



P KALIKA NOTES

[For NET/GATE/SET/NBHM/JAM/CUCET/MSc/PhD Exams...etc.]



CALCULUS

[Handwritten Study Material with solved examples]

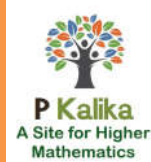


P. Kalika & K Munesh
(NET(JRF), GATE, SET, INSPIRE)
Email: maths.whisperer@gmail.com

No. of Pages:

P Kalika Maths (www.pkalika.in)

(A Hub of Handwritten Study Materials/Solutions for CSIR-NET(JRF),
GATE, SET, JAM, CUCET, NBHM, MSc/PhD Entrances, ...etc)



Your Remarks/NotesAbbreviation

- ✓ w.r.t = with respect to
- ✓ f^n = function
- ✓ d.n.e = does not exist
- ✓ pt. = point.
- ✓ homo. = homogeneous
- ✓ diff' = differentiation
- ✓ D.E = differential Equation
- ✓ p.D = Partial derivative
- ✓ s.t.b = said to be
- ✓ nb.d = neighbourhood
- ✓ max. = maxima/maximum
- ✓ min. = minima/minimum
- ✓ b/w = between
- ✓ A/c = According to
- ✓ Vol. = Volume
- ✓ b.d.d = bounded
- ✓ cts. = continuous

Index(Calculus for CSIR-NET, GATE, JAM, SET, PSC)

1. Function of Two Variables**(04)**
 - a. Basic Definitions & Terminology
2. Limit of function
 - a. Repeated Limit, Simultaneous Limit
 - b. Algebra of limits
3. Continuity of function of two variables **(16)**
4. Directional & Partial Derivatives **(20)**
5. Partial Differentiation/ Derivatives **(24)**
 - a. Homogeneous Function
 - b. Euler's Theorem on Hom. Functions
6. Total Derivative/Derivatives **(29)**
7. Differentiability **(35)**
 - a. Necessary & Sufficient Condition
 - b. How to Check Differentiability ?
 - c. Young's Theorem & Schwartz's Theorem
 - d. Jacobian Matrix
8. CSIR-NET Asked Problems **(45)**
9. Maxima & Minima for function of Single Variables **(49)**
10. Extreme Points of the function of two variables **(52)**
 - a. Local minima/Local maxima/Extreme Point
 - b. Saddle Point/Stationary Point/Critical Point
11. Lagranges's Method for Maxima & Minima **(68)**
12. Multiple Integrals **(72)**
 - a. Line/Single Integral
 - b. Area/Sectoral Area
 - c. Volume of 3D body
 - d. Volume of solid of Revolution
 - e. Surface Area of Solid Revolution
13. Double Integral **(88)**
 - a. Change of order of Integration
 - b. Application of double Integral
14. Triple Integral **(103)**
 - a. Change of Variables(Cartesian to Polar Form)
 - b. Application of Triple Integral **(108)**
15. Gauss Theorem **(111)**
16. Stoke's Theorem
17. Green's Theorem **(118)**
18. Surface Integral
19. Application of Double Integral
 - a. Area enclosed within a closed curve in XY-plane
 - b. Surface Area of 3D-Objects
 - c. Volume enclosed by 3D-Object

Some Solved Problems of JAM, NET & GATE (127)

Partial Differentiation

Partial Derivatives: Let $z = f(x, y)$ be a f^n of two variables x & y . Then partial derivative of $z (= f)$ w.r.t x , at point (a, b) is defined as —

$$z_x = f_x = \left. \frac{\partial f}{\partial x} \right|_{(a,b)} = \lim_{h \rightarrow 0} \frac{f(a+h, b) - f(a, b)}{h}$$

and

$$z_y = f_y = \left. \frac{\partial f}{\partial y} \right|_{(a,b)} = \lim_{k \rightarrow 0} \frac{f(x, y+k) - f(x, y)}{k} \Big|_{(a,b)}$$
$$= \lim_{k \rightarrow 0} \frac{f(a, b+k) - f(a, b)}{k}$$

Provided, the limit exists.

