{———} (4)$$

Putting the values of r and t in eq(1), we get

$$R\left(\frac{dp - s\,dy}{dx}\right) + Ss + T\left(\frac{dq - s\,dx}{dy}\right) = V$$

$$\Rightarrow R(dp - s\,dy)dy + Ss\,dx\,dy + T(dq - s\,dx)dx = V\,dx\,dy$$

$$\Rightarrow (R\,dp\,dy - V\,dx\,dy + T\,dq\,dx) - s(R\,dy^2 - S\,dx\,dy + T\,dx^2) = 0 \quad \text{——— (5)}$$

If some relation between x, y, z, p, q makes each of the bracketted expressions vanish the relation will satisfy (5)

i.e. $R\,dp\,dy - V\,dx\,dy + T\,dq\,dx = 0$ ——— (6)

and $R\,dy^2 - S\,dx\,dy + T\,dx^2 = 0$ ——— (7)

These two equations are known as Monge's subsidiary equation.

eq (7) can be resolved into two linear equation in dx and dy

say $dy - m_1 dx = 0$ ——— (8)

and $dy - m_2 dx = 0$ ——— (9)

Now from (8) and (6), combined with (if necessary) $dz = p\,dx + q\,dy$ obtain two integrals

$u_1 = a$ and $v_1 = b$

$\Rightarrow u_1 = f_1(v_1)$, ——— (10)

is the solution and called an intermediate integral.

Similarly from (9) and (6), obtain another intermediate integral $u_2 = f_2(v_2)$ ——— (11)

Solving (10) and (11) to find the value of p and q and then substituting its values in eq (2) and then by integrating we get required solution of (1).

Q:- Solve $r = a^2 t$

Sol:- Given, $r = a^2 t$

$$\Rightarrow r - a^2 t = 0 \quad \text{——— (1)}$$

$\therefore R = 1,\ S = 0,\ T = -a^2,\ V = 0$

Then the Monge's subsidiary equations are

$$R\,dp\,dy - V\,dx\,dy + T\,dq\,dx = 0$$

and $R\,dy^2 - S\,dx\,dy + T\,dx^2 = 0$

$$\Rightarrow dp\,dy - a^2 dq\,dx = 0 \quad \text{——— (2)}$$

and $dy^2 - a^2 dx^2 = 0$ ——— (3)

$$\Rightarrow (dy - a\,dx)(dy + a\,dx) = 0$$

$$\Rightarrow \underbrace{dy - a\,dx = 0}_{\text{(4)}} \text{ and } dy + a\,dx = 0 \quad \text{——— (5)}$$

On integrating eq(4), we get

$$y - ax = c_1 \quad \text{——— (6)}$$

using eq (4) in (2), we have

$$a\,dp\,dx - a^2 dq\,dx = 0$$

$$\Rightarrow (dp - a\,dq)\,a\,dx = 0$$

$$\Rightarrow dp - a\,dq = 0$$

On integrating we get

$$p - aq = c_2 \quad \text{——— (7)}$$

From eq(6) and (7), we get

$$p - aq = f_1(y - ax) \quad \text{——— (8)}$$

Now, on integrating eq(5), we get

$$y + ax = c_3 \quad \text{———} (9)$$

Using eq(5) in (2), we have

$$-a\,dp\,dx - a^2 dq\,dx = 0$$

$$\Rightarrow -(dp + a\,dq)\,a\,dx = 0$$

$$\Rightarrow dp + a\,dq = 0$$

On integrating, we get

$$p + aq = c_4 \quad \text{———} (10)$$

From eq(9) and (10), we have

$$p + aq = f_2(y + ax) \quad \text{———} (11)$$

Solving eq(8) and (11), we get

$$p = \frac{1}{2}\left[f_1(y - ax) + f_2(y + ax)\right]$$

$$q = \frac{1}{2a}\left[f_2(y + ax) - f_1(y - ax)\right]$$

$\therefore$ from $dz = p\,dx + q\,dy$

$$\Rightarrow dz = \frac{1}{2}\left[f_1(y - ax) + f_2(y + ax)\right]dx + \frac{1}{2a}\left[f_2(y + ax) - f_1(y - ax)\right]dy$$

$$= \frac{1}{2a}\left[\{f_1(y - ax) + f_2(y + ax)\}\,a\,dx + \{f_2(y + ax) - f_1(y - ax)\}\,dy\right]$$

$$= \frac{1}{2a}\left[f_1(y - ax)(a\,dx - dy) + f_2(y + ax)(a\,dx + dy)\right]$$

$$= \frac{1}{2a}\left[f_2(y + ax)(dy + a\,dx) - f_1(y - ax)(dy - a\,dx)\right]$$

On integrating, we get

$$\boxed{z = \phi_1(y + ax) + \phi_2(y - ax)}$$

Q:-2 $r-(\cos^2 x)t+p\tan x=0$

Sol:- Given, $r-(\cos^2 x)t+p\tan x=0$ —— (1)

$\therefore R=1,\ S=0,\ T=-\cos^2 x,\ V=-p\tan x$

then the Monge's subsidiary equations are

$$R\,dp\,dy - V\,dx\,dy + T\,dq\,dx = 0$$

and $R\,dy^2 - S\,dx\,dy + T\,dx^2 = 0$

$\Rightarrow dp\,dy + p\tan x\,dx\,dy - \cos^2 x\,dq\,dx = 0$ —— (2)

and $dy^2 - \cos^2 x\,dx^2 = 0$ —— (3)

$\Rightarrow (dy-\cos x\,dx)(dy+\cos x\,dx)=0$

$\Rightarrow dy - \cos x\,dx = 0$ —— (4)

& $dy + \cos x\,dx = 0$ —— (5)

On integrating eq(4), we get

$$y-\sin x = c_1 \quad \text{—— (6)}$$

Using eq(4) in eq(2), we have

$$\cos x\,dp\,dx + p\tan x\cos x\,dx^2 - \cos^2 x\,dq\,dx = 0$$

$\Rightarrow (dp + p\tan x\,dx - \cos x\,dq)\cos x\,dx = 0$

$\Rightarrow dp + p\tan x\,dx - \cos x\,dq = 0$

$\Rightarrow \sec x\,dp + p\sec x\cdot\tan x\,dx - dq = 0$

$\Rightarrow d(p\sec x) - dq = 0$

On integrating, we get

$$p\sec x - q = c_2 \quad \text{—— (7)}$$

From eq (6) and (7), we get

$$p\sec x - q = f_1(y-\sin x) \quad \text{—— (8)}$$

Now, on integrating eq(5), we get

$$y+\sin x = c_3 \quad \text{---} (9)$$

Using eq(5) in eq(2), we have

$$-\cos x\, dp\, dx - p\tan x \cos x\, dx^2 - \cos^2 x\, dq\, dx = 0$$

$$\Rightarrow -\cos x\, dx(dp + p\tan x\, dx + \cos x\, dq) = 0$$

$$\Rightarrow dp + p\tan x\, dx + \cos x\, dq = 0$$

$$\Rightarrow \sec x\, dp + p\sec x \cdot \tan x\, dx + dq = 0$$

$$\Rightarrow d(p\sec x) + dq = 0$$

On integrating, we get

$$p\sec x + q = c_4 \quad \text{---} (10)$$

From eq(9) and (10), we get

$$p\sec x + q = f_2(y+\sin x) \quad \text{---} (11)$$

On solving (8) and (11), we have

$$p = \frac{1}{2\sec x}\left[f_1(y-\sin x) + f_2(y+\sin x)\right]$$

and $q = \frac{1}{2}\left[f_2(y+\sin x) - f_1(y-\sin x)\right]$

$\therefore$ From $dz = p\,dx + q\,dy$, we have

$$dz = \frac{1}{2\sec x}\left[f_1(y-\sin x) + f_2(y+\sin x)\right]dx + \frac{1}{2}\left[f_2(y+\sin x) - f_1(y-\sin x)\right]dy$$

$$= \frac{1}{2}\cos x\left[f_1(y-\sin x) + f_2(y+\sin x)\right]dx + \frac{1}{2}\left[-f_1(y-\sin x) + f_2(y+\sin x)\right]dy$$

$$= \frac{-1}{2} f_1(y-\sin x)(dy - \cos x\, dx) + \frac{1}{2} f_2(y+\sin x)(dy + \cos x\, dx)$$

$$= \frac{-1}{2} f_1(y-\sin x)\, d(y-\sin x) + \frac{1}{2} f_2(y+\sin x)\, d(y+\sin x)$$

On integrating we get,

$$\boxed{z = \phi_1(y-\sin x) + \phi_2(y+\sin x)}$$

Q-3 $t - r\sec^4 y = 2q\tan y$.

Sol:- Given, $t - r\sec^4 y = 2q\tan y$ ——— (1)

$\therefore R = -\sec^4 y,\ S = 0,\ T = 1,\ V = 2q\tan y$

then the Moonge's subsidiary equations are

$$R\,dp\,dy - V\,dx\,dy + T\,dq\,dx = 0$$

and $R\,dy^2 - S\,dx\,dy + T\,dx^2 = 0$

$\Rightarrow -\sec^4 y\, dp\,dy - 2q\tan y\,dx\,dy + dq\,dx = 0$ ——— (2)

and $-\sec^4 y\, dy^2 + dx^2 = 0$ ——— (3)

eq(3) $\Rightarrow (dx - \sec^2 y\,dy)(dx + \sec^2 y\,dy) = 0$

$\Rightarrow dx - \sec^2 y\,dy = 0$ ——— (4)

and $dx + \sec^2 y\,dy = 0$ ——— (5)

On integrating eq(4), we get,

$$x - \tan y = c_1 \quad \text{——— (6)}$$

Using eq(4) in eq(2), we have

$-\sec^4 y\,dp\,dy - 2q\tan y\sec^2 y\,dy^2 + \sec^2 y\,dq\,dy = 0$

$\Rightarrow (-\sec^2 y\,dp - 2q\tan y\,dy + dq)\sec^2 y\,dy = 0$

$\Rightarrow -\sec^2 y\,dp - 2q\tan y\,dy + dq = 0$

$\Rightarrow -dp - 2q\sin y\cdot\cos y\,dy + \cos^2 y\,dq = 0$

$\Rightarrow dp - (\cos^2 y\,dq - 2q\sin y\cos y\,dy) = 0$

$\Rightarrow dp - d(q\cos^2 y) = 0$

$\Rightarrow p - q\cos^2 y = c_2$ ——— (7)

$\therefore$ From eq(6) & (7), we get

$$p - q\cos^2 y = f_1(x - \tan y) \quad \text{——} (8)$$

On integrating eq(5), we get

$$x + \tan y = c_3 \quad \text{——} (9)$$

Using eq(5) in eq(2), we have

$$-\sec^4 y\, dp\, dy + 2q \tan y \sec^2 y\, dy^2 - \sec^2 y\, dq\, dy = 0$$

$$\Rightarrow (-\sec^2 y\, dp + 2q \tan y\, dy - dq)\sec^2 y\, dy = 0$$

$$\Rightarrow -\sec^2 y\, dp + 2q \tan y\, dy - dq = 0$$

$$\Rightarrow -dp + 2q \sin y \cos y\, dy - \cos^2 y\, dq = 0$$

$$\Rightarrow dp + \cos^2 y\, dq - 2q \sin y \cos y\, dy = 0$$

$$\Rightarrow dp + d(q\cos^2 y) = 0$$

$$\Rightarrow p + q\cos^2 y = c_4 \quad \text{——} (10)$$

From eqs (9) and (10), we get

$$p + q\cos^2 y = f_2(x + \tan y) \quad \text{——} (11)$$

On solving (8) and (11), we have

$$p = \frac{1}{2}\left[f_1(x - \tan y) + f_2(x + \tan y)\right]$$

$$q = \frac{1}{2\cos^2 y}\left[f_2(x + \tan y) - f_1(x - \tan y)\right]$$

$\therefore$ From $dz = p\,dx + q\,dy$, we have

$$dz = \frac{1}{2}\left[f_1(x - \tan y) + f_2(x + \tan y)\right]dx + \frac{1}{2\cos^2 y}\left[f_2(x + \tan y) - f_1(x - \tan y)\right]dy$$

$$= \frac{1}{2}\left[f_1(x - \tan y)(dx - \sec^2 y\, dy) + f_2(x + \tan y)(dx + \sec^2 y\, dy)\right]$$

$$\Rightarrow dz = \frac{1}{2} f_1(x-\tan y)\, d(x-\tan y) + \frac{1}{2} f_2(x+\tan y)\, d(x+\tan y)$$

On integrating we get,

$$\boxed{z = \phi_1(x-\tan y) + \phi_2(x+\tan y)}$$

Q-4 $r = t$ (wave equation)

Sol:- Given, $r = t$

$$\Rightarrow r - t = 0 \quad \text{——— (1)}$$

$\therefore R = 1,\ S = 0,\ T = -1,\ V = 0$

then Monge's subsidiary equations are

$$R\,dp\,dy - V\,dx\,dy + T\,dq\,dx = 0$$

and $R\,dy^2 - S\,dx\,dy + T\,dx^2 = 0$

$$\Rightarrow dp\,dy - dq\,dx = 0 \quad \text{——— (2)}$$

and $dy^2 - dx^2 = 0$ ——— (3)

eq(3) $\Rightarrow (dy - dx)(dy + dx) = 0$

$$\Rightarrow dy - dx = 0 \text{ ——— (4) and } dy + dx = 0 \text{ ——— (5)}$$

On integrating eq(4), we get

$$y - x = c_1 \quad \text{——— (6)}$$

Using eq(4) in eq(2), we have

$$dp\,dx - dq\,dx = 0$$

$$\Rightarrow dp - dq = 0$$

$$\Rightarrow p - q = c_2 \quad \text{——— (7)}$$

From eq(6) & (7), we get

$$p - q = f_1(y - x) \quad \text{——— (8)}$$

On integrating eq(5), we get

$$y+x=c_3 \quad \text{——— (9)}$$

Using eq(5) in (2), we have

$$-dp\,dx - dq\,dx = 0$$

$$\Rightarrow dp + dq = 0$$

$$\Rightarrow p+q = c_4 \quad \text{——— (10)}$$

From eq(9) & (10), we get

$$p+q = f_2(y+x) \quad \text{——— (11)}$$

On solving (8) and (11), we have

$$p = \frac{1}{2}\left[f_1(y-x) + f_2(y+x)\right]$$

$$q = \frac{1}{2}\left[f_2(y+x) - f_1(y-x)\right]$$

From $dz = p\,dx + q\,dy$, we get

$$dz = \frac{1}{2}\left[\{f_1(y-x) + f_2(y+x)\}dx + \{f_2(y+x) - f_1(y-x)\}dy\right]$$

$$= \frac{1}{2}\left[f_1(y-x)(dx-dy) + f_2(y+x)(dx+dy)\right]$$

On integrating we get

$$\boxed{z = \phi_1(y-x) + \phi_2(y+x)}$$

* Cauchy's problem for 2nd order pde

Let us consider a second order pde as

$$Rr + Ss + Tt + f(x,y,z,p,q) = 0 \quad \text{——— (1)}$$

where R, S, T are function of x and y. The Cauchy's problem consists of the problem of determining the solution of (1) with the help of some condition on z.

Ex:- To determine solution of $\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial y^2}$ with the following data $z(x,0) = f(x)$, $z_y(x,0) = g(x)$.

Wave Equation

An equation of the form

$$\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2} \quad \text{or} \quad \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

is called wave equation where $z = z(x,t)$ or $u = u(x,t)$

Q:- Obtain the fundamental solution of the wave equation $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$ by separation of variable method.

Sol:- Given wave equation is,

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \quad \text{——— (1)}$$

we seek the solution in the form

$$u(x,t) = X(x) \cdot T(t) \quad \text{——— (2)}$$

Putting this value of $u(x,t)$ in (1), we get

$$X''T = \frac{1}{c^2} X T''$$

$$\Rightarrow \frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = \lambda \text{ (say)} \quad \text{——— (3)}$$

where λ is constant independent of x and t.

Then eq(3) gives two ODE's

$$X'' = \lambda X \quad \text{and} \quad T'' = c^2 \lambda T \quad \text{——— (4)}$$

Now we consider the following cases

<u>case-(i)</u> If $\lambda = 0$, then eq(4) becomes

$$X'' = 0 \quad \text{and} \quad T'' = 0$$

$$\Rightarrow X = a_1 x + a_2 \quad \text{and} \quad T = b_1 t + b_2$$

$\therefore$ From (2), the solution of (1) is,

$$u(x,t) = (a_1 x + a_2)(b_1 t + b_2) \quad \text{——— (5)}$$

<u>case-(ii)</u> If $\lambda = k^2 > 0$, then eq(4) becomes

$$X'' = k^2 X \quad \text{and} \quad T'' = k^2 c^2 T$$

$$\Rightarrow X = a_1 e^{kx} + a_2 e^{-kx} \quad \text{and} \quad T = b_1 e^{kct} + b_2 e^{-kct}$$

Then from eq(2), solution of (1) is,

$$u(x,t) = (a_1 e^{kx} + a_2 e^{-kx})(b_1 e^{kct} + b_2 e^{-kct}) \quad \text{——— (6)}$$

<u>case-(iii)</u> If $\lambda = -k^2 < 0$, then eq(4) becomes

$$X'' = -k^2 X \quad \text{and} \quad T'' = -k^2 c^2 T$$

$$\Rightarrow X = (a_1 \cos kx + a_2 \sin kx) \quad \text{and} \quad T = (b_1 \cos kct + b_2 \sin kct)$$

$\therefore$ Solution of (1) is,

$$u(x,t) = (a_1 \cos kx + a_2 \sin kx)(b_1 \cos kct + b_2 \sin kct) \quad \text{——— (7)}$$

Since the wave phenomenon is periodic, so the solution (5) and (6) are not accepted as it has function of x, e^{x} which are not periodic.

Hence eq(7) is required solution of (1).

Q:- Find the deflection $u(x,t)$ of a vibrating string of length l under the boundary condition $u(0,t) = u(l,t) = 0$; $t > 0$ and the initial conditions $u(x,0) = f(x)$, $u_t(x,0) = g(x)$.

Sol:- Let us consider the wave equation as,

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \quad \text{——— (1)}$$

with boundary condition $u(0,t) = u(l,t) = 0$ ——— (2)

and with initial conditions $u(x,0) = f(x)$, $u_t(x,0) = g(x)$ ——— (3)

we seek the solution of (1) in form

$$u(x,t) = X(x) \cdot T(t) \quad \text{——— (4)}$$

∴ By putting $u(x,t)$ from (4) in (1), we get

$$X'' T = \frac{1}{c^2} X T''$$

$$\Rightarrow \frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = \lambda \text{ (say)}$$

$$\Rightarrow X'' = \lambda X \text{ and } T'' = \lambda c^2 T \quad \text{——— (5)}$$

we consider the following cases :

<u>case-(i)</u> If $\lambda = 0$, then eq (5) becomes

$$X'' = 0 \text{ and } T'' = 0$$

$$\Rightarrow X = a_1 x + a_2 \text{ and } T = b_1 t + b_2$$

∴ Solution of (1) is

$$u(x,t) = (a_1 x + a_2)(b_1 t + b_2)$$

$\Rightarrow u(x,t) = 0$ [because $u(0,t) = u(l,t) = 0$]

or solution is not periodic so it is rejected.

<u>case-(ii)</u> If $\lambda = K^2 > 0$, then eq(5) becomes,

$$X'' = K^2 X \quad \text{and} \quad T'' = c^2K^2T$$

$$\Rightarrow X = a_1 e^{Kx} + a_2 e^{-Kx} \quad \text{and} \quad T = b_1 e^{Kct} + b_2 e^{-Kct}$$

∴ Solution of (1) is,

$$u(x,t) = (a_1 e^{Kx} + a_2 e^{-Kx})(b_1 e^{Kct} + b_2 e^{-Kct})$$

∵ $u(0,t) = 0$ and $u(l,t) = 0 \Rightarrow a_1 = a_2 = 0$

$\Rightarrow u(x,t) = 0$

or solution is not periodic hence, rejected.

<u>case-(iii)</u> If $\lambda = -K^2$, then eq(5) becomes

$$X'' = -K^2 X \quad \text{and} \quad T'' = -c^2K^2T$$

$$\Rightarrow X = a_1 \cos Kx + a_2 \sin Kx \quad \text{and} \quad T = b_1 \cos Kct + b_2 \sin Kct$$

∴ Solution of (1) is,

$$u(x,t) = (a_1 \cos Kx + a_2 \sin Kx)(b_1 \cos Kct + b_2 \sin Kct)$$
$$= XT \quad \text{------ (6)}$$

∵ $u(0,t) = 0 \Rightarrow a_1 T = 0 \Rightarrow a_1 = 0$

and $u(l,t) = 0 \Rightarrow a_2 \sin(Kl) \cdot T = 0$

$\Rightarrow \sin(Kl) = 0$

$\Rightarrow Kl = n\pi$

$\Rightarrow K = \frac{n\pi}{l}, \; n \in \mathbb{N}$

∴ eq (6) becomes

$$u(x,t) = a_2 \sin\left(\frac{n\pi x}{l}\right)\left\{b_1 \cos\left(\frac{n\pi ct}{l}\right) + b_2 \sin\left(\frac{n\pi ct}{l}\right)\right\}$$

$$\Rightarrow u_n(x,t) = \sin\left(\frac{n\pi x}{l}\right)\left\{a_n \cos\left(\frac{n\pi ct}{l}\right) + b_n \sin\left(\frac{n\pi ct}{l}\right)\right\}$$

$\therefore$ By the principle of superposition the series solution can be taken as,

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{l}\right) \left\{ a_n \cos\left(\frac{n\pi ct}{l}\right) + b_n \sin\left(\frac{n\pi ct}{l}\right) \right\} \quad \text{——— (7)}$$

$\because u(x,0) = f(x)$

$$\therefore f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{l}\right)$$

which is Fourier sine series

$\therefore$ The Fourier coefficient a_n is given by,

$$a_n = \frac{2}{l}\int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx \quad \text{——— (8)}$$

Again differentiating eq(7) partially w.r.t 't' we get

$$u_t(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{l}\right) \left\{ -\frac{n\pi c}{l} a_n \sin\left(\frac{n\pi ct}{l}\right) + \frac{n\pi c}{l} b_n \cos\left(\frac{n\pi ct}{l}\right) \right\}$$

$\because u_t(x,0) = g(x)$

$$\therefore g(x) = \sum_{n=1}^{\infty} \frac{n\pi c}{l} b_n \sin\left(\frac{n\pi x}{l}\right)$$

Again, which is Fourier sine series.

So Fourier coefficient b_n is given by,

$$\frac{n\pi c}{l} b_n = \frac{2}{l}\int_0^l g(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

$$\Rightarrow b_n = \frac{2}{n\pi c}\int_0^l g(x) \sin\left(\frac{n\pi x}{l}\right) dx \quad \text{——— (9)}$$

Hence eq(7) is the series solution of the problem where the coefficient a_n and b_n are given by (8) and (9) respectively.

Q:- Find the Fourier series solution of the problem of vibrating string.

Q:- Solve the problem of vibrating string by Fourier series method.

Prob-1 Solve the equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ under the boundary condition $u(0,t) = u(1,t) = 0$ and initial condition $u(x,0) = A\sin(\pi x)$ and $u_t(x,0) = 0$.

Sol:- Given that $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$ ——— (1)

with boundary condition $u(0,t) = u(l,t) = 0$ ——— (2)

and initial condition, $u(x,0) = A\sin(\pi x)$ and $u_t(x,0) = 0$ ——— (3)

then we have to find solution of (1) in the form

$$u(x,t) = X(x)\cdot T(t) \quad \text{——— (4)}$$

Putting this value of $u(x,t)$ in (1), we get

$$X'' T = \frac{1}{c^2} X T''$$

$$\Rightarrow \frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = \lambda \text{ (say)} \quad \text{——— (5)}$$

where λ is constant independent of x and t.

then eq(5) gives two ODE's

$$X'' = \lambda X \text{ and } T'' = c^2\lambda T \quad \text{——— (6)}$$

Now, we consider the following cases.

case-(i) If $\lambda = 0$, then eq(6) becomes

$$X'' = 0 \text{ and } T'' = 0$$

$$\Rightarrow X = a_1 x + a_2 \text{ and } T = b_1 t + b_2$$

$\therefore$ From (4), the solution of (1) is,

$$u(x,t) = (a_1 x + a_2)(b_1 t + b_2) \qquad \text{——— (7)}$$

Using (2), we get

$$u(x,t) = 0$$

So it is rejected.

<u>case-(ii)</u> If $\lambda = K^2 > 0$, then eq (6) becomes

$$X'' = K^2 X \quad \text{and} \quad T'' = c^2 K^2 T$$

$$\Rightarrow X = a_1 e^{Kx} + a_2 e^{-Kx} \quad \text{and} \quad T = b_1 e^{Kct} + b_2 e^{-Kct}$$

$\therefore$ Solution of (1) is

$$u(x,t) = (a_1 e^{Kx} + a_2 e^{-Kx})(b_1 e^{Kct} + b_2 e^{-Kct})$$

$\because u(0,t) = u(l,t) = 0 \Rightarrow a_1 = a_2 = 0$

$\Rightarrow u(x,t) = 0$

So it is rejected.

<u>case-(iii)</u> If $\lambda = -K^2$, then eq(6) becomes

$$X'' = -K^2 X \quad \text{and} \quad T'' = -c^2 K^2 T$$

$$\Rightarrow X = a_1 \cos Kx + a_2 \sin Kx \quad \text{and} \quad T = b_1 \cos Kct + b_2 \sin Kct$$

$\therefore$ Solution of (1) is

$$u(x,t) = (a_1 \cos Kx + a_2 \sin Kx)(b_1 \cos Kct + b_2 \sin Kct)$$
$$= XT \qquad \text{——— (8)}$$

$\because u(0,t) = 0 \Rightarrow a_1 T = 0 \Rightarrow a_1 = 0$

and $u(l,t) = 0 \Rightarrow a_2 \sin(Kl) T = 0$

$\Rightarrow \sin(Kl) = 0$

$\Rightarrow Kl = n\pi \Rightarrow K = \frac{n\pi}{l}, \ n \in \mathbb{N}$

$\therefore$ eq(8) becomes

$$u(x,t) = a_2 \sin\left(\frac{n\pi x}{l}\right)\left\{ b_1 \cos\left(\frac{n\pi ct}{l}\right) + b_2 \sin\left(\frac{n\pi ct}{l}\right) \right\}$$

$$\Rightarrow u_n(x,t) = \sin\left(\frac{n\pi x}{l}\right)\left\{a_n \cos\left(\frac{n\pi c t}{l}\right) + b_n \sin\left(\frac{n\pi c t}{l}\right)\right\} \quad \text{---(9)}$$

Differentiating eq(9), partially w.r.t 't' we get

$$\frac{\partial u_n}{\partial t} = \sin(n\pi x)\left\{-a_n \cdot n\pi c \sin(n\pi c t) + b_n \cdot n\pi c \cos(n\pi c t)\right\}$$

Since $u_t(x,0) = 0$

$$\Rightarrow \sin(n\pi x)\{b_n \cdot n\pi c\} = 0 \quad \Rightarrow b_n = 0$$

$\therefore$ eq (9) $\Rightarrow$

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin(n\pi x) \cdot \cos(n\pi c t)$$

Now put $t = 0$ then

$$u(x,0) = A \sin(\pi x)$$

$$\Rightarrow A \sin(\pi x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x)$$

$\Rightarrow n = 1$ and $a_n = A$

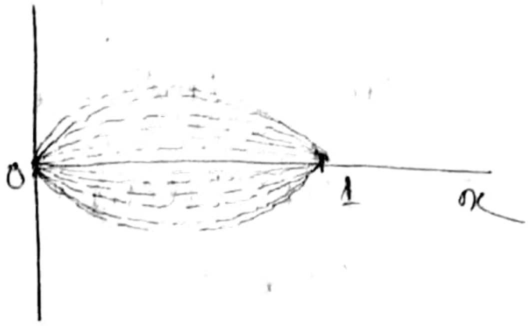
$\therefore$ Solution of (1) is

$$\boxed{u(x,t) = A \sin(\pi x) \cos(\pi c t)}$$

<u>Prob-2</u> A tightly stretched string with fixed end point $x=0$, $x=1$ is initially in a position given by $u = \sin^3(\pi x)$, if it released from rest form, then find the displacement $u(x,t)$.

Sol:- The wave equation for the vibrating string is

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \quad \text{——— (1)}$$



with the boundary condition

$$u(0,t) = u(1,t) = 0 \quad \text{——— (2)}$$

and initial condition $u(x,0) = \sin^3 \pi x$, $u_t(x,0) = 0$ ——— (3)

we seek the solution of (1) in the form

$$u(x,t) = X(x)\cdot T(t) \quad \text{——— (4)}$$

By putting $u(x,t)$ from (4) in (1), we get

$$X''T = \frac{1}{c^2} X T''$$

$$\Rightarrow \frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = \lambda \text{ (say)}$$

$$\Rightarrow X'' = \lambda X \text{ and } T'' = c^2\lambda T \quad \text{——— (5)}$$

we consider the following cases:-

case-(i) If $\lambda = 0$, then eq(5) becomes

$$X'' = 0 \text{ and } T'' = 0$$

$$\Rightarrow X = a_1 x + a_2 \text{ and } T = b_1 t + b_2$$

$\therefore$ Solution of (1) is,

$$u(x,t) = (a_1 x + a_2)(b_1 t + b_2)$$

$\because u(0,t) = u(1,t) = 0$

$$\Rightarrow u(x,t) = 0$$

So it is rejected.

case-(ii) If $\lambda = k^2 > 0$, then eq(5) becomes

$$X'' = k^2 X \text{ and } T'' = c^2 k^2 T$$

$$\Rightarrow X = a_1 e^{kx} + a_2 e^{-kx} \text{ and } T = b_1 e^{ckt} + b_2 e^{-ckt}$$

$\therefore$ Solution of (1) is,

$$u(x,t) = (a_1 e^{kx} + a_2 e^{-kx})(b_1 e^{kct} + b_2 e^{-kct})$$

$\because u(0,t) = u(1,t) = 0 \Rightarrow a_1 = a_2 = 0$

$\Rightarrow u(x,t) = 0$

$\therefore$ It is rejected.

<u>case-(iii)</u> If $\lambda = -k^2 < 0$, then eq(5) becomes

$$X'' = -k^2 X \text{ and } T'' = -k^2 c^2 T$$

$\Rightarrow X = a_1 \cos kx + a_2 \sin kx$ and $T = b_1 \cos kct + b_2 \sin kct$

$\therefore$ Solution of (1) is

$$u(x,t) = (a_1 \cos kx + a_2 \sin kx)(b_1 \cos kct + b_2 \sin kct) = XT$$

$\because u(0,t) = 0 \Rightarrow a_1 T = 0 \Rightarrow a_1 = 0$

and $u(1,t) = 0 \Rightarrow a_2 \sin k \cdot T = 0$

$\Rightarrow \sin k = 0 \Rightarrow k = n\pi,\ n \in \mathbb{N}$

$$\therefore u(x,t) = a_2 \sin(n\pi x)(b_1 \cos n\pi ct + b_2 \sin n\pi ct) \quad \text{--- (6)}$$

Differentiating eq(6) w.r.t 't', we get

$$\frac{\partial u}{\partial t} = a_2 \sin(n\pi x)\{-b_1 n\pi c \sin(n\pi ct) + b_2 n\pi c \cos(n\pi ct)\}$$

$\because u_t(x,0) = 0$

$\Rightarrow a_2 \sin(n\pi x)\, b_2 \cdot n\pi c = 0$

$\Rightarrow b_2 = 0$

$\therefore$ eq(6) $\Rightarrow u(x,t) = a_2 \sin(n\pi x) \cdot b_1 \cos(n\pi c t)$

$\Rightarrow u_n(x,t) = a_n \sin(n\pi x) \cdot \cos(n\pi c t)$

$\therefore$ By the principle of superposition, we take the solution of the form

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin(n\pi x) \cos(n\pi c t) \quad \text{——— (7)}$$

Also $\because u(x,0) = \sin^3(\pi x)$

$$\therefore \sin^3(\pi x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x)$$

$$\Rightarrow \frac{3}{4}\sin(\pi x) - \frac{1}{4}\sin(3\pi x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x)$$

$\therefore n=1$ and $n=3$ and $a_1 = \frac{3}{4}$, $a_3 = \frac{-1}{4}$, $a_n = 0 \ \forall n \neq 1, 3$

$\therefore$ Solution of (1) is

$$\boxed{u(x,t) = \frac{3}{4}\sin(\pi x) \cdot \cos(\pi c t) - \frac{1}{4}\sin(3\pi x)\cos(3\pi c t)}$$

<u>Prob-3</u>:- A string is fastened to two fixed point which are l distance apart to each other, then set vibrating. Find the deflection $u(x,t)$ if the string is initially released from the position $u(x,0) = K(lx + x^2)$.

