Complex Analysis

[For NET/GATE/NBHM/SET/MSc/PSC, ...etc.]

(Handwritten Study Material)

P. Kalika & K. Munesh
(NET(JRF), GATE, SET)
Email: maths.whisperer@gmail.com

No. of Pages:

P Kalika Maths (www.pkalika.in)
(A Hub of Handwritten Study Materials/Solutions for CSIR-NET(JRF), GATE, SET, JAM, CUCET, NBHM, MSc/PhD Entrances, etc.)

→ Every time Question based on bilinear map & analyticity has been asked in CSIR-NET.

→ Questions based on Complex Integration, Residue, Singularity, meromorphic f^n, Analytic f^n, ... etc are very very important for NET & GATE.

→ Mostly time in GATE, Que. based on Bilinear map Cauchy Principle Value, Conformal map, .. etc comes.

Important Topics for NET & GATE

- Inequality (M-L, Cauchy, ..)
- Power Series
- Radius of Convergence
- Complex Integration
- Singularities
- Analyticity/Analytic
- Max./min. modulus Principle
- Sachwarz Lemma
- Residue thm
- Mobius transf
- Bilinear map.
- Rouche's Theorem

- Marks in CSIR-NET (28 ~ 34.50 marks)

Teacher's Signature : P. Kalika

First 30 pages are just basic Introduction and definition

Index start after page-32

Complex Analysis (Index)

Short Notes

(P. Kalika)

Introduction

Note:- (1) The concept $\sqrt{-1}$ is meaningless in the set of real numbers.

(2) Euler (1707 - 1783), Introduced the symbol i with the property $i^2 = -1$.

(3) It was gauss (1777 - 1855), who first studied that an algebraic equation with real co-efficients has complex root of the form $a + ib$, a, b are reals.

→ Euler called i as imaginary unit and a no. of the form $a+ib$, $a, b \in \mathbb{R}$ is called complex no.

→ Write $z = ~~a+ib~~ x+ib$, then z is called complex variable symbolically $z = (x, y)$, $x = R(z)$, $y = I(z)$. If $x = 0$, then $z = iy$ is called pure imaginary number. Complex conjugate of z is denoted ~~by~~ as $\bar{z}$ and defined by $\bar{z} = x - iy$.

$$R(z) = x = \frac{z+\bar{z}}{2}, \quad I(z) = y = \frac{z-\bar{z}}{2i} \text{ (verify)}$$

$$\frac{z+\bar{z}}{2} = \frac{(x+iy)+(x-iy)}{2} = \frac{2x}{2} = x = Re(z)$$

$$\frac{z-\bar{z}}{2i} = \frac{(x+iy)-(x-iy)}{2i} = \frac{+2iy}{2i} = y = Im(z)$$

z_1 & z_2 are s.t.b equal iff $x_1 = x_2$ & $y_1 = y_2$.

The prases "greater than" or "less than" have no meaning in complex no.

Fundamental operations

(i) Addition: $(a_1+ib_1)+(a_2+ib_2) = (a_1+a_2)+i(b_1+b_2)$

(ii) Substraction: $(a_1+ib_1)-(a_2+ib_2) = (a_1-a_2)+i(b_1-b_2)$

(iii) Multiplicatn $(a_1+ib_1)\times(a_2+ib_2) = (a_1a_2-b_1b_2)+i(b_1a_2+a_1b_2)$

(iv) Divi. $\dfrac{a_1+ib_1}{a_2+ib_2} = \dfrac{(a_1+ib_1)(a_2-ib_2)}{(a_2+ib_2)(a_2-ib_2)}$

$$= \frac{(a_1a_2+b_1b_2)+i(b_1a_2-a_1b_2)}{a_2^2+b_2^2}$$

$$= \left(\frac{a_1a_2+b_1b_2}{a_2^2+b_2^2}\right)+i\left(\frac{b_1a_2-a_1b_2}{a_2^2+b_2^2}\right)$$

Absolute Value (Modulus)

Absolute value of a complex No. $z=a+ib$ is denoted by $|z|=|a+ib|$ and defined as $|z|=|a+ib|=\sqrt{a^2+b^2}$

Note:- $|z|^2=a^2+b^2=(a+ib)(a-ib)=z\bar{z}$ $\quad (|z|^2=z\bar{z})$

6

GEOMETRIC REPRESENTATION OF COMPEX No.

let $z=x+iy$ be a complex No. then z can be written as ordered pair of real no.s x & y i.e $z=(x,y)$. This form of z shows that z can be represented by a point P with cordinates x & y relative to co-ordinates axis x & y.