(Note: $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ are called the first order partial derivative)

Second order Partial derivative

$$(i), \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial x} \right) \quad (ii), \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial y} \right)$$

$$(iii), \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial x} \right) = \frac{\partial^2 z}{\partial y \partial x} \quad (iv), \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial y} \right) = \frac{\partial^2 z}{\partial x \partial y}$$

Note: Continuity of a f^n at a point, doesn't imply that partial derivative of that f^n at that pt. will exist.

Example (1) $f(x, y) = \sqrt{|x|} + \sqrt{|y|}$

Hint: Here, $f(x, y)$ is cts in whole $\mathbb{R}^2$.

Also $f_x(0, 0) = \lim_{h \rightarrow 0} \frac{\sqrt{|h|} + 0}{h} = \lim_{h \rightarrow 0} \frac{1}{\pm\sqrt{|h|}} = \text{does not exist}$

likewise $f_y(0, 0) = \lim_{h \rightarrow 0} \frac{\sqrt{0} + \sqrt{|h|}}{h} = \text{d.n.e}$

Example. Find the first order Partial Derivative (P.D) —

(i). $x^3 + y^3 - 3axy$

(ii). $\log(x^2 + y^2)$.

(i). Let $z = x^3 + y^3 - 3axy$

$$\therefore \frac{\partial z}{\partial x} = 3x^2 - 3ay \quad (\text{treat } y \text{ as a constant in } z)$$

$$+ \frac{\partial z}{\partial y} = 3y^2 - 3ax \quad (\text{treat } x \text{ as a constant in } z)$$

(ii). Let $z = \log(x^2 + y^2)$. then

$$\frac{\partial z}{\partial x} = z_x = \frac{1}{x^2 + y^2} \cdot 2x = \frac{2x}{x^2 + y^2}$$

$$+ \frac{\partial z}{\partial y} = z_y = \frac{1}{x^2 + y^2} \cdot 2y = \frac{2y}{x^2 + y^2}$$

Homogeneous Function

A fⁿ $z = f(x, y)$ is called Homogeneous function of degree n , if it is expressible as

$$\boxed{z = x^n g\left(\frac{y}{x}\right)}$$

(OR: If $f(kx, ky) = k^n f(x, y)$ for some constant k
then $f(x, y)$ is a homogeneous fⁿ of degree n)

Note: If $f(x, y)$ is a homo. fⁿ with degree n , then

$$f(x, y) = \begin{cases} x^n \phi\left(\frac{y}{x}\right) \\ y^n \psi\left(\frac{x}{y}\right) \end{cases}$$

Example: (a) $ax^2 + 2hxy + by^2$

$$(b) \quad ax^2 + 2hxy + by^2 = x^2 \left[a + 2h\left(\frac{y}{x}\right) + b\left(\frac{y}{x}\right)^2 \right]$$

is homo. fⁿ of degree 2.

Euler's Theorem on Homogeneous Functions

If $z = f(x, y)$ is a homo. fⁿ of x & y of degree ' n ' then

(a)

$$\boxed{x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = n z}$$

Proof: Hint: Since, $z = f(x, y)$ is a homo. fⁿ of degree ' n '
 $\Rightarrow z = x^n g(y/x)$

$$\text{So, } \frac{\partial z}{\partial x} = n x^{n-1} g(y/x) + x^n g'(y/x) (-y/x^2)$$

$$\& \frac{\partial z}{\partial y} = x^n g'(y/x) \cdot 1/x$$

$$\begin{aligned} \therefore x \left(\frac{\partial z}{\partial x} \right) + y \left(\frac{\partial z}{\partial y} \right) &= x \left[n x^{n-1} g(y/x) - y x^{n-2} g'(y/x) \right] + y x^{n-1} g'(y/x) \\ &= n x^n g(y/x) = n z \end{aligned}$$

$$\Rightarrow \boxed{x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = n z}$$

$$(b) \boxed{x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = n(n-1) z}$$

$$\approx \text{or } (x z_x + y z_y)^2 = n(n-1) z$$

Proof:

diff (a) w.r.t x and y , we get —

$$\frac{\partial z}{\partial x} + x \frac{\partial^2 z}{\partial x^2} + y \frac{\partial^2 z}{\partial x \partial y} = n \frac{\partial z}{\partial x} \quad \text{--- (i)}$$

$$\& x \frac{\partial^2 z}{\partial y \partial x} + \frac{\partial z}{\partial y} + y \frac{\partial^2 z}{\partial y^2} = n \frac{\partial z}{\partial y} \quad \text{--- (ii)}$$