<u>Sol</u>:- The wave equation for the vibrating string is given by, $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$ ——— (1)

with the boundary condition $u(0,t) = u(l,t) = 0$ ——— (2)

and initial condition $u(x,0) = K(lx - x^2)$, $u_t(x,0) = 0$ ——— (3)

we seek the solution of (1) in the form

$$u(x,t) = X(x) \cdot T(t) \quad \text{——— (4)}$$

Using (4) in (1), we get

$$X''T = \frac{1}{c^2} X T''$$

$$\Rightarrow \frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = \lambda \text{ (say)}$$

$$\Rightarrow X'' = \lambda X \quad \text{and} \quad T'' = c^2 \lambda T \quad \text{——— (5)}$$

we consider the following cases :-

<u>case-(i)</u> If $\lambda = 0$, then eq(5) becomes

$$X'' = 0 \quad \text{and} \quad T'' = 0$$

$$\Rightarrow X = a_1 x + a_2 \quad \text{and} \quad T = b_1 t + b_2$$

$\therefore$ Solution of (1) is,

$$u(x,t) = (a_1 x + a_2)(b_1 t + b_2)$$

$\because u(0,t) = u(l,t) = 0$

$\Rightarrow u(x,t) = 0$

$\therefore$ So it is rejected.

<u>case-(ii)</u> If $\lambda = k^2 > 0$, then eq(5) becomes

$$X'' = k^2 X \quad \text{and} \quad T'' = c^2 k^2 T$$

$$\Rightarrow X = a_1 e^{kx} + a_2 e^{-kx} \quad \text{and} \quad T = b_1 e^{kct} + b_2 e^{-kct}$$

$\therefore$ Solution of (1) is

$$u(x,t) = (a_1 e^{kx} + a_2 e^{-kx})(b_1 e^{kct} + b_2 e^{-kct})$$

$\because u(0,t) = u(l,t) = 0 \Rightarrow a_1 = a_2 = 0$

$\therefore u(x,t) = 0$

So it is rejected.

<u>case-(iii)</u> If $\lambda = -k^2 < 0$, then eq(5) becomes

$$X'' = -k^2 X \quad \text{and} \quad T'' = -k^2 c^2 T$$

$$\Rightarrow X = a_1 \cos kx + a_2 \sin kx \quad \text{and} \quad T = b_1 \cos kct + b_2 \sin kct$$

$\therefore$ Solution of (1) is,

$$u(x,t) = (a_1 \cos Kx + a_2 \sin Kx)(b_1 \cos Kct + b_2 \sin Kct)$$
$$= XT \quad \text{------ (6)}$$

$u(0,t) = 0 \Rightarrow a_1 T = 0 \Rightarrow a_1 = 0$

$u(l,t) = 0 \Rightarrow a_2 \sin(Kl)\, T = 0$

$\Rightarrow \sin(Kl) = 0 \Rightarrow Kl = n\pi$

$\Rightarrow K = \frac{n\pi}{l}, \; n \in \mathbb{N}$

$$\therefore u(x,t) = a_2 \sin\left(\frac{n\pi x}{l}\right)\left\{ b_1 \cos\left(\frac{n\pi ct}{l}\right) + b_2 \sin\left(\frac{n\pi ct}{l}\right)\right\} \quad \text{------ (7)}$$

Differentiating eq(7) partially w.r.t 't' we get

$$u_t(x,t) = a_2 \sin\left(\frac{n\pi x}{l}\right)\left\{ -b_1 \frac{n\pi c}{l} \sin\left(\frac{n\pi ct}{l}\right) + b_2 \cdot \frac{n\pi c}{l} \cos\left(\frac{n\pi ct}{l}\right)\right\}$$

$u_t(x,0) = 0 \Rightarrow a_2 \sin\left(\frac{n\pi x}{l}\right) \cdot b_2 \cdot \frac{n\pi c}{l} = 0$

$\Rightarrow b_2 = 0$

* From eq(6) calculate $u_t(x,t)$ then write $u(x,t)$

$\therefore$ eq(7) gives

$$u(x,t) = a_2 \sin\left(\frac{n\pi x}{l}\right) b_1 \cos\left(\frac{n\pi ct}{l}\right)$$

$$\Rightarrow u_n(x,t) = a_n \sin\left(\frac{n\pi x}{l}\right) \cos\left(\frac{n\pi ct}{l}\right)$$

By the principle of superposition, we take the solution is of the form

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{l}\right) \cos\left(\frac{n\pi ct}{l}\right) \quad \text{------ (8)}$$

Also $u(x,0) = K(lx - x^2)$

$\therefore$ eq (8) $\Rightarrow$ $$K(lx - x^2) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{l}\right)$$

which is Fourier sine series

The Fourier coefficient a_n is given by,

$$a_n = \frac{2}{l}\int_0^l K(lx - x^2) \sin\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{2K}{l}\int_0^l (lx - x^2) \sin\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{2K}{l}\left\{\left[(lx - x^2)\left\{\frac{-\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)}\right\}\right]_0^l + \int_0^l \frac{l}{n\pi}\cos\left(\frac{n\pi x}{l}\right)\cdot(l - 2x)\,dx\right\}$$

$$= \frac{2K}{l}\left\{0 - 0 + \frac{l}{n\pi}\int_0^l (l - 2x)\cos\left(\frac{n\pi x}{l}\right) dx\right\}$$

$$= \frac{2K}{l}\cdot\frac{l}{n\pi}\left\{\left[(l - 2x)\frac{l}{n\pi}\cdot\sin\left(\frac{n\pi x}{l}\right)\right]_0^l - \int_0^l \frac{l}{n\pi}\cdot\sin\left(\frac{n\pi x}{l}\right)(-2)\,dx\right\}$$

$$= \frac{2K}{n\pi}\left\{0 - 0 + \frac{2l}{n\pi}\cdot\frac{l}{n\pi}\left[-\cos\left(\frac{n\pi x}{l}\right)\right]_0^l\right\}$$

$$= \frac{4Kl^2}{n^3\pi^3}\left\{-\cos(n\pi) - (-1)\right\}$$

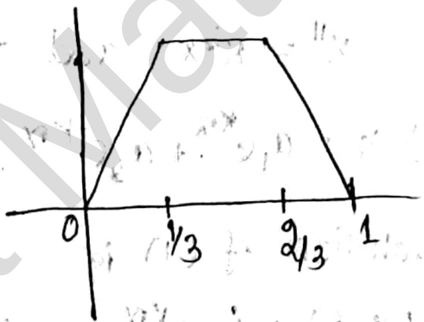
$$= \frac{4Kl^2}{n^3\pi^3}(1 - \cos(n\pi))$$

$\therefore$ Solution of (1) is given by,

$$u(x,t) = \sum_{n=1}^{\infty} \frac{4Kl^2}{n^3\pi^3}(1-\cos(n\pi))\sin\left(\frac{n\pi x}{l}\right)\cdot\cos\left(\frac{n\pi ct}{l}\right)$$

$$\Rightarrow \boxed{u(x,t) = \frac{4Kl^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}(1-\cos(n\pi))\sin\left(\frac{n\pi x}{l}\right)\cdot\cos\left(\frac{n\pi ct}{l}\right)}$$

Prob-4 Suppose a string has $c=2$ and $l=1$, the ends are fixed. The segment of string between $1/3$ and $2/3$ is lifted horizontally and string is released from rest form that position, then describe the motion of string.



Sol:- Given boundary value problem is,

$$\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2} \quad \text{——— (1)}$$

with boundary condition $u(0,t) = u(1,t) = 0$ ——— (2)

and initial condition, $u(x,0) = f(x)$ and $u_t(x,0) = 0$ ——— (3)

where $f(x) = \begin{cases} x : & 0 \le x \le \frac{1}{3} \\ \frac{1}{3} : & 1/3 \le x \le 2/3 \\ 1-x : & 2/3 \le x \le 1 \end{cases}$

we seek the solution of (1) in the form

$$u(x,t) = X(x)\cdot T(t) \quad \text{——— (4)}$$

Using (4) in (1), we get

$$XT'' = 4X''T$$

$$\Rightarrow \frac{X''}{X} = \frac{1}{4}\frac{T''}{T} = \lambda \text{ (say)}$$

$$\Rightarrow X'' = \lambda X \text{ and } T'' = 4\lambda T \quad \text{——— (5)}$$

We consider the following cases :-

case-(i) If $\lambda = 0$, then eq (5) becomes

$$X'' = 0 \text{ and } T'' = 0$$

$$\Rightarrow X = a_1 x + a_2 \text{ and } T = b_1 t + b_2$$

$\therefore$ Solution of (1) is

$$u(x,t) = (a_1 x + a_2)(b_1 t + b_2)$$

$\because$ $u(0,t) = u(1,t) = 0$

$\Rightarrow u(x,t) = 0$

$\therefore$ It is rejected.

case-(ii) If $\lambda = k^2 > 0$, then eq (5) becomes

$$X'' = k^2 X \text{ and } T'' = 4k^2 T$$

$$\Rightarrow X = a_1 e^{kx} + a_2 e^{-kx} \text{ and } T = b_1 e^{2kt} + b_2 e^{-2kt}$$

$\therefore$ Solution of (1) is

$$u(x,t) = (a_1 e^{kx} + a_2 e^{-kx})(b_1 e^{2kt} + b_2 e^{-2kt})$$

$u(0,t) = u(1,t) = 0 \Rightarrow a_1 = a_2 = 0$

$\Rightarrow u(x,t) = 0$

hence rejected.

case-(iii) If $\lambda = -k^2 < 0$, then eq (5) becomes

$$X'' = -k^2 X \text{ and } T'' = -4k^2 T$$

$$\Rightarrow X = (a_1 \cos kx + a_2 \sin kx) \text{ and } T = b_1 \cos 2kt + b_2 \sin 2kt$$

$\therefore$ Solution of (1) is,

$$u(x,t) = (a_1 \cos kx + a_2 \sin kx)(b_1 \cos 2kt + b_2 \sin 2kt)$$

$$= XT \qquad \text{——— (6)}$$

$u(0,t)=0 \Rightarrow a_1 T=0 \Rightarrow a_1=0$ ——— (7)

$u(1,t)=0 \Rightarrow a_2 \sin(k)\cdot T=0$

$\Rightarrow \sin k=0 \Rightarrow k=n\pi,\quad n\in\mathbb{N}$ ——— (8)

Now ~~$u(x,t) = a_2 \sin kx\,(b_1\cos 2kt + b_2 \sin 2kt)$~~

Differentiating eq(6) partially w.r.t 't' we get

$u_t(x,t) = X(-b_1\cdot 2k\sin 2kt + b_2\cdot 2k\cos 2kt)$

$u_t(x,0)=0 \Rightarrow X(b_2\cdot 2k)=0$

$\Rightarrow b_2=0$ ——— (9)

Using (7),(8),(9) in eq(6), we get

$u(x,t) = a_2 \sin(n\pi x)\{b_1 \cos(2n\pi t)\}$

$\Rightarrow u_n(x,t) = a_n \sin(n\pi x)\cos(2n\pi t)$

$\therefore$ By the principle of superposition, we take the solution of the form

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin(n\pi x)\cos(2n\pi t) \quad \text{——— (10)}$$

Since $u(x,0)=f(x)$

$$\Rightarrow f(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x)$$

which is Fourier sine series

$\therefore$ The Fourier co-efficient a_n is given by,

$$a_n = 2\int_0^1 f(x)\sin(n\pi x)\,dx$$

$$= 2\int_0^{1/3} x\sin(n\pi x)\,dx + 2\int_{1/3}^{2/3} \tfrac{1}{3}\sin(n\pi x)\,dx$$

$$+ 2\int_{2/3}^{1}(1-x)\sin(n\pi x)\,dx$$

$$= 2\left\{\left[-x\,\frac{\cos n\pi x}{n\pi}\right]_0^{1/3} + \int_0^{1/3}\frac{\cos n\pi x}{n\pi}\,dx\right\}$$

$$+ 2\left\{\frac{-1}{3}\left[\frac{\cos(n\pi x)}{n\pi}\right]_{1/3}^{2/3}\right\}$$

$$+ 2\left\{\left[-(1-x)\,\frac{\cos(n\pi x)}{n\pi}\right]_{2/3}^{1} - \int_{2/3}^{1}\frac{\cos(n\pi x)}{n\pi}\,dx\right\}$$

$$= 2\left\{\frac{-1}{3n\pi}\cos\frac{n\pi}{3} + \frac{1}{n^2\pi^2}\left[\sin(n\pi x)\right]_0^{1/3}\right\}$$

$$-\frac{2}{3n\pi}\left\{\cos\left(\frac{2n\pi}{3}\right) - \cos\left(\frac{n\pi}{3}\right)\right\}$$

$$+\frac{2}{n\pi}\cdot\frac{1}{3}\cos\left(\frac{2n\pi}{3}\right) - \frac{2}{n^2\pi^2}\left[\sin(n\pi x)\right]_{2/3}^{1}$$

$$= \frac{-2}{3n\pi}\cancel{\cos\left(\frac{n\pi}{3}\right)} + \frac{2}{n^2\pi^2}\sin\left(\frac{n\pi}{3}\right) - \frac{2}{3n\pi}\cancel{\cos\left(\frac{2n\pi}{3}\right)}$$

$$+\frac{2}{3n\pi}\cancel{\cos\left(\frac{n\pi}{3}\right)} + \frac{2}{3n\pi}\cancel{\cos\left(\frac{2n\pi}{3}\right)} + \frac{2}{n^2\pi^2}\sin\left(\frac{2n\pi}{3}\right)$$

$$= \frac{2}{n^2\pi^2}\left\{\sin\left(\frac{n\pi}{3}\right) + \sin\left(\frac{2n\pi}{3}\right)\right\}$$

$\therefore$ Solution of (1) is given by,

$$\boxed{u(x,t) = \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left(\sin\left(\frac{n\pi}{3}\right) + \sin\left(\frac{2n\pi}{3}\right)\right)\cdot \sin(n\pi x)\cdot\cos(2n\pi t)}$$

Prob-5 Consider the problem $u_{tt} = 9u_{xx}$ with boundary condition $u(0,t) = u(\pi,t) = 0$ and initial condition $u(x,0) = x^2(\pi - x)$ and $u_t(x,0) = \sin x$ for $0 < x < \pi$.

Sol:- Given problem, $u_{tt} = 9u_{xx}$ ——— (1)

with boundary condition $u(0,t) = u(\pi,t) = 0$ ——— (2)

and initial condition $u(x,0) = x^2(\pi - x)$ and $u_t(x,0) = \sin x$ ——— (3)

we seek the solution of (1) in the form

$$u(x,t) = X(x) \cdot T(t) \quad \text{——— (4)}$$

Using (4) in (1), we get

$$XT'' = 9X''T$$

$$\Rightarrow \frac{X''}{X} = \frac{1}{9}\frac{T''}{T} = \lambda \text{ (say)}$$

$$\Rightarrow X'' = \lambda X \text{ and } T'' = 9\lambda T \quad \text{——— (5)}$$

we consider the following cases:

case-(i) If $\lambda = 0$, then eq(5) becomes

$$X'' = 0 \text{ and } T'' = 0$$

$$\Rightarrow X = a_1 x + a_2 \text{ and } T = b_1 t + b_2$$

$\therefore$ Solution of (1) is,

$$u(x,t) = (a_1 x + a_2)(b_1 t + b_2)$$

$\because$ $u(0,t) = u(\pi,t) = 0 \Rightarrow u(x,t) = 0$

hence rejected.

case-(ii) If $\lambda = k^2 > 0$, then eq(5) becomes

$$X'' = k^2 X \text{ and } T'' = 9k^2 T$$

$$\Rightarrow X = a_1 e^{kx} + a_2 e^{-kx} \text{ and } T = b_1 e^{3kt} + b_2 e^{-3kt}$$

$\therefore$ Solution of (1) is,

$$u(x,t) = (a_1 e^{kx} + a_2 e^{-kx})(b_1 e^{3kt} + b_2 e^{-3kt})$$

$u(0,t) = u(\pi,t) = 0 \Rightarrow a_1 = a_2 = 0 \Rightarrow u(x,t) = 0$. hence rejected

Case – (iii) If $\lambda = -k^2 < 0$, then eq (5) becomes

$$X'' = -k^2 X \quad \text{and} \quad T'' = -9k^2 T$$

$$\Rightarrow X = a_1 \cos kx + a_2 \sin kx \quad \text{and} \quad T = b_1 \cos 3kt + b_2 \sin 3kt$$

$\therefore$ Solution of (1) is,

$$u(x,t) = (a_1 \cos kx + a_2 \sin kx)(b_1 \cos 3kt + b_2 \sin 3kt) = XT \quad \text{——— (6)}$$

$$u(0,t) = 0 \Rightarrow a_1 T = 0 \Rightarrow a_1 = 0 \quad \text{——— (7)}$$

$$u(\pi,t) = 0 \Rightarrow a \sin(k\pi)\, T = 0$$
$$\Rightarrow \sin(k\pi) = 0 \Rightarrow k\pi = n\pi,\ n \in \mathbb{N}$$
$$\Rightarrow k = n,\ n \in \mathbb{N} \quad \text{——— (8)}$$

~~Differentiating (6) partially w.r.t 't', we get~~

$\cancel{u_t(x,t) = X(-3b_1 k \sin 3kt + 3kb_2 \cos 3kt)}$

$\cancel{u_t(x,0) = \sin x}$

$\cancel{\Rightarrow X}$

$$\therefore\ u(x,t) = a_2 \sin(nx)\{b_1 \cos(3nt) + b_2 \sin(3nt)\}$$

$$\Rightarrow u_n(x,t) = a_n \sin(nx)\cos(3nt) + b_n \sin(nx)\sin(3nt)$$

$\therefore$ By the principle of superposition, we take the solution of the form

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin(nx)\cos(3nt) + b_n \sin(nx)\sin(3nt) \quad \text{——— (9)}$$

since $u(x,0) = x^2(\pi - x)$

$$\Rightarrow x^2(\pi - x) = \sum_{n=1}^{\infty} a_n \sin(nx)$$

which is Fourier sine series.

$\therefore$ The Fourier coefficient a_n is given by,

$$a_n = \frac{2}{\pi}\int_0^{\pi} x^2(\pi - x)\sin(nx)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \pi x^2 \sin(nx)\,dx - \frac{2}{\pi}\int_0^{\pi} x^3 \sin(nx)\,dx$$

$$= 2\left\{\left[-x^2\frac{\cos(nx)}{n}\right]_0^{\pi} + \int_0^{\pi}\frac{\cos(nx)}{n}\cdot 2x\,dx\right\}$$

$$-\frac{2}{\pi}\left\{\left[-x^3\frac{\cos(nx)}{n}\right]_0^{\pi} + \int_0^{\pi} 3x^2\frac{\cos(nx)}{n}\,dx\right\}$$

$$= 2\left\{\frac{-\pi^2}{n}\cos(n\pi) + \frac{2}{n}\left(x\frac{\sin(nx)}{n}\Big|_0^{\pi} + \frac{\cos nx}{n^2}\Big|_0^{\pi}\right)\right\}$$

$$-\frac{2}{\pi}\left\{\frac{-\pi^3}{n}\cos(n\pi) + \frac{3}{n}\left(\left[\frac{x^2\sin(nx)}{n}\right]_0^{\pi} - \int_0^{\pi} 2x\frac{\sin(nx)}{n}\,dx\right)\right\}$$

$$= \frac{-2\pi^2}{n}\cancel{\cos}(n\pi) + \frac{2}{n^3}\{\cos(n\pi) - 1\} + \frac{2\pi^2}{n}\cancel{\cos}(n\pi)$$

$$+ \frac{6}{n\pi}\cdot\frac{2}{n}\left\{\left[-x\frac{\cos(nx)}{n}\right]_0^{\pi} + \left[\frac{\sin(nx)}{n^2}\right]_0^{\pi}\right\}$$

$$= \frac{2}{n^3}\{\cos(n\pi) - 1\} + \frac{12}{n^2\pi}\left(-\pi\frac{\cos(n\pi)}{n}\right)$$

$$= \frac{2}{n^3}\{\cos(n\pi) - 1\} - \frac{12}{n^3}\cos(n\pi)$$

$$= \frac{-10}{n^3}\cos(n\pi) - \frac{2}{n^3}$$

Differentiating eq(9) w.r.t 't' we get

$$u_t(x,t) = \sum_{n=1}^{\infty} -a_n \cdot 3n \sin(nx)\sin(3nt) + 3nb_n \sin(nx)\cos(3nt)$$

Since $u_t(x,0) = \sin x$

$$\Rightarrow \sin x = \sum_{n=1}^{\infty} 3nb_n \sin(nx)$$

$$\Rightarrow n = 1 \text{ and } 3n\,b_n = 1$$

$$\Rightarrow 3b_n = 1$$

$$\Rightarrow b_n = \frac{1}{3}$$

$\therefore$ Solution of (1) is,

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \left\{\frac{-10}{n^3}\cos(n\pi) - \frac{2}{n^3}\right\}\sin(nx)\cos(3nt) + \frac{1}{3}\sin(nx)\sin(3nt)}$$

D Alembert's Solution :-

Let us consider a wave equation,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{for } -\infty < x < \infty, \; t > 0 \quad \text{——— (1)}$$

with $u(x,0) = \phi(x)$ and $u_t(x,0) = \psi(x)$; $-\infty < x < \infty$.

Since there are no end points, so no boundary condition.

$\therefore$ We have to find solution of (1) with initial condition (i.e. initial position and initial velocity) only.

Let solution of (1) be

$$u(x,t) = f_1(x-ct) + f_2(x+ct) \quad \text{——— (2)}$$

$\because u(x,0) = \phi(x)$

$$\Rightarrow f_1(x) + f_2(x) = \phi(x) \quad \text{——— (3)}$$

Differentiating eq(2) partially w.r.t 't' we get,

$$u_t(x,t) = -c f_1'(x-ct) + c f_2'(x+ct)$$

$\because u_t(x,0) = \psi(x)$

$$\Rightarrow -c f_1'(x) + c f_2'(x) = \psi(x)$$

$$\Rightarrow -f_1'(x) + f_2'(x) = \frac{1}{c}\psi(x)$$

On integrating, we get

$$-f_1(x) + f_2(x) = \frac{1}{c}\int_0^x \psi(s)\,ds \quad \text{——— (4)}$$

Adding eq(3) and (4), we get

$$2f_2(x) = \phi(x) + \frac{1}{c}\int_0^x \psi(s)\,ds$$

$$\Rightarrow f_2(x) = \frac{1}{2}\phi(x) + \frac{1}{2c}\int_0^x \psi(s)\,ds$$

From eq(3) and (4), we get

$$2f_1(x) = \phi(x) - \frac{1}{c}\int_0^x \psi(s)\,ds$$

$$\Rightarrow f_1(x) = \frac{1}{2}\phi(x) - \frac{1}{2c}\int_0^x \psi(s)\,ds$$

$\therefore$ from (2),

$$u(x,t) = \frac{1}{2}\phi(x-ct) - \frac{1}{2c}\int_0^{x-ct} \psi(s)\,ds + \frac{1}{2}\phi(x+ct) + \frac{1}{2c}\int_0^{x+ct} \psi(s)\,ds$$

$$= \frac{1}{2}\left[\phi(x-ct) + \phi(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{0} \psi(s)\,ds + \frac{1}{2c}\int_0^{x+ct} \psi(s)\,ds$$

$$\therefore \boxed{u(x,t) = \frac{1}{2}\left[\phi(x-ct) + \phi(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds}$$

Prob-1 Solve $\frac{\partial^2 u}{\partial t^2} = 9\frac{\partial^2 u}{\partial x^2}$; $-\infty < x < \infty$, $t > 0$
with initial condition $u(x,0) = \cos x$, $u_t(x,0) = \sin 2x$.

Sol:- Given, $u_{tt} = 9u_{xx}$; $-\infty < x < \infty$, $t > 0$ —— (1)
with initial condition $u(x,0) = \cos x$, $u_t(x,0) = \sin 2x$ —— (2)
Here $c = 3$, $\phi(x) = \cos x$, $\psi(x) = \sin 2x$
and we know that the solution of (1) with initial condition (2) is given by,

$$u(x,t) = \frac{1}{2}\left[\phi(x-ct) + \phi(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds$$

$$\Rightarrow u(x,t) = \frac{1}{2}\left[\cos(x-3t) + \cos(x+3t)\right] + \frac{1}{6}\int_{x-3t}^{x+3t} \sin(2s)\,ds$$

$$\Rightarrow u(x,t) = \frac{1}{2}\left[\cos(x-3t)+\cos(x+3t)\right] - \frac{1}{12}\left[\cos(2s)\right]_{x-3t}^{x+3t}$$

$$= \frac{1}{2}\cdot 2\cos(x)\cdot\cos(3t) - \frac{1}{12}\left[\cos 2(x+3t) - \cos 2(x-3t)\right]$$

$$= \cos(x)\cos(3t) - \frac{1}{12}(-2)\sin(2x)\cdot\sin(6t)$$

$$= \cos(x)\cdot\cos(3t) + \frac{1}{6}\sin(2x)\cdot\sin(6t)$$

i.e. $\boxed{u(x,t) = \cos(x)\cdot\cos(3t) + \frac{1}{6}\sin(2x)\cdot\sin(6t)}$

Theorem:- Continuous dependence on initial data

Let $u_1(x,t)$ be a solution of wave equation with initial condition $u(x,0) = \phi_1(x)$ and $u_t(x,0) = \psi_1(x)$. Also let $u_2(x,t)$ be an other solution of that wave equation with initial condition $u(x,0) = \phi_2(x)$ and $u_t(x,0) = \psi_2(x)$. then for $\epsilon > 0$ and $T > 0$, $\exists\ \delta > 0$ such that

$|u_1(x,t) - u_2(x,t)| < \epsilon$, whenever $|\phi_1(x) - \phi_2(x)| < \delta$ and $|\psi_1(x) - \psi_2(x)| < \delta$ $\forall x$, and $0 \le t \le T$.

Proof:- Since $u_1(x,t)$ and $u_2(x,t)$ are solution of wave equation $u_{xx} = \frac{1}{c^2}u_{tt}$; $-\infty < x < \infty$, $t > 0$, then,

$$u_1(x,t) = \frac{1}{2}\left[\phi_1(x-ct) + \phi_1(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi_1(s)\,ds$$

& $$u_2(x,t) = \frac{1}{2}\left[\phi_2(x-ct) + \phi_2(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi_2(s)\,ds$$

Now if $\delta > 0$ such that it satisfy the inequalities $|\phi_1(x) - \phi_2(x)| < \delta$ and $|\psi_1(x) - \psi_2(x)| < \delta$, then

$$|u_1(x,t) - u_2(x,t)| \le \frac{1}{2}|\phi_1(x-ct) - \phi_2(x-ct)| + \frac{1}{2}|\phi_1(x+ct) - \phi_2(x+ct)| + \frac{1}{2c}\int_{x-ct}^{x+ct}|\psi_1(s) - \psi_2(s)|\,ds$$

$$< \frac{1}{2}\delta + \frac{1}{2}\delta + \frac{1}{2c}\cdot\delta\int_{x-ct}^{x+ct}ds$$

$$= \delta + \frac{1}{2c}\cdot\delta\cdot 2ct \le \delta + \delta T \qquad [\because 0 \le t \le T]$$

$= (1+T)\delta$

$\Rightarrow |u_1(x,t) - u_2(x,t)| < \epsilon$ whenever $(1+T)\delta < \epsilon$

$\Rightarrow 0 < \delta < \frac{\epsilon}{1+T}$

i.e. $|u_1(x,t) - u_2(x,t)| < \epsilon$ whenever

$|\phi_1(x) - \phi_2(x)| < \delta$ and $|\psi_1(x) - \psi_2(x)| < \delta$.

Q:-1 Let $u(x,t)$ be the solution of the initial value problem $u_{tt} = u_{xx}$ with $u(x,0) = x^3$ and $u_t(x,0) = \sin x$, then $u(\pi,\pi)$ is

(i) $4\pi^3$ (ii) π^3 (iii) 0 (iv) 1

Sol:- Given $u_{tt} = u_{xx}$ ——(1) with initial condition

$u(x,0) = x^3$ and $u_t(x,0) = \sin x$ ——(2)

Here $c = 1$, $\phi(x) = x^3$, $\psi(x) = \sin x$ and we know that the solution of (1) with initial condition (2) is given by,

$$u(x,t) = \frac{1}{2}\left[\phi(x-ct) + \phi(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds$$

$$= \frac{1}{2}\left[(x-t)^3 + (x+t)^3\right] + \frac{1}{2}\int_{x-t}^{x+t} \sin s\,ds$$

$$= \frac{1}{2}\left[x^3 - \cancel{t^3} - 3x^2t + 3xt^2 + x^3 + \cancel{t^3} + \cancel{3x^2t} + 3xt^2\right] + \left(\frac{-1}{2}\right)\left[\cos s\right]_{x-t}^{x+t}$$

$$= \frac{1}{2}(2x^3 + 6xt^2) - \frac{1}{2}\left[\cos(x+t) - \cos(x-t)\right]$$

$$= x^3 + 3xt^2 - \frac{1}{2}(-2)\sin(x)\cdot\sin(t)$$

$\Rightarrow u(x,t) = x^3 + 3xt^2 + \sin(x)\cdot\sin(t)$

$\Rightarrow u(\pi,\pi) = \pi^3 + 3\pi^3 = 4\pi^3$

2) Solve the wave equation $u_{xx} - u_{yy} = 0$ $(-\infty < x < \infty)$ with initial condition $u(x,0) = 0$, $u_y(x,0) = x$, then $u(x,y)$ is

(i) $\frac{x}{y}$ (ii) $xy + \frac{x}{y}$ (iii) 0 (iv) xy

Sol:- Given, $u_{xx} - u_{yy} = 0$

$\Rightarrow u_{yy} = u_{xx}$ ——— (1)

with initial condition $u(x,0) = 0$, $u_y(x,0) = x$ ——— (2)

Here $c = 1$, $\phi(x) = 0$ and $\psi(x) = x$

We know that the solution of (1) with initial condition (2) is given by,

$$u(x,y) = \frac{1}{2}\left[\phi(x-cy) + \phi(x+cy)\right] + \frac{1}{2c}\int_{x-cy}^{x+cy} \psi(s)\,ds$$

$$= \frac{1}{2}(0+0) + \frac{1}{2}\int_{x-y}^{x+y} s\,ds$$

$$= \frac{1}{2}\left[\frac{s^2}{2}\right]_{x-y}^{x+y} = \frac{1}{4}\left[(x+y)^2 - (x-y)^2\right]$$

$$= \frac{1}{4}\left[x^2 + 2xy + y^2 - x^2 + 2xy - y^2\right]$$

$$= xy.$$

Alternate method

3) $u_{xx} - u_{tt} = 0$ $(-\infty < x < \infty)$, $t > 0$
with initial condition $u(x,0) = b$ and $u_t(x,0) = \sin x$ then find $u(x,t)$.

Sol:- Given $u_{xx} = u_{tt}$ ——— (1)

with $u(x,0) = b$ ——— (2) and $u_t(x,0) = \sin x$ ——— (3)

general solution of (1) is,

$u(x,t) = f(x-t) + g(x+t) \quad [\because c = t] \quad \text{——} (4)$

$\because u(x,0) = b$

$\Rightarrow b = f(x) + g(x)$

$\Rightarrow f'(x) + g'(x) = 0 \quad \text{——} (5)$

differentiate eq(4) partially w.r.t 't', we get

$u_t(x,t) = -f'(x-t) + g'(x+t)$

$\Rightarrow u_t(x,0) = -f'(x) + g'(x)$

$\Rightarrow \sin x = -f'(x) + g'(x) \quad \text{——} (6)$

Adding (5) and (6), we get

$2g'(x) = \sin x$

$\Rightarrow g'(x) = \frac{\sin x}{2}$

$\Rightarrow g(x) = -\frac{\cos x}{2} + c_1$

from (5), $f'(x) = -g'(x) = -\frac{\sin x}{2}$

$\Rightarrow f(x) = \frac{\cos x}{2} + c_2$

Putting these values in (4), we get

$u(x,t) = \frac{1}{2}\cos(x-t) - \frac{1}{2}\cos(x+t) + c_1 + c_2$

$\Rightarrow u(x,t) = \frac{1}{2} \cdot 2\sin x \cdot \sin t + c_1 + c_2$

$\Rightarrow u(x,t) = \sin x \cdot \sin t + c_1 + c_2$

Also given $u(x,0) = b$

$\Rightarrow c_1 + c_2 = b$

$\therefore \boxed{u(x,t) = b + \sin x \cdot \sin t}$

Characteristic Triangles :-

In characteristic triangle we have to find a solution of wave equation $u(x,t)$ at some point (x_0, t_0) in the domain.