"To each complex No. z, there corresponds one and only one pt. in the xy-plane & conversly to each pt. in the xy-plane. there corresponds one and only one complex No."

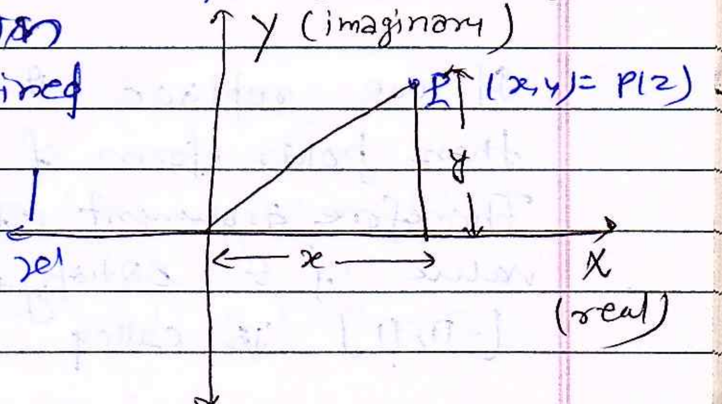
Due to this fact of complex no. z is reffered to a pt. z in the xy-plane. The xy-plane is called complex plane / Gaussion plane / Argand plane

Complex No. $x+iy$ is called Complex Co-ordinate (x,y) and x, y axes are called real & imaginary axes. The representation of a complex No. in the Argand plane is called Argand diagram

Distance b/n two no. in complex plane is defined as $d=|z_1-z_2|$

$$=|(x_1-x_2)+i(y_1-y_2)|$$

$$=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$$



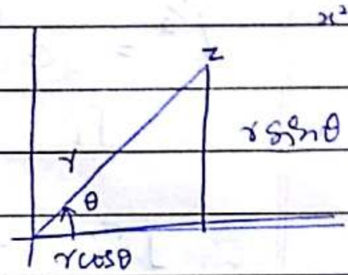
* * *

Motivation: Look at $x^2+1=0$, has no real roots

$\mathbb{C} := \{a+ib : a, b \in \mathbb{R}\}$ where $i=\sqrt{-1}$ | $i^2=-1$ $\Rightarrow i^2+1=0$, i is root of $x^2+1=0$

$z = \underline{a+ib}$, $\bar{z} = a-ib$

$|z| = \sqrt{a^2+b^2}$, $z = re^{i\theta}$



$$z = r\cos\theta + ri\sin\theta$$
$$= r(\cos\theta + i\sin\theta)$$
$$= \underline{re^{i\theta}}$$

Square root of a complex root :-

If $\alpha + i\beta \in \mathbb{C}$, then $\sqrt{\alpha+i\beta} = ??$

let $x+iy = \sqrt{\alpha+i\beta}$

$$\Rightarrow (x+iy)^2 = \alpha + i\beta$$
$$\Rightarrow x^2-y^2+i2xy = \alpha+i\beta$$
$$\Rightarrow x^2-y^2 = \alpha \quad \text{and} \quad 2xy = \beta$$
$$\Rightarrow (x^2+y^2)^2 = (x^2-y^2)^2 + 4x^2y^2$$
$$= \alpha^2+\beta^2$$
$$x^2+\cancel{y^2} = \sqrt{\alpha^2+\beta^2}$$
$$x^2-\cancel{y^2} = \alpha$$
$$x^2 = \frac{1}{2}\left[\alpha + \sqrt{\alpha^2+\beta^2}\right]$$
$$\& \quad y^2 = \frac{1}{2}\left(\alpha+\sqrt{\alpha^2+\beta^2}\right) - \alpha$$
$$= \frac{1}{2}\left(\sqrt{\alpha^2+\beta^2} - \alpha\right)$$

i.e. $\sqrt{\alpha+i\beta} = x+iy$

$$= \sqrt{\tfrac{1}{2}}\left\{\sqrt{\alpha+\sqrt{\alpha^2+\beta^2}} + i\sqrt{\sqrt{\alpha^2+\beta^2}-\alpha}\right\}$$

