PhD admission Jul-Nov 2022
Department of Mathematics, IIT Madras

Date: 30/May/2022 **Duration: 9:30 am to 11:30 am** **36 Marks**

If you copy an answer from others, the provider and receiver shall get zero marks for that question.

Write your name and application number on each page of your answer. Write the answers on white paper(s) using a black/blue inked pen.

Answer any 6 questions from the following list.

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1. Let M_n be the vector space of $n \times n$ real matrices. Let $f \colon M_n \to \mathbb{R}$ be a linear map. Prove or disprove that there exists a matrix $B \in M_n$ such that

$$f(A) = tr(BA), \ \forall\, A \in M_n,$$

where tr is the normalized trace ($tr(1) = 1$) on M_n. (6)

2. Find the steady-state temperature distribution inside the annular region

$$C := \left\{(x, y) \in \mathbb{R}^2 \mid \sqrt{2} < x^2 + by^2 < 2\right\}$$

whose outer edge is heat insulated and the inner edge is kept at the temperature $\sin^2 \theta$. (6)

3. Let p be a prime number and $R = \frac{\mathbb{Z}}{p\mathbb{Z}}$. Find all prime numbers p for which $R_p = \frac{R[x]}{(x^3-1)}$ and $R'_p = \frac{R[x]}{(x-2)(x-3)(x-5)}$ are isomorphic as rings. (6)

4. Find the exact solution $y(x)$ of the initial value problem

$$y' = y^2, \quad y(0) = 1.$$

Starting with $y_0(x) = 1$, apply Picard's method to calculate $y_1(x), y_2(x), y_3(x)$ and show that, for $|x| < 1$

$$|y(x) - y_3(x)| \leq \frac{x^4}{1 - x}.$$

(6)

5. (a) Construct a one-one map from $\mathbb{Q}$ to $\mathbb{N}$, where $\mathbb{Q}$ is the set of rationals and $\mathbb{N}$ is the set of natural numbers.

   (b) Is it possible to construct a family of subsets of the natural numbers $\mathbb{N}$ that forms a total order under inclusion (subset relation) but has uncountably many members? Justify. (2+4)

6. (a) Let $f(z) = u(x, y) + \sqrt{-1}v(x, y)$ be an entire function on $\mathbb{C}$ such that $u^2 < v^2 + 2022$ where $z = x + \sqrt{-1}y$. What can you say about the function f? Justify.

   (b) Find all Möbius transformation which carries the circle $|z| = 2$ into $|z+1| = 1$, the point -2 into the origin, and the origin into $\sqrt{-1}$ $(\in \mathbb{C})$.

(2+4)

7. Define $f_n\colon [0,1] \to \mathbb{R}$ by $f_n(x) = \frac{x^n}{1+x^n}$.

   (a) Prove that the sequence $\{f_n\}$ converges pointwise on $[0,1]$.

   (b) Let
   $$f(x) = \lim_{n\to\infty} f_n(x)$$
   for $x \in [0,1]$. Prove that f_n converges to f uniformly on $[0,\alpha]$ where $0 < \alpha < 1$.

   (c) Prove or disprove that f_n converges to f uniformly on $[0,1]$.

(1+ 3+2)

8. Let $a > b > 1$. Consider the subset
$$S = \{(u,v,x,y) \in \mathbb{R}^4 \mid (u^2+v^2-a)^2 + (x^2+y^2-b) = 1\}.$$
Prove or disprove that S is homeomorphic to the product $(S^1)^3$ where S^1 is the unit circle. Justify. (6)

9. Let U and V be two dependent discrete random variables, each being uniformly distributed on $\{1,2,...,k\}$. Let W be another random variable having the same uniform distribution but independent of U and V. Define a random variable $X := (V+W) \mod k$. Prove or disprove the following statements with clarification.

   (a) X is uniformly distributed on $\{0,1,2,...,k-1\}$.

   (b) U and X are independent.

(3+3)