As we know that the characteristic equation of wave equation

$$u_{tt} = c^2 u_{xx}$$

are straight lines

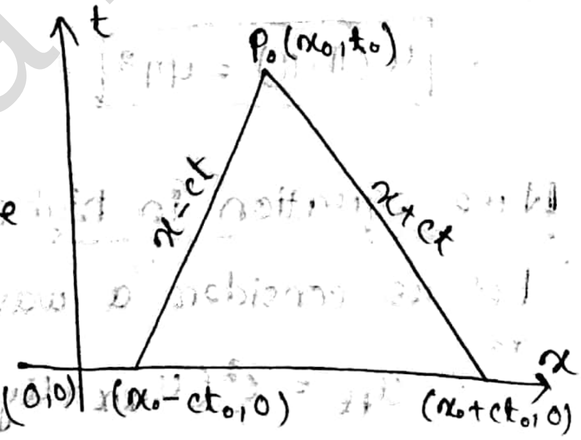
$$x - ct = c_1 \quad \text{and} \quad x + ct = c_2$$

having slope $\frac{1}{c}$ and $\frac{-1}{c}$ respectively in the xt-plane, there are exactly two characteristic curves through any point $P_0(x_0, t_0)$ with $t_0 > 0$.

These two characteristic curves $x-ct$ and $x+ct$ intersecting x-axis at $(x_0 - ct_0, 0)$ and $(x_0 + ct_0, 0)$ respectively.

these two points together with P_0 are the vertices of the characteristic triangle at $P_0(x_0, t_0)$

the base of the triangle is the interval $[x_0 - ct_0, x_0 + ct_0]$ on the x-axis.



If we considering a cauchy problem on the entire real line then we know that the solution at P_0 is

$$u(x_0, t_0) = \frac{1}{2}\left[\phi(x_0 - ct_0) + \phi(x_0 + ct_0)\right] + \frac{1}{2c}\int_{x_0 - ct_0}^{x_0 + ct_0} \psi(s)\, ds$$

the solution value depends on two things.

(i) the value of ϕ at the base vertices $x_0 - ct_0$ and $x_0 + ct_0$ of the characteristic triangle at (x_0, t_0) and

(ii) the value of ψ on the entire base of the triangle

Prob:- If $u(x,t)$ be solution of wave equation $u_{xx} = u_{tt}$ with $u(x,0) = x^3$ and $u_t(x,0) = \sin x$, then find $u(\pi,\pi) = ?$

Sol:- Here $c = 1$, $\phi(x) = x^3$, $\psi(x) = \sin x$

$x_0 = \pi$, $t_0 = \pi$

then $u(x_0,t_0) = \frac{1}{2}\left[\phi(x_0 - ct_0) + \phi(x_0 + ct_0)\right] + \frac{1}{2c}\int_{x_0 - ct_0}^{x_0+ct_0} \psi(s)\,ds$

$$\Rightarrow u(\pi,\pi) = \frac{1}{2}\left[\phi(0) + \phi(2\pi)\right] + \frac{1}{2}\int_0^{2\pi} \sin s\, ds$$

$$= \frac{1}{2}\left[0 + (2\pi)^3\right] + \frac{1}{2}\left[-\cos(s)\right]_0^{2\pi}$$

$$= 4\pi^3 - \frac{1}{2}(1-1)$$

$$= 4\pi^3 + 0 = 4\pi^3$$

$$\therefore \boxed{u(\pi,\pi) = 4\pi^3}$$

Wave equation in higher dimensions

Let us consider a wave equation in two dimension as

$$u_{tt} = c^2(u_{xx} + u_{yy}) \;;\; 0 < x < L,\; 0 < y < K,\; t > 0 \quad \text{——— (1)}$$

with boundary conditions

$u(x,0,t) = u(x,K,t) = 0$ for $0 < x < L$, $t > 0$

$u(0,y,t) = u(L,y,t) = 0$ for $0 < y < K$, $t > 0$

and initial conditions

$u(x,y,0) = \phi(x,y)$ and $u_t(x,y,0) = 0$ for $0 < x < L$, $0 < y < K$

We seek the solution of the form

$$u(x,y,t) = X(x)\cdot Y(y)\cdot T(t) \quad \text{——— (2)}$$

then eq(1) becomes,

$$XYT'' = c^2(X''YT + XY''T)$$

then $\dfrac{T''}{c^2T} = \dfrac{X''}{X} + \dfrac{Y''}{Y}$

$$\Rightarrow \frac{T''}{c^2T} - \frac{Y''}{Y} = \frac{X''}{X} = -\lambda \text{ (say)}$$

$$\Rightarrow X'' + \lambda X = 0 \text{ and } \frac{T''}{c^2T} - \frac{Y''}{Y} = -\lambda$$

$$\Rightarrow X'' + \lambda X = 0 \text{ and } \frac{T''}{c^2T} + \lambda = \frac{Y''}{Y} = -\mu \text{ (say)}$$

$$\Rightarrow X'' + \lambda X = 0 \text{ and } Y'' + \mu Y = 0 \text{ and } T'' + (\lambda+\mu)c^2T = 0 \quad \text{——— (3)}$$

From the boundary conditions,

$$X(0) = X(L) = 0 \quad \text{and} \quad Y(0) = Y(K) = 0$$

Now, the problems for X and Y are

$$\left.\begin{array}{l} X'' + \lambda X = 0 \;;\; X(0) = X(L) = 0 \\ \text{and } Y'' + \mu Y = 0 \;;\; Y(0) = Y(K) = 0 \end{array}\right\} \text{——— (4)}$$

Consider $X'' + \lambda X = 0 \;;\; X(0) = X(L) = 0$ ——— (4a)

<u>case-(i)</u> If $\lambda = 0$, then eq (4a) becomes

$$X'' = 0$$
$$\Rightarrow X = a_1x + a_2$$

$X(0) = 0 \Rightarrow a_2 = 0$

$X(L) = 0 \Rightarrow a_1L = 0 \Rightarrow a_1 = 0$

$\therefore X = 0$, hence rejected.

Case-(ii) If $\lambda = -m^2 < 0$, then eq(4a) becomes,

$$X'' - m^2 X = 0$$
$$\Rightarrow X'' = m^2 X$$
$$\Rightarrow X = a_1 e^{mx} + a_2 e^{-mx}$$

$X(0) = 0 \Rightarrow a_1 + a_2 = 0 \Rightarrow a_2 = -a_1$

$X(L) = 0 \Rightarrow a_1 e^{mL} - a_1 e^{-mL} = 0$

$\Rightarrow a_1 (e^{mL} - e^{-mL}) = 0$

$\Rightarrow a_1 = 0$

$\Rightarrow a_2 = 0$

$\therefore X = 0$, hence rejected.

Case-(iii) If $\lambda = m^2 > 0$, then eq(4a) becomes

$$X'' + m^2 X = 0$$
$$\Rightarrow X'' = -m^2 X$$
$$\Rightarrow X = a_1 \cos mx + a_2 \sin mx$$

$X(0) = 0 \Rightarrow a_1 = 0$

$X(L) = 0 \Rightarrow a_2 \sin mL = 0$

$\Rightarrow mL = n\pi$

$\Rightarrow m = \frac{n\pi}{L}$, $n \in \mathbb{N}$

$\therefore \lambda_n = \frac{n^2\pi^2}{L^2}$, $X_n(x) = \sin\left(\frac{n\pi x}{L}\right)$; $n \in \mathbb{N}$

Similarly, $\mu_m = \frac{m^2\pi^2}{K^2}$, $Y_m(y) = \sin\left(\frac{m\pi y}{K}\right)$; $m \in \mathbb{N}$

Here $m, n \in \mathbb{N}$ are independent.

Now from eq(3), we get

$$T'' + \left(\frac{n^2\pi^2}{L^2} + \frac{m^2\pi^2}{K^2}\right) c^2 T = 0 \quad \text{with } T'(0) = 0$$

$\Rightarrow T'' + a_{nm}^2 c^2 T = 0$, where $a_{nm}^2 = \frac{n^2\pi^2}{L^2} + \frac{m^2\pi^2}{K^2}$

$\Rightarrow T = b_1 \cos(a_{nm} ct) + b_2 \sin(a_{nm} ct)$

$\Rightarrow T' = -a_{nm} \cdot c \cdot b_1 \sin(a_{nm} ct) + a_{nm} \cdot c \cdot b_2 \cos(a_{nm} ct)$

$\because T'(0) = 0 \Rightarrow b_2 = 0$

$\therefore T(t) = b_1 \cos(a_{nm} ct)$

$\therefore$ From eq(2), we get

$$u_{nm}(x,y,t) = b_{nm} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{K}\right) \cos(a_{nm} ct) \, ; \quad n, m \in \mathbb{N}$$

which satisfies the two dimensional wave equation and the boundary conditions as well as initial condition.

$\therefore$ By principle of superposition of these function we get,

$$u(x,y,t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} b_{nm} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{K}\right) \cos(a_{nm} cT) \quad \text{---} (5)$$

Also, $\because u(x,y,0) = \phi(x,y)$

$$\Rightarrow \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} b_{nm} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{K}\right) = \phi(x,y)$$

which is double fourier sine series.

$\therefore$ The Fourier coefficient b_{nm} is given by,

$$b_{nm} = \frac{4}{LK} \int_0^K \int_0^L \phi(x,y) \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{K}\right) dx\, dy \quad \text{---} (6)$$

Hence eq (5) is required solution of (1), where b_{nm} is given in (6).

Prob-1 Solve wave equation in two dimension as

$$u_{tt} = c^2(u_{xx} + u_{yy}) \quad \text{——— (1)}$$

with boundary condition

$$u(x,0,t) = u(x,\pi,t) = 0, \quad 0 < x < \pi, \quad t > 0$$

$$u(0,y,t) = u(\pi,y,t) = 0, \quad 0 < y < \pi, \quad t > 0$$

and initial condition

$$u(x,y,0) = xy(\pi - x)(\pi - y), \quad u_t(x,y,0) = 0.$$

Sol:- We know that the solution of eq(1) is,

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} b_{nm} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{K}\right)\cos(a_{nm}ct) \quad \text{——— (2)}$$

where $a_{nm}^2 = \frac{n^2\pi^2}{L^2} + \frac{m^2\pi^2}{K^2}$

Here $L = \pi$, $K = \pi$

$\therefore$ eq(2) $\Rightarrow$ $u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} b_{nm}\sin(nx)\sin(my)\cos(a_{nm}ct)$

where $a_{nm}^2 = n^2 + m^2$ ——— (3)

Since, $u(x,y,0) = xy(\pi - x)(\pi - y)$

$$\therefore \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} b_{nm}\sin(nx)\cdot\sin(my) = xy(\pi - x)(\pi - y)$$

which is double Fourier sine series.

$\therefore$ The Fourier coefficient b_{nm} is given by,

$$b_{nm} = \frac{4}{LK}\int_0^L\int_0^K xy(\pi - x)(\pi - y)\sin(nx)\sin(my)\,dx\,dy$$

$$= \frac{4}{\pi^2}\int_0^\pi\int_0^\pi xy(\pi - x)(\pi - y)\sin(nx)\sin(my)\,dx\,dy$$

$$= \frac{4}{\pi^2} \int_0^\pi x(\pi - x)\sin(nx)\,dx \int_0^\pi y(\pi - y)\sin(my)\,dy$$

$$= \frac{4}{\pi^2} I_1 I_2$$

where $I_1 = \int_0^\pi x(\pi - x)\sin(nx)\,dx$

& $I_2 = \int_0^\pi y(\pi - y)\sin(my)\,dy$

Now, $I_1 = \int_0^\pi (\pi x - x^2)\sin(nx)\,dx$

$$= \left[(\pi x - x^2)\left\{\frac{-\cos(nx)}{n}\right\}\right]_0^\pi + \int_0^\pi \frac{\cos(nx)}{n}(\pi - 2x)\,dx$$

$$= \frac{1}{n}\left\{\left[(\pi - 2x)\frac{\sin(nx)}{n}\right]_0^\pi - \int_0^\pi \frac{\sin(nx)}{n}\cdot(-2)\right\}$$

$$= \frac{2}{n^2}\left[\frac{-\cos(nx)}{n}\right]_0^\pi$$

$$= \frac{-2}{n^3}\{(-1)^n - 1\}$$

Similarly, $I_2 = \int_0^\pi y(\pi - y)\sin(my)\,dy$

$$= \frac{-2}{m^3}\{(-1)^m - 1\}$$

$\therefore\ b_{nm} = \frac{16}{\pi^2 n^3 m^3}\{(-1)^n - 1\}\{(-1)^m - 1\}$

$\therefore$ eq(3) $\Rightarrow$

$$u(x,y,t) = \frac{16}{\pi^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\{(-1)^n - 1\}\{(-1)^m - 1\}}{n^3 m^3}\sin(nx)\sin(my)\cos(a_{nm}ct),$$

where $a_{nm} = \sqrt{n^2 + m^2}$

Prob-2 $c=3,\ L=3,\ K=6,\ \phi(x,y)=\sin\left(\frac{\pi x}{3}\right)y(6-y)$

Sol:- The wave equation in two dimension as

$$u_{tt}=c^2(u_{xx}+u_{yy}) \quad \text{——— (1)}$$

We know that the solution of eq(1) is,

$$u(x,y,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} b_{nm}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{K}\right)\cos(a_{nm}ct) \quad \text{——— (2)}$$

where $a_{nm}^2=\frac{n^2\pi^2}{L^2}+\frac{m^2\pi^2}{K^2}$

Given, $L=3,\ K=6$

$\therefore$ eq(2) $\Rightarrow$

$$u(x,y,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} b_{nm}\sin\left(\frac{n\pi x}{3}\right)\sin\left(\frac{m\pi y}{6}\right)\cos(a_{nm}3t)$$

$$\text{where } a_{nm}^2=\frac{n^2\pi^2}{9}+\frac{m^2\pi^2}{36} \quad \text{——— (3)}$$

Since $u(x,y,0)=\phi(x,y)=\sin\left(\frac{\pi x}{3}\right)y(6-y)$

$$\therefore \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} b_{nm}\sin\left(\frac{n\pi x}{3}\right)\sin\left(\frac{m\pi y}{6}\right)=\sin\left(\frac{\pi x}{3}\right)y(6-y)$$

which is double Fourier sine series

$\therefore$ The fourier coefficient b_{nm} is given by,

$$b_{nm}=\frac{4}{LK}\int_0^L\int_0^K \sin\left(\frac{\pi x}{3}\right)y(6-y)\sin\left(\frac{n\pi x}{3}\right)\sin\left(\frac{m\pi y}{6}\right)dx\,dy$$

$$=\frac{4}{18}\int_0^3\int_0^6 \sin\left(\frac{\pi x}{3}\right)y(6-y)\sin\left(\frac{n\pi x}{3}\right)\sin\left(\frac{m\pi y}{6}\right)dx\,dy$$

$$=\frac{2}{9}\int_0^3 \sin\left(\frac{\pi x}{3}\right)\sin\left(\frac{n\pi x}{3}\right)dx\int_0^6 y(6-y)\sin\left(\frac{m\pi y}{6}\right)dy$$

$= \frac{2}{9} I_1 I_2$,

where $I_1 = \int_0^3 \sin\left(\frac{\pi x}{3}\right)\sin\left(\frac{n\pi x}{3}\right)dx$, $I_2 = \int_0^6 y(6-y)\sin\left(\frac{m\pi y}{6}\right)dy$

$$I_1 = \int_0^3 \sin\left(\frac{\pi x}{3}\right)\sin\left(\frac{n\pi x}{3}\right)dx$$

$$= \frac{1}{2}\int_0^3 2\sin\left(\frac{\pi x}{3}\right)\sin\left(\frac{n\pi x}{3}\right)dx$$

$$= \frac{1}{2}\int_0^3 \left\{\cos\left(\frac{\pi x - n\pi x}{3}\right) - \cos\left(\frac{\pi x + n\pi x}{3}\right)\right\}dx$$

$$= \frac{1}{2}\left[\frac{\sin\left(\frac{\pi x - n\pi x}{3}\right)}{\left(\frac{\pi - n\pi}{3}\right)} - \frac{\sin\left(\frac{\pi x + n\pi x}{3}\right)}{\left(\frac{\pi + n\pi}{3}\right)}\right]_0^3$$

$$= \frac{1}{2}\left\{\frac{3}{\pi - n\pi}\sin\{(1-n)3\pi\} - \frac{3}{\pi + n\pi}\sin\{(1+n)3\pi\}\right\}$$

$$= \frac{1}{2}\times 0 = 0$$

For $n=1$, $I_1 = \int_0^3 \sin^2\left(\frac{\pi x}{3}\right) = \frac{1}{2}\int_0^3 \left\{1 - \cos\left(\frac{2\pi x}{3}\right)\right\}dx$

$$= \frac{1}{2}\left[x - \sin\left(\frac{2\pi x}{3}\right)\cdot\frac{3}{2\pi}\right]_0^3 = \frac{3}{2}$$

$$I_2 = \int_0^6 y(6-y)\sin\left(\frac{m\pi y}{6}\right)dy$$

$$= \int_0^6 (6y - y^2)\sin\left(\frac{m\pi y}{6}\right)dy$$

$$= \left[(6y-y^2)\cdot\left(\frac{-6}{m\pi}\right)\cos\left(\frac{m\pi y}{6}\right)\right]_0^6 + \frac{6}{m\pi}\int_0^6 \cos\left(\frac{m\pi y}{6}\right)(6-2y)\,dy$$

$$= \frac{6}{m\pi}\left\{\left[(6-2y)\cdot\frac{6}{m\pi}\sin\left(\frac{m\pi y}{6}\right)\right]_0^6 - \frac{6}{m\pi}\int_0^6 \sin\left(\frac{m\pi y}{6}\right)(-2)\,dy\right\}$$

$$= \frac{-72}{m^2\pi^2}\cdot\frac{6}{m\pi}\left[\cos\left(\frac{m\pi y}{6}\right)\right]_0^6$$

$$= \frac{-432}{m^3\pi^3}\left\{(-1)^m - 1\right\}$$

$$\therefore\ b_{nm} = \frac{\cancel{2}}{\cancel{9}_3}\times\frac{\cancel{3}}{\cancel{2}}\times\frac{(-432)}{m^3\pi^3}\left\{(-1)^m - 1\right\}$$

$$= \frac{-144}{m^3\pi^3}\left\{(-1)^m - 1\right\}$$

$\therefore$ eq(3) $\Rightarrow$

$$u(x,y,t) = \sum_{m=1}^{\infty}\frac{-144}{m^3\pi^3}\left\{(-1)^m - 1\right\}\sin\left(\frac{\pi x}{3}\right)\sin\left(\frac{m\pi y}{6}\right)\cos(a_{1m}3t),$$

$$\text{where } a_{1m}^2 = \frac{n^2\pi^2}{9} + \frac{m^2\pi^2}{36}$$

$$= \frac{144}{\pi^3}\sum_{m=1}^{\infty}\frac{1}{m^3}\left\{1-(-1)^m\right\}\sin\left(\frac{\pi x}{3}\right)\sin\left(\frac{m\pi y}{6}\right)\cos(a_{1m}3t)$$

$$\text{where } a_{1m} = \sqrt{\frac{\pi^2}{9} + \frac{m^2\pi^2}{36}}$$

Prob-3 $c=2$, $L=\pi$, $K=2\pi$, $\phi(x,y)=x^2(\pi-x)y^2(2\pi-y)$

Sol:- The wave equation in two dimension is

$$u_{tt}=c^2(u_{xx}+u_{yy}) \quad \text{—— (1)}$$

$$\Rightarrow u_{tt}=4(u_{xx}+u_{yy})$$

We know that the solution of eq(1) is,

$$u(x,y,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} b_{nm}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{K}\right)\cos(a_{nm}ct) \quad \text{—— (2)}$$

where $a_{nm}^2=\dfrac{n^2\pi^2}{L^2}+\dfrac{m^2\pi^2}{K^2}$

Given $L=\pi$ and $K=2\pi$

$$\therefore \text{eq(2)} \Rightarrow u(x,y,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} b_{nm}\sin(nx)\sin\left(\frac{my}{2}\right)\cos(a_{nm}2t)$$

$$\text{where } a_{nm}^2=n^2+\frac{m^2}{4} \quad \text{—— (3)}$$

Since, $u(x,y,0)=\phi(x,y)=x^2(\pi-x)y^2(2\pi-y)$

$$\therefore \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} b_{nm}\sin(nx)\sin\left(\frac{my}{2}\right)=x^2(\pi-x)y^2(2\pi-y)$$

which is double Fourier sine series.

$\therefore$ The Fourier coefficient b_{nm} is given by,

$$b_{nm}=\frac{4}{LK}\int_0^L\int_0^K x^2(\pi-x)y^2(2\pi-y)\sin(nx)\sin\left(\frac{my}{2}\right)dx\,dy$$

$$=\frac{4}{2\pi^2}\int_0^{\pi}\int_0^{2\pi} x^2(\pi-x)y^2(2\pi-y)\sin(nx)\sin\left(\frac{my}{2}\right)dx\,dy$$

$$=\frac{4}{2\pi^2}\int_0^{\pi} x^2(\pi-x)\sin(nx)\,dx\int_0^{2\pi} y^2(2\pi-y)\sin\left(\frac{my}{2}\right)dy$$

$$=\frac{4}{2\pi^2}I_1 I_2$$

where

$$I_1 = \int_0^\pi x^2(\pi - x)\sin(nx)\,dx$$

$$= \int_0^\pi (x^2\pi - x^3)\sin(nx)\,dx$$

$$= \left[(x^2\pi - x^3)\left\{-\frac{\cos(nx)}{n}\right\}\right]_0^\pi + \int_0^\pi \frac{\cos(nx)}{n}(2x\pi - 3x^2)\,dx$$

$$= \frac{1}{n}\left\{\left[(2x\pi - 3x^2)\frac{\sin(nx)}{n}\right]_0^\pi - \int_0^\pi \frac{\sin(nx)}{n}(2\pi - 6x)\,dx\right\}$$

$$= \frac{-1}{n^2}\left\{\left[(2\pi - 6x)\left\{-\frac{\cos(nx)}{n}\right\}\right]_0^\pi + \int_0^\pi \frac{\cos(nx)}{n}(-6)\,dx\right\}$$

$$= \frac{1}{n^2}\left\{\left((-4\pi)\frac{(-1)^n}{n} - \frac{2\pi}{n}\right) + \frac{6}{n^3}\left[\frac{\sin(nx)}{n}\right]_0^\pi\right\}$$

$$= \frac{-4\pi(-1)^n}{n^3} - \frac{2\pi}{n^3}$$

$$I_2 = \int_0^{2\pi} y^2(2\pi - y)\sin\left(\frac{my}{2}\right)dy$$

$$= \int_0^{2\pi}(2\pi y^2 - y^3)\sin\left(\frac{my}{2}\right)dy$$

$$= \left[(2\pi y^2 - y^3)\left\{-\frac{2}{m}\cos\left(\frac{my}{2}\right)\right\}\right]_0^{2\pi} + \int_0^{2\pi}\frac{2}{m}\cos\left(\frac{my}{2}\right)(4\pi y - 3y^2)\,dy$$

$$= \frac{2}{m}\left\{\left[(4\pi y - 3y^2)\cdot\frac{2}{m}\cdot\sin\left(\frac{my}{2}\right)\right]_0^{2\pi} - \int_0^{2\pi}\frac{2}{m}\sin\left(\frac{my}{2}\right)(4\pi - 6y)\,dy\right\}$$

$$= \frac{-4}{m^2}\left\{\left[(4\pi - 6y)\cdot\frac{(-2)}{m}\cos\left(\frac{my}{2}\right)\right]_0^{2\pi} - \frac{12}{m}\int_0^{2\pi}\cos\left(\frac{my}{2}\right)dy\right\}$$

$$= \frac{8\{(-8\pi)(-1)^m - 4\pi\}}{m^3} + \frac{48\left[2\sin\left(\frac{my}{2}\right)\right]_0^{2\pi}}{m^3}$$

$$= (64\pi(-1)^m - 32\pi)/m^3$$

$$\therefore b_{nm} = \frac{4}{2\pi^2}\left\{\frac{4\pi(-1)^n}{n^3} + \frac{2\pi}{n^3}\right\}\left\{\frac{64\pi(-1)^m + 32\pi}{m^3}\right\}$$

$$= \frac{2}{\pi^2}\left\{\frac{4\pi(-1)^n}{n^3} + \frac{2\pi}{n^3}\right\}\left\{\frac{64\pi(-1)^m + 32\pi}{m^3}\right\}$$

$$= \frac{2}{\pi^2}\cdot\frac{2\pi}{n^3}\left\{2(-1)^n + 1\right\}\cdot\frac{32\pi\left\{2(-1)^m + 1\right\}}{m^3}$$

$$= \frac{128}{m^3 n^3}\left(2(-1)^n + 1\right)\left(2(-1)^m + 1\right)$$

$\therefore$ eq(3) $\Rightarrow$

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{128}{m^3 n^3}\left(2(-1)^n - 1\right)\left(2(-1)^{m+1} - 1\right)$$
$$\sin(nx)\sin\left(\frac{my}{2}\right)\cos(a_{nm}\,2t),$$

where $a_{nm} = \sqrt{n^2 + \frac{m^2}{4}}$.

The Poisson integral Solution:-

We will solve the Cauchy problem for the wave equation in three dimensions as

$$u_{tt} = u_{xx} + u_{yy} + u_{zz} \quad \text{——— (1)}$$

with initial displacement $u(x,y,z,0) = \phi(x,y,z)$

and initial velocity $u_t(x,y,z,0) = \psi(x,y,z,0)$

In order to show $u(x,y,z,t)$ is solution of eq(1) we solve the following pde with initial velocity $\eta(x,y,z)$ and initial displacement 0.

i.e. $v_{tt} = v_{xx} + v_{yy} + v_{zz}$ ——— (2)

with $v(x,y,z,0) = 0$, $v_t(x,y,z,0) = \eta(x,y,z)$.

Theorem:- We claim that if v is a solution of (2) then $u = v_t$ is a solution of (1) with initial displacement $\eta(x,y,z)$ and initial velocity 0.

Proof:- For that let $u = v_t$, then we first show that u satisfies the wave equation.

$$\therefore \quad u_{tt} = (v_t)_{tt} = (v_{tt})_t$$
$$= (v_{xx} + v_{yy} + v_{zz})_t$$
$$= (v_t)_{xx} + (v_t)_{yy} + (v_t)_{zz}$$
$$\Rightarrow u_{tt} = u_{xx} + u_{yy} + u_{zz}$$

i.e. $u = v_t$ satisfies the wave equation.

Now, the initial condition are

$$u(x,y,z,0) = v_t(x,y,z,0) = \eta(x,y,z)$$

and $u_t(x,y,z,0) = v_{tt}(x,y,z,0) = (v_{xx} + v_{yy} + v_{zz})_{t=0}$

$$\Rightarrow u_t(x,y,z,0) = 0 \quad [\text{because of } v(x,y,z,0) = 0]$$

Hence, if u_ψ is solution of (2) with velocity $\psi(x,y,z)$ and u_ϕ is solution of (2) with initial velocity $\phi(x,y,z)$, then

$$u(x,y,z,t) = \frac{\partial}{\partial t}(u_\phi) + u_\psi \quad \text{—— (3)}$$

is integral formula for the solution of (2).

* Kirchhoff's integral solution:-

Let $\psi(x)$ be a continuous function with continuous first and second order partial derivatives exists for all (x,y,z) then for all real x, y, z and $t>0$ the solution of (2) is,

$$u(x,y,z,t) = \frac{1}{4\pi t} \iint_{S(x,y,z,t)} \psi(X,Y,Z)\, d\sigma_t \quad \text{——(4)}$$

in which the integral is a surface integral over the sphere $S(x,y,z,t)$ of radius t about (x,y,z) and (X,Y,Z) is a variable of integration on $S(x,y,z,t)$. $S(x,y,z,t)$ consists of all points (X,Y,Z) with

$$(X-x)^2 + (Y-y)^2 + (Z-z)^2 = t^2$$

The integral equation (4) is known as Kirchhoff's integral for wave equation.

Proof:-

We first show that u satisfies the wave equation for that let $a=(x,y,z)$ be any arbitary point in 3-space and $A=(X,Y,Z)$ be an integration variable, then $u(x,y,z,t) = u(a,t)$

we denote $S(x,y,z,t) = S_t$

Let U be a sphere of radius 1 about the origin and let $d\sigma$ be the differential element of the surface area on U, while $d\sigma_t$ be the differential element of the surface area of S_t ; then

$$d\sigma_t = t^2 d\sigma$$

let η_t be the unit outward normal vector on S_t and n be the unit outward normal vector on U.

then from eq(4),

$$u(a,t) = \frac{1}{4\pi t}\iint_{S_t} \psi(A)\, d\sigma_t \quad ; \quad A = (X,Y,Z),\ a = (x,y,z)$$

(and in unit sphere $A = a + t\bar{a}$)

$$\Rightarrow u(a,t) = \frac{t^2}{4\pi t}\iint_{U} \psi(A)\, d\sigma$$

$$\Rightarrow u(a,t) = \frac{t}{4\pi}\iint_{U} \psi(A)\, d\sigma \quad \text{---} (5)$$

From eq(5), we get

$$u_{xx} + u_{yy} + u_{zz} = \frac{t}{4\pi}\iint_{U} (\psi_{xx} + \psi_{yy} + \psi_{zz})_A\, d\sigma$$

$$= \frac{1}{4\pi t}\iint_{S_t} \big(\psi_{xx}(A) + \psi_{yy}(A) + \psi_{zz}(A)\big)\, d\sigma_t \quad \text{---} (6)$$

Also from eq(5), we have

$$u_t(a,t) = \frac{1}{4\pi}\iint_{U} \psi(A)\, d\sigma + \frac{t}{4\pi}\iint_{U} \nabla\psi(A)\cdot n\, d\sigma \quad \text{---} (*)$$

$$\Rightarrow u_t(a,t) = \frac{1}{t}u(a,t) + \frac{1}{4\pi t}\iint_{S_t} \nabla\psi(A)\cdot n_t\, d\sigma_t \quad \text{---} (7)$$

where ∇ is the gradient operator.

$\therefore$ By Gauss divergence theorem,

$$\iint_{S_t} \nabla\psi(A)\cdot n_t\, d\sigma_t = \iiint_{B_t} \text{div}(\nabla\psi)\, dv$$

where B_t is open ball of radius t about (x,y,z)

$$\Rightarrow \iint_{S_t} \nabla\psi(A)\cdot n_t\, d\sigma_t = \iiint_{B_t} (\psi_{xx} + \psi_{yy} + \psi_{zz})\, dv$$

$\therefore$ eq(7) $\Rightarrow$ $u_t(a,t) = \frac{1}{t}u(a,t) + \frac{1}{4\pi t}\iiint\limits_{B_t}(\psi_{xx}+\psi_{yy}+\psi_{zz})\,dv$

$\Rightarrow u_t(a,t) = \frac{1}{t}u(a,t) + \frac{1}{4\pi t}I$ ——— (8)

where $I = \iiint\limits_{B_t}(\psi_{xx}+\psi_{yy}+\psi_{zz})\,dv$

$\therefore$ from eq(8), we have

$$u_{tt}(a,t) = \frac{-1}{t^2}u(a,t) + \frac{1}{t}u_t(a,t) + \frac{1}{4\pi t}I_t - \frac{1}{4\pi t^2}I$$

$$= \frac{1}{t}\left(\frac{-1}{t}u(a,t) + u_t(a,t) - \frac{1}{4\pi t}I\right) + \frac{1}{4\pi t}I_t$$

$\Rightarrow u_{tt}(a,t) = \frac{1}{4\pi t}I_t$ ——— (9)

$\because$ $I = \iiint\limits_{B_t}(\psi_{xx}+\psi_{yy}+\psi_{zz})\,dv$

$\Rightarrow I_t = \iint\limits_{S_t}(\psi_{xx}+\psi_{yy}+\psi_{zz})\,d\sigma_t$

$\therefore$ eq(9) $\Rightarrow$

$u_{tt}(a,t) = \frac{1}{4\pi t}\iint\limits_{S_t}(\psi_{xx}+\psi_{yy}+\psi_{zz})\,d\sigma_t$

$\Rightarrow \boxed{u_{tt} = u_{xx}+u_{yy}+u_{zz}}$ [using (6)]

$\therefore$ The function defined in (3) satisfies wave equation. Now, we have to show that $u(x,y,z,t)$ defined in (3) also satisfies initial condition $u(x,y,z,0)=0$ and $u_t(x,y,z,0) = \psi(x,y,z)$.