Now multiply (i) by x , & (ii) by y and then adding, we get

$$\left(x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} \right) + \left(x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} \right) = n \left(x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} \right)$$

$$(\because \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x})$$

Now, using (a), we get —

$$nz + x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = n^2 z$$

Hence

$$\boxed{x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = n^2 z - nz = n(n-1)z}$$

Examples (i). $x^2 + xy + y^2$, degree $n=2$

(ii). $2x^3 + xy^2 + z^3$, degree $n=3$

(iii). $\frac{xy^2 - y^3}{2x + 3y}$, degree, $n=3-1=2$

Example: $f(x,y) = xy^3 + \frac{(x^2+y^2)^3}{2xy} + (x^2-y^2) \sin yx$

Solⁿ

→ homo. f^n with degree $n=4$

Then,

$$x^2 f_{xx} + 2xy f_{xy} + y^2 f_{yy} = 4(4-1)f(x,y) = 12f(x,y).$$

H.W

Example: Consider $f(x,y) = \tan^{-1} \frac{x+y}{(x^2+y^2)^{3/2}}$ (This is NOT homo. in (x,y))

Solⁿ

a Find —

(a). $xf_x + yf_y$

(b) $x^2 f_{xx} + 2xy f_{xy} + y^2 f_{yy}$

Note: If $u(x, y) = f(x, y) + g(x, y) + h(x, y)$, where f, g & h are homo. fⁿ with degree m, n & p respectively, then—

$$(a). x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = mf + ng + ph$$

$$(b). x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = m(m-1)f + n(n-1)g + p(p-1)h$$

If $f(u)$ is a homo. fⁿ in two variables x & y with degree 'n', then —

$$(a). x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{n f(u)}{f'(u)} = F(u)$$

$$(b). x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = F(u) [F'(u) - 1]$$

Note: If f_{xy} and f_{yx} at any point exists, then they may or may not be equal.

Example: If $u = \tan^{-1}(y/x)$, show that —

$$(i). x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 0 \quad (ii). x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 0$$

Solⁿ: $\frac{\partial u}{\partial x} = \frac{1}{1 + (y/x)^2} \left(\frac{-y}{x} \right) = -\frac{y}{x^2 + y^2}$

$$\frac{\partial u}{\partial y} = \frac{1}{1 + (y/x)^2} \left(\frac{1}{x} \right) = \frac{x}{x^2 + y^2}$$

$$\therefore x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 0 \quad \text{————— (i')}$$

Now, diffⁿ (i), partially w.r.t. x & y respectively, we get —

$$\frac{\partial u}{\partial x} + x \frac{\partial^2 u}{\partial x^2} + y \frac{\partial^2 u}{\partial x \partial y} = 0 \quad \text{————— (ii)}$$

$$x \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial y} + y \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{————— (iii)}$$

Multiplying (ii) by x , (iii) by y , then adding, we get —

$$x^2 u_{xx} + y u_y + x^2 u_{xx} + 2xy u_{xy} + y^2 u_{yy} = 0$$

Hence, $x^2 u_{xx} + 2xy u_{xy} + y^2 u_{yy} = 0$

Total Derivative

Theorem: Let $z = f(x, y)$ possess cts. 1st order partial derivative where $x = \phi(t)$ & $y = \psi(t)$ possess cts derivatives. Then the fn $z = f(x, y) = f(\phi(t), \psi(t))$ has total derivative

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

($\frac{dz}{dt}$ is called the total derivative of z)

