P KALIKA NOTES

[For NET/GATE/SET/NBHM/JAM/CUCET/MSc/PhD Exams…etc.]

CALCULUS

[Handwritten Study Material with solved examples]

P. Kalika & K Munesh
(NET(JRF), GATE, SET, INSPIRE)
Email: maths.whisperer@gmail.com

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P Kalika Maths (www.pkalika.in)

(A Hub of Handwritten Study Materials/Solutions for CSIR-NET(JRF), GATE, SET, JAM, CUCET, NBHM, MSc/PhD Entrances, ...etc)

P Kalika
A Site for Higher Mathematics

Your Remarks/Notes

Abbreviation

- w.r.t = with respect to
- f^n = function
- d.n.e = does not exist
- pt. = point.
- homo. = homogeneous
- diff' = differentation
- D.E = differential Equation
- P.D = Partial derivative
- s.t.b = said to be
- nb.d = neighbourhood
- max. = maxima/maximum
- min. = minima/minimum
- b/w = between
- A/c = According to
- Vol. = Volume
- b.dd = bounded
- cts. = continuous

Index(Calculus for CSIR-NET, GATE, JAM, SET, PSC)

Partial Differentiation

Partial Derivatives: Let $z = f(x,y)$ be a f^n of two variables x & y. Then partial derivative of $z(=f)$ w.r.t x, at point (a,b) is defined as —

$$z_x = f_x = \left.\frac{\partial f}{\partial x}\right|_{(a,b)} = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

and

$$z_y = f_y = \left.\frac{\partial f}{\partial y}\right|_{(a,b)} = \lim_{k \to 0} \left.\frac{f(x,y+k) - f(x,y)}{k}\right|_{(a,b)}$$

$$= \lim_{k \to 0} \frac{f(a,b+k) - f(a,b)}{k}$$

Provided, the limit exists.

(Note: $\partial f/\partial x$, $\partial f/\partial y$ are called the first order partial derivative)

Second order Partial derivative

(i). $\dfrac{\partial^2 z}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial x}\right)$ (ii). $\dfrac{\partial^2 z}{\partial y^2} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial y}\right)$

(iii). $\dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial x}\right) = \dfrac{\partial^2 z}{\partial y \partial x}$ (iv). $\dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial y}\right) = \dfrac{\partial^2 z}{\partial x \partial y}$

Note: Continuity of a f^n at a point, doesn't imply that partial derivative of that f^n at that pt. will exists.

Example ① $f(x,y) = \sqrt{|x|} + \sqrt{|y|}$

Hint: Here, $f(x,y)$ is cts in whole $\mathbb{R}^2$.

Also $f_x(0,0) = \lim_{h\to 0} \dfrac{\sqrt{|h|} + 0}{h} = \lim_{h \to 0} \dfrac{1}{\pm\sqrt{|h|}}$ = doesnot exist

I'ly $f_y(0,0) = \lim_{h \to 0} \dfrac{0 + \sqrt{|h|}}{h}$ = d.n.e

Example. Find the first order Partial Derivative (P.D)

(i). x^3+y^3-3axy (ii). $\log(x^2+y^2)$.

(i). Let $z = x^3+y^3-3axy$

$\therefore \dfrac{\partial z}{\partial x} = 3x^2 - 3ay$ (treat y as a constant in z)

& $\dfrac{\partial z}{\partial y} = 3y^2 - 3ax$ (treat x as a constant in z)

(ii). Let $z = \log(x^2+y^2)$. then

$$\frac{\partial z}{\partial x} = z_x = \frac{1}{x^2+y^2}\cdot 2x = \frac{2x}{x^2+y^2}$$

$$\& \ \frac{\partial z}{\partial y} = z_y = \frac{1}{x^2+y^2}\cdot 2y = \frac{2y}{x^2+y^2}$$

Homogeneous Function

A f^n $z = f(x,y)$ is called Homogeneous function of degree n, if is is expressible as

$$\boxed{z = x^n g(y/x)}$$

(OR: If $f(kx, ky) = k^n f(x,y)$ for some constant k then $f(x,y)$ is a homogeneous f^n of degree n)

Note: If $f(x,y)$ is a homo. f^n with degree 'n', then

$$f(x,y) = \begin{cases} x^n \phi(y/x) \\ y^n \psi(x/y) \end{cases}$$

Example: (a) $ax^2+2hxy+by^2$

as $ax^2+2hxy+by^2 = x^2\left[a + 2h(y/x) + b(y/x)^2\right]$

is homo. f^n of degree 2.