From eq(5),

$$u(a,t) = \frac{t}{4\pi}\iint_U \psi(A)\,d\sigma$$

$$\Rightarrow u(a,t) = 0$$

$$\Rightarrow u(x,y,z,0) = 0$$

and from eq(*), we have

$$u_t(a,t) = \frac{1}{4\pi}\iint_U \psi(a+t\bar{a})\,d\sigma + \frac{t}{4\pi}\iint_U \nabla\psi(A)\cdot n\,d\sigma$$

$$\Rightarrow u_t(a,0) = \frac{1}{4\pi}\iint_U \psi(a)\,d\sigma$$

$$\Rightarrow u_t(a,0) = \frac{1}{4\pi}\psi(a)\iint_U d\sigma = \frac{1}{4\pi}\psi(a)\cdot 4\pi$$

$$\Rightarrow u_t(a,0) = \psi(a)$$

$$\Rightarrow u_t(x,y,z,0) = \psi(x,y,z).$$ Proved

Poisson's formula:-

From eq(3), the solution of (1) is given by,

$$u(a,t) = \frac{1}{4\pi}\frac{\partial}{\partial t}\left[\frac{1}{t}\iint_{S_t}\phi(A)\,d\sigma_t\right] + \frac{1}{4\pi t}\iint_{S_t}\psi(A)\,d\sigma_t$$

is known as Poisson's formula.

Heat Equation

Solve the heat equation $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial u}{\partial t}$ by separation of variable method.

Sol:- The one dimensional heat equation is

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial u}{\partial t} \quad \text{——— (1)}$$

We seek the solution of (1) in the form

$$u(x,t) = X(x)\cdot T(t) \quad \text{——— (2)}$$

Putting the value of $u(x,t)$ from (2) in (1), we get

$$X''T = \frac{1}{c^2} XT'$$

$$\Rightarrow \frac{X''}{X} = \frac{1}{c^2}\frac{T'}{T} = \lambda \text{ (say)}$$

$$\Rightarrow X'' - \lambda X = 0 \text{ and } T' = c^2\lambda T \quad \text{——— (3)}$$

then there are three cases arise

case-(I) If $\lambda = 0$, then (3) becomes

$$X'' = 0 \text{ and } T' = 0$$

$$\Rightarrow X = a_1 x + a_2 \text{ and } T = b_1$$

$\therefore$ From (2),

$$u(x,t) = (a_1 x + a_2)\, b_1 = c_1 x + c_2 \quad \text{——— (4)}$$

case-(II) If $\lambda = k^2 > 0$, then from (3), we have

$$X'' = k^2 X \text{ and } T' = c^2k^2T$$

$$\Rightarrow X = a_1 e^{kx} + a_2 e^{-kx}, \quad T = b_1 e^{c^2k^2t}$$

$\therefore$ From (2), solution of (1) is

$$u(x,t) = (a_1 e^{kx} + a_2 e^{-kx})\, b_1 e^{c^2k^2t}$$

$$\Rightarrow u(x,t) = (c_1 e^{kx} + c_2 e^{-kx})\, e^{c^2k^2t} \quad \text{——— (5)}$$

Case-(III) :- If $\lambda = -K^2 < 0$, then from (3), we have

$$X'' = -K^2 X \quad , \quad T' = -c^2K^2 T$$

$$\Rightarrow X = a_1 \cos Kx + a_2 \sin Kx \quad , \quad T = b_1 e^{-c^2K^2t}$$

$\therefore$ Solution of (1) is,

$$u(x,t) = (a_1 \cos Kx + a_2 \sin Kx)\, b_1 e^{-c^2K^2t}$$

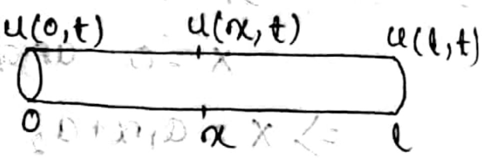
$$\Rightarrow u(x,t) = (c_1 \cos Kx + c_2 \sin Kx) e^{-c^2K^2t} \qquad \text{——— (6)}$$

Hence, $u(x,t)$ given in eq (4), (5), (6) comprises the solution of heat equation in cartesian form.

Ends are kept at zero temperature

Q:- Solve the heat equation $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial u}{\partial t}$ where the ends $x=0$ and $x=l$ of the bar are kept at zero temperature i.e. $u(0,t) = u(l,t) = 0$

Sol:- The given pde is,



$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial u}{\partial t} \qquad \text{——— (1)}$$

with boundary condition

$$u(0,t) = 0 = u(l,t) \qquad \text{——— (2)}$$

then we seek the solution of (1) in the form

$$u(x,t) = X(x) \cdot T(t) \qquad \text{——— (3)}$$

then by putting $u(x,t)$ from (3) in (1), we get

$$X''T = \frac{1}{c^2} XT'$$

$$\Rightarrow \frac{X''}{X} = \frac{1}{c^2}\frac{T'}{T} = \lambda \text{ (say)}$$

$$\Rightarrow X'' = \lambda X \quad \text{and} \quad T' = c^2 \lambda T \qquad \text{——— (4)}$$

<u>case-(i)</u> If $\lambda = 0$, then from eq(4), we get

$X'' = 0$ and $T' = 0$

$\Rightarrow X = a_1 x + a_2$ and $T = b_1$

$\therefore$ The complete solution is,

$u(x,t) = (a_1 x + a_2) b_1$

$\Rightarrow u(x,t) = c_1 x + c_2$

$\because u(0,t) = 0 \Rightarrow c_2 = 0$

and $u(l,t) = 0 \Rightarrow c_1 = 0$

$\therefore u(x,t) = 0 \Rightarrow$ no solution exist in this case.

<u>case-(ii)</u> If $\lambda = k^2 > 0$, then from eq(4), we get

$X'' = k^2 X$ and $T' = k^2 c^2 T$

$\Rightarrow X = a_1 e^{kx} + a_2 e^{-kx}$ and $T = b_1 e^{k^2 c^2 t}$

$\therefore$ The complete solution is,

$u(x,t) = (a_1 e^{kx} + a_2 e^{-kx}) b_1 e^{k^2 c^2 t}$

$\Rightarrow u(x,t) = (c_1 e^{kx} + c_2 e^{-kx}) e^{k^2 c^2 t}$

$$\left.\begin{aligned} &\because u(0,t) = 0 \Rightarrow c_1 + c_2 = 0 \Rightarrow c_2 = -c_1 \\ &\text{and } u(l,t) = 0 \Rightarrow c_1 e^{kl} + c_2 e^{-kl} = 0 \end{aligned}\right\}$$

$\Rightarrow c_1 e^{kl} - c_1 e^{-kl} = 0$

$\Rightarrow c_1 (e^{kl} - e^{-kl}) = 0$

$\Rightarrow c_1 = 0$

$\Rightarrow c_2 = 0$

$\therefore u(x,t) = 0 \Rightarrow$ no solution.

<u>case-(iii)</u>. If $\lambda = -k^2 < 0$, then from eq(4), we get

$X'' = -k^2 X$ and $T' = -k^2 c^2 T$

$\Rightarrow X = a_1 \cos kx + a_2 \sin kx$ and $T = b_1 e^{-k^2 c^2 t}$

$\Rightarrow u(x,t) = (a_1 \cos kx + a_2 \sin kx) b_1 e^{-k^2 c^2 t}$

$$\Rightarrow u(x,t) = (c_1 \cos Kx + c_2 \sin Kx)\, e^{-K^2c^2t}$$

$\because u(0,t) = 0 \Rightarrow c_1 = 0$

and $u(l,t) = 0 \Rightarrow c_2 \sin Kl = 0$

$$\Rightarrow K = \frac{n\pi}{l} \quad ; n \in \mathbb{N}$$

$$\therefore u(x,t) = c_n \sin\left(\frac{n\pi x}{l}\right) e^{-c^2 \frac{n^2\pi^2 t}{l^2}} \quad ; n \in \mathbb{N} \quad \text{——(5)}$$

Hence, the required solution of (1) is given in (5).

Q:- Find the temperature function of the rod of length l which is insulated laterally and whose ends are kept at zero temperature ~~in the bar~~ and $f(x)$ is the initial temperature in the bar (rod)

OR

Obtain the series solution of the heat equation $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial u}{\partial t}$ with boundary condition $u(0,t) = 0 = u(l,t)$ and the initial condition $u(x,0) = f(x)$.

Sol:- The given pde is, $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial u}{\partial t}$ ——(1)

with boundary condition,

$$u(0,t) = 0 = u(l,t) \quad \text{——(2)}$$

then we seek the solution of (1) in the form

$$u(x,t) = X(x) \cdot T(t) \quad \text{——(3)}$$

then by putting $u(x,t)$ from (3) in (1), we get

$$X''T = \frac{1}{c^2} X T'$$

$$\Rightarrow \frac{X''}{X} = \frac{1}{c^2}\frac{T'}{T} = \lambda \text{ (say)}$$

$\Rightarrow X'' = \lambda X$ and $T' = c^2 \lambda T$ —— (4)

case-(i) If $\lambda = 0$, then from eq(4), we get

$X'' = 0$ and $T' = 0$

$\Rightarrow X = a_1 x + a_2$ and $T = b_1$

$\therefore$ The complete solution is,

$u(x,t) = (a_1 x + a_2) b_1$

$\Rightarrow u(x,t) = c_1 x + c_2$

$\because u(0,t) = 0 \Rightarrow c_2 = 0$

and $u(l,t) = 0 \Rightarrow c_1 = 0$

$\therefore u(x,t) = 0 \Rightarrow$ no solution exist in this case.

case-(ii) If $\lambda = k^2 > 0$, then from eq(4), we get

$X'' = k^2 X$ and $T' = k^2 c^2 T$

$\Rightarrow X = a_1 e^{kx} + a_2 e^{-kx}$ and $T = b_1 e^{k^2 c^2 t}$

$\therefore$ The complete solution is,

$u(x,t) = (a_1 e^{kx} + a_2 e^{-kx}) b_1 e^{k^2 c^2 t}$

$\Rightarrow u(x,t) = (c_1 e^{kx} + c_2 e^{-kx}) e^{k^2 c^2 t}$

$\because u(0,t) = 0 \Rightarrow c_1 + c_2 = 0 \Rightarrow c_2 = -c_1$

and $u(l,t) = 0 \Rightarrow c_1 e^{kl} + c_2 e^{-kl} = 0$

$\Rightarrow c_1 (e^{kl} - e^{-kl}) = 0$

$\Rightarrow c_1 = 0$

$\Rightarrow c_2 = 0$

$\therefore u(x,t) = 0 \Rightarrow$ no solution.

Case-(iii) Let $\lambda = -k^2 < 0$, then from eq(4), we get

$$X'' = -k^2 X \quad \text{and} \quad T' = -k^2c^2T$$

$$\Rightarrow X = a_1 \cos kx + a_2 \sin kx \quad \text{and} \quad T = b_1 e^{-k^2c^2t}$$

$$\Rightarrow u(x,t) = (a_1 \cos kx + a_2 \sin kx)\, b_1 e^{-k^2c^2t}$$

$$\Rightarrow u(x,t) = (c_1 \cos kx + c_2 \sin kx) e^{-k^2c^2t}$$

$\because u(0,t) = 0 \Rightarrow c_1 = 0$

and $u(l,t) = 0 \Rightarrow c_2 \sin kl = 0$

$$\Rightarrow k = \frac{n\pi}{l} \; ; n \in \mathbb{N}$$

$$\therefore u(x,t) = c_n \sin\left(\frac{n\pi x}{l}\right) e^{-c^2n^2\pi^2t/l^2} \; ; n \in \mathbb{N} \quad \text{—— (5)}$$

By the principle of superposition, we assume the series solution in the form

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{l}\right) e^{-n^2\pi^2c^2t/l^2} \quad \text{—— (6)}$$

$\because u(x,0) = f(x)$

$$\therefore f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{l}\right)$$

which is Fourier sine series, then the Fourier coefficient c_n is given by,

$$c_n = \frac{2}{l}\int_0^l f(x) \cdot \sin\left(\frac{n\pi x}{l}\right) dx \quad \text{—— (7)}$$

Hence eq(6) is the required solution, where c_n is given in (7).

Q:-4 Solve the equation $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$ given that $u(x,t)=0$ when $t=\infty$ and $u(x,t)=0$ when $x=0$ and $x=l$.

Sol:- The given pde is, $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$ ——— (1)

with boundary conditions $u(0,t)=0=u(l,t)$ ——(2)

and $u(x,t)=0$ when $t=\infty$.

We seek the solution of (1) in the form

$u(x,t) = X(x)\cdot T(t)$ ——— (3)

By putting $u(x,t)$ from (3) in (1), we get

$X''T = XT'$

$\Rightarrow \frac{X''}{X} = \frac{T'}{T} = \lambda$ (say)

$\Rightarrow X'' = \lambda X$ and $T' = \lambda T$ ——— (4)

case-(i) If $\lambda=0$, then from eq(4), we get

$X''=0$ and $T'=0$

$\Rightarrow X = a_1 x + a_2$ and $T = b_1$

$\therefore$ The complete solution is,

$u(x,t) = X(x)\cdot T(t)$

$= (a_1 x + a_2) b_1 = c_1 x + c_2$

$\because u(0,t)=0 \Rightarrow c_2=0$

and $u(l,t)=0 \Rightarrow c_1=0$

$\therefore u(x,t)=0$

$\Rightarrow$ no solution exist in this case.

Case-(ii) If $\lambda = k^2 > 0$, then from eq(4), we get,

$$X'' = k^2 X \quad \text{and} \quad T' = k^2 T$$

$$\Rightarrow X = a_1 e^{kx} + a_2 e^{-kx} \quad \text{and} \quad T = b_1 e^{k^2 t}$$

$\therefore$ The complete solution is,

$$u(x,t) = (a_1 e^{kx} + a_2 e^{-kx}) b_1 e^{k^2 t}$$

$$= (c_1 e^{kx} + c_2 e^{-kx}) e^{k^2 t}$$

$\because u(0,t) = 0 \Rightarrow c_1 + c_2 = 0 \Rightarrow c_2 = -c_1$

and $u(l,t) = 0 \Rightarrow c_1 e^{kl} + c_2 e^{-kl} = 0$

$$\Rightarrow c_1 e^{kl} - c_1 e^{-kl} = 0$$

$$\Rightarrow c_1 = 0$$

$$\Rightarrow c_2 = 0$$

$\therefore u(x,t) = 0 \Rightarrow$ no solution.

Case-(iii) When $\lambda = -k^2 < 0$, then from eq(4), we get

$$X'' = -k^2 X \quad \text{and} \quad T' = -k^2 T$$

$$\Rightarrow X = a_1 \cos kx + a_2 \sin kx \quad \text{and} \quad T = c_1 e^{-k^2 t}$$

$$\Rightarrow u(x,t) = (a_1 \cos kx + a_2 \sin kx) c_1 e^{-k^2 t}$$

$$\Rightarrow u(x,t) = (b_1 \cos kx + b_2 \sin kx) e^{-k^2 t}$$

$\because u(0,t) = 0 \Rightarrow b_1 = 0$

and $u(l,t) = 0 \Rightarrow b_2 \sin kl = 0$

$$\Rightarrow k = \frac{n\pi}{l} \; ; n \in N$$

$$\therefore u(x,t) = b_n \sin\left(\frac{n\pi x}{l}\right) e^{-n^2\pi^2 t/l^2} \; ; n \in N$$

By the principle of superposition, the solution is given by

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) e^{-n^2\pi^2 t/l^2}$$

It satisfies the condition $u(x,t) = 0$ as $t \to \infty$.

5) Suppose we have a homogeneous bar of length L with ends kept at zero temperature and initial temperature function

$$u(x,0) = f(x) = \begin{cases} x, & \text{if } 0 \le x \le L/2 \\ L-x, & \text{if } L/2 \le x \le L \end{cases}$$

Sol:- We know that the solution of heat equation is given by,

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-n^2\pi^2c^2t/L^2} \quad ; n \in N \qquad \text{---(1)}$$

$\because u(x,0) = f(x)$

$$\therefore f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

which is Fourier sine series.

Hence, the Fourier coefficient b_n is given by,

$$b_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{2}{L}\int_0^{L/2} x \sin\left(\frac{n\pi x}{L}\right) dx + \frac{2}{L}\int_{L/2}^{L} (L-x)\sin\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{2}{L}\left[\frac{-x\cos\left(\frac{n\pi x}{L}\right)}{\frac{n\pi}{L}} + \frac{\sin\left(\frac{n\pi x}{L}\right)}{\frac{n^2\pi^2}{L^2}}\right]_0^{L/2}$$

$$+ \frac{2}{L}\left[\frac{-(L-x)\cos\left(\frac{n\pi x}{L}\right)}{\frac{n\pi}{L}} - \frac{\sin\left(\frac{n\pi x}{L}\right)}{\frac{n^2\pi^2}{L^2}}\right]_{L/2}^{L}$$

$$= \frac{2}{L}\left[-\frac{L}{n\pi}x\cos\left(\frac{n\pi x}{L}\right)+\frac{L^2}{n^2\pi^2}\sin\left(\frac{n\pi x}{L}\right)\right]_0^{L/2}$$

$$+\frac{2}{n\pi}\left[-(L-x)\cos\left(\frac{n\pi x}{L}\right)-\frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\right]_{L/2}^{L}$$

$$= \frac{2}{n\pi}\left[-\frac{L}{2}\cos\left(\frac{n\pi}{2}\right)+\frac{L}{n\pi}\sin\left(\frac{n\pi}{2}\right)\right]$$

$$+\frac{2}{n\pi}\left[0+\frac{L}{2}\cos\left(\frac{n\pi}{2}\right)+\frac{L}{n\pi}\sin\left(\frac{n\pi}{2}\right)\right]$$

$$= \frac{4L}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

$\therefore$ From eq(1), we get

$$u(x,t)=\sum_{n=1}^{\infty}\frac{4L}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi x}{L}\right)e^{-n^2\pi^2c^2t/L^2}$$

$$\Rightarrow \boxed{u(x,t)=\frac{4L}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi x}{L}\right)e^{-n^2\pi^2c^2t/L^2}}$$

Q:-6 Solve the heat equation $\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t}$ with length of bar is $l=1$ and ends are kept at zero temperature with initial temperature

$$u(x,0)=f(x)=x\sin(\pi x)$$

Sol:- We know that the solution of heat equation is given by,

$$u(x,t)=\sum_{n=1}^{\infty}b_n\sin\left(\frac{n\pi x}{l}\right)e^{-n^2\pi^2c^2t/l^2} \quad ; n\in\mathbb{N}$$

Here $l=1$, $c^2=1$

$$\Rightarrow u(x,t)=\sum_{n=1}^{\infty}b_n\sin(n\pi x)\,e^{-n^2\pi^2t} \quad ; n\in\mathbb{N} \quad \text{——(1)}$$

$\because u(x,0)=f(x)=x\sin\pi x$

$$\therefore x\sin\pi x=\sum_{n=1}^{\infty}b_n\sin(n\pi x)$$

which is fourier sine series.

Hence, the fourier coefficient b_n is given by,

$$b_n = \frac{2}{1}\int_0^1 f(x)\sin(n\pi x)\,dx$$

$$= 2\int_0^1 x\sin(\pi x)\cdot\sin(n\pi x)\,dx$$

For $n=1$

$$b_1 = 2\int_0^1 x\sin^2\pi x\,dx$$

$$= \int_0^1 x(1-\cos 2\pi x)\,dx$$

$$= \int_0^1 x\,dx - \int_0^1 \cos(2\pi x)\,dx$$

$$= \left[\frac{x^2}{2}\right]_0^1 - \left[\frac{\sin(2\pi x)}{2\pi}\right]_0^1$$

$$= \frac{1}{2}$$

For $n=2,3,4,\ldots$

$$b_n = 2\int_0^1 x\sin(\pi x)\cdot\sin(n\pi x)\,dx$$

$$= \int_0^1 x\{\cos(\pi x - n\pi x) - \cos(\pi x + n\pi x)\}\,dx$$

$$= \int_0^1 x\cos\{(\pi - n\pi)x\}\,dx - \int_0^1 x\cos\{(\pi+n\pi)x\}\,dx$$

$$= \left[\frac{x\sin\{(\pi-n\pi)x\}}{(\pi-n\pi)}\right]_0^1 + \left[\frac{\{\cos(\pi-n\pi)x\}}{(\pi-n\pi)^2}\right]_0^1$$

$$- \left[\frac{x\sin\{(\pi+n\pi)x\}}{(\pi+n\pi)}\right]_0^1 - \left[\frac{\cos\{(\pi+n\pi)x\}}{(\pi+n\pi)^2}\right]_0^1$$

$$= \frac{1}{(\pi-n\pi)^2}\{(-1)^{1-n}-1\} - \frac{1}{(\pi+n\pi)^2}\{(-1)^{1+n}-1\}$$

$$= \frac{(\pi+n\pi)^2\{(-1)^{1-n}-1\} - (\pi-n\pi)^2\{(-1)^{1+n}-1\}}{(\pi-n\pi)^2(\pi+n\pi)^2}$$

$$= \frac{(\pi+n\pi)^2((-1)^{n-1}-1) - (\pi-n\pi)^2((-1)^{1+n}-1)}{(\pi-n\pi)^2(\pi+n\pi)^2} \quad [\because (-1)^{1-n} = (-1)^{n-1}]$$

$$= \frac{\pi^2(1+n)^2\{(-1)^n(-1)^{-1}-1\} - \pi^2(1-n)^2\{(-1)(-1)^n-1\}}{\pi^2(1-n)^2(1+n)^2\pi^2}$$

$$= \frac{-\pi^2(1+n)^2\{1+(-1)^n\} + \pi^2(1-n)^2(1+(-1)^n)}{\pi^4(1-n^2)^2}$$

$$= \frac{\pi^2(1+(-1)^n)\{(1-n)^2-(1+n)^2\}}{\pi^4(n^2-1)^2}$$

$$= \frac{\pi^2(1+(-1)^n)(-4n)}{\pi^4(n^2-1)^2}$$

$$= \frac{-4n(1+(-1)^n)}{\pi^2(n^2-1)^2} \quad \text{———— (2)}$$

$\therefore$ $u(x,t) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)\, e^{-n^2\pi^2 t}$; ~~Ans~~

where b_n is given by, .

$$b_n = \begin{cases} \frac{1}{2} & , \text{if } n=1 \\ \frac{-4n(1+(-1)^n)}{\pi^2(n^2-1)^2} & , \text{if } n=2,3,\ldots \end{cases}$$

* Insulated Ends

If the ends of the bar are insulated, then the flow of energy across the ends is zero.

In this type of problem the boundary condition is,

$u_x(0,t)=0$ and $u_x(l,t)=0$

Since flow of energy or flow of heat $=-K\frac{\partial u}{\partial x}$

where K is thermal conductivity, and flow of energy at ends are kept at zero.

i.e. $-K\frac{\partial u}{\partial x}(0,t)=0$ and $-K\frac{\partial u}{\partial x}(l,t)=0$

$\Rightarrow u_x(0,t)=0$ and $u_x(l,t)=0$

1) Solve the heat equation $\frac{\partial^2 u}{\partial x^2}=\frac{1}{c^2}\frac{\partial u}{\partial t}$ with boundary condition $u_x(0,t)=0=u_x(l,t)$; $t>0$ and initial condition $u(x,0)=f(x)$; $0<x<l$.

Sol:- Given pde is $\frac{\partial^2 u}{\partial x^2}=\frac{1}{c^2}\frac{\partial u}{\partial t}$ ——— (1)

with boundary condition $u_x(0,t)=0=u_x(l,t)$ —(2)

and initial condition $u(x,0)=f(x)$ ——(3)

We seek the solution of the form

$$u(x,t)=X(x)\cdot T(t) \quad \text{—— (4)}$$

By putting the value of $u(x,t)$ from (4) in (1), we get $X''T=\frac{1}{c^2}XT'$

$\Rightarrow \frac{X''}{X}=\frac{1}{c^2}\frac{T'}{T}=\lambda$ (say)

$\Rightarrow X''=\lambda X$ and $T'=c^2\lambda T$ ——— (5)

case-(i) If $\lambda=0$, then from eq (5),

$X''=0$ and $T'=0$

$\Rightarrow X=a_1x+a_2$ and $T=b_1$

$\Rightarrow u(x,t)=(a_1x+a_2)b_1=c_1x+c_2$

$\Rightarrow u_x(x,t) = c_1$

$u_x(0,t) = u_x(l,t) = 0 \Rightarrow c_1 = 0$

$\therefore \boxed{u(x,t) = c_2}$

<u>case-(ii)</u> If $\lambda > 0$ i.e. $\lambda = k^2$, then

$X'' = k^2 X$ and $T' = c^2k^2T$

$\Rightarrow X = a_1 e^{kx} + a_2 e^{-kx}$ and $T = b_1 e^{c^2k^2t}$

$\therefore u(x,t) = (a_1 e^{kx} + a_2 e^{-kx}) b_1 e^{c^2k^2t}$

$= (c_1 e^{kx} + c_2 e^{-kx}) e^{c^2k^2t}$

$\Rightarrow u_x(x,t) = (c_1 k e^{kx} - c_2 k e^{-kx}) e^{c^2k^2t}$

$\because u_x(0,t) = 0 \Rightarrow c_1 - c_2 = 0 \Rightarrow c_2 = c_1$

and $u_x(l,t) = 0 \Rightarrow c_1 k e^{kl} - k c_2 e^{-kl} = 0$ $\Big\} \Rightarrow c_1 = c_2 = 0$

$\Rightarrow c_1 k (e^{kl} - e^{-kl}) = 0$

$\Rightarrow c_1 = 0$

$\Rightarrow c_2 = 0$

$\therefore u(x,t) = 0$

<u>case-(iii)</u> If $\lambda < 0$ i.e. $\lambda = -k^2$, then

$X'' = -k^2 X$ and $T' = -c^2k^2T$

$\Rightarrow X = a_1 \cos kx + a_2 \sin kx$ and $T = b_1 e^{-c^2k^2t}$

$\Rightarrow u(x,t) = (a_1 \cos kx + a_2 \sin kx) b_1 e^{-c^2k^2t}$

$= (c_1 \cos kx + c_2 \sin kx) e^{-c^2k^2t}$

$\Rightarrow u_x(x,t) = (-kc_1 \sin kx + kc_2 \cos kx) e^{-c^2k^2t}$

$\because u_x(0,t) = 0 \Rightarrow c_2 = 0$

and $u_x(l,t) = 0 \Rightarrow \sin kl = 0$

$\Rightarrow k = \frac{n\pi}{l}$; $n \in \mathbb{N}$

$\therefore u_n(x,t) = c_n \cos\left(\frac{n\pi x}{l}\right) e^{-c^2 n^2 \pi^2 t / l^2}$; $n \in \mathbb{N}$

By principle of superposition,

$$u(x,t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi x}{l}\right) e^{-c^2 n^2 \pi^2 t / l^2} \quad \text{——(6)}$$

Also $\because u(x,0) = f(x)$

$$\Rightarrow f(x) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi x}{l}\right)$$

which is Fourier cosine series.

then the Fourier coefficient c_n is given by,

$$c_n = \frac{2}{l} \int_0^l f(x) \cos\left(\frac{n\pi x}{l}\right) dx \; ; n = 0, 1, 2, \ldots \quad \text{——(7)}$$

Hence solution of (1) is given in (6) where c_n is in (7) and $c_0 = \frac{2}{l}\int_0^l f(x)\,dx$.

2) Suppose a homogeneous bar of length π has insulated ends and $c = 2$ for the material of the bar. Suppose the initial temperature is,

$$f(x) = \begin{cases} 0, & 0 \le x < \pi/2 \\ 50, & \pi/2 \le x < \pi \end{cases}$$

Sol:- $c_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi/2} f(x)\,dx + \frac{2}{\pi}\int_{\pi/2}^{\pi} f(x)\,dx$

$$= \frac{2}{\pi}\int_0^{\pi/2} 0\,dx + \frac{2}{\pi}\int_{\pi/2}^{\pi} 50\,dx$$

$$= 50$$

and $c_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos\left(\frac{n\pi x}{\pi}\right)dx \;;\; n=1,2,3,\ldots$

$$= \frac{2}{\pi}\int_0^{\pi/2} 0\cdot\cos(nx)\,dx + \frac{2}{\pi}\int_{\pi/2}^{\pi} 50\cos(nx)\,dx$$

$$= \frac{100}{\pi}\left[\frac{\sin(nx)}{n}\right]_{\pi/2}^{\pi}$$

$$= -\frac{100}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

$\therefore$ Solution of (1) is

$$\boxed{u(x,t) = 25 - \sum_{n=1}^{\infty}\frac{100}{n\pi}\sin\left(\frac{n\pi}{2}\right)\cos(nx)\,e^{-4n^2t}}$$

* Ends at different temperature

Suppose the heat equation is,

$$u_t = c^2 u_{xx} \text{ for } 0<x<l,\ t>0 \quad \text{—— (1)}$$

with boundary condition $u(0,t)=A$, $u(l,t)=B$ for $t>0$ and initial condition $u(x,0)=f(x)$ for $0<x<l$. Here A and B are non-negative numbers and at least one of them is non-zero. Since ends of the bar are maintained at different temperatures, so these are non-homogeneous boundary condition.

Now, we transform the given pde (heat equation) into another heat equation with boundary condition zero.

For that let $u(x,t) = U(x,t)+\psi(x)$ —— (2)

then $u_t = U_t$ and $u_{xx} = U_{xx} + \psi''(x)$

$\therefore$ From eq(1), we get

$$U_t = c^2\left(U_{xx} + \psi''(x)\right)$$

$$\Rightarrow U_t = c^2 U_{xx} + c^2\psi''(x) \quad \text{—— (3)}$$

We choose $\psi(x)$ such that $c^2\psi''(x) = 0$

i.e. $\psi''(x) = 0$

$$\Rightarrow \psi(x) = Cx + D \quad \text{—— (4)}$$

Now, the boundary condition

$u(0,t) = A$

$\Rightarrow U(0,t) + \psi(0) = A$

$\Rightarrow U(0,t) = 0$ if $\psi(0) = A$

i.e. $U(0,t) = 0$ if $D = A$ [From (4)]

and $u(l,t) = B$

$\Rightarrow U(l,t) + \psi(l) = B$

$\Rightarrow U(l,t) = 0$ if $\psi(l) = B$

i.e. $U(l,t) = 0$ if $Cl + D = B$

$$\Rightarrow C = \frac{1}{l}(B-D) = \frac{1}{l}(B-A)$$

$$\therefore \psi(x) = \frac{1}{l}(B-A)x + A$$

then the transformed heat equation is given from (3)

$$\left.\begin{aligned} & U_t = c^2 U_{xx} \\ & \text{with boundary condition } U(0,t) = 0 = U(l,t) \\ & \text{and initial condition } U(x,0) = u(x,0) - \psi(x) \\ & \qquad = f(x) - \frac{1}{l}(B-A)x - A \end{aligned}\right\} \text{—(5)}$$

Q:-1 Solve the problem

$$\left.\begin{aligned} &u_t = 7u_{xx} \text{ for } 0<x<5,\ t>0 \\ &u(0,t)=1,\ u(5,t)=4 \text{ for } t>0 \\ &\text{and } u(x,0)=f(x)=\begin{cases} 3-x & \text{for } 0<x\le 3 \\ 10(x-3) & \text{for } 3\le x<5 \end{cases} \end{aligned}\right\} \text{---(1)}$$

Sol:- Here, $c^2=7$, $l=5$, $A=1$, $B=4$

then by putting

$$u(x,t) = U(x,t) + \psi(x)$$

where $\psi(x) = \frac{1}{l}(B-A)x + A$

$$= \frac{1}{5}(3)x + 1 = \frac{3x}{5}+1$$

the pde (1) transformed into

$$\left.\begin{aligned} &U_t = 7U_{xx} \text{ for } 0<x<5,\ t>0 \\ &U(0,t)=U(5,t)=0 \\ &\text{and } U(x,0) = u(x,0) - \frac{3x}{5} - 1 \end{aligned}\right\} \text{-(2)}$$

$$\Rightarrow U(x,0) = g(x) = \begin{cases} 2-\frac{8x}{5} & \text{if } 0<x\le 3 \\ -31+\frac{47x}{5} & \text{if } 3\le x<5 \end{cases}$$

then the solution of the problem (2) is,

$$U(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{5}\right) e^{-7n^2\pi^2 t/25} \quad ; n\in\mathbb{N} \text{ ---(3)}$$

Also, since $U(x,0) = g(x)$

$$\Rightarrow g(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{5}\right)$$

which is Fourier sine series.