Total differential Coefficient

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

Note: (i) If $f(x, y) = c$ is an implicit fn, then

$$\frac{dy}{dx} = -\frac{f_x}{f_y} \quad \left(\text{i.e. } f_x dx + f_y dy = 0 \right)$$

& $f_y \neq 0$

Corollary: If $z = f(x, y)$, where $x = \phi(u, v)$ & $y = \psi(u, v)$ then —

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$\& \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}$$

Example: Find $\frac{dz}{dt}$, when —

(i)

(a). $z = x^2 - y^2$, $x = e^t \cos t$, $y = e^t \sin t$

(b). $z = e^x \sin y$, $x = \log t$, $y = t^2$.

Solⁿ (i) (a). $\therefore \frac{\partial z}{\partial x} = 2x$, $\frac{\partial z}{\partial y} = -2y$

$$\& \frac{dx}{dt} = -e^t \sin t + e^t \cos t, \quad \frac{dy}{dt} = e^t \sin t + e^t \cos t$$

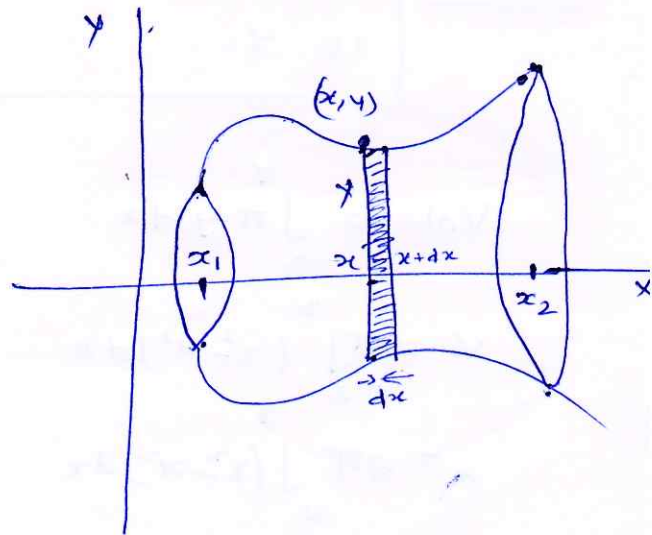
$$\therefore \frac{dz}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 2x e^t (\cos t - \sin t) + (-2y) \cdot e^t (\sin t + \cos t)$$



Volume of solid generated by revolving

$$\text{Vol. (elementary)} = \pi y^2 dx$$

$$\Rightarrow \boxed{V = \int_{x_1}^{x_2} \pi y^2 dx}$$



(I). Volume of solid obtained by revolution of curve —

(I). $y = f(x)$ about x -axis is —

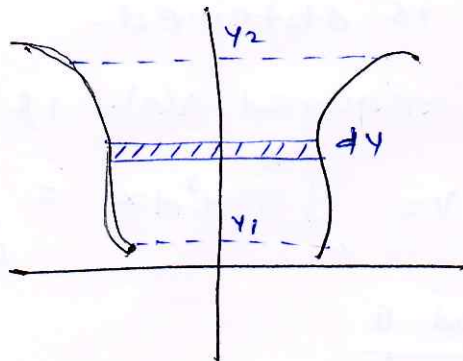
$$V = \int_{x_1}^{x_2} \pi y^2 dx$$

$$\boxed{V = \int_{x_1}^{x_2} \pi (f(x))^2 dx}$$

(II). $y = f(x)$ (or $x = g(y)$) about y -axis

$$V = \int_{y_1}^{y_2} \pi x^2 dy$$

$$\boxed{V = \int_{y_1}^{y_2} \pi (g(y))^2 dy}$$



Example: Find the volume of sphere, whose radius is r .