Euler's Theorem on Homogeneous Functions

If $z = f(x,y)$ is a homo. f^n of x & y of degree 'n'
then

ⓐ $$\boxed{x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz}$$

~~Hint~~ Proof: Since, $z = f(x,y)$ is a homo. f^n of degree n

$\Rightarrow \quad z = x^n g(y/x)$

so, $$\frac{\partial z}{\partial x} = nx^{n-1}g(y/x) + x^n \cdot g'(y/x)(-y/x^2)$$

& $$\frac{\partial z}{\partial y} = x^n g'(y/x)\cdot 1/x$$

$$\therefore x\left(\frac{\partial z}{\partial x}\right) + y\left(\frac{\partial z}{\partial y}\right) = \left[nx^n g(y/x) - y x^{n-1} g'(y/x)\right] + y x^{n-1} g'(y/x)$$

$$= nx^n g(y/x) = nz$$

$$\Rightarrow \boxed{x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz}$$

ⓑ $$\boxed{x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2\frac{\partial^2 f}{\partial y^2} = n(n-1)f}$$

or $$(xz_x + yz_y)^2 \equiv n(n-1)z$$

Proof:

diff ⓐ w.r.t x and y, we get—

$$\frac{\partial z}{\partial x} + x\frac{\partial^2 z}{\partial x^2} + y\frac{\partial^2 z}{\partial x \partial y} = n\frac{\partial z}{\partial x} \quad \text{——} \quad (i)$$

& $$x\frac{\partial^2 z}{\partial y \partial x} + \frac{\partial z}{\partial y} + y\frac{\partial^2 z}{\partial y^2} = n\frac{\partial z}{\partial x} \quad \text{——} \quad (ii)$$

Now multiply (i) by x, & (ii) by y and then adding, we get

$$\left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right) + \left(x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2}\right) = n\left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right)$$

$$\left(\because \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}\right).$$

Now, using (a), we get—

$$nz + x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} = n^2 z$$

Hence

$$\boxed{x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} = n^2 z - nz = n(n-1)z}$$

Examples (i). $x^2 + xy + y^2$, degree $n = 2$

(ii). $2x^3 + xy^2 + z^3$, degree $n = 3$

(iii). $\dfrac{xy^2 - y^3}{2x + 3y}$, degree, $n = 3 - 1 = 2$

Examples: $f(x,y) = xy^3 + \dfrac{(x^2+y^2)^3}{2xy} + (x^2 - y^2)\sin y/x$

Solⁿ → homo. fⁿ with degree $n = 4$

Then,

$$x^2 f_{xx} + 2xy f_{xy} + y^2 f_{yy} = 4(4-1) f(x,y)$$
$$= 12 f(x,y).$$

H.W

Example: Consider $f(x,y) = \tan^{-1}\dfrac{x+y}{(x^2+y^2)^{3/2}}$ (This is NOT homo. in (x,y))

Solⁿ: Find—

(a). $x f_x + y f_y$

(b) $x^2 f_{xx} + 2xy f_{xy} + y^2 f_{yy}$

Note: If $u(x,y) = f(x,y) + g(x,y) + h(x,y)$, where f, g & h are homo. f^n with degree m, n & p respectively, then —

(a). $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = mf + ng + ph$

(b). $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = m(m-1)f + n(n-1)g + p(p-1)h$

If $f(u)$ is a homo. f^n in two variables x & y with degree 'n', then —

(a). $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = n\dfrac{f(u)}{f'(u)} = F(u)$

(b). $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = F(u)[F'(u) - 1]$

Note: If f_{xy} and f_{yx} at any point exists, then they may or may not be equal.

Example: If $u = \tan^{-1}(y/x)$, show that —

(i). $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0$ (ii). $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = 0$

Sol^n: $\dfrac{\partial u}{\partial x} = \dfrac{1}{1 + (y/x)^2}\left(\dfrac{-y}{x}\right) = -\dfrac{y}{x^2+y^2}$

$\dfrac{\partial u}{\partial y} = \dfrac{1}{1 + (y/x)^2}\left(\dfrac{1}{x}\right) = \dfrac{x}{x^2+y^2}$

$\therefore x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0$ ——— (1')

Now, diff^n (1) partially w.r.t. x & y respectively, we get —

$\dfrac{\partial u}{\partial x} + x\dfrac{\partial^2 u}{\partial x^2} + y\dfrac{\partial^2 u}{\partial x \partial y} = 0$ ——— (ii)