$\therefore$ The Fourier coefficient b_n is given by,

$$b_n = \frac{2}{l}\int_0^l g(x)\sin\left(\frac{n\pi x}{5}\right)dx$$

$$= \frac{2}{5}\int_0^5 g(x)\sin\left(\frac{n\pi x}{5}\right)dx$$

$$= \frac{2}{5}\int_0^3 g(x)\sin\left(\frac{n\pi x}{5}\right)dx + \frac{2}{5}\int_3^5 g(x)\sin\left(\frac{n\pi x}{5}\right)dx$$

$$= \frac{2}{5}\int_0^3 \left(2-\frac{8x}{5}\right)\sin\left(\frac{n\pi x}{5}\right)dx + \frac{2}{5}\int_3^5 \left(-31+\frac{47x}{5}\right)\sin\left(\frac{n\pi x}{5}\right)dx$$

$$= \frac{2}{5}\left\{\left[\left(2-\frac{8x}{5}\right)\left\{\frac{-\cos\left(\frac{n\pi x}{5}\right)}{\frac{n\pi}{5}}\right\}\right]_0^3 + \frac{5}{n\pi}\int_0^3 \cos\left(\frac{n\pi x}{5}\right)\left(-\frac{8}{5}\right)dx\right\}$$

$$+ \frac{2}{5}\left\{\left[\left(-31+\frac{47x}{5}\right)\left\{\frac{-\cos\left(\frac{n\pi x}{5}\right)}{\frac{n\pi}{5}}\right\}\right]_3^5 + \frac{5}{n\pi}\int_3^5 \cos\left(\frac{n\pi x}{5}\right)\cdot\frac{47}{5}\,dx\right\}$$

$$= -\frac{2}{5}\left(2-\frac{24}{5}\right)\cdot\frac{5}{n\pi}\cos\left(\frac{3n\pi}{5}\right) + \frac{2}{5}\cdot 2\cdot\frac{5}{n\pi} - \frac{16}{5n\pi}\cdot\frac{5}{n\pi}\left[\sin\left(\frac{n\pi x}{5}\right)\right]_0^3$$

$$- \frac{2}{5}(-31+47)\cdot\frac{5}{n\pi}\cos(n\pi) + \frac{2}{5}\left(-31+\frac{141}{5}\right)\cdot\frac{5}{n\pi}\cos\left(\frac{3n\pi}{5}\right)$$

$$+ \frac{94}{5n\pi}\cdot\frac{5}{n\pi}\left[\sin\left(\frac{n\pi x}{5}\right)\right]_3^5$$

$$= \frac{28}{5n\pi}\cancel{\cos}\left(\frac{3n\pi}{5}\right) + \frac{4}{n\pi} - \frac{16}{n^2\pi^2}\sin\left(\frac{3n\pi}{5}\right) - \frac{32}{n\pi}\cos(n\pi)$$

$$- \frac{28}{5n\pi}\cancel{\cos}\left(\frac{3n\pi}{5}\right) - \frac{94}{n^2\pi^2}\sin\left(\frac{3n\pi}{5}\right)$$

$$= \frac{-110}{n^2\pi^2}\sin\left(\frac{3n\pi}{5}\right) + \frac{4}{n\pi} - \frac{32}{n\pi}(-1)^n$$

i.e. $U(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{5}\right)e^{-7n^2\pi^2 t/25}$; $n\in\mathbb{N}$

where $b_n = \frac{-110}{n^2\pi^2}\sin\left(\frac{3n\pi}{5}\right) + \frac{4}{n\pi} - \frac{32}{n\pi}(-1)^n$

$\Rightarrow u(x,t) = U(x,t) + \psi(x)$

$\Rightarrow \boxed{u(x,t) = U(x,t) + \frac{3x}{5} + 1}$ is required solution for pde (1)

Q:- Solve $u_t = K u_{xx}$ with the given boundary and initial conditions.

(1) $u(x,0) = \sin(\pi x)$, $u(0,t) = u(1,t) = 0$, $t>0$, $0<x<1$

(2) $u_x(0,t) = u_x(4,t) = 0$; $t>0$ and $u(x,0) = x^2$; $0<x<4$

(3) $u(0,t) = 3$, $u(5,t) = \sqrt{7}$; $t>0$ and $u(x,0) = x^2$; $0<x<5$.

Sol:-1 Given, pde $u_t = K u_{xx}$

with initial and boundary conditions

$$u(x,0) = \sin(\pi x)$$

and $u(0,t) = u(1,t) = 0$, $t>0$, $0<x<1$ — (1)

We know that the solution of (1) is given by,

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)\, e^{-n^2\pi^2 K t} \quad ; n \in \mathbb{N}$$

$\because u(x,0) = \sin(\pi x)$

$$\therefore \sin(\pi x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)$$

which is Fourier sine series.

Hence, the Fourier co-efficient b_n is given by,

$$b_n = \frac{2}{1}\int_0^1 \sin(\pi x)\sin(n\pi x)\,dx$$

$$= \int_0^1 2\sin(\pi x)\cdot\sin(n\pi x)\,dx$$

$$= \int_0^1 \{\cos(\pi x - n\pi x) - \cos(\pi x + n\pi x)\}\,dx$$

$$= \int_0^1 \cos(1-n)\pi x\,dx - \int_0^1 \cos(1+n)\pi x\,dx$$

$$= \left[\frac{\sin(1-n)\pi x}{(1-n)}\right]_0^1 - \left[\frac{\sin(1+n)\pi x}{(1+n)}\right]_0^1$$

$$= 0 \quad , \text{ for } n = 2,3,4,\ldots$$

<u>For $n=1$</u>

$$b_1 = 2\int_0^1 \sin^2(\pi x)\,dx$$

$$= \int_0^1 \{1 - \cos(2\pi x)\}\,dx$$

$$= \int_0^1 dx - \int_0^1 \cos(2\pi x)\,dx$$

$$= 1 - \left[\frac{\sin(2\pi x)}{2\pi}\right]_0^1 = 1$$

$$\therefore\ b_n = \begin{cases} 0 & , \text{ for } n = 2,3,4,\ldots \\ 1 & , \text{ for } n = 1 \end{cases}$$

$$\therefore \boxed{u(x,t) = \sin(\pi x)\cdot e^{-\pi^2 kt}}$$

<u>Sol:-2</u> Given, pde $u_t = k u_{xx}$

with initial and boundary conditions,

$$\left.\begin{aligned} &u(x,0) = x^2 \ , \ 0 < x < 4 \\ \text{and } \ &u_x(0,t) = u_x(4,t) = 0 \ ; \ t > 0 \end{aligned}\right\} \quad -(1)$$

The solution of (1) is given by,

$$u(x,t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi x}{4}\right) e^{-kn^2\pi^2 t/16}$$

$\because u(x,0) = x^2$

$$\Rightarrow \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi x}{4}\right) = x^2$$

which is Fourier cosine series.

The Fourier coefficients are given by,

$$c_0 = \frac{2}{4}\int_0^4 x^2\,dx = \frac{2}{4}\left[\frac{x^3}{3}\right]_0^4 = \frac{2}{\cancel{12}_6} \times \cancel{64}^{32} = \frac{32}{3}$$

$$c_n = \frac{2}{4}\int_0^4 x^2 \cos\left(\frac{n\pi x}{4}\right) dx$$

$$= \frac{1}{2}\left\{\left[x^2 \frac{\sin\left(\frac{n\pi x}{4}\right)}{\frac{n\pi}{4}}\right]_0^4 - \frac{4}{n\pi}\int_0^4 \sin\left(\frac{n\pi x}{4}\right)\cdot 2x\,dx\right\}$$

$$= 0 - \frac{4}{n\pi}\left\{\left[-x\cdot\frac{4}{n\pi}\cos\left(\frac{n\pi x}{4}\right)\right]_0^4 + \left(\frac{4}{n\pi}\right)^2\left[\sin\left(\frac{n\pi x}{4}\right)\right]_0^4\right\}$$

$$= \frac{16}{n^2\pi^2} \times 4\cos(n\pi)$$

$$= \frac{64}{n^2\pi^2}(-1)^n$$

$$\therefore \boxed{u(x,t) = \frac{16}{3} + \sum_{n=1}^{\infty} \frac{64}{n^2\pi^2}(-1)^n \cos\left(\frac{n\pi x}{4}\right) e^{-Kn^2\pi^2 t/16}}$$

Sol:-3 Given pde $u_t = K u_{xx}$

with boundary and initial conditions
$u(0,t) = 3,\ u(5,t) = \sqrt{7},\ t > 0$
and $u(x,0) = x^2,\ 0 < x < 5$ ——(1)

Here $c^2 = K,\ l = 5,\ A = 3,\ B = \sqrt{7}$

then by putting

$$u(x,t) = U(x,t) + \psi(x)$$

where $\psi(x) = \frac{1}{l}(B-A)x + A$

$$= \frac{(\sqrt{7}-3)x}{5} + 3$$

then pde (1) transformed into

$$\left.\begin{aligned} &U_t = K U_{xx} \quad \text{for } 0<x<5, \; t>0 \\ &U(0,t) = U(5,t) = 0 \\ &\text{and } U(x,0) = u(x,0) - \psi(x) \\ &\qquad = x^2 - \frac{(\sqrt{7}-3)x}{5} - 3 \end{aligned}\right\} \quad \text{---}(2)$$

~~$\Rightarrow U(x,0) =$~~

then the solution of the problem (2) is,

$$U(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{5}\right) e^{-Kn^2\pi^2 t/25} \quad ; n \in \mathbb{N} \quad \text{---}(3)$$

Also, since $U(x,0) = x^2 - \frac{(\sqrt{7}-3)x}{5} - 3$

$$\Rightarrow x^2 - \frac{(\sqrt{7}-3)x}{5} - 3 = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{5}\right)$$

which is Fourier sine series.

∴ The Fourier coefficient b_n is given by,

$$b_n = \frac{2}{5}\int_0^5 \left(x^2 - \frac{(\sqrt{7}-3)x}{5} - 3\right) \sin\left(\frac{n\pi x}{5}\right) dx$$

$$\cancel{= \frac{2}{5}\int_0^5 x^2 \sin\left(\frac{n\pi x}{5}\right) dx - \frac{2(\sqrt{7}-3)}{25}}$$

$$= \frac{2}{5}\left\{\left[-\left(x^2 - \frac{(\sqrt{7}-3)x}{5} - 3\right)\cdot \frac{5}{n\pi}\cos\left(\frac{n\pi x}{5}\right)\right]_0^5 + \frac{5}{n\pi}\int_0^5 \cos\left(\frac{n\pi x}{5}\right)\left(2x - \frac{\sqrt{7}-3}{5}\right) dx\right\}$$

$$= \frac{-2}{n\pi}(25 - \sqrt{7} + 3 - 3)\cos(n\pi) + \frac{2}{n\pi}(-3)$$

$$+ \frac{2}{n\pi}\left\{\left[\left(2x - \frac{\sqrt{7}-3}{5}\right)\cdot \frac{5}{n\pi}\sin\left(\frac{n\pi x}{5}\right)\right]_0^5 - \frac{5}{n\pi}\int_0^5 \sin\left(\frac{n\pi x}{5}\right)\cdot 2\, dx\right\}$$

$$= \frac{(2\sqrt{7}-50)}{n\pi}\cos(n\pi) - \frac{6}{n\pi} - \frac{20}{n^2\pi^2}\left[\frac{-\cos\left(\frac{n\pi x}{5}\right)}{\left(\frac{n\pi}{5}\right)}\right]_0^5$$

$$= \frac{(2\sqrt{7}-50)}{n\pi}\cos(n\pi) - \frac{6}{n\pi} + \frac{100}{n^3\pi^3}\left(\cos(n\pi) - 1\right)$$

$$= \left(\frac{(2\sqrt{7}-50)}{n\pi} + \frac{100}{n^3\pi^3}\right)(-1)^n - \frac{6}{n\pi} - \frac{100}{n^3\pi^3}$$

$$\therefore U(x,t) = \sum_{n=1}^{\infty}\left[\left(\frac{(2\sqrt{7}-50)}{n\pi} + \frac{100}{n^3\pi^3}\right)(-1)^n - \frac{6}{n\pi} - \frac{100}{n^3\pi^3}\right]$$

$$\sin\left(\frac{n\pi x}{5}\right) e^{-kn^2\pi^2 t/25}$$

The solution of (1) is given by,

$$u(x,t) = U(x,t) + \psi(x)$$

$$\therefore u(x,t) = \sum_{n=1}^{\infty}\left[\left(\frac{(2\sqrt{7}-50)}{n\pi} + \frac{100}{n^3\pi^3}\right)(-1)^n - \frac{6}{n\pi} - \frac{100}{n^3\pi^3}\right]\sin\left(\frac{n\pi x}{5}\right) e^{-kn^2\pi^2 t/25}$$

$$+ \frac{(\sqrt{7}-3)x}{5} + 3.$$

A Non-homogeneous Problem

Consider the non-homogeneous initial boundary value problem

$$\left.\begin{array}{l} u_t = c^2 u_{xx} + F(x,t) \quad ; 0<x<l,\ t>0 \\ \text{with boundary condition } u(0,t) = u(l,t) = 0 \ ; t>0 \\ \text{and initial condition } u(x,0) = f(x),\ 0<x<l \end{array}\right\} \text{—(1)}$$

which is non homogeneous because of $F(x,t)$.

If $F(x,t)=0$ then the problem (1) has a solution of the form

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) e^{-n^2\pi^2c^2t/l^2} \quad ; n \in N$$

where b_n are the Fourier sine coefficient of the initial temperature function on the interval.

This suggested that we attempt a solution of the present problem of the form

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) \sin\left(\frac{n\pi x}{l}\right) \quad \text{——— (2)}$$

Now, we have to determine the function $T_n(t)$ so that (2) is a solution of the problem (1).

If t is fixed then eq(2) is Fourier sine series of $u(x,t)$ as a function of x on $[0,l]$

$\therefore$ $T_n(t)$ is nth fourier sine coefficient of $u(x,t)$ on $[0,l]$

i.e. $$T_n(t) = \frac{2}{l}\int_0^l u(x,t)\sin\left(\frac{n\pi x}{l}\right)dx \quad \text{——(3)}$$

Assume that for any $t \geq 0$, $F(x,t)$ as a function of x, can also be expanded in a Fourier sine series on $[0,l]$.

i.e. $$F(x,t) = \sum_{n=1}^{\infty} B_n(t) \sin\left(\frac{n\pi x}{l}\right) \quad \text{—— (4)}$$

where the Fourier coefficient is,

$$\boxed{B_n(t) = \frac{2}{l}\int_0^l F(x,t)\sin\left(\frac{n\pi x}{l}\right)dx} \quad \text{---(5)}$$

Now from eq(3), we have

$$T_n'(t) = \frac{2}{l}\int_0^l u_t(x,t)\sin\left(\frac{n\pi x}{l}\right)dx \quad \text{---(6)}$$

$$\Rightarrow T_n'(t) = \frac{2c^2}{l}\int_0^l u_{xx}(x,t)\sin\left(\frac{n\pi x}{l}\right)dx + \frac{2}{l}\int_0^l F(x,t)\sin\left(\frac{n\pi x}{l}\right)dx \quad \text{[Using (1)]}$$

$$\Rightarrow T_n'(t) = \frac{2c^2}{l}\int_0^l u_{xx}(x,t)\sin\left(\frac{n\pi x}{l}\right)dx + B_n(t) \quad \text{[Using (5)]}$$

---(7)

Now, $\int_0^l u_{xx}(x,t)\sin\left(\frac{n\pi x}{l}\right)dx$

$$= \left[\sin\left(\frac{n\pi x}{l}\right)u_x(x,t)\right]_0^l - \int_0^l u_x(x,t)\cdot\frac{n\pi}{l}\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= 0 - \int_0^l \frac{n\pi}{l}u_x(x,t)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \left[-\frac{n\pi}{l}u(x,t)\cos\left(\frac{n\pi x}{l}\right)\right]_0^l + \frac{n\pi}{l}\int_0^l u(x,t)\left(-\frac{n\pi}{l}\right)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= 0 + \left(-\frac{n^2\pi^2}{l^2}\right)\int_0^l u(x,t)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= -\frac{n^2\pi^2}{l^2}\cdot\frac{l}{2}T_n(t)$$

$$= -\frac{n^2\pi^2}{2l}T_n(t)$$

$\therefore$ From eq(7), we get

$$T_n'(t) = \frac{-n^2\pi^2c^2}{l^2} T_n(t) + B_n(t)$$

$$\Rightarrow T_n'(t) + \frac{n^2\pi^2c^2}{l^2} T_n(t) = B_n(t) \quad \text{------} \quad (8)$$

which is ode for $T_n(t)$; $n = 1, 2, 3, \ldots$

with initial condition,

$$T_n(0) = \frac{2}{l}\int_0^l u(x,0)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$\Rightarrow T_n(0) = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx = b_n$$

$\therefore$ Solution of (8) is given by,

$$T_n(t)\, e^{n^2\pi^2c^2t/l^2} = \int_0^t B_n(t)\, e^{n^2\pi^2c^2t/l^2}\,dt + b_n$$

$$\Rightarrow T_n(t) = e^{-n^2\pi^2c^2t/l^2}\int_0^t B_n(\tau)\, e^{n^2\pi^2c^2\tau/l^2}\,d\tau + b_n e^{-n^2\pi^2c^2t/l^2}$$

$$\Rightarrow T_n(t) = \int_0^t \left\{B_n(\tau)\, e^{n^2\pi^2c^2(\tau-t)/l^2}\,d\tau\right\} + b_n e^{-n^2\pi^2c^2t/l^2} \quad \text{------} (9)$$

$\therefore$ From eq(2), the solution of given problem is,

$$\boxed{\begin{gathered} u(x,t) = \sum_{n=1}^{\infty}\left(\int_0^t B_n(\tau)\, e^{n^2\pi^2c^2(\tau-t)/l^2}\,d\tau\right)\sin\left(\frac{n\pi x}{l}\right) \\ + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) e^{-n^2\pi^2c^2t/l^2} \quad \text{------} (10) \\ \text{where } B_n(t) \text{ is given in (5) and} \\ b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx \end{gathered}}$$

2) Solve, $u_t = c^2 u_{xx} + x\sin(t)$; $0 < x < l$, $t > 0$
with $u(0,t) = u(l,t) = 0$; $t > 0$
and $u(x,0) = 1$; $0 < x < l$.

Sol:- Here $F(x,t) = x\sin(t)$

$$\therefore \quad B_n(t) = \frac{2}{l}\int_0^l F(x,t)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\int_0^l x\sin(t)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2\sin(t)}{l}\int_0^l x\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2\sin(t)}{l}\left\{\left[x\left\{\frac{-\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)}\right\}\right]_0^l + \left(\frac{l}{n\pi}\right)^2\left[\sin\left(\frac{n\pi x}{l}\right)\right]_0^l\right\}$$

$$= \frac{2\sin(t)}{l}\left\{-l\cdot\frac{l}{n\pi}(-1)^n + \left(\frac{l}{n\pi}\right)^2\cdot 0\right\}$$

$$= \frac{2l\sin(t)}{n\pi}(-1)^{n+1}$$

Also, $\int_0^t B_n(\tau)\, e^{n^2\pi^2c^2(\tau-t)/l^2}\, d\tau$

$$= \int_0^t \frac{2l\sin(\tau)}{n\pi}(-1)^{n+1} e^{n^2\pi^2c^2\tau/l^2}\cdot e^{-n^2\pi^2c^2t/l^2}\, d\tau$$

$$= \frac{2l}{n\pi}(-1)^{n+1}\cdot e^{-n^2\pi^2c^2t/l^2}\int_0^t \sin(\tau)\cdot e^{n^2\pi^2c^2\tau/l^2}\, d\tau$$

Let $I = \int_0^t \sin\tau\, e^{n^2\pi^2c^2\tau/l^2}\, d\tau$

$$= \left[\sin\tau\cdot\frac{e^{n^2\pi^2c^2\tau/l^2}}{\frac{n^2\pi^2c^2}{l^2}}\right]_0^t - \frac{l^2}{n^2\pi^2c^2}\int_0^t e^{n^2\pi^2c^2\tau/l^2}\cos\tau\, d\tau$$

$$= \frac{l^2}{n^2\pi^2c^2}\sin t\cdot e^{n^2\pi^2c^2t/l^2}$$
$$- \frac{l^2}{n^2\pi^2c^2}\left\{\left[\cos\tau\cdot\frac{e^{n^2\pi^2c^2\tau/l^2}}{\frac{n^2\pi^2c^2}{l^2}}\right]_0^t + \frac{l^2}{n^2\pi^2c^2}\int_0^t e^{n^2\pi^2c^2\tau/l^2}\cdot\sin\tau\, d\tau\right.$$

$$= \frac{l^2}{n^2\pi^2c^2}\sin t\cdot e^{n^2\pi^2c^2t/l^2} - \frac{l^4}{n^4\pi^4c^4}\left(\cos t\, e^{n^2\pi^2c^2t/l^2} - 1\right)$$
$$- \frac{l^4}{n^4\pi^4c^4} I$$

$$\Rightarrow \left(1+\frac{l^4}{n^4\pi^4c^4}\right) I = \frac{l^2}{n^2\pi^2c^2}\sin t\cdot e^{n^2\pi^2c^2t/l^2}$$
$$- \frac{l^4}{n^4\pi^4c^4}\left(\cos t\cdot e^{n^2\pi^2c^2t/l^2} - 1\right)$$

$$\Rightarrow I = \left(1+\frac{l^4}{n^4\pi^4c^4}\right)^{-1}\left\{\frac{l^2}{n^2\pi^2c^2}\sin t\, e^{n^2\pi^2c^2t/l^2} - \frac{l^4}{n^4\pi^4c^4}\left(\cos t\, e^{n^2\pi^2c^2t/l^2} - 1\right)\right\}$$

$$\therefore \int_0^t B_n(\tau) e^{n^2\pi^2c^2(\tau-t)/l^2}\, d\tau$$

$$= \frac{2l}{n\pi}(-1)^{n+1} e^{-n^2\pi^2c^2t/l^2}\left\{\left(1+\frac{l^4}{n^4\pi^4c^4}\right)^{-1}\left(\frac{l^2}{n^2\pi^2c^2}\sin t\, e^{n^2\pi^2c^2t/l^2} - \frac{l^4}{n^4\pi^4c^4}\left(\cos t\, e^{n^2\pi^2c^2t/l^2}-1\right)\right)\right\}$$

$$= \frac{2l}{n\pi}(-1)^{n+1}\left(1+\frac{l^4}{n^4\pi^4c^4}\right)^{-1}\left(\frac{l^2}{n^2\pi^2c^2}\sin t - \frac{l^4}{n^4\pi^4c^4}\cos t + \frac{l^4}{n^4\pi^4c^4}\right)$$

and $b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$

$$= \frac{2}{l}\int_0^l 1\cdot\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\left[\frac{-\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)}\right]_0^l$$

$$= -\frac{2}{l}\cdot\frac{l}{n\pi}\left(\cos(n\pi)-1\right)$$

$$= \frac{2}{n\pi}\left(1-(-1)^n\right)$$

Hence the solution of the problem is,

$$u(x,t) = \sum_{n=1}^{\infty}\frac{2l}{n\pi}(-1)^{n+1}\left(1+\frac{l^4}{n^4\pi^4c^4}\right)^{-1}\left(\frac{l^2\sin t}{n^2\pi^2c^2} - \frac{l^4\cos t}{n^4\pi^4c^4} + \frac{l^4}{n^4\pi^4c^4}\right)\sin\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty}\frac{2}{n\pi}\left(1+(-1)^n\right)\sin\left(\frac{n\pi x}{l}\right)e^{-n^2\pi^2c^2t/l^2}.$$

Heat equation in two space variables

The equation of the form

$$u_t = c^2(u_{xx} + u_{yy})$$

is called heat equation in two space variables, where x and y are independent variables.

We solve two dimensional heat equation by using separation of variable method.

Prob-1 Solve $u_t = K(u_{xx} + u_{yy})$; $0<x<a,\ 0<y<b,\ t>0$

$u(x,0,t) = u(x,b,t) = 0$; $0<x<a,\ t>0$

$u(0,y,t) = u(a,y,t) = 0$; $0<y<b,\ t>0$

and $u(x,y,0) = f(x,y)$

Sol:- Given, heat equation is,

$$\left.\begin{aligned} &u_t = K(u_{xx} + u_{yy}) \\ &\text{with } u(x,0,t) = u(x,b,t) = 0 \\ &\qquad u(0,y,t) = u(a,y,t) = 0 \\ &\text{and } u(x,y,0) = f(x,y) \end{aligned}\right\} \quad \text{---} (1)$$

we seek the solution of the form

$$u(x,y,t) = X(x)\cdot Y(y)\cdot T(t) \quad \text{---} (2)$$

And from eq(1), we get

$$XYT' = K(X''YT + XY''T)$$

$$\Rightarrow \frac{T'}{KT} = \frac{X''}{X} + \frac{Y''}{Y}$$

$$\Rightarrow \frac{T'}{KT} - \frac{Y''}{Y} = \frac{X''}{X} = -\lambda \text{ (say)}$$

$$\Rightarrow X'' + \lambda X = 0 \quad \text{and} \quad \frac{T'}{KT} + \lambda = \frac{Y''}{Y} = -\mu \text{ (say)}$$

$$\Rightarrow X'' + \lambda X = 0 \quad \text{and} \quad Y'' + \mu Y = 0 \quad \text{and} \quad T' + K(\lambda+\mu)T = 0 \quad \text{--- (3)}$$

From the boundary condition

$$X(0) = X(a) = 0 \quad \text{and} \quad Y(0) = Y(b) = 0$$

Now the problems on X and Y are

$$\left.\begin{array}{l} X'' + \lambda X = 0 \;;\; X(0) = X(a) = 0 \\ \text{and } Y'' + \mu Y = 0 \;;\; Y(0) = Y(b) = 0 \end{array}\right\} \text{--- (4)}$$

then solving,

$$X'' + \lambda X = 0 \;,\; X(0) = X(a) = 0$$

case-(i) For $\lambda = 0$

$$X'' = 0$$

$$\Rightarrow X = a_1 x + a_2$$

$$X(0) = a_2 = 0$$

$$X(a) = a_1 a = 0 \Rightarrow a_1 = 0$$

$\therefore X = 0$

hence the solution is rejected.

case-(ii) For $\lambda < 0$ let $\lambda = -m^2$

$$X'' - m^2 X = 0$$

$$\Rightarrow X'' = m^2 X$$

$$\therefore X = a_1 e^{mx} + a_2 e^{-mx}$$

$$X(0) = a_1 + a_2 = 0 \Rightarrow a_2 = -a_1$$

$$X(a) = a_1 e^{ma} - a_1 e^{-ma} = 0$$

$$\Rightarrow a_1 = 0$$

$$\Rightarrow a_2 = 0$$

$\therefore X = 0$

hence, the solution is rejected.

Case-(iii) when $\lambda > 0$, $\lambda = m^2$

$$X'' + m^2 X = 0$$

$$\Rightarrow X = a_1 \cos mx + a_2 \sin mx$$

$$X(0) = a_1 = 0$$

$$X(a) = a_2 \sin ma = 0$$

$$\Rightarrow \sin ma = 0$$

$$\Rightarrow ma = n\pi \quad , n \in \mathbb{N}$$

$$\Rightarrow m = \frac{n\pi}{a}$$

$\therefore$ Corresponding eigen values and eigen vectors are $\lambda_n = \frac{n^2\pi^2}{a^2}$; $X_n(x) = \sin\left(\frac{n\pi x}{a}\right)$

Similarly for $Y'' + \mu Y = 0$, $Y(0) = Y(b) = 0$, we get

$$\mu_m = \frac{m^2\pi^2}{b^2} \; ; \; Y_m(y) = \sin\left(\frac{m\pi y}{b}\right)$$

Here $m, n \in \mathbb{N}$ are independent.

Now from eq(3), we get

$$T'(t) + K\left(\frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2}\right) T(t) = 0$$

$$\Rightarrow T'(t) + K a_{nm} T(t) = 0$$

where $a_{nm} = \frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2}$

$$\Rightarrow T_{nm}(t) = e^{-a_{nm}Kt}$$

Hence, solution of (1) for each $n, m \in \mathbb{N}$ is,

$$u_{nm}(x,y,t) = \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) e^{-a_{nm}Kt}$$

$\therefore$ By principle of superposition, we get

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{nm}\, u_{nm}(x,y,t) \quad \text{---(5)}$$

Also $\because$ $u(x,y,0) = f(x,y)$

$$\Rightarrow f(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{nm} \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)$$

which is double Fourier sine series

Therefor the Fourier coefficient C_{nm} is given by,

$$C_{nm} = \frac{4}{ab}\int_0^a\int_0^b f(x,y)\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)dx\,dy \quad \text{---(6)}$$

Hence eq(5) is required solution of (1) where C_{nm} is given in (6).