Complex → Radian

$\because e^{i\theta} = \cos\theta + i\sin\theta$

M.C Lemon

Page No. 07
Date 30 10 17

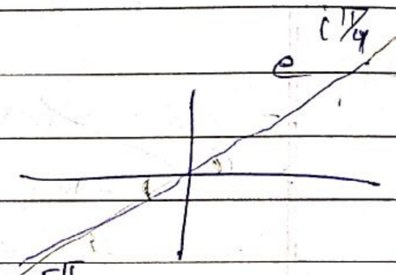
(1) $\sqrt{i} = ??$

$i = e^{i\pi/2} = e^{i(\pi/2 + 2k\pi)}, k \in N$

$i^{1/2} = e^{\frac{i(\pi/2 + 2k\pi)}{2}} ; k = 0, 1$

$\therefore \sqrt{i} = e^{i\pi/4}, e^{i\left(\frac{\pi/2 + 2\pi}{2}\right)}$

$\therefore \sqrt{i} = e^{i\pi/4}, e^{i5\pi/4}$



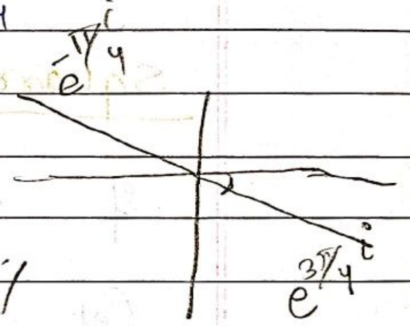
(2) $\sqrt{-i}$

$-i = e^{-i\pi/2}$

$(-i)^{1/2} = e^{\frac{i(-\pi/2 + 2k\pi)}{2}}, k = 0, 1$

$= e^{i(-\pi/4 + 0)}, e^{i(-\pi/4 + \pi)}$

$= \ldots$



(3) $\sqrt{\frac{1 - i\sqrt{3}}{2}}$, let $\frac{1 - i\sqrt{3}}{2} = \frac{1}{2} - i\frac{\sqrt{3}}{2} = \cos\pi/3 - i\sin\pi/3 = \cos 5\pi/3 + i\sin 5\pi/3 = e^{i5\pi/3}$

$= e^{i(\frac{5\pi}{3} + 2k\pi)}$

$\left(\frac{1 - i\sqrt{3}}{2}\right)^{1/2} = e^{\frac{i}{2}(5\pi/3 + 2k\pi)} ; k = 0, 1$

$\therefore \left(\frac{1 - i\sqrt{3}}{2}\right)^{1/2} = e^{i5\pi/6}, e^{11\pi/6}$

(4) If $G = \left\{ \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} ; \alpha, \beta \in \mathbb{R} \right\}$ then $G \cong \mathbb{C}$ (T/F)

(True)

Soln: G is group under addn.

$\phi: \mathbb{C} \to G \quad \phi(\alpha + i\beta) = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$ (Check 1-1, onto)

Let $\phi(\alpha + i\beta) = \phi(x + iy)$

(3) $f(z) = \dfrac{e^z}{z-1}$, $0 < |z-1|$, i.e. Centre = 1

$= \dfrac{1}{z-1} e^{(z-1)+1} = \dfrac{e}{z-1}\left[e^{z-1}\right]$

$= \dfrac{e}{z-1}\left[1 + (z-1) + \dfrac{(z-1)^2}{2!} + \dfrac{(z-1)^3}{3!} + \cdots\right]$

$= \underbrace{\dfrac{e}{z-1}}_{\text{P.P (finite)}} + \underbrace{e + \dfrac{(z-1)e}{2!} + \dfrac{e(z-1)^2}{3!} + \cdots}_{\text{R.P (infinite)}}$

Res. at $z=1$, = Co.eff. of $\frac{1}{z-1}$ = e

(N) About Singularity

In above eg. if No. of terms in —

(1). P.P has no term → Removable Sing.

(2). P.P has finite term (say 4) → Pole of the order 4

(3). P.P. has infinite No. of term ⇒ Essential singularity.

eg (4) $f(z) = z\cos\frac{1}{z}$, $0 < |z|$

$= z\left[1 - \dfrac{1}{z^2 2!} + \dfrac{1}{z^4 4!} - \dfrac{1}{z^6 6!} + \cdots\right]$

$= \underbrace{z}_{\text{RP}} \underbrace{- \dfrac{1}{z\,2!} + \dfrac{1}{z^3 4!} - \dfrac{1}{z^5 6!} + \cdots}_{\text{P.P (Infinite)}}$

So, at $z=0$, $f(z)$ has essential singularity

∴ Res at $z=0$ = co-eff of $\frac{1}{z} = -\frac{1}{2}$

(5) $f(z) = \frac{1}{z(z-3)}$: $0 < |z| < 3$

$\frac{1}{z}\left[\frac{1}{z-3}\right] = \frac{-1}{3z}\left[\frac{1}{1-z/3}\right]$