$x\dfrac{\partial^2 u}{\partial x \partial y} + \dfrac{\partial u}{\partial y} + y\dfrac{\partial^2 u}{\partial y^2} = 0$ ——— (iii)

Multiplying (ii) by x, (iii). by y, then adding, we get —

$xu_x + yu_y + x^2u_{xx} + 2xy\,u_{xy} + y^2u_{yy} = 0$

Hence, $\boxed{x^2u_{xx} + 2xy\,u_{xy} + y^2u_{yy} = 0}$

Total Derivative

Theorem: Let $z = f(x,y)$ possess cts. 1^{st} order partial derivative where $x = \phi(t)$ & $y = \psi(t)$ possess cts derivatives. Then the f^n

$z = f(x,y) = f(\phi(t), \psi(t))$ has total derivative

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

($\frac{dz}{dt}$ is called the total derivative of z)

Total differential Co-efficient

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

Note: (1) If $f(x,y) = c$ is an implicit f^n. then

$$\frac{dy}{dx} = \frac{-f_x}{f_y} \qquad \left(\text{i.e } f_x dx + f_y dy = 0 \ \& \ f_y \neq 0\right)$$

Corollary: If $z = f(x,y)$, where $x = \phi(u,v)$, & $y = \psi(u,v)$ then —

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$\& \quad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}$$

Example: (1) Find $\frac{dz}{dt}$, when —

(a). $z = x^2 - y^2$, $x = e^t \cos t$, $y = e^t \sin t$

(b). $z = e^x \sin y$, $x = \log t$, $y = t^2$.

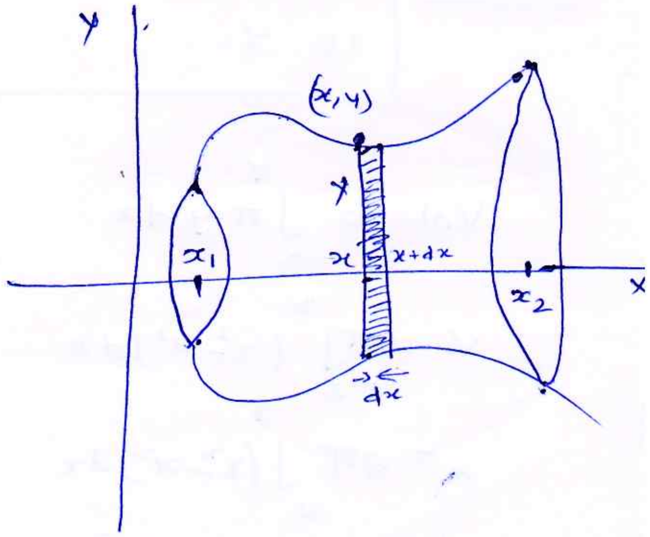
Sol^n (1)(a). $\because \frac{\partial z}{\partial x} = 2x$, $\frac{\partial z}{\partial y} = -2y$

& $\frac{dx}{dt} = -e^t \sin t + e^t \cos t$, $\frac{dy}{dt} = e^t \sin t + e^t \cos t$

$\therefore \frac{dz}{dt} = z_x \frac{dx}{dt} + z_y \frac{dy}{dt} = 2x e^t (\cos t - \sin t) + (-2y) . e^t (\sin t + \cos t)$

Volume of Solid generated by revolving

Vol.(elementry) $= \pi y^2 dx$



$$\Rightarrow \boxed{V = \int_{x_1}^{x_2} \pi y^2 dx}$$

(I). Volume of Solid obtained by revolution of curve —

(I). $y = f(x)$ about x-axis is —

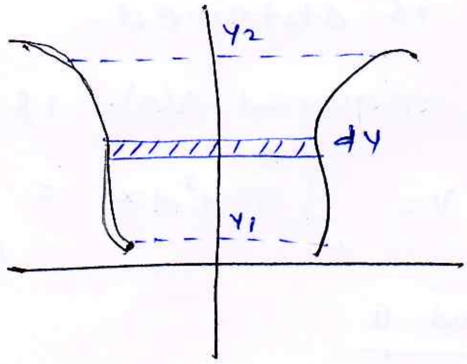
$$V = \int_{x_1}^{x_2} \pi y^2 dx$$

$$\boxed{V = \int_{x_1}^{x_2} \pi (f(x))^2 dx}$$

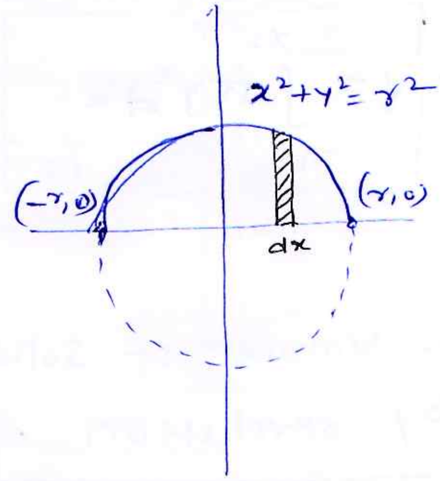
(II). $y = f(x)$ (or $x = g(y)$) about y-axis