1) Solve heat equation $u_t = K(u_{xx} + u_{yy})$ ---(1) where the ends are kept at zero temperature and initial temperature is,

$$u(x,y,0) = x(a-x)\,y\cos\left(\frac{\pi y}{2b}\right)$$

$(0<x<a,\ 0<y<b,\ t>0)$

Sol:- $f(x,y) = x(a-x)\,y\cos\left(\frac{\pi y}{2b}\right)$

Here $C_{nm} = \frac{4}{ab}\int_0^a\int_0^b x(a-x)\,y\cos\left(\frac{\pi y}{2b}\right)\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)dx\,dy$ ---(2)

$$= \frac{4}{ab}\int_0^a x(a-x)\sin\left(\frac{n\pi x}{a}\right)dx\int_0^b y\cos\left(\frac{\pi y}{2b}\right)\sin\left(\frac{m\pi y}{b}\right)dy$$

Now, $\int_0^a x(a-x)\sin\left(\frac{n\pi x}{a}\right)dx$

$$= \int_0^a (ax-x^2)\sin\left(\frac{n\pi x}{a}\right)dx$$

$$= \left[(ax-x^2)\frac{-\cos\left(\frac{n\pi x}{a}\right)}{\left(\frac{n\pi}{a}\right)}\right]_0^a + \frac{a}{n\pi}\int_0^a \cos\left(\frac{n\pi x}{a}\right)(a-2x)dx$$

$$= \frac{a}{n\pi}\left[(a-2x)\frac{\sin\left(\frac{n\pi x}{a}\right)}{\frac{n\pi}{a}}\right]_0^a - \frac{a^2}{n^2\pi^2}\int_0^a \sin\left(\frac{n\pi x}{a}\right)(-2)dx$$

$$= \frac{2a^2}{n^2\pi^2}\left[\frac{-\cos\left(\frac{n\pi x}{a}\right)}{\left(\frac{n\pi}{a}\right)}\right]_0^a$$

$$= \frac{-2a^3}{n^3\pi^3}\left((-1)^n-1\right) = \frac{2a^3\left(1-(-1)^n\right)}{n^3\pi^3}$$

Now, $\int_0^b y\cos\left(\frac{\pi y}{2b}\right)\sin\left(\frac{m\pi y}{b}\right)dy$

$$= \frac{1}{2}\int_0^b 2y\sin\left(\frac{m\pi y}{b}\right)\cos\left(\frac{\pi y}{2b}\right)dy$$

$$= \frac{1}{2}\int_0^b y\left\{\sin\left(\frac{2m\pi y+\pi y}{2b}\right)+\sin\left(\frac{2m\pi y-\pi y}{2b}\right)\right\}dy$$

$$= \frac{1}{2}\int_0^b y\sin\left(\left(\frac{2m+1}{2b}\right)\pi y\right)dy + \frac{1}{2}\int_0^b y\sin\left(\frac{(2m-1)\pi y}{2b}\right)dy$$

$$= \frac{1}{2}\left\{\left[y\cdot\frac{2b}{(2m+1)\pi}\left\{-\cos\left(\frac{(2m+1)\pi y}{2b}\right)\right\}\right]_0^b + \left(\frac{2b}{(2m+1)\pi}\right)^2\left[\sin\left(\frac{(2m+1)\pi y}{2b}\right)\right]_0^b\right\}$$

$$+\frac{1}{2}\left\{\left[y\cdot\frac{2b}{(2m-1)\pi}\left\{-\cos\left(\frac{(2m-1)\pi y}{2b}\right)\right\}\right]_0^b + \left(\frac{2b}{(2m-1)\pi}\right)^2\left[\sin\left(\frac{(2m-1)\pi y}{2b}\right)\right]_0^b\right\}$$

$$= \frac{1}{2}\left(0-0+\frac{4b^2}{(2m+1)^2\pi^2}(-1)^m\right) + \frac{1}{2}\left(\frac{4b^2}{(2m-1)^2\pi^2}(-1)^{m+1}\right)$$

$$= \frac{2b^2}{(2m+1)^2\pi^2}(-1)^m + \frac{2b^2}{(2m-1)^2\pi^2}(-1)^{m+1}$$

$$= \frac{2b^2(-1)^m}{\pi^2}\left(\frac{1}{(2m+1)^2} - \frac{1}{(2m-1)^2}\right)$$

$$= \frac{2b^2(-1)^m}{\pi^2} \cdot \frac{(2m-1)^2 - (2m+1)^2}{(4m^2-1)^2}$$

$$= \frac{2b^2(-1)^m}{\pi^2} \frac{(\cancel{4m^2} - 4m + \cancel{1} - \cancel{4m^2} - 4m - \cancel{1})}{(4m^2-1)^2}$$

$$= \frac{2b^2(-1)^m}{\pi^2} \frac{(-8m)}{(4m^2-1)^2}$$

$$= \frac{16b^2 m(-1)^{m+1}}{\pi^2(4m^2-1)^2}$$

$\therefore$ From (2),

$$C_{nm} = \frac{4}{ab} \frac{2a^3(1-(-1)^n)}{n^3\pi^3} \cdot \frac{16b^2 m(-1)^{m+1}}{\pi^2(4m^2-1)^2}$$

$$= \frac{128a^2bm(1-(-1)^n)(-1)^{m+1}}{n^3\pi^5(4m^2-1)^2}$$

$\therefore$ Solution of (1) is,

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{128a^2bm(1-(-1)^n)(-1)^{m+1}}{n^3\pi^5(4m^2-1)^2} \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right) \cdot e^{-a_{nm}kt}$$

where $a_{nm} = \dfrac{n^2\pi^2}{a^2} + \dfrac{m^2\pi^2}{b^2}$

2) Solve heat equation

$$u_t = K(u_{xx} + u_{yy}) \quad \text{———} (1)$$

where $a=\pi$, $b=\pi$, $K=1$, $f(x,y) = \sin(x)\, y(\pi-y)$

Sol:- The solution is given by,

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{nm} u_{nm}(x,y,t) \quad \text{———} (2)$$

where $u_{nm} = \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right) e^{-a_{nm}Kt}$

$$= \sin(nx)\sin(my)\, e^{-a_{nm}t},$$

where $a_{nm} = \frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2} = n^2 + m^2$

and $C_{nm} = \frac{4}{ab}\int_0^a\int_0^b f(x,y)\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)dx\,dy$

$$= \frac{4}{\pi^2}\int_0^{\pi}\int_0^{\pi} \sin(x)\, y(\pi-y)\sin(nx)\sin(my)\,dx\,dy$$

$$= \frac{4}{\pi^2}\int_0^{\pi}\sin(x)\sin(nx)\,dx\int_0^{\pi} y(\pi-y)\sin(my)\,dy$$

Now,

$$\int_0^{\pi}\sin(x)\sin(nx)\,dx = \frac{1}{2}\int_0^{\pi} 2\sin(nx)\sin(x)\,dx$$

$$= \frac{1}{2}\int_0^{\pi}\{\cos(nx-x) - \cos(nx+x)\}\,dx$$

$$= \frac{1}{2}\left[\frac{\sin(nx-x)}{n-1} - \frac{\sin(nx+x)}{n+1}\right]_0^{\pi}$$

$= 0$ for $n = 2, 3, \ldots$

For $n=1$,

$$\int_0^\pi \sin(x)\sin(nx)\,dx = \int_0^\pi \sin^2 x\,dx$$

$$= \frac{1}{2}\int_0^\pi (1-\cos 2x)\,dx$$

$$= \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right]_0^\pi$$

$$= \frac{1}{2}\cdot\pi = \frac{\pi}{2}$$

and, $\int_0^\pi y(\pi-y)\sin(my)\,dy$

$$= \int_0^\pi (y\pi - y^2)\sin(my)\,dy$$

$$= \left[(y\pi-y^2)\left\{-\frac{\cos(my)}{m}\right\}\right]_0^\pi + \frac{1}{m}\int_0^\pi \cos(my)(\pi-2y)\,dy$$

$$= \frac{1}{m}\left\{\left[(\pi-2y)\frac{\sin(my)}{m}\right]_0^\pi - \frac{1}{m}\int_0^\pi \sin(my)(-2)\,dy\right\}$$

$$= \frac{2}{m^2}\left[\frac{-\cos(my)}{m}\right]_0^\pi = \frac{-2}{m^3}\left((-1)^m-1\right) = \frac{2}{m^3}\left(1-(-1)^m\right)$$

$$\therefore\ C_{nm} = \begin{cases} \frac{4}{\pi^2}\cdot\frac{\pi}{2}\cdot\frac{2}{m^3}\left(1-(-1)^m\right) &, \text{if } n=1 \\ 0 &, \text{if } n=2,3,\ldots \ \cancel{\text{otherwise}} \end{cases}$$

$$= \begin{cases} \frac{4}{\pi m^3}\left(1-(-1)^m\right) &, \text{if } n=1 \\ 0 &, \cancel{\text{otherwise}} \text{ if } n=2,3,\ldots \end{cases}$$

From (2),

$$u(x,y,t) = \sum_{m=1}^{\infty} \frac{4}{\pi m^3}\left(1-(-1)^m\right)\cdot\sin(x)\sin(my)e^{-a_{1m}t}$$

where $a_{1m} = 1+m^2$

3) Solve heat equation,

$$u_t = K(u_{xx} + u_{yy}) \quad \text{——— (1)}$$

where $a = 1$, $b = 1$, $K = 1$, $f(x,y) = xy$

Sol:- The solution is given by,

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{nm} u_{nm}(x,y,t) \quad \text{——— (2)}$$

where $u_{nm} = \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right) e^{-a_{nm}Kt}$

$$= \sin(n\pi x)\sin(m\pi y)\, e^{-a_{nm}t}$$

where $a_{nm} = \frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2} = (n^2+m^2)\pi^2$

and $C_{nm} = \frac{4}{ab}\int_0^a\int_0^b f(x,y)\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)dx\,dy$

$$= 4\int_0^1\int_0^1 xy\sin(n\pi x)\sin(m\pi y)\,dx\,dy$$

$$= 4\int_0^1 x\sin(n\pi x)\,dx\int_0^1 y\sin(m\pi y)\,dy$$

$$\int_0^1 x\sin(n\pi x)\,dx = \left[x\left\{\frac{-\cos(n\pi x)}{n\pi}\right\}\right]_0^1 + \frac{1}{n^2\pi^2}\left[\sin(n\pi x)\right]_0^1$$

$$= -\frac{1}{n\pi}(-1)^n = \frac{(-1)^{n+1}}{n\pi}$$

Similarly,

$$\int_0^y y\sin(m\pi y)\,dy = \frac{(-1)^{m+1}}{m\pi}$$

$$\therefore\ C_{nm} = 4\cdot\frac{(-1)^{n+1}}{n\pi}\cdot\frac{(-1)^{m+1}}{m\pi}$$

$$= \frac{4}{nm\pi^2}(-1)^{n+m+2} = \frac{4(-1)^{n+m}}{nm\pi^2}$$

$\therefore$ From (2), $u(x,y,t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{4(-1)^{n+m}}{nm\pi^2} \sin(n\pi x)\sin(m\pi y) \, e^{-a_{nm}t}$,

where $a_{nm} = (n^2+m^2)\pi^2$.

Theorem – Derive the equation of conduction of heat in a rod.

OR

Derive one dimensional heat equation in cartesian coordinates.

OR

Derive one dimensional heat flow equation.

OR

Derive the PDE $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial u}{\partial t}$ for one-dimensional heat flow.

Proof:- * Formula

① Heat energy = $m\sigma t$

m = mass

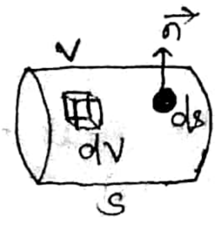
σ = specific heat

t = temperature

② Heat flux = $-K\vec{\nabla}u$, K = thermal conductivity.

First we obtain the three dimensional heat conduction equation. For that we consider the heat flow in a homogeneous isotropic solid body.

In the body we consider a volume V enclosed by a surface S.



$\therefore$ the heat energy inside the volume element dv is given by,

$\sigma \rho u \, dv$

where, σ = specific heat of material of the body.

ρ = density of the body

$u(x,y,z,t)$ = temperature of the body at (x,y,z) in time t

$\Rightarrow$ Total heat energy inside v is $= \iiint_V \sigma \rho u \, dv$

$\Rightarrow$ the rate of decrease of heat energy inside v is

$$= -\iiint_V \sigma \rho \frac{\partial u}{\partial t} dv \quad \text{——— (1)}$$

Now heat flux through the surface elements ds of s is

$$= -K \vec{\nabla} u$$

$\Rightarrow$ the heat flux along the outward normal vector $\vec{n}$ through ds is $= -K \vec{\nabla} u \cdot \vec{n} \, ds$

where, $\vec{n}$ is outward unit normal vector on ds and K is thermal conductivity.

$\Rightarrow$ Hence, total outward heat flux through S is

$$= -\iint_S K \vec{\nabla} u \cdot \vec{n} \, ds$$

By Gauss divergence theorem, we have.

total outward heat flux through $S = -\iiint_V \vec{\nabla} \cdot (K \vec{\nabla} u) dv$ ——— (2)

$\therefore$ By principle of conservation of energy

$$-\iiint_V \sigma \rho \frac{\partial u}{\partial t} dv = -\iiint_V K \nabla^2 u \, dv$$

$$\Rightarrow \sigma \rho \frac{\partial u}{\partial t} = K \nabla^2 u$$

$$\Rightarrow \frac{\partial u}{\partial t} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right); \quad c^2 = \frac{K}{\sigma \rho} \text{ is}$$

called thermal diffusibility.

Hence, the heat flow in one dimensional i.e. in a rod/bar along x-axis is given when u depends only on x and independent from y and z.

$\Rightarrow$ One dimensional heat equation is

$$\boxed{\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}}$$

Laplace's Equation

Definition:- An equation of the form

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{——} (1)$$

is called two dimensional Laplace equation in cartesian co-ordinates.
where u is dependent variable and x, y are independent variables.

Equation (1) can also be written as

$$\nabla^2 u(x,y) = 0$$

where ∇ is del or gradient operator, it is also called nabla operator

* In three dimensional Laplaces equation is given by

$$\nabla^2 u(x,y,z) = 0$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

Definition:- Harmonic function

* A function that satisfying Laplace equation in two or more dimensions is called a harmonic function.
* Sum of two harmonic functions are again harmonic function.
* constant multiple of Harmonic function is again harmonic function.

Example:- If $f(z) = z^4 = u + iv$ is analytic function in complex analysis then both u and v are harmonic functions.

Here $u = x^4 - 6x^2y^2 + y^4$

$v = 4x^3y - 4xy^3$

$$\frac{\partial u}{\partial x} = 4x^3 - 12xy^2 , \quad \frac{\partial u}{\partial y} = -12x^2y + 4y^3$$

$$\frac{\partial^2 u}{\partial x^2} = 12x^2 - 12y^2 \qquad \frac{\partial^2 u}{\partial y^2} = -12x^2 + 12y^2$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\frac{\partial v}{\partial x} = 12x^2y - 4y^3 , \quad \frac{\partial v}{\partial y} = 4x^3 - 12xy^2$$

$$\frac{\partial^2 v}{\partial x^2} = 24xy \qquad \frac{\partial^2 v}{\partial y^2} = -24xy$$

$$\therefore \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Hence, both u and v are harmonic function.

Q:- Solve Laplace equation $\nabla^2 u(x,y) = 0$ by using separation of variable method.

Sol:- We have Laplace equation

$$\nabla^2 u(x,y) = 0$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{------} (1)$$

We assume the solution of (1) of the form

$$u(x,y) = X(x) \cdot Y(y) \quad \text{------} (2)$$

then by putting $u(x,y)$ in (1), we get

$$X''Y + XY'' = 0$$

$$\Rightarrow \frac{X''}{X} + \frac{Y''}{Y} = 0$$

$$\Rightarrow \frac{X''}{X} = -\frac{Y''}{Y} = \lambda \text{ (say)}$$

$$\Rightarrow X'' = \lambda X \text{ and } Y'' = -\lambda Y \quad \text{------} (3)$$

Then there are three cases arise

case-1 If $\lambda = 0$, then eq(3) $\Rightarrow$

$$X'' = 0 \text{ and } Y'' = 0$$

$$\Rightarrow X = a_1 x + a_2 \text{ and } Y = b_1 y + b_2$$

$\therefore$ Solution of (1) is,

$$u(x,y) = (a_1 x + a_2)(b_1 y + b_2) \quad \text{——— (4)}$$

case-2:- If $\lambda = k^2 > 0$, then eq(3) becomes

$$X'' = k^2 X \text{ and } Y'' = -k^2 Y$$

$$\Rightarrow X = a_1 e^{kx} + a_2 e^{-kx} \text{ and } Y = b_1 \cos ky + b_2 \sin ky$$

$\therefore$ Solution of (1) is,

$$u(x,y) = (a_1 e^{kx} + a_2 e^{-kx})(b_1 \cos ky + b_2 \sin ky)$$

case-3:- Let $\lambda = -k^2 < 0$, then eq(3) becomes,

$$X'' = -k^2 X \text{ and } Y'' = k^2 Y$$

$$\Rightarrow X = (a_1 \cos kx + a_2 \sin kx) \text{ and } Y = b_1 e^{ky} + b_2 e^{-ky}$$

$\therefore$ Solution of (1) is,

$$u(x,y) = (a_1 \cos kx + a_2 \sin kx)(b_1 e^{ky} + b_2 e^{-ky}) \quad \text{——— (6)}$$

Hence, the solution of (1) is given in eq(4), (5), (6).

* Laplace equation in polar coordinate

We know that $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ ——— (1)

is Laplace equation in cartesian co-ordinate. then we want to transform it into polar coordinates $U(r,\theta)$ by using transformation

$$x = r\cos\theta \text{ and } y = r\sin\theta$$

then $u(x,y)$ becomes

$$U(r,\theta) = u(r\cos\theta, r\sin\theta)$$

$$\Rightarrow U_r = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$= \frac{\partial u}{\partial x}\cdot\cos\theta + \frac{\partial u}{\partial y}\cdot\sin\theta$$

$$\Rightarrow U_{rr} = \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial x}\cos\theta\right) + \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial y}\sin\theta\right)$$

$$= \left(\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)\cdot\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)\cdot\frac{\partial y}{\partial r}\right)\cos\theta$$

$$+ \left(\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)\cdot\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)\cdot\frac{\partial y}{\partial r}\right)\sin\theta$$

$$\Rightarrow U_{rr} = \frac{\partial^2 u}{\partial x^2}\cos^2\theta + \frac{\partial^2 u}{\partial x\partial y}\sin\theta\cdot\cos\theta + \frac{\partial^2 u}{\partial x\partial y}\cos\theta\cdot\sin\theta + \frac{\partial^2 u}{\partial y^2}\sin^2\theta$$

Now, $U_\theta = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \theta}$

$$= \frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y}(r\cos\theta)$$

$$\Rightarrow U_{\theta\theta} = -\frac{\partial u}{\partial x}\, r\cos\theta - \frac{\partial u}{\partial y}\, r\sin\theta$$

$$- r\sin\theta\left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)\cdot\frac{\partial x}{\partial \theta} + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)\cdot\frac{\partial y}{\partial \theta}\right]$$

$$+ r\cos\theta\left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)\cdot\frac{\partial x}{\partial \theta} + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)\cdot\frac{\partial y}{\partial \theta}\right]$$

$$= -rU_r - r\sin\theta\left[\frac{\partial^2 u}{\partial x^2}(-r\sin\theta) + \frac{\partial^2 u}{\partial x\partial y}(r\cos\theta)\right]$$

$$+ r\cos\theta\left[\frac{\partial^2 u}{\partial x\partial y}(-r\sin\theta) + \frac{\partial^2 u}{\partial y^2}(r\cos\theta)\right]$$

$$= -rU_r + r^2\left[\frac{\partial^2 u}{\partial x^2}\sin^2\theta - 2\frac{\partial^2 u}{\partial x\partial y}\sin\theta\cdot\cos\theta + \frac{\partial^2 u}{\partial y^2}\cos^2\theta\right]$$

$$\therefore U_{rr} + \frac{1}{r^2}U_{\theta\theta} = u_{xx} + u_{yy} - \frac{1}{r}U_r$$

$$\Rightarrow U_{rr} + \frac{1}{r^2}U_{\theta\theta} = \frac{-1}{r}U_r \quad [\text{Using } (1)]$$

$$\Rightarrow \boxed{U_{rr} + \frac{1}{r}U_r + \frac{1}{r^2}U_{\theta\theta} = 0}$$

which is Laplaces equation in polar coordinate.

Dirichlet Problem for a rectangle :-

Let D be a set of points in the plane, bounded by a curve C. A Dirichlet problem for D consists of finding a function that is harmonic on C. and solution of the boundary value problem

$$\nabla^2 u(x,y) = 0 \text{ for } (x,y) \in D \text{ with}$$

boundary condition $u(x,y) = g(x,y)$ for (x,y) on C

Q:-1 Let R be the rectangle consisting of all (x,y) with $0 \le x \le 3$, $0 \le y \le 7$. then solve the Dirichlet problem

$$\nabla^2 u(x,y) = 0 \text{ for } (x,y) \text{ in } R \quad \text{——(1)}$$

$$u(x,0) = 0 \text{ for } 0 \le x \le 3$$

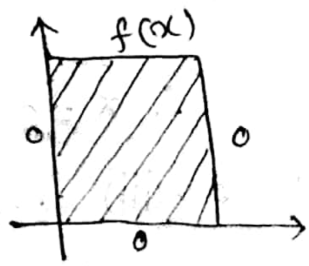
$$u(0,y) = u(3,y) = 0 \text{ for } 0 < y < 7$$

$$u(x,7) = f(x) = x\sinh(3-x) \text{ for } 0 < x < 3.$$

Sol:- We assume the solution of (1) is of the form

$$u(x,y) = X(x)\cdot Y(y) \quad \text{——— (2)}$$

$\therefore$ From eq(1), we get



$$X''Y + XY'' = 0$$

$$\Rightarrow \cancel{\frac{X''}{X} + \frac{Y''}{Y} = 0}$$

$$\Rightarrow \frac{X''}{X} = -\frac{Y''}{Y} = -\lambda \text{ (say)}$$

$\Rightarrow X'' = -\lambda X$ and $Y'' = +\lambda Y$ ——— (3)

and the boundary condition become,

$X(0) = X(3) = 0$ and $Y(0) = 0$

Hence, the ODE's are

$$\left.\begin{aligned} X'' &= -\lambda X \,;\ X(0) = X(3) = 0 \\ \text{and } Y'' &= +\lambda Y \,;\ Y(0) = 0 \end{aligned}\right\} \text{ ——— (4)}$$

~~the problem for X has eigen values~~

Now let's solve the problem for X.

case-(i) For $\lambda = 0$

$X'' = 0$

$\Rightarrow X = a_1 x + a_2$

$X(0) = 0 \Rightarrow a_2 = 0$

$X(3) = 0 \Rightarrow a_1 = 0$

$\therefore X = 0$, hence rejected

case-(ii) For $\lambda = -K^2 < 0$

$X'' = K^2 X$

$\Rightarrow X = a_1 e^{Kx} + a_2 e^{-Kx}$

$X(0) = 0 \Rightarrow a_1 + a_2 = 0 \Rightarrow a_2 = -a_1$

$X(3) = 0 \Rightarrow a_1 e^{3K} - a_1 e^{-3K} = 0$

$\Rightarrow a_1 = 0$

$\therefore a_2 = 0$

$\therefore X = 0$, hence rejected.

case-(iii) For $\lambda = K^2 > 0$

$X'' = -K^2 X$

$\Rightarrow X = a_1 \cos Kx + a_2 \sin Kx$

$X(0)=0 \Rightarrow a_1=0$

$X(3)=0 \Rightarrow a_2 \sin 3K = 0$

$\Rightarrow K = \frac{n\pi}{3}$; $n \in \mathbb{N}$

$\therefore$ The problem for X has eigen value and eigen functions are $\lambda_n = \frac{n^2\pi^2}{9}$ and $X_n(x) = \sin\left(\frac{n\pi x}{3}\right)$; $n \in \mathbb{N}$ (neglecting the constant)

$\therefore$ Problem for Y from eq(4) becomes

$Y''(y) - \frac{n^2\pi^2}{9} Y(y) = 0$; $Y(0)=0$

$\Rightarrow Y(y) = ae^{\frac{n\pi y}{3}} + be^{-n\pi y/3}$

$\because Y(0)=0 \Rightarrow a+b=0 \Rightarrow b=-a$

$\therefore Y(y) = ae^{n\pi y/3} - ae^{-n\pi y/3}$

$\Rightarrow Y_n(y) = \sinh\left(\frac{n\pi y}{3}\right)$; $n \in \mathbb{N}$

$\therefore$ From eq(2), the solution is,

$u_n(x,y) = c_n \sin\left(\frac{n\pi x}{3}\right) \cdot \sinh\left(\frac{n\pi y}{3}\right)$

which are harmonic on the rectangle that satisfy boundary condition on left, right and lower side.

Let $u(x,y) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{3}\right) \cdot \sinh\left(\frac{n\pi y}{3}\right)$

$\because u(x,7) = f(x) = x\sinh(3-x)$

$\Rightarrow x\sinh(3-x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{3}\right) \sinh\left(\frac{7n\pi}{3}\right)$

which is Fourier sine series.

Therefore Fourier co-efficient $C_n \sinh\left(\frac{7n\pi}{3}\right)$ is given by,

$$C_n \sinh\left(\frac{7n\pi}{3}\right) = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2}{3}\int_0^3 x\sinh(3-x)\sin\left(\frac{n\pi x}{3}\right)dx$$

$$= \frac{2}{3}\left\{\left[x\int \sinh(3-x)\sin\left(\frac{n\pi x}{3}\right)dx\right]_0^3 - \int_0^3\left(\int \sinh(3-x)\sin\left(\frac{n\pi x}{3}\right)dx\right)\cdot 1\, dx\right\}$$

Let $I = \int \sinh(3-x)\sin\left(\frac{n\pi x}{3}\right)dx$

$$\cancel{= -\sinh(3-x)\cos\left(\frac{n\pi x}{3}\right)}$$

$$= \left[-\sin\left(\frac{n\pi x}{3}\right)\cosh(3-x)\right]$$

$$+ \int \cosh(3-x)\cdot\cos\left(\frac{n\pi x}{3}\right)\cdot\frac{n\pi}{3}\,dx$$

$$= -\sin\left(\frac{n\pi x}{3}\right)\cosh(3-x)$$

$$+ \frac{n\pi}{3}\int \cosh(3-x)\cos\left(\frac{n\pi x}{3}\right)dx$$

$$= -\sin\left(\frac{n\pi x}{3}\right)\cosh(3-x)$$

$$+ \frac{n\pi}{3}\left\{\left[-\cos\left(\frac{n\pi x}{3}\right)\sinh(3-x)\right] - \int \sinh(3-x)\sin\left(\frac{n\pi x}{3}\right)\cdot\frac{n\pi}{3}dx\right\}$$

$$= -\sin\left(\frac{n\pi x}{3}\right)\cosh(3-x) - \frac{n\pi}{3}\cos\left(\frac{n\pi x}{3}\right)\sinh(3-x)$$

$$- \frac{n^2\pi^2}{9} I$$

$$\Rightarrow I\left(1+\frac{n^2\pi^2}{9}\right) = -\sin\left(\frac{n\pi x}{3}\right)\cosh(3-x)$$

$$- \frac{n\pi}{3}\cos\left(\frac{n\pi x}{3}\right)\sinh(3-x)$$

$$\Rightarrow I = \left(\frac{-9}{n^2\pi^2+9}\right)\left(\sin\left(\frac{n\pi x}{3}\right)\cosh(3-x) + \frac{n\pi}{3}\cos\left(\frac{n\pi x}{3}\right)\sinh(3-x)\right)$$

$$\therefore\ C_n \sinh\left(\frac{7n\pi}{3}\right) = \frac{2}{3}\left\{\left[x\left(\frac{-9}{n^2\pi^2+9}\right)\left(\sin\left(\frac{n\pi x}{3}\right)\cosh(3-x)\right.\right.\right.$$

$$\left.\left. + \frac{n\pi}{3}\cos\left(\frac{n\pi x}{3}\right)\sinh(3-x)\right)\right]_0^3$$

$$- \int_0^3 \left(\frac{-9}{n^2\pi^2+9}\right)\left(\sin\left(\frac{n\pi x}{3}\right)\cosh(3-x)\right.$$

$$\left. + \frac{n\pi}{3}\cos\left(\frac{n\pi x}{3}\right)\sinh(3-x)\right)dx$$

$$= \left(\frac{6}{n^2\pi^2+9}\right)\int_0^3 \sin\left(\frac{n\pi x}{3}\right)\cosh(3-x)\,dx$$

$$+ \frac{2n\pi}{n^2\pi^2+9}\int_0^3 \cos\left(\frac{n\pi x}{3}\right)\sinh(3-x)\,dx$$

Let $I_1 = \int_0^3 \sin\left(\frac{n\pi x}{3}\right)\cosh(3-x)\,dx$

$$= \left[-\sin\left(\frac{n\pi x}{3}\right)\sinh(3-x)\right]_0^3 + \int_0^3 \sinh(3-x)\cdot\frac{n\pi}{3}\cos\left(\frac{n\pi x}{3}\right)dx$$

$$= \frac{n\pi}{3}\int_0^3 \sinh(3-x)\cdot\cos\left(\frac{n\pi x}{3}\right)dx$$

$$\therefore\ C_n \sinh\left(\frac{7n\pi}{3}\right) = \frac{\cancel{2}n\pi}{n^2\pi^2+9}\int_0^3 \cos\left(\frac{n\pi x}{3}\right)\sinh(3-x)\,dx$$

Let $I_2 = \int_0^3 \cos\left(\frac{n\pi x}{3}\right)\cdot\sinh(3-x)\,dx$

$$= \left[-\cos\left(\frac{n\pi x}{3}\right)\cosh(3-x)\right]_0^3$$

$$- \int_0^3 \cosh(3-x)\cdot\sin\left(\frac{n\pi x}{3}\right)\cdot\frac{n\pi}{3}\,dx$$

$$= \left(-\cos(n\pi) + 1\cdot\cosh(3)\right)$$

$$-\frac{n\pi}{3}\left\{\left[\sin\left(\frac{n\pi x}{3}\right)\cdot\left(-\sinh(3-x)\right)\right]_0^3 + \int_0^3 \sinh(3-x)\cdot\frac{n\pi}{3}\cos\left(\frac{n\pi x}{3}\right)dx\right\}$$

$$\Rightarrow I_2\left(1+\frac{n^2\pi^2}{9}\right) = -\cos(n\pi) + \cosh(3)$$

$$\Rightarrow I_2 = \frac{9}{n^2\pi^2+9}\left(\cosh(3) - (-1)^n\right)$$

$$\therefore C_n \sinh\left(\frac{7n\pi}{3}\right) = \frac{36n\pi}{(n^2\pi^2+9)^2}\left(\cosh(3) - (-1)^n\right)$$

$$\Rightarrow C_n = \frac{36n\pi}{(n^2\pi^2+9)^2 \sinh\left(\frac{7n\pi}{3}\right)}\left(\cosh(3) - (-1)^n\right)$$

Hence, solution of (1) is,

$$u(x,y) = \sum_{n=1}^{\infty} \frac{36n\pi}{(n^2\pi^2+9)^2 \sinh\left(\frac{7n\pi}{3}\right)}\left(\cosh(3) - (-1)^n\right) \sin\left(\frac{n\pi x}{3}\right) \cdot \sinh\left(\frac{n\pi y}{3}\right)$$

2) Solve the Dirichlet problem for the indicated rectangle and boundary conditions.

$u(0,y) = u(1,y) = 0$ for $0<y<\pi$

$u(x,0) = \sin(\pi x)$

$u(x,\pi) = 0$ for $0<x<1$.

Sol:- Consider the Dirichlet problem,

$$\nabla^2 u(x,y) = 0 \quad \text{for } (x,y) \text{ in } R \quad \text{——(1)}$$

We assume the solution of (1) is of the form

$$u(x,y) = X(x)\cdot Y(y) \quad \text{——(2)}$$

Using (2) in (1), we get

$$X''Y + XY'' = 0$$

$$\Rightarrow \frac{X''}{X} = \frac{-Y''}{Y} = -\lambda \text{ (say)}$$

$$\Rightarrow X'' = -\lambda X \text{ and } Y'' = \lambda Y \quad \text{---} (3)$$

and the boundary conditions become

$$X(0) = X(1) = 0 \text{ and } Y(0) = \sin(\pi x)$$

Hence, the ODE's are

$$\left.\begin{aligned} &X'' = -\lambda X \ ; \ X(0) = X(1) = 0 \\ \text{and } &Y'' = \lambda Y \ ; \ Y(\pi) = \overset{0}{\cancel{\sin(\pi x)}} \end{aligned}\right\} \quad \text{---} (4)$$

Now let's solve the problem for X.

<u>case-(i)</u> <u>For $\lambda = 0$</u>

$$X'' = 0$$

$$\Rightarrow X = a_1 x + a_2$$

$$X(0) = X(1) = 0 \Rightarrow a_1 = a_2 = 0$$

$\therefore X = 0$, hence rejected.

<u>case-(ii)</u> <u>For $\lambda = -k^2 < 0$</u>

$$X'' = k^2 X$$

$$\Rightarrow X = a_1 e^{kx} + a_2 e^{-kx}$$

$$X(0) = X(1) = 0 \Rightarrow a_1 = a_2 = 0$$

$\therefore X = 0$, hence rejected.