NB : make $|z| < 1$, for which take largest as common like above (Here $3 > 2$ so, take 3 common)
(as $a < |z| < b \Rightarrow a < |z|$, $\frac{a}{|z|} < 1$ or $\frac{|z|}{b} < 1$)

$= \frac{-1}{3z}\left[1 + \frac{z}{3} + \frac{z^2}{9} + \frac{z^3}{27} + \cdots\right]$

$= \underbrace{-\frac{1}{3z}}_{PP} \underbrace{- \frac{1}{9} - \frac{z}{27} - \frac{z^2}{81} - \cdots}_{RP}$

Singularity : Pole of order 1 $(z=0)$
Residue : at $z=0$, $= -1/3$

(6) $f(z) = \frac{1}{z(z-3)}$, $|z| > 3$ $\Rightarrow \frac{3}{|z|} < 1$

$= \frac{1}{z}\left[\frac{1}{z-3}\right] = \frac{1}{z}\cdot\frac{1}{z}\left[\frac{1}{1-3/z}\right] = \frac{1}{z^2}\left[1-3/z\right]^{-1}$

$= \frac{1}{z^2}\left[1 + \frac{3}{z} + \left(\frac{3}{z}\right)^2 + \left(\frac{3}{z}\right)^3 + \cdots\right]$

$= \frac{1}{z^2} + \frac{3}{z^3} + \frac{3^2}{z^3} + \cdots$

Singularity : Essential sing. (bcz infinite term)
Residue : at $z=0$, $= 0$

(7) $f(z) = \frac{1}{(z-1)(z+2)}$: $0 < |z-1| < 1$

$= \frac{1}{(z-1)}\left[\frac{1}{3+(z-1)}\right] = \frac{1}{3(z-1)}\left[\frac{1}{1+\frac{(z-1)}{3}}\right]$

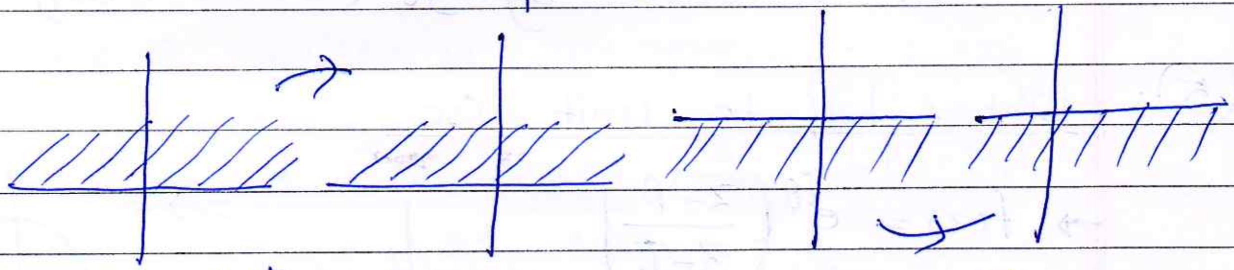
$= \frac{1}{3(z-1)}\left[1 - \left(\frac{z-1}{3}\right) + \left(\frac{z-1}{3}\right)^2 - \cdots\right]$

w.1 for exam

Important Results (B.T)

1. The most general B.T. that maps upper half (lower half) plane onto itself is given by

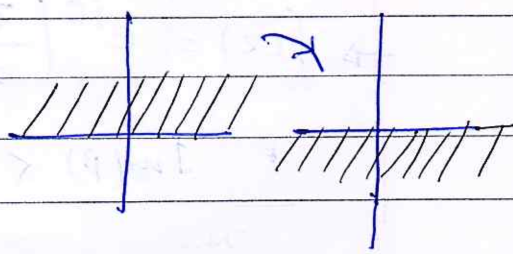
$$f(z) = \frac{az+b}{cz+d} \quad : a, b, c, d \in \mathbb{R}$$
$$\& \ ad - bc > 0$$



$H^+ \to H^+$ $\qquad$ $H^- \to H^-$

2. The most general B.T. that maps upper half (Lower half) to Lower half (Upper half) is given by —

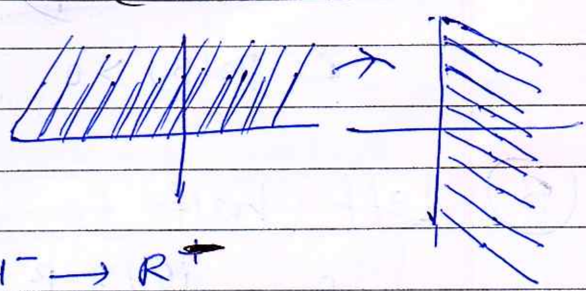
$$f(z) = \frac{az+b}{cz+d} \quad : a, b, c, d \in \mathbb{R}$$
$$\& \ ad - bc < 0$$



i.e if $\underline{|D| < 0}$; $H^+ \to H^-$ & $H^- \to H^+$

3. Upper (lower) half to Right ~~Left~~ (~~Right~~ Left) half.

$$f(z) = e^{-i\pi/2} \cdot \frac{az+b}{cz+d}$$
$$, \ ad - bc > 0$$



then $H^+ \to R^+$ & $H^- \to R^-$

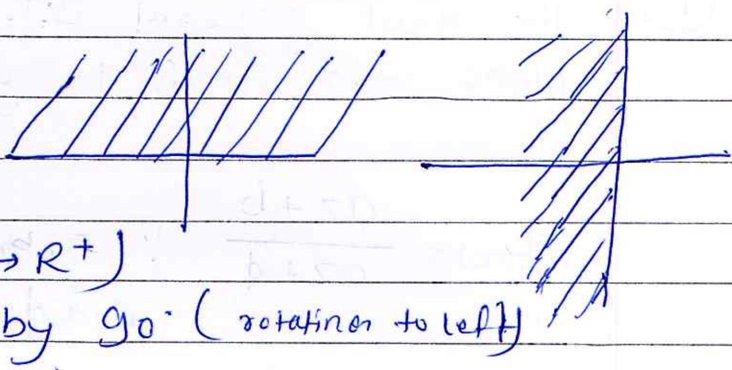
→ i.e Rotation by 90° to ~~left~~ → Right side.
i.e rotate by $\underline{-90°}$. (↻)

P. Kalika

(4). Upper (Lower) half to Left (Right) half —

$$f(z) = e^{i\pi/2}\,\frac{az+b}{cz+d}$$

$ad - bc > 0$



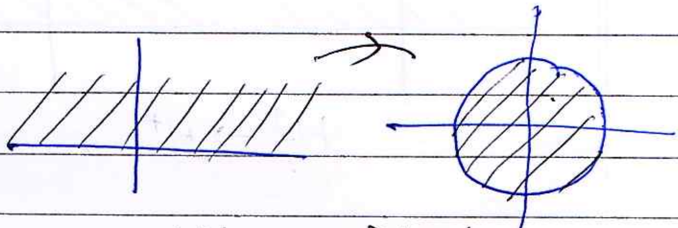
$(H^+ \to R^-,\ H^- \to R^+)$

— i.e Rotation by 90° (rotation to left)

(5). Upper half to Unit disc

$\to$ $f(z) = e^{i\theta}\left(\dfrac{z-\beta}{z-\bar{\beta}}\right)$.

$Im(\beta) > 0$

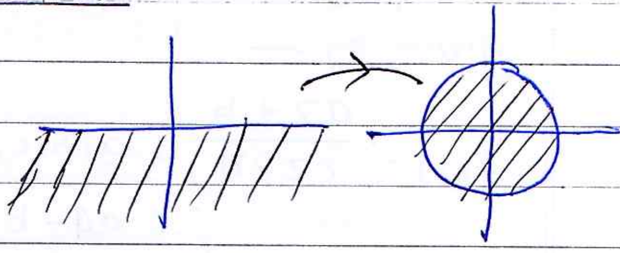


$H^+ \longrightarrow D(0,1)$

(6). Lower half to Unit disc.

$\to$ $f(z) = e^{i\theta}\left(\dfrac{z-\beta}{z-\bar{\beta}}\right)$

& $Im(\beta) < 0$

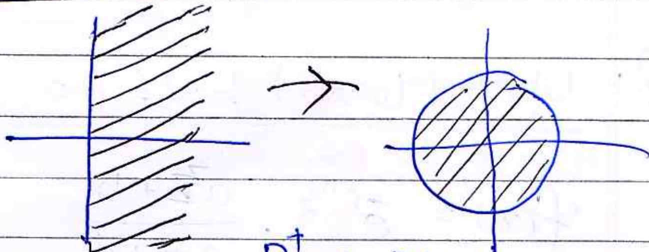


$H^- \to D(0,1)$

(7). Right half to Unit disc.

$\to$ $f(z) = e^{i\theta}\left(\dfrac{z-\beta}{z+\bar{\beta}}\right)$.

& $Re(\beta) > 0$