$$V = \int_{y_1}^{y_2} \pi x^2 dy$$

$$\boxed{V = \int_{y_1}^{y_2} \pi (g(y))^2 dy}$$



Example: Find the volume of sphere, whose radius is r.



$$\text{Vol. } V = \int_{-r}^{r} \pi y^2 \, dx$$

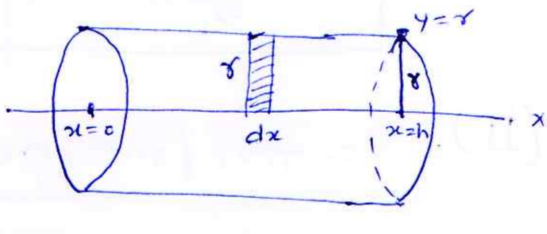
$$V = \pi \int_{-r}^{r} (r^2 - x^2)\, dx$$

$$= 2\pi \int_{0}^{r} (r^2 - x^2)\, dx$$

$$= 2\pi \left[r^3 - \frac{r^3}{3} \right]$$

$$= 2\pi \cdot \frac{2r^3}{3} = \frac{4}{3}\pi r^3.$$

Example: Using method of volume of solid revolution. Find vol. of cylinder whose radius is r & height h.



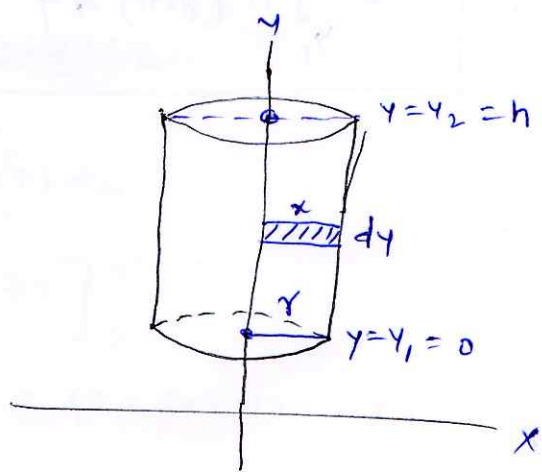
Sol^n let line $y=r$ is revolved about x-axis from $x=0$ to $x=h$, then cylinder of radius r & height h is obtained.

So, required Vol. is

$$V = \int_0^h \pi y^2 \, dx = \int_0^h \pi r^2 dx = \pi r^2 h.$$

Method - II

H.W. when revolved about y-axis.



so, required vol. is

$$V = \int_0^h \pi x^2 dy = \int_0^h \pi r^2 dy$$

$$= \pi r^2 h.$$

Q^n The maximum value of $f(x,y) = 49 - x^2 - y^2$ on the line $x + 3y = 10$ is $\underline{39}$.

Sol^n

Let $f(x,y) = 49 - x^2 - y^2$ and

$g(x,y) = x + 3y - 10$ ——— (*)

Then, By using Lagrange Multiplier λ, we have

$$F(x,y,\lambda) = f(x,y) + \lambda\, g(x,y)$$

$$\Rightarrow F(x,y,\lambda) = (49 - x^2 - y^2) + \lambda(x + 3y - 10).$$

Now,

$F_x = -2x + \lambda$, $F_y = -2y + 3\lambda$

So, $F_x = 0 \Rightarrow \lambda = 2x$ ——— (I)

& $F_y = 0 \Rightarrow \lambda = \frac{2}{3}y$ ——— (II)

From eq^n (I) & (II), we get

$$2x = \frac{2}{3}y \Rightarrow 3x = y$$

Now, Putting $y = 3x$ in eq^n (*), we get

$$x + 3(3x) - 10 = 0$$

$$\Rightarrow x = 1$$

and $y = 3x = 3 \cdot 1 = 3$

So, point is $(1,3)$

And maximum value of $f(x,y)$ is at $(1,3)$ i.e.

$f(1,3) = 49 - 1 - 9 = 39$ $\underline{\text{Ans}}$