<u>case-(iii)</u> <u>For $\lambda = k^2 > 0$</u>

$$X'' = -k^2 X$$

$$\Rightarrow X = a_1 \cos kx + a_2 \sin kx$$

$$X(0) = 0 \Rightarrow a_1 = 0$$

$$X(1) = 0 \Rightarrow a_2 \sin k = 0$$

$$\Rightarrow k = n\pi \ ; \ n \in \mathbb{N}$$

$\therefore$ The problem for X has eigen value and eigen functions are $\lambda_n = n^2\pi^2$ and $X_n(x) = \sin(n\pi x)$, $n \in \mathbb{N}$

$\therefore$ Problem for Y from eq(4) becomes,

$$Y'' - n^2\pi^2 Y = 0 \quad ; \; Y(\pi) = \overset{0}{\cancel{\sin(\pi x)}}$$

$$\Rightarrow Y = ae^{n\pi y} + be^{-n\pi y}$$

$$\because \cancel{Y(0) = \sin(\pi x) = a+b}$$

$$\cancel{\Rightarrow b = \sin(\pi x) - a}$$

$$\therefore \cancel{Y = ae^{n\pi y} + \{\sin(\pi x) - a\} e^{-n\pi y}}$$

$$\cancel{= ae^{n\pi y} - ae^{-n\pi y} + \sin(\pi x)\, e^{-n\pi y}}$$

$$Y(\pi) = 0$$

$$\Rightarrow ae^{n\pi^2} + be^{-n\pi^2} = 0$$

$$\Rightarrow be^{-n\pi^2} = -ae^{n\pi^2}$$

$$\Rightarrow b = -ae^{2n\pi^2}$$

$$\therefore Y(y) = ae^{n\pi y} - ae^{2n\pi^2} e^{-n\pi y}$$

$$= ae^{n\pi^2}\left(e^{-n\pi^2} \cdot e^{n\pi y} - e^{n\pi^2} \cdot e^{-n\pi y}\right)$$

$$= \frac{2ae^{n\pi^2}}{2}\left(e^{n\pi(y-\pi)} - e^{-n\pi(y-\pi)}\right)$$

$$= 2ae^{n\pi^2} \sinh\left(n\pi(y-\pi)\right)$$

$$\therefore Y_n(y) = e^{n\pi^2} \sinh\left(n\pi(y-\pi)\right)$$

$\therefore$ From (2), the solution of (1) is,

$$u_n(x,y) = c_n \sin(n\pi x) \cdot e^{n\pi^2} \sinh\left(n\pi(y-\pi)\right)$$

which are harmonic on the rectangle that satisfy boundary condition on left, right and upper side

Let $u(x,y) = \sum_{n=1}^{\infty} c_n \sin(n\pi x)\, e^{n\pi^2} \sinh(n\pi(y-\pi))$

$\therefore u(x,0) = \sin \pi x$

$$\Rightarrow \sin(\pi x) = \sum_{n=1}^{\infty} c_n \sin(n\pi x)\, e^{n\pi^2} \sinh(-n\pi^2)$$

$$= \sum_{n=1}^{\infty} -c_n \sin(n\pi x)\, e^{n\pi^2} \sinh(n\pi^2)$$

which is Fourier sine series

Therefore Fourier coefficient $-c_n \sinh(n\pi^2)\, e^{n\pi^2}$ is given by,

$$-c_n e^{n\pi^2} \sinh(n\pi^2) = \frac{2}{1}\int_0^1 f(x) \sin(n\pi x)\, dx$$

$$= 2\int_0^1 \sin(\pi x)\cdot \sin(n\pi x)\, dx$$

$$= \int_0^1 \{\cos(n\pi x - \pi x) - \cos(n\pi x + \pi x)\} dx$$

$$= \left[\frac{\sin(n\pi - \pi)x}{(n\pi - \pi)} - \frac{\sin(n\pi + \pi)x}{(n\pi + \pi)}\right]_0^1$$

$$= 0$$

For $n=1$

$$-c_1 e^{\pi^2} \sinh(\pi^2) = \int_0^1 2\sin^2(\pi x)\, dx$$

$$= \int_0^1 \{1 - \cos(2\pi x)\} dx$$

$$= x - \frac{\sin(2\pi x)}{2\pi}\Big|_0^1$$

$$= 1$$

$$\Rightarrow c_1 = \frac{-1}{e^{\pi^2}\sinh(\pi^2)} = -\operatorname{cosech}(\pi^2)\, e^{-\pi^2}$$

Hence, solution of (1) is,

$$u(x,y) = -\text{cosech}(\pi^2) e^{-\pi^2} \sinh(\pi(y-\pi)) \sin(\pi x) e^{\pi^2}$$

$$\therefore \boxed{u(x,y) = -\text{cosech}(\pi^2) \sinh(\pi(y-\pi)) \sin(\pi x)}$$

Dirichlet Problem for a disk:-

Let D be a disk of radius ρ about the origin. Using polar coordinates we will solve the Dirichlet problem.

$$\left.\begin{aligned} \nabla^2 u(r,\theta) &= u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \quad \text{for } 0 \le r < \rho \\ & \qquad\qquad -\pi \le \theta \le \pi \\ u(\rho,\theta) &= f(\theta) \quad \text{for } -\pi \le \theta \le \pi \end{aligned}\right\} \text{---(1)}$$

then the solution of (1) is,

$$u(r,\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left(a_n r^n \cos(n\theta) + b_n r^n \sin(n\theta)\right) \quad \text{---- (2)}$$

Since $u(\rho,\theta) = f(\theta)$

$$\Rightarrow f(\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left(a_n \rho^n \cos(n\theta) + b_n \rho^n \sin(n\theta)\right)$$

which is a fourier series of $f(\theta)$ on $[-\pi,\pi]$

$$\therefore \left.\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\, d\theta \\ a_n \rho^n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos(n\theta)\, d\theta \\ b_n \rho^n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin(n\theta)\, d\theta \end{aligned}\right\} \text{------ (3)}$$

Hence, eq (2) is required solution where eq (3) gives constant terms of (2)

Q:-3 Solve the Dirichlet problem

$$\nabla^2 u(x,y) = 0 \quad \text{for} \quad x^2+y^2<12$$

$$u(x,y) = x^2 - y \quad \text{for} \quad x^2+y^2 = 12$$

Sol:- Let $x = r\cos\theta$, $y = r\sin\theta$

and the problem is such that $\rho = 2\sqrt{3}$ about $(0,0)$.

$\therefore x^2 - y = \rho^2\cos^2\theta - \rho\sin\theta$

$$f(\theta) = 12\cos^2\theta - 2\sqrt{3}\sin\theta$$

Hence, the problem in polar form is,

$$\nabla^2 u(r,\theta) = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \text{ for } r^2<12, \; -\pi \le \theta \le \pi$$

$$u(2\sqrt{3},\theta) = f(\theta) = 12\cos^2\theta - 2\sqrt{3}\sin\theta \quad \text{——— (1)}$$

$\therefore$ Solution of (1) is given by,

$$u(r,\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n r^n\cos(n\theta) + b_n r^n\sin(n\theta)\right) \quad \text{——— (2)}$$

where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}(12\cos^2\theta - 2\sqrt{3}\sin\theta)\,d\theta$$

$$= \frac{12}{\pi}\int_{-\pi}^{\pi}\frac{1+\cos 2\theta}{2}\,d\theta - \frac{2\sqrt{3}}{\pi}\int_{-\pi}^{\pi}\sin\theta\,d\theta$$

$$= \frac{6}{\pi}\left[\theta + \frac{\sin 2\theta}{2}\right]_{-\pi}^{\pi} + \frac{2\sqrt{3}}{\pi}\left[\cos\theta\right]_{\pi}^{\pi}$$

$$= 12$$

$$a_n = \frac{1}{\pi(2\sqrt{3})^n}\int_{-\pi}^{\pi} f(\theta)\cos(n\theta)\,d\theta$$

$$= \frac{1}{\pi(2\sqrt{3})^n}\int_{-\pi}^{\pi}(12\cos^2\theta - 2\sqrt{3}\sin\theta)\cos(n\theta)\,d\theta$$

$$= \frac{12}{\pi(2\sqrt{3})^n}\int_{-\pi}^{\pi}\cos^2\theta\cdot\cos(n\theta)\,d\theta - \frac{\sqrt{3}}{\pi(2\sqrt{3})^n}\int_{-\pi}^{\pi}2\sin\theta\cdot\cos(n\theta)\,d\theta$$

$$= \frac{6}{\pi(2\sqrt{3})^n}\int_{-\pi}^{\pi}(1+\cos 2\theta)\cos(n\theta)\,d\theta - 0 \quad [\because 2\sin\theta\cos(n\theta) \text{ is odd fun}^n]$$

$$= \frac{6}{\pi(2\sqrt{3})^n}\left\{\int_{-\pi}^{\pi}\cos(n\theta)\,d\theta + \int_{-\pi}^{\pi}\cos 2\theta\cdot\cos(n\theta)\,d\theta\right\}$$

$$= \frac{6}{\pi(2\sqrt{3})^n}\left[\frac{\sin(n\theta)}{n}\right]_{-\pi}^{\pi} + \frac{3}{\pi(2\sqrt{3})^n}\int_{-\pi}^{\pi}\left(\cos(n\theta+2\theta) + \cos(n\theta-2\theta)\right)d\theta$$

$$= \frac{3}{\pi(2\sqrt{3})^n}\left[\frac{\sin(n\theta+2\theta)}{n+2} + \frac{\sin(n-2)\theta}{n-2}\right]_{-\pi}^{\pi}$$

$= 0$ for $n \neq 2$

<u>For $n=2$</u>

$$a_n = a_2 = \frac{1}{\pi(2\sqrt{3})^2}\int_{-\pi}^{\pi}(12\cos^2\theta - 2\sqrt{3}\sin\theta)\cos(2\theta)\,d\theta$$

$$= \frac{1}{12\pi}\int_{-\pi}^{\pi}12\cos^2\theta\cos 2\theta\,d\theta - \frac{1}{2\sqrt{3}\pi}\int_{-\pi}^{\pi}\sin\theta\cdot\cos 2\theta$$

$$= \frac{6}{12\pi}\int_{-\pi}^{\pi}(1+\cos 2\theta)\cos 2\theta\,d\theta$$

$$= \frac{1}{2\pi}\left[\frac{\sin 2\theta}{2}\right]_{-\pi}^{\pi} + \frac{3}{12\pi}\int_{-\pi}^{\pi}(1+\cos 4\theta)\,d\theta$$

$$= \frac{1}{4\pi}\left[\theta + \frac{\sin 4\theta}{4}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{4\pi}\cdot 2\pi = \frac{1}{2}$$

$$\therefore a_n = \begin{cases} 0, & \text{for } n \neq 2 \\ \frac{1}{2}, & \text{for } n = 2 \end{cases}$$

$$b_n = \frac{1}{\pi(2\sqrt{3})^n}\int_{-\pi}^{\pi} f(\theta)\sin(n\theta)\,d\theta$$

$$= \frac{1}{\pi(2\sqrt{3})^n}\int_{-\pi}^{\pi}(12\cos^2\theta - 2\sqrt{3}\sin\theta)\sin(n\theta)\,d\theta$$

$$= \frac{1}{\pi(2\sqrt{3})^n}\int_{-\pi}^{\pi} 12\cos^2\theta\cdot\sin(n\theta)\,d\theta - \frac{\sqrt{3}}{\pi(2\sqrt{3})^n}\int_{-\pi}^{\pi} 2\sin\theta\cdot\sin(n\theta)\,d\theta$$

$$= 0 - \frac{\sqrt{3}}{\pi(2\sqrt{3})^n}\int_{-\pi}^{\pi}[\cos(n\theta-\theta) - \cos(n\theta+\theta)]\,d\theta$$

$$= \frac{-\sqrt{3}}{\pi(2\sqrt{3})^n}\left[\frac{\sin(n-1)\theta}{n-1} - \frac{\sin(n+1)\theta}{n+1}\right]_{-\pi}^{\pi}$$

$$= 0 \quad \text{for } n \neq 1$$

<u>For $n=1$</u>

$$b_n = b_1 = \frac{1}{2\sqrt{3}\pi}\int_{-\pi}^{\pi}(12\cos^2\theta - 2\sqrt{3}\sin\theta)\sin\theta\,d\theta$$

$$= \frac{1}{2\sqrt{3}\pi}\int_{-\pi}^{\pi} 12\cos^2\theta\sin\theta\,d\theta - \frac{1}{\pi}\int_{-\pi}^{\pi}\sin^2\theta\,d\theta$$

$$= 0 - \frac{1}{2\pi}\int_{-\pi}^{\pi}(1-\cos 2\theta)\,d\theta$$

$$= \frac{-1}{2\pi}\left[\theta - \frac{\sin 2\theta}{2}\right]_{-\pi}^{\pi} = \frac{-1}{2\pi}\cdot 2\pi = -1$$

$$\therefore b_n = \begin{cases} 0, & \text{for } n \neq 1 \\ -1, & \text{for } n = 1. \end{cases}$$

$\therefore$ Solution ~~of (2)~~ is,

$$u(r,\theta) = 6 + \frac{1}{2} r^2 \cos 2\theta - r\sin\theta \quad \text{in polar form.}$$

$$\therefore \boxed{u(x,y) = 6 + \frac{1}{2}(x^2 - y^2) - y} \quad \text{in cartesian form.}$$

Poisson's Integral Solutions

We will write an integral solution of the Dirichlet problem for a disk about the origin, starting with a disk of radius 1.

Insert the integrals for the Fourier coefficient of $f(\theta)$ into equation

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n r^n \cos(n\theta) + b_n r^n \sin(n\theta)\right) \quad \text{---(1)}$$

with $\rho = 1$, then

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\xi)\,d\xi + \sum_{n=1}^{\infty} r^n \left(a_n \cos(n\theta) + b_n \sin(n\theta)\right)$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\xi)\,d\xi + \sum_{n=1}^{\infty} \frac{r^n}{\pi}\left[\int_{-\pi}^{\pi} f(\xi)\cos(n\xi)\,d\xi \,\cos(n\theta) + \int_{-\pi}^{\pi} f(\xi)\sin(n\xi)\,d\xi \,\sin(n\theta)\right]$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left[1 + 2\sum_{n=1}^{\infty} r^n \left(\cos(n\xi)\cos(n\theta) + \sin(n\xi)\sin(n\theta)\right)\right] f(\xi)\,d\xi$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[1+2\sum_{n=1}^{\infty} r^n \cos(n(\theta-\xi))\right] f(\xi)\, d\xi$$

then the quantity

$$\boxed{P(r,\eta) = \frac{1}{2\pi}\left[1+2\sum_{n=1}^{\infty} r^n \cos(n\eta)\right]} \quad \text{—— (2)}$$

is called the Poisson Kernel.

Hence, the solution is,

$$u(r,\theta) = \int_{-\pi}^{\pi} P(r,\theta-\xi) f(\xi)\, d\xi \quad \text{—— (3)}$$

Now since, $e^{i\theta} = \cos\theta + i\sin\theta$

and $z = re^{i\theta}$

$$\Rightarrow z^n = r^n e^{in\theta} = r^n\cos(n\theta) + i r^n \sin(n\theta)$$

$\therefore$ $r^n\cos(n\theta)$ is real part of z^n.

i.e. $r^n\cos(n\theta) = Re(z^n)$

$\therefore$ From eq (2)

$$1+2\sum_{n=1}^{\infty} r^n\cos(n\eta) = Re\left(1+2\sum_{n=1}^{\infty} z^n\right)$$

$$= Re\left(1+2\frac{z}{1-z}\right)$$

$$= Re\left(\frac{1+z}{1-z}\right)$$

$$= Re\left(\frac{1+re^{i\eta}}{1-re^{i\eta}}\right) \quad \text{—— (4)}$$

$$\frac{1+re^{i\eta}}{1-re^{i\eta}} = \left(\frac{1+re^{i\eta}}{1-re^{i\eta}}\right)\left(\frac{1-re^{-i\eta}}{1-re^{-i\eta}}\right)$$

$$= \frac{1-r^2+r(e^{i\eta}-e^{-i\eta})}{1+r^2-r(e^{i\eta}+e^{-i\eta})} = \frac{1-r^2+2ir\sin(\eta)}{1+r^2-2r\cos(\eta)}$$

∴ From eq(4), we get

$$1+2\sum_{n=1}^{\infty} r^n \cos(n\eta) = Re\left(\frac{1+re^{i\eta}}{1-re^{i\eta}}\right)$$

$$= \frac{1-r^2}{1+r^2-2r\cos(\eta)}$$

∴ From eq(2), we get

$$P(r,\eta) = \frac{1}{2\pi}\cdot\frac{1-r^2}{1+r^2-2r\cos(\eta)}$$

Hence eq(3) ⇒

$$\boxed{u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1-r^2}{1+r^2-2r\cos(\theta-\xi)}f(\xi)\,d\xi}$$

which is Poisson integral formula for the Dirichlet problem for the unit disk. Hence, the solution of the Dirichlet problem on the disk of radius ρ about origin is

$$\boxed{u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{(\rho^2-r^2)}{\rho^2+r^2-2\rho r\cos(\theta-\xi)}f(\xi)\,d\xi}$$

which is known as Poisson integral solution of the Dirichlet problem for a disk.

Neumann Problem :-

A problem in two dimensions of the type

$\nabla^2 u = 0$ on Ω (domain)

and $\frac{\partial u}{\partial n} = f(x,y)$ on $\partial\Omega$ (boundary)

is known as Neumann problem on the given domain Ω.

Neumann Problem for a rectangle

Let Ω be a set of points in the plane, bounded by a curve $\partial\Omega$. A Neumann problem for Ω consisting of finding a function which is harmonic on Ω and satisfy the boundary condition on $\partial\Omega$.

i.e.
$$\left.\begin{aligned} &\nabla^2 u(x,y) = 0 \quad \text{for } 0<x<L,\ 0<y<K \\ &\text{and } u_y(x,0) = u_y(x,K) = 0 \quad \text{for } 0<x<L \\ &u_x(0,y) = 0,\ u_x(L,y) = g(y) \quad \text{for } 0<y<K \end{aligned}\right\} (1)$$

Sol:- Let $u(x,y) = X(x)\cdot Y(y)$ be solution of (1)

then eq(1) reduces to

$$X''Y + XY'' = 0$$

$$\Rightarrow \frac{X''}{X} = \frac{-Y''}{Y} = \lambda \text{ (say)}$$

$$\Rightarrow X'' - \lambda X = 0 \text{ and } Y'' + \lambda Y = 0,$$ where λ is separation constant

and boundary conditions are

$u_y(x,0) = X(x)\,Y'(0) = 0 \Rightarrow Y'(0) = 0$

$u_y(x,K) = X(x)\cdot Y'(K) = 0 \Rightarrow Y'(K) = 0$

$u_x(0,y) = X'(0)\cdot Y(y) = 0 \Rightarrow X'(0) = 0$

$\therefore$ ODE's are

$$X'' - \lambda X = 0 \;, \quad X'(0) = 0$$

and $Y'' + \lambda Y = 0 \;, \quad Y'(0) = 0 \,, \; Y'(K) = 0$

So, the problem for Y has eigen values and eigen vectors are

$$\lambda_n = \frac{n^2\pi^2}{K^2} \;; \quad Y_n(y) = \cos\left(\frac{n\pi y}{K}\right) \;; \; n \in \mathbb{N} \cup \{0\}$$

Problem for X becomes,

$$X'' - \frac{n^2\pi^2}{K} X = 0 \;, \quad X'(0) = 0$$

If $n = 0$, then $X(x) = cx + d$

and $\because X'(0) = 0 \Rightarrow c = 0$

i.e. $X(x) = d$ (constant)

And if $n \in \mathbb{N}$, then

$$X(x) = c e^{n\pi x/K} + d e^{-n\pi x/K}$$

$\because X'(0) = 0 \Rightarrow c - d = 0 \Rightarrow c = d$

$$\therefore X_n(x) = c\left(e^{n\pi x/K} + e^{-n\pi x/K}\right)$$
$$= 2c \cosh\left(\frac{n\pi x}{K}\right)$$

$\therefore$ Solution is $u_0(x,y) = \text{constant}$ for $n = 0$

and $u_n(x,y) = a_n \cos\left(\frac{n\pi y}{K}\right) \cdot \cosh\left(\frac{n\pi x}{K}\right)$

$\Rightarrow$ By principle of superposition the solution of (1) can be written as,

$$u(x,y) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi y}{K}\right) \cdot \cosh\left(\frac{n\pi x}{K}\right) \quad \text{--- (2)}$$

$$\Rightarrow u_x(x,y) = \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi y}{K}\right)\cdot\left(\frac{n\pi}{K}\right)\sinh\left(\frac{n\pi x}{K}\right)$$

$\because u_x(L,y) = g(y)$

$$\therefore g(y) = \sum_{n=1}^{\infty} a_n\left(\frac{n\pi}{K}\right)\cos\left(\frac{n\pi y}{K}\right)\sinh\left(\frac{n\pi L}{K}\right)$$

which is Fourier cosine series.

$\therefore$ Fourier coefficient is given by

$$a_n \frac{n\pi}{K}\sinh\left(\frac{n\pi L}{K}\right) = \frac{2}{K}\int_0^K g(y)\cos\left(\frac{n\pi y}{K}\right)dy$$

$$\Rightarrow a_n = \frac{2}{n\pi \sinh\left(\frac{n\pi L}{K}\right)}\int_0^K g(y)\cos\left(\frac{n\pi y}{K}\right)dy \quad \text{---(3)}$$

Hence eq(2) is solution of (1), where constant is in (3).

Neumann Problem for a disk

We will solve the Neumann problem for a disk about origin

i.e. $\nabla^2 u(r,\theta) = 0$ for $0 \le r < \rho$, $-\pi \le \theta \le \pi$ ---(1)

$$\frac{\partial u}{\partial r}(\rho,\theta) = f(\theta) \text{ for } -\pi \le \theta \le \pi$$

As with the Dirichlet problem for a disk solution of (1) is,

$$u(r,\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n r^n \cos(n\theta) + b_n r^n \sin(n\theta)\right) \text{---(2)}$$

$$\Rightarrow u_r(r,\theta) = \sum_{n=1}^{\infty}\left(a_n n r^{n-1}\cos(n\theta) + b_n n r^{n-1}\sin(n\theta)\right)$$

$\because u_r(\rho,\theta) = f(\theta)$

$$\Rightarrow f(\theta) = \sum_{n=1}^{\infty}\left(a_n n \rho^{n-1}\cos(n\theta) + b_n n \rho^{n-1}\sin(n\theta)\right)$$

which is Fourier expansion of $f(\theta)$ on $[-\pi,\pi]$

$\therefore$ Fourier coefficients are

$$a_n n \rho^{n-1} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos(n\theta)\,d\theta$$

and $$b_n n \rho^{n-1} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin(n\theta)\,d\theta$$

$$\left.\begin{aligned} \Rightarrow a_n &= \frac{1}{n\pi\rho^{n-1}}\int_{-\pi}^{\pi} f(\theta)\cos(n\theta)\,d\theta \\ \text{and } b_n &= \frac{1}{n\pi\rho^{n-1}}\int_{-\pi}^{\pi} f(\theta)\sin(n\theta)\,d\theta \end{aligned}\right\} \text{——} (3)$$

$\therefore$ From eq (2), solution is

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{r^n}{n\pi\rho^{n-1}}\left[\int_{-\pi}^{\pi} f(\xi)\cos(n\xi)\,d\xi\,\cos(n\theta) + \int_{-\pi}^{\pi} f(\xi)\sin(n\xi)\,d\xi\,\sin(n\theta)\right]$$

$$= \frac{a_0}{2} + \frac{\rho}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{r}{\rho}\right)^n \int_{-\pi}^{\pi}\left[\cos(n\xi)\cos(n\theta) + \sin(n\xi)\sin(n\theta)\right] f(\xi)\,d\xi$$

$$\Rightarrow u(r,\theta) = \frac{a_0}{2} + \frac{\rho}{\pi}\int_{-\pi}^{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{r}{\rho}\right)^n \cos(n(\theta-\xi))\, f(\xi)\,d\xi \quad \text{——} (4)$$

Let $z = Re^{i\eta} \Rightarrow z^n = R^n(\cos(n\eta) + i\sin(n\eta))$

$$\Rightarrow \sum_{n=1}^{\infty} R^n\cos(n\eta) = \sum_{n=1}^{\infty} \mathrm{Re}\,(z^n)$$

$$= \mathrm{Re}\left[\sum_{n=1}^{\infty} z^n\right] \quad \text{——} (5)$$

Now, since

$$\sum_{n=1}^{\infty} z^n = \frac{z}{1-z}$$

$$\Rightarrow \frac{1}{2}+\sum_{n=1}^{\infty} z^n = \frac{1}{2}+\frac{z}{1-z} = \frac{1+z}{2(1-z)}$$

Multiplying numerator and denominator on RHS side by $(1-\bar{z})$, we get

$$\frac{1}{2}+\sum_{n=1}^{\infty} z^n = \frac{1}{2}\,\frac{(1+z)(1-\bar{z})}{\cancel{2}(1-z)(1-\bar{z})}$$

$$= \frac{1}{2}\,\frac{(1-\bar{z}+z-z\bar{z})}{(1-z-\bar{z}+z\bar{z})}$$

$$\Rightarrow \frac{1}{2}+\sum_{n=1}^{\infty} z^n = \frac{1}{2}\left(\frac{1+(-2iR\sin(\eta))-R^2}{1-(2R\cos(\eta))+R^2}\right)$$

$$\Rightarrow Re\left(\frac{1}{2}+\sum_{n=1}^{\infty} z^n\right) = \frac{1}{2}\times\frac{(1-R^2)}{1+R^2-2R\cos(\eta)}$$

$$\Rightarrow Re\left(\sum_{n=1}^{\infty} z^n\right) = \frac{(1-R^2)}{2(1+R^2-2R\cos(\eta))}-\frac{1}{2}$$

$$\Rightarrow Re\left(\sum_{n=1}^{\infty} z^n\right) = \frac{1-R^2-1-R^2+2R\cos(\eta)}{2(1+R^2-2R\cos(\eta))}$$

$$\Rightarrow Re\left(\sum_{n=1}^{\infty} z^n\right) = \frac{R\cos(\eta)-R^2}{1+R^2-2R\cos(\eta)} \longleftarrow (6)$$

Using eq(6) in (5), we get,

$$\sum_{n=1}^{\infty} R^n\cos(n\eta) = \frac{R\cos(\eta)-R^2}{1+R^2-2R\cos(\eta)}$$

$$\Rightarrow \cancel{\int_0^R} \sum_{n=1}^{\infty} R^{n-1}\cos(n\eta) = \frac{\cos(\eta)-R}{1+R^2-2R\cos(\eta)}$$

$$\Rightarrow \int_0^R \sum_{n=1}^{\infty} R^{n-1}\cos(n\eta)\,dR = \int_0^R \frac{\cos(\eta)-R}{1+R^2-2R\cos(\eta)}\,dR$$

$$\Rightarrow \sum_{n=1}^{\infty}\int_0^R R^{n-1}\cos(n\eta)\,dR = \int_0^R \frac{\cos(\eta)-R}{1+R^2-2R\cos(\eta)}\,dR$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{R^n}{n} \cos(n\eta) = -\frac{1}{2} \log\left(1+R^2-2R\cos(\eta)\right)$$

Put $R = \frac{r}{\rho}$ and $\eta = \theta - \xi$, then

$$\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{r}{\rho}\right)^n \cos(n(\theta-\xi)) = -\frac{1}{2} \log\left(1+\frac{r^2}{\rho^2}-2\frac{r}{\rho}\cos(\theta-\xi)\right)$$

Hence, from eq(4), solution is,

$$\boxed{u(r,\theta) = \frac{1}{2}a_0 - \frac{\rho}{2\pi}\int_{-\pi}^{\pi} \log\left(1+\frac{r^2}{\rho^2}-2\frac{r}{\rho}\cos(\theta-\xi)\right) f(\xi)\, d\xi}$$

in which a_0 is arbitary constant.

Q.1 Solve

$$\left.\begin{aligned} &\nabla^2 u = 0 \text{ for } 0<x<1,\ 0<y<1 \\ &u_x(0,y) = u_x(1,y) = 0 \text{ for } 0<y<1 \\ &u_y(x,0) = 4\cos(\pi x),\ u_y(x,1) = 0 \text{ for } 0<x<1 \end{aligned}\right\} \text{--(1)}$$

Sol:- Let $u(x,y) = X(x)\cdot Y(y)$ be solution of (1), then eq(1), reduces to

$$X''Y + XY'' = 0$$

$$\Rightarrow \frac{X''}{X} = -\frac{Y''}{Y} = -\lambda \text{ (say)}$$

$$\Rightarrow X'' + \lambda X = 0 \quad \text{and} \quad Y'' - \lambda Y = 0$$

and the boundary conditions are,

$$u_x(0,y) = X'(0)\cdot Y(y) = 0 \Rightarrow X'(0) = 0$$

$$u_x(1,y) = X'(1)\cdot Y(y) = 0 \Rightarrow X'(1) = 0$$

$$u_y(x,1) = X(x)\cdot Y'(1) = 0 \Rightarrow Y'(1) = 0$$

$\therefore$ ODE's are

$$X'' + \lambda X = 0,\quad X'(0) = X'(1) = 0$$

and $Y'' - \lambda Y = 0$, $Y'(1) = 0$

Solution for problem of X.

case-I If $\lambda = 0$

$\Rightarrow X = a_1x + a_2$

$\Rightarrow X' = a_1$

$X'(0) = 0 \Rightarrow a_1 = 0$

$X'(1) = 0 \Rightarrow a_1 = 0$

$\therefore X = a_2$ is a solution.

case-II If $\lambda = -\mu^2 < 0$, then

$X'' - \mu^2 X = 0$

$\therefore X = a_1 e^{\mu x} + a_2 e^{-\mu x}$

$\Rightarrow X' = a_1 \mu e^{\mu x} - a_2 \mu e^{-\mu x}$

$X'(0) = 0$

$\Rightarrow a_1\mu - a_2\mu = 0 \Rightarrow a_1 = a_2$

$X'(1) = 0$

$\Rightarrow a_1 \mu e^{\mu} - a_2 \mu e^{-\mu} = 0$

$\Rightarrow a_1 \mu (e^{\mu} - e^{-\mu}) = 0$

$\Rightarrow a_1 = 0$

$\therefore a_2 = 0$

$\therefore X = 0$, hence rejected.

case-III If $\lambda = \mu^2 > 0$, then

$X'' + \mu^2 X = 0$

$\Rightarrow X = a_1 \cos \mu x + a_2 \sin \mu x$

$\Rightarrow X' = -a_1 \mu \sin \mu x + a_2 \mu \cos \mu x$

$X'(0) = 0 \Rightarrow a_2 = 0$

$X'(1) = 0 \Rightarrow -a_1 \mu \sin \mu = 0$

$\Rightarrow \mu = n\pi$

$\therefore X = a_1 \cos(n\pi x)$ and $\lambda_n = n^2\pi^2$

So the problem for X has eigen values and eigen vectors are,

$\lambda_n = n^2\pi^2$; $X_n(x) = \cos(n\pi x)$, $n \in \mathbb{N} \cup \{0\}$

$\therefore$ Problem for Y becomes,

$$Y'' - n^2\pi^2 Y = 0 \quad , Y'(1) = 0$$

If $n = 0$, then $Y(y) = cy + d$

$$\Rightarrow Y' = c$$

$$\because Y'(1) = 0 \Rightarrow c = 0$$

i.e. $Y(y) = d$ (constant)

and if $n \in \mathbb{N}$, then $Y = ce^{n\pi y} + de^{-n\pi y}$

$$Y' = cn\pi e^{n\pi y} - dn\pi e^{-n\pi y}$$

$$Y'(1) = cn\pi e^{n\pi} - dn\pi e^{-n\pi} = 0$$

$$\Rightarrow n\pi(ce^{n\pi} - de^{-n\pi}) = 0$$

$$\Rightarrow ce^{n\pi} - de^{-n\pi} = 0$$

$$\Rightarrow d = ce^{2n\pi}$$

$$\therefore Y = ce^{n\pi y} + ce^{2n\pi - n\pi y}$$

$$= ce^{n\pi}(e^{-n\pi} \cdot e^{n\pi y} + e^{n\pi} \cdot e^{-n\pi y})$$

$$= ce^{n\pi}\left(e^{n\pi(y-1)} + e^{-n\pi(y-1)}\right)$$

$$= 2ce^{n\pi}\cosh(n\pi(y-1))$$

$$\therefore Y_n(y) = e^{n\pi}\cosh(n\pi(y-1))$$

$$u_0(x,y) = \text{constant}$$

$$u_n(x,y) = a_n\cos(n\pi x) \cdot e^{n\pi}\cosh(n\pi(y-1))$$

By principle of superposition, the solution of (1) can be written as,

$$u(x,y) = a_0 + \sum_{n=1}^{\infty} a_n\cos(n\pi x) \cdot e^{n\pi}\cosh(n\pi(y-1))$$

$$\Rightarrow u_y(x,y) = \sum_{n=1}^{\infty} a_n\cos(n\pi x)\, e^{n\pi}\sinh(n\pi(y-1)) \cdot n\pi$$

$\therefore u_y(x,0) = 4\cos(\pi x)$

$$\Rightarrow 4\cos(\pi x) = \sum_{n=1}^{\infty} a_n \cos(n\pi x)\, e^{n\pi} n\pi \sinh(-n\pi)$$

comparing, we get

$n=1$ and $\quad -a_n e^{n\pi}\cdot n\pi \sinh(n\pi) = 4$

$$\Rightarrow a_n = \frac{-4}{n\pi e^{n\pi}\sinh(n\pi)}, \text{ for } n=1$$

$$\Rightarrow a_1 = \frac{-4}{\pi e^{\pi}\sinh(\pi)}$$

$$\therefore u(x,y) = a_0 - \frac{4}{\pi e^{\pi}\sinh(\pi)} \cos(\pi x)\, e^{\pi} \cosh(\pi(y-1))$$

$$= a_0 - \frac{4}{\pi \sinh(\pi)} \cos(\pi x)\cosh(\pi(y-1))$$

Q:-2 Solve $\nabla^2 u(r,\theta) = 0$ for $0 \le r < \rho$, $-\pi \le \theta \le \pi$

$\frac{\partial u}{\partial r}(\rho,\theta) = \sin(3\theta)$ for $-\pi \le \theta \le \pi$

Sol:- ~~Since~~ solution of Laplace equation is

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n r^n \cos(n\theta) + b_n r^n \sin(n\theta)\right) \quad \text{——— (1)}$$

where, $a_n = \frac{1}{n\pi\rho^n}\int_{-\pi}^{\pi} f(\theta)\cos(n\theta)\,d\theta$

$$= \frac{1}{n\pi\rho^n}\int_{-\pi}^{\pi}\sin(3\theta)\cos(n\theta)\,d\theta$$

$$= \frac{1}{2n\pi\rho^n}\int_{-\pi}^{\pi}\left(\sin((n+3)\theta) - \sin((n-3)\theta)\right)d\theta$$

$$= \frac{1}{2n\pi\rho^n}\cdot 0 = 0 \quad \text{if } n \ne 3$$

if $n=3$,

$$a_3 = \frac{1}{3\pi\rho^3}\int_{-\pi}^{\pi}\sin 3\theta\cdot\cos 3\theta\, d\theta = 0$$

$$b_n = \frac{1}{n\pi\rho^n}\int_{-\pi}^{\pi}\sin(3\theta)\sin(n\theta)\,d\theta$$

$$= \frac{1}{2n\pi\rho^n}\int_{-\pi}^{\pi}\left(\cos((n-3)\theta) - \cos((n+3)\theta)\right)d\theta$$

$$= \frac{1}{2n\pi\rho^n}\left[\frac{\sin(n-3)\theta}{n-3} - \frac{\sin(n+3)\theta}{n+3}\right]_{-\pi}^{\pi}$$

$$= 0 \quad , \text{ for } n \neq 3$$

if $n=3$,

$$b_3 = \frac{1}{3\pi\rho^3}\int_{-\pi}^{\pi}\sin^2 3\theta\,d\theta$$

$$= \frac{1}{3\pi\rho^3}\cdot\frac{1}{2}\int_{-\pi}^{\pi}[1-\cos(6\theta)]\,d\theta$$

$$= \frac{1}{6\pi\rho^3}\left[\theta - \frac{\sin(6\theta)}{6}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{6\pi\rho^3}\cdot 2\pi = \frac{1}{3\rho^3}$$

$\therefore$ Solution is

$$u(r,\theta) = \frac{a_0}{2} + \frac{r^3}{3\rho^2}\sin(3\theta).$$

Q:-3 By converting the problem to polar coordinat[es]

solve, $\nabla^2 u(x,y) = 0$ for $x^2+y^2 < 9$

$$\frac{\partial u}{\partial n} = 4xy \quad \text{if} \quad x^2+y^2 = 9$$

Sol:- $\frac{\partial u}{\partial n} = 4xy$ if $x^2+y^2 = 9 = 3^2$

$$\Rightarrow \frac{\partial u}{\partial r} = 4r\cos\theta \cdot r\sin\theta = 18\sin 2\theta, \text{ for } -\pi \le \theta \le$$

Solution of Laplace equation is.