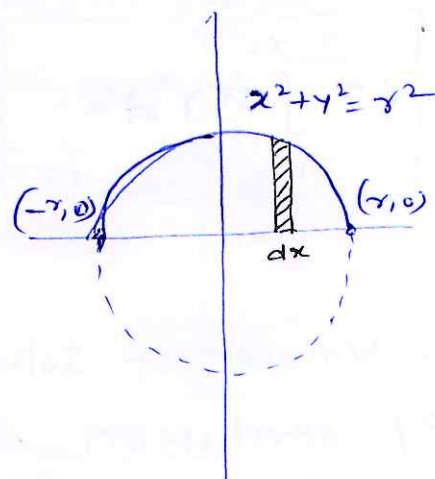
$$\text{Vol. } V = \int_{-r}^r \pi y^2 dx$$

$$V = \pi \int_{-r}^r (r^2 - x^2) dx$$

$$= 2\pi \int_0^r (r^2 - x^2) dx$$

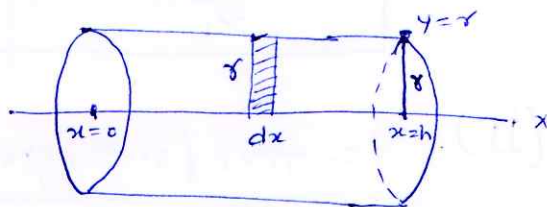
$$= 2\pi \left[\frac{r^3}{3} - \frac{x^3}{3} \right]$$

$$= 2\pi \cdot \frac{2r^3}{3} = \frac{4}{3} \pi r^3.$$



Example Using method of volume of solid revolution. Find vol. of cylinder whose radius is r & height h .

Solⁿ let line $y=r$ is revolved about x -axis from $x=0$ to $x=h$, then cylinder of radius r & height h is obtained.



So, required Vol. is

$$V = \int_0^h \pi y^2 dx = \int_0^h \pi r^2 dx = \pi r^2 h.$$

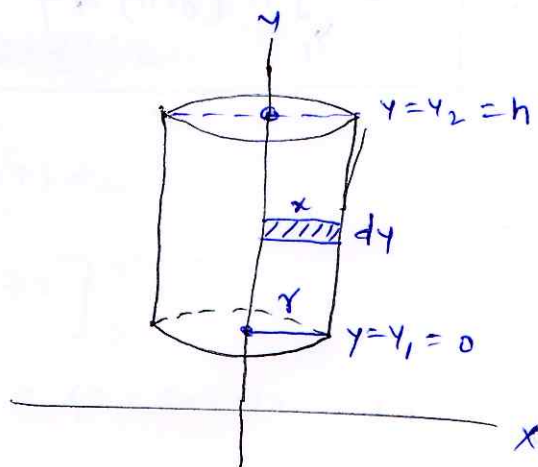
Method-II

H.W when revolved about y -axis.

so, required vol. is

$$V = \int_0^h \pi x^2 dy = \int_0^h \pi r^2 dy$$

$$= \pi r^2 h.$$



Qⁿ

The maximum value of $f(x, y) = 49 - x^2 - y^2$ on the line $x + 3y = 10$ is 39.

Solⁿ

Let $f(x, y) = 49 - x^2 - y^2$ and

$g(x, y) = x + 3y - 10$ ———— (*)

Then, By using Lagrange Multiplier λ , we have

$$F(x, y, \lambda) = f(x, y) + \lambda g(x, y)$$

$$\Rightarrow F(x, y, \lambda) = (49 - x^2 - y^2) + \lambda (x + 3y - 10)$$

Now,

$$F_x = -2x + \lambda, \quad F_y = -2y + 3\lambda$$

So, $F_x = 0 \Rightarrow \lambda = 2x$ ———— ①

& $F_y = 0 \Rightarrow \lambda = \frac{2}{3}y$ ———— ②

From eqⁿ ① & ②, we get

$$2x = \frac{2}{3}y \Rightarrow 3x = y$$

Now, Putting $y = 3x$ in eqⁿ (*), we get

$$x + 3(3x) - 10 = 0$$

$$\Rightarrow x = 1$$

$$\text{and } y = 3x = 3 \cdot 1 = 3$$

So, point is $(1, 3)$

And maximum value of $f(x, y)$ is at $(1, 3)$ i.e.

$$f(1, 3) = 49 - 1 - 9 = 39 \quad \underline{\text{Ans}}$$