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n r^n\cos(n\theta) + b_n r^n\sin(n\theta)\right)$$

where $a_n = \frac{1}{n\pi \rho^n} \int_{-\pi}^{\pi} f(\theta) \cos(n\theta)\, d\theta$

$$= \frac{1}{n\pi 3^n} \int_{-\pi}^{\pi} 18 \sin(2\theta) \cos(n\theta)\, d\theta$$

$$= \frac{9}{n\pi 3^n} \int_{-\pi}^{\pi} \left(\sin((2+n)\theta) + \sin((2-n)\theta)\right) d\theta$$

$$= \frac{9}{n\pi 3^n} \cdot 0 = 0 \quad \text{for } n \neq 2$$

$$a_2 = \frac{1}{2\pi 3^2} \int_{-\pi}^{\pi} 18 \sin(2\theta) \cos(2\theta)\, d\theta$$

$$= 0$$

$$b_n = \frac{1}{n\pi 3^n} \int_{-\pi}^{\pi} 18 \sin(2\theta) \sin(n\theta)\, d\theta$$

$$= \frac{9}{n\pi 3^n} \int_{-\pi}^{\pi} \left[\cos((n-2)\theta) - \cos((n+2)\theta)\right] d\theta$$

$$= \frac{9}{n\pi 3^n} \cdot 0 = 0 \quad \text{for } n \neq 2$$

$$b_2 = \frac{1}{2\pi \cdot 3^2} \int_{-\pi}^{\pi} 18 \sin^2 2\theta \, d\theta$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} (1 - \cos 4\theta)\, d\theta$$

$$= \frac{1}{2\pi} \cdot 2\pi = 1$$

Solution is,

$$\therefore u(r,\theta) = \frac{a_0}{2} + r^2 \sin(2\theta).$$

Solution by Eigen function expansion

Let us consider a differential equation

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y = f(x) \quad \text{——(1)}$$

OR, $(py')' + q(x)y = f(x)$

$\Rightarrow Ly = f(x)$,

where $L = (py')' + q$ ——(2)

this is called Sturm-Liouville operator.

then we want to solve

$$Ly = \lambda y \quad \text{——(3)}$$

where λ is a parameter

$\Rightarrow (py')' + q\cdot y - \lambda y = 0 ; \; x \in (a, b)$ ——(4)

where $p(x)$, $p'(x)$, $q(x)$ are continuous functions of x on $[a, b]$.

We are also given

$$\left.\begin{aligned} a_1 y(a) + a_2 y'(a) = 0 \\ b_1 y(b) + b_2 y'(b) = 0 \end{aligned}\right\} \text{——(5)}$$

thus we want to find out those value of λ for which non-trivial solution is obtained, this problem is called Sturm-Liouville problem.

those values of λ for which eq(4) has non-trivial solution are called eigen values and corresponding solutions are called eigen functions.

There are two types of Sturm-Liouville problem on $[a, b]$.

① Regular Sturm-Liouville problem

A differential equation

$$(py')' + q\cdot y - \lambda y = 0 \quad \text{on } [a,b] \qquad \text{—— (1)}$$

with $p(a)$ and $p(b)$ both non-zero finite and the boundary conditions (where a, b both are finite)

$$a_1 y(a) + a_2 y'(a) = 0$$
$$b_1 y(b) + b_2 y'(b) = 0$$

is regular Sturm-Liouville problem.

② Singular Sturm-Liouville problem

A problem (1) which is not regular is called singular Sturm-Liouville problem.

i.e. $p(a)=0$ and $p(b)=0$

Boundary condition:-

① Dirichlet boundary conditions:-

$$y(a)=0 \text{ and } y(b)=0$$

② Neumann boundary conditions:-

$$y'(a)=0 \text{ and } y'(b)=0$$

③ Mixed boundary conditions:-

$$a_1 y(a) + a_2 y'(a) = 0$$
$$b_1 y(b) + b_2 y'(b) = 0$$

④ Periodic boundary conditions:-

$$y(a) = y(b), \quad y'(a) = y'(b).$$

Q:- Let us consider a differential equation

$$y'' + \lambda y = 0 \;;\; x \in (0, l) \,,\; l > 0 \quad \text{--- (1)}$$

and dirichlet boundary condition

$$y(0) = 0 = y(l)$$

then find the values of λ for which we get non-trivial solution of (1).

Sol:- We have $y'' + \lambda y = 0$

then we shall consider 3 cases for different values of λ.

case-(i) $\lambda = 0$, then from (1)

$$y'' = 0 \Rightarrow y = ax + b,$$

where a, b are constants.

$\because y(0) = 0 \Rightarrow b = 0$

and $y(l) = 0 \Rightarrow a = 0$

i.e. $y = 0$, which is trivial solution.

so we reject $\lambda = 0$.

$\Rightarrow \lambda = 0$ is not an eigen value of (1).

case-(ii) If $\lambda < 0$ i.e. let $\lambda = -k^2$, $k \neq 0$

then from (1)

$$y'' - k^2 y = 0 \Rightarrow y = ae^{kx} + be^{-kx}$$

$$\left.\begin{aligned} &\because y(0) = 0 \Rightarrow a + b = 0 \\ &\text{and } y(l) = 0 \Rightarrow ae^{kl} + be^{-kl} = 0 \end{aligned}\right\} \Rightarrow a = b = 0$$

$\Rightarrow y = 0$, which is trivial solution.

$\therefore \lambda < 0$ is not an eigen value of (1)

Case-(iii) If $\lambda > 0$ i.e. let $\lambda = K^2$, $K \neq 0$

then eq(1) $\Rightarrow y'' + K^2 y = 0$

$\Rightarrow y = a\cos Kx + b\sin Kx$

where a, b are constants.

$\because y(0) = 0 \Rightarrow a = 0$

and $y(l) = 0 \Rightarrow b\sin(Kl) = 0 \Rightarrow Kl = n\pi$

$\Rightarrow K = \frac{n\pi}{l}$; $n \in \mathbb{N}$

$\Rightarrow \lambda = \frac{n^2\pi^2}{l^2}$; $n \in \mathbb{N}$

and $y_n = B_n \sin\left(\frac{n\pi x}{l}\right)$; $n \in \mathbb{N}$

$\Rightarrow \lambda = \frac{n^2\pi^2}{l^2}$ is eigen values and $y_n(x) = B_n \sin\left(\frac{n\pi x}{l}\right)$ are eigen functions of (1).

Note:- ① If $\lambda_1, \lambda_2, \lambda_3, \ldots$ are eigen values of Sturm-Liouville boundary value problem then we have

$\lambda_1 < \lambda_2 < \lambda_3 < \ldots$

i.e. eigen values of Sturm-Liouville BVP are ordered.

② Eigen value of Sturm-Liouville BVP are countable and real.

Q-2 $y'' + \lambda y = 0$; $x \in (0, l)$, $l > 0$ and finite

$y(0) = 0$, $y'(l) = 0$ (mixed boundary condition)

Sol:- $\lambda_n = \frac{(2n+1)^2 \pi^2}{4l^2}$; $n = 0, 1, 2, \ldots = \mathbb{N} \cup \{0\}$

$y_n(x) = B_n \sin\left(\frac{(2n+1)\pi x}{2l}\right)$; $n \in \mathbb{N} \cup \{0\}$

3) $y'' + \lambda y = 0$; $x \in (0, l)$, $l > 0$

$y'(0) = 0$, $y'(l) = 0$ (Neumann boundary condition)

Sol:- $\lambda_n = \frac{n^2\pi^2}{l^2}$; $n \in \mathbb{N} \cup \{0\}$

$y_n(x) = A_n \cos\left(\frac{n\pi x}{l}\right)$, $n \in \mathbb{N} \cup \{0\}$

(4) $y'' + \lambda y = 0$; $x \in (0, l)$, $l > 0$

$y(0) = y(l)$ and $y'(0) = y'(l)$ (Periodic Boundary condition)

Sol:- $\lambda_n = \frac{4n^2\pi^2}{l^2}$; $n \in \mathbb{N} \cup \{0\}$

$y_n(x) = A_n \cos\left(\frac{2n\pi x}{l}\right) + B_n \sin\left(\frac{2n\pi x}{l}\right)$

So, in this case, for one eigen value we get two eigen function which are L.I and orthogonal

Q:- $y'' + \lambda y = 0$, $x \in (0, 2\pi)$

$y(0) = 0$, $y(2\pi) = 0$ then the BVP has

(i) Non trivial solⁿ for any value of λ.

(ii) Trivial solⁿ for any value of λ.

(iii) Non trivial solⁿ for countable values of λ.

(iv) None of these.

Q:- $x^2 y'' + x y' + \lambda y = 0$; $x \in (0, e^{\pi})$

$y(1) = 0$, $y(e^{\pi}) = 0$

then find eigen value and eigen function.

Q:- $y'' + 2y' + \lambda y = 0$; $y(0) = 0$, $y(\pi) = 0$

then which of the following is/are correct

(i) $\lambda = 1$ is not an eigen value.

(ii) $\lambda = 0$ is an eigen value.

(iii) there is no eigen value λ such that $\lambda < 1$

(iv) nth positive eigen value is $\lambda_n = n^2\pi^2 + 1$ with corresponding eigen function $y_n(x) = e^{-x} \sin(n\pi x)$.

Sol:- $y'' + 2y' + \lambda y = 0$ ——— (1)

$\therefore$ A.E is $m^2 + 2m + \lambda = 0$

$$\Rightarrow m = \frac{-1 \pm \sqrt{4-4\lambda}}{2} = -1 \pm \sqrt{1-\lambda}$$

$\therefore$ Sol of (1) exists if $1-\lambda < 0 \Rightarrow \lambda > 1$

and solⁿ is

$$y = e^{-x}\left[A\cos\sqrt{\lambda-1}\,x + B\sin\sqrt{\lambda-1}\,x\right]$$

$\because y(0) = 0 \Rightarrow A = 0$

and $y(1) = 0 \Rightarrow \frac{1}{e} B\sin\sqrt{\lambda-1} = 0$

$$\Rightarrow \sin\sqrt{\lambda-1} = \sin(n\pi)$$
$$\Rightarrow \sqrt{\lambda-1} = n\pi$$
$$\Rightarrow \lambda = n^2\pi^2 + 1$$

i.e. $\lambda_n = n^2\pi^2 + 1$, $n \in \mathbb{N}$

and eigen function is,

$$y_n(x) = B_n e^{-x} \sin(n\pi x) \; ; n \in \mathbb{N}$$

Green's function

Let us consider a differential equation with variable coefficient as

$$\frac{d}{dx}\left(P(x)\frac{dy}{dx}\right) + q(x)\cdot y = f(x) \; ; x \in (a,b) \quad \text{——— (1)}$$

with boundary condition

$$\left.\begin{aligned} a_1 y(a) + a_2 y'(a) &= 0 \\ b_1 y(b) + b_2 y'(b) &= 0 \end{aligned}\right\} \quad \text{——— (2)}$$

then we solve eq(1) with the help of Green's function.

and solution of (1) is,

$$y(x) = \int_a^b G(x,t)\, f(t)\, dt$$

where $G(x,t)$ is known as Green's function.

From eq (1)

$$P(x)y'' + P'(x)y' + q(x)y = f(x)$$

$$\Rightarrow y'' + \frac{P'(x)}{P(x)}y' + \frac{q(x)}{P(x)}y = \frac{f(x)}{P(x)} \longleftarrow (3)$$

First we have to find two LI solution of homogeneous part of (3) i.e. $u(x)$ and $v(x)$ st $u(x)$ satisfies boundary condition at $x=a$ and $v(x)$ " " " " $x=b$.

then solution of (1) using variation of parameter can be written as,

$$y(x) = c_1(x)u(x) + c_2(x)v(x)$$

where, $c_1'(x) = \dfrac{-v(x)f(x)}{W(x)P(x)}$

and $c_2'(x) = \dfrac{u(x)f(x)}{W(x)P(x)}$

$W(x)$ is Wronskian of $u(x)$ and $v(x)$.

$$\Rightarrow c_1(x) = -\int_b^x \frac{v(x)f(x)}{W(x)P(x)}dx = -\int_b^x \frac{v(t)f(t)}{W(t)P(t)}dt$$

and $c_2(x) = \displaystyle\int_a^x \frac{u(t)f(t)}{W(t)P(t)}dt$

$\therefore$ Solution is

$$y(x) = \int_a^x \frac{v(x)u(t)f(t)}{W(t)P(t)}dt - \int_b^x \frac{u(x)v(t)f(t)}{W(t)P(t)}dt$$

$$= \int_a^x \frac{v(x)u(t)}{W(t)P(t)}f(t)\,dt + \int_x^b \frac{u(x)v(t)}{W(t)P(t)}f(t)\,dt$$

$$y(x) = \int_a^b G(x,t) f(t)\, dt$$

where $G(x,t) = \begin{cases} \dfrac{v(x)u(t)}{W(t)P(t)} ; & a < t < x \\ \dfrac{u(x)v(t)}{W(t)P(t)} ; & x < t < b \end{cases}$

is called Green's function.

Method of constructing Green's function

Step-1:- Find two L.I solution of homogeneous equation. Let $u(x)$ be solution which satisfies boundary condition at $x=a$.

and let $v(x)$ be the solution which satisfies boundary condition at $x=b$

Step-2:- Find $W(x)$ of $u(x)$ and $v(x)$ and then find $W(x)\cdot P(x)$.

Step-3:- Construct Green's function as

$$G(x,t) = \begin{cases} \dfrac{v(x)u(t)}{W(t)P(t)} ; & a < t < x \\ \dfrac{u(x)v(t)}{W(t)P(t)} ; & x < t < b \end{cases}$$

1) Given $y'' = f(x)$; $x \in (0,1)$

with boundary condition $y(0)=0$, $y(1)=0$.

Sol:- $y'' = f(x)$, $x \in (0,1)$ —— (1)

$y(0)=0$, $y(1)=0$

$\therefore$ Solution of homogeneous part of (1) is

$y = A + Bx$

$\because y(0)=0 \Rightarrow A=0$

$\Rightarrow y = Bx \Rightarrow y = x$ if $B=1$

$\Rightarrow u(x) = x$

Also $y(1)=0 \Rightarrow A+B=0 \Rightarrow B=-A$

$\Rightarrow y(x) = A(1-x)$

$\Rightarrow v(x) = 1-x$

$\therefore u(x)=x$ and $v(x)=1-x$ are two L.I. solution of $y''=0$ such that $u(x)$ satisfies $y(0)=0$ and $v(x)$ satisfies $y(1)=0$.

Now, $W(x) = \begin{vmatrix} u(x) & v(x) \\ u'(x) & v'(x) \end{vmatrix} = \begin{vmatrix} x & 1-x \\ t & -1 \end{vmatrix}$

$\Rightarrow W(x) = -1 \neq 0$

Here $P(x)=1$

$\therefore W(x)P(x) = -1$

$$\Rightarrow G(x,t) = \begin{cases} \dfrac{(1-x)t}{(-1)} & ; t<x \\ \dfrac{x(1-t)}{(-1)} & ; t>x \end{cases}$$

$$\Rightarrow G(x,t) = \begin{cases} t(x-1) & ; t<x \\ x(t-1) & ; t>x \end{cases}$$

which is required Green's function.

$\therefore$ Solution of (1) is,

$$y(x) = \int_0^1 G(x,t)f(t)\,dt$$

$$\Rightarrow y(x) = \int_0^x t(x-1)f(t)\,dt + \int_x^1 x(t-1)f(t)\,dt$$

$$\Rightarrow \boxed{y(x) = (x-1)\int_0^x t\,f(t)\,dt + x\int_x^1 (t-1)f(t)\,dt}$$

Q:2 $x^2y'' - 2xy' + 2y = x$ ——— (1)

with boundary condition $y(1)=0$, $y(2)=0$

Sol:- Solution of homogeneous part of (1) is

$y(x) = c_1 x + c_2 x^2$

$\because y(1)=0 \Rightarrow c_1 + c_2 = 0 \Rightarrow c_1 = -c_2$

$\therefore y(x) = c_1(x - x^2)$

$\Rightarrow u(x) = x - x^2$

and since $y(2)=0 \Rightarrow 2c_1 + 4c_2 = 0$

$\Rightarrow c_1 = -2c_2$

$\therefore y(x) = c_2(x^2 - 2x)$

$\Rightarrow v(x) = x^2 - 2x$

Now, $W(x) = \begin{vmatrix} x-x^2 & x^2-2x \\ 1-2x & 2x-2 \end{vmatrix} = -x^2$

Here $P(x) = x^2$

$\therefore P(t)W(t) = -t^4$

$\Rightarrow$ Green's function is,

$$G(x,t) = \begin{cases} \dfrac{v(x)u(t)}{W(t)P(t)} & ; t<x \\[2ex] \dfrac{u(x)\,v(t)}{W(t)\,P(t)} & ; t>x \end{cases}$$

$$= \begin{cases} \dfrac{(x^2-2x)(t-t^2)}{-t^4} & ; t<x \\[2ex] \dfrac{(x-x^2)(t^2-2t)}{-t^4} & ; t>x \end{cases}$$

Note:- Green's function is symmetric for self adjoint sturm- Liouville problem form

i.e. for $\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right)+q(x)y=f(x)$

$$\Rightarrow G(x,t)=G(t,x)$$

Q:- $y''-\frac{1}{x}y'=1$; $y(0)=0=y(1)$ then the Green's function is given by,

(i) $G(x,t)=\begin{cases}\frac{1}{2}t(x^2-1) & ; t<x \\ \frac{1}{2t}x^2(t^2-1) & ; t>x\end{cases}$

(ii) $G(x,t)=\begin{cases}\frac{1}{2}x^2t & ; t<x \\ \frac{1}{2}(t^2-1)x & ; t>x\end{cases}$

(iii) $G(x,t)=\begin{cases}\frac{1}{2}t(x-1) & ; t<x \\ \frac{1}{2t}(x^2-t) & ; t>x\end{cases}$

(iv) $G(x,t)=\begin{cases}\frac{1}{2}t(x^2-x) & ; t<x \\ \frac{1}{2t}(x-1) & ; t>x\end{cases}$

Properties:-

① $G(x,t)$ is continuous function $\forall x\in(a,b)$ and $\forall t\in(a,b)$.

② $\frac{\partial G}{\partial x}$ and $\frac{\partial^2 G}{\partial x^2}$ are also continuous $\forall x\in(a,b)$ and $\forall t\in(a,b)$ except at $x=t$

③ $\frac{\partial G}{\partial x}$ has discontinuity at $x=t$

i.e. $\left(\frac{\partial G}{\partial x}\right)_{x>t}-\left(\frac{\partial G}{\partial x}\right)_{x<t}=\frac{1}{p(t)}$

(Jump discontinuity)

(4) $G(x,t)$ satisfies boundary conditions at $x=a$ and $x=b$.

Non-homogeneous boundary condition

Given, $\frac{d}{dx}\left(P(x)\frac{dy}{dx}\right)+q(x)\cdot y=f(x)$ —— (1) ; $a<x<b$

with non-homogeneous boundary condition

$$\left.\begin{aligned} a_1 y(a)+a_2 y'(a)&=\alpha \\ b_1 y(b)+b_2 y'(b)&=\beta \end{aligned}\right\} \text{—— (2)} \quad ,\alpha\neq 0,\beta\neq 0.$$

We shall first convert non-homogeneous boundary condition to homogeneous boundary condition by using transformation

$$z=y+A+Bx$$

then dependent variable y changed to z. and independent variable remains same.

then eq (1) becomes,

$$\frac{d}{dx}\left[P(x)\frac{dz}{dx}\right]+q(x)\cdot z=\phi(x) \quad \text{—— (3)}$$

with boundary condition

$$\left.\begin{aligned} a_1 z(a)+a_2 z'(a)&=0 \\ b_1 z(b)+b_2 z'(b)&=0 \end{aligned}\right\} \text{—— (4)}$$

Now, we shall solve this to get solution of (1)

Q:-1 Solve $\frac{d^2y}{dx^2}=f(x)$ —— (1) ; $y(0)=\alpha,\ y'(1)=\beta$ —— (2)

Sol:- Let $z=y+A+Bx$

$$\Rightarrow y=z-A-Bx$$

$$\Rightarrow \frac{dy}{dx}=\frac{dz}{dx}-B$$

$$\Rightarrow \frac{d^2y}{dx^2}=\frac{d^2z}{dx^2}$$

$\therefore$ From (1), we get $\frac{d^2z}{dx^2}=f(x)$ —— (3)

and $z(0) = y(0) + A = \alpha + A \Rightarrow z(0) = 0$ if $A = -\alpha$

and $z'(1) = y'(1) + B = \beta + B \Rightarrow z'(1) = 0$ if $B = -\beta$

i.e. by the transformation $z = y - \alpha - \beta x$

then eq(1) reduces to,

$$\frac{d^2 z}{dx^2} = f(x) \text{ with } z(0) = 0,\ z'(1) = 0 \quad \text{—(4)}$$

Solution of homogeneous part of (3) is

$$z = A + Bx$$

$\because z(0) = 0 \Rightarrow A = 0 \Rightarrow z(x) = Bx$
$\Rightarrow u(x) = x$

and $z'(1) = 0 \Rightarrow B = 0 \Rightarrow z(x) = A$
$\Rightarrow v(x) = 1$

$\therefore W(x) = \begin{vmatrix} x & 1 \\ 1 & 0 \end{vmatrix} = -1$

$$\Rightarrow G(x,t) = \begin{cases} \dfrac{1 \cdot t}{(-1) \cdot 1} & ; t < x \\[2ex] \dfrac{x \cdot 1}{(-1) \cdot 1} & ; t > x \end{cases}$$

$$\Rightarrow G(x,t) = \begin{cases} -t & ; t < x \\ -x & ; t > x \end{cases}$$

$\therefore$ Solution of (3) is,

$$z(x) = \int_0^1 G(x,t) f(t)\, dt$$

$$\Rightarrow z(x) = \int_0^x (-t) f(t)\, dt + \int_x^1 (-x) f(t)\, dt$$

$\therefore y(x) = z(x) + \alpha + \beta x$

is required solution of (1) with (2).

Q-2 Solve $y'' = x$ with boundary condition $y(0) = 1$ and $y'(1) = 2$ by using Green's function.

Sol:- $y(x) = \frac{x^3}{6} + \frac{3x}{2} + 1$

Q.3 Given $xy'' + y' = 0$

y is bounded as $x \to 0$ and $y(1) = y'(1)$

construct Green's function.

Sol:- $xy'' + y' = 0$ ——— (1)

solution is $y = A + B\log x$

$\because y(1) = y'(1)$

$\Rightarrow A = B$ ——— (2)

Let us consider the Green's be

$$G(x,t) = \begin{cases} a + b\log x & ; 0 < x < t \\ c + d\log x & ; 1 > x > t \end{cases}$$

(I) Since $G(x,t)$ is continuous $\forall x, t \in (a,b)$

$\Rightarrow a + b\log t = c + d\log t$

$\Rightarrow (a-c) + (b-d)\log(t) = 0$ ——— (3)

(II) $\left(\frac{\partial G}{\partial x}\right)_{x>t} - \left(\frac{\partial G}{\partial x}\right)_{x<t} = \frac{1}{P(t)}$

$\Rightarrow \left(\frac{d}{x}\right)_{x>t} - \left(\frac{b}{x}\right)_{x<t} = \frac{1}{t}$

$\Rightarrow \frac{1}{t}(d-b) = \frac{1}{t}$

$\Rightarrow d - b = 1$ ——— (4)

(III) $G_1(x,t)$ satisfies boundary condition at $x=0$ and $G_2(x,t)$ satisfies boundary condition at $x=1$.

$\therefore\ b=0 \quad \text{——} (5)$

and $G_2(1,t) = \left(\frac{\partial G_2}{\partial x}\right)_{(1,t)}$

$\Rightarrow\ c + d\log(1) = \left(\frac{d}{(1)}\right)$

$\Rightarrow\ c=d \quad \text{——} (6)$

$\therefore$ From (3), (4), (5) and (6), we get

$a = 1+\log(t)\ ,\ b=0\ ,\ c=d=1$

$$\therefore\ G(x,t) = \begin{cases} 1+\log(t) & ; \ 0<x<t \\ 1+\log(x) & ; \ 1>x>t \end{cases}$$

Subject wise Marks Weightage of CSIR-NET Examination

(Maximum Marks: 200)

Subject(Mathematics)	Marks Range (Min. ~ Max.)	No. of Que. (Min. ~ Max.)	Important Topics (Note Here)
UNIT-I			
Real Analysis	(45.25 ~ 73.25)	15 ~ 20	
Linear Algebra	(41.25 ~ 75.00)	15 ~ 20	
UNIT-II			
Abstract Algebra	(25.00 ~ 45.25)	6 ~ 8	
Number Theory	(3 ~ 07.75)	1 ~ 2	
Complex Analysis	(25.00 ~ 34.50)	5 ~ 8	
Topolgy	(3 ~ 07.75)	1 ~ 2	
UNIT-III			
Ordinary Differential Equation	(15.50 ~ 25.00)	4 ~ 7	
Partial Differential Eqn.(PDE)	(20.20 ~ 25.00)	4 ~ 7	
Dynamical System	(0 ~ 07.75)	0 ~ 2	
Numerical Analysis(NA)	(3 ~ 12.50)	1 ~ 3	
Calculua of Variation (COV)	(3 ~ 12.50)	1 ~ 3	
Integral Equation(I.E)	(3 ~ 12.50)	1 ~ 3	
Classical Mechanics	(0 ~ 07.75)	0 ~ 2	
UNIT-IV			
Probability & Statistics			
Markov Chain	(3 - 12.50)	1 to 3	
Operation Research(LPP)	(0 ~ 07.75)	0 ~ 2	
TOTAL			

CSIR-NET Exam. Paper Structure(Total Marks = 200)

PARTS	Total Que.	To Attempt	Max. Mark	Negative	Major Part
PART - A	20 (2 Marks)	15	30	0.50 Neg.	General
PART - B	40 (3 Marks)	25	75	0.75 Neg.	Pure Maths
	UNIT I - IV				
PART - C	60 (4.75 Mark)	20	95	No Neg.	Pure Maths
	UNIT I - IV				
UNIT - I	***Real Analysis & Linear Algebra***				
UNIT II-1V	***Complex Anal., Modern Alg.,ODE,PDE***				

**** Prepare Accordingly***

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CSIR-NET Year wise Cut-off (Subject : Mathematics)

	Category	General		EWS		OBC		SC		ST		PwD	
Year		JRF	LS(NET)	JRF	LS(NET)	JRF	LS(NET)	JRF	LS(NET)	JRF	LS(NET)	JRF	LS(NET)
2020	December												
	June	114	102.6	102.76	92.476	101.5	91.35	80.5	72.45	61.76	55.576	57.5	51.75
2019	December	107.3	96.54	96.26	86.64	92.5	83.26	70.26	63.25	55	50	50	50
	June	111.5	100.36	93.26	83.94	97.76	87.98	75.5	67.96	61	54.9	57	50
2018	December	97.26	87.54	-	-	82	73.8	63.76	57.38	50.5	50	50	50
	June	112.5	101.26	-	-	94.76	85.28	74	66.6	55.5	50	50	50
2017	December	96.76	87.08	-	-	81.5	73.36	62.5	56.26	50	50	50.26	50
	June	100.8	90.68	-	-	85.76	77.18	68.26	61.48	50	50	50	50
2016	December	119	107.1	-	-	100	90	78.5	70.66	55.26	50	52	50
	June	109.8	98.78	-	-	94.76	85.28	75.26	67.74	50	50	51.5	50
2015	December	109.8	98.78	-	-	95.5	85.96	77.26	69.54	51.26	50	64	51.08
	June	106.3	95.64	-	-	84.5	81.5	72.24	65.04	51	50	77.26	69.54
2014	December												
	June												

Category	General		EWS		OBC		SC		ST		PwD	
Your Target	119	107.5	103	93	102	92	81	73	62	56	78	70

For CSIR-NET, GATE, SET, … etc
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Some Useful Links:

1. Free Maths Study Materials (https://pkalika.in/2020/04/06/free-maths-study-materials/**)**

2. BSc/MSc Free Study Materials (https://pkalika.in/2019/10/14/study-material/**)**

3. PhD/MSc Entrance Exam Que. Paper: (https://pkalika.in/que-papers-collection/**)**
[CSIR-NET, GATE(MA), BHU, CUCET,IIT, JAM(MA), NBHM, ...etc]

4. CSIR-NET Maths Que. Paper: (https://pkalika.in/2020/03/30/csir-net-previous-yr-papers/**)**
[Upto Lastest CSIR NET Exams]

5. PhD/JRF Position Interview Asked Questions:
(https://pkalika.in/phd-interview-asked-questions/)

6. List of Maths Suggested Books (https://pkalika.in/suggested-books-for-mathematics/)

7. CSIR-NET Mathematics Details Syllabus (https://wp.me/p6gYUB-Fc)

8. CSIR-NET, GATE, PhD Exams, ...etc Study Materials & Solutions
https://pkalika.in/kalika-notes-centre/

9. CSIR-NET, GATE, ... Solutions (https://wp.me/P6gYUB-1eP)

10. Topic-wise Video Lectures (Free Crash Course)
https://www.youtube.com/pkalika/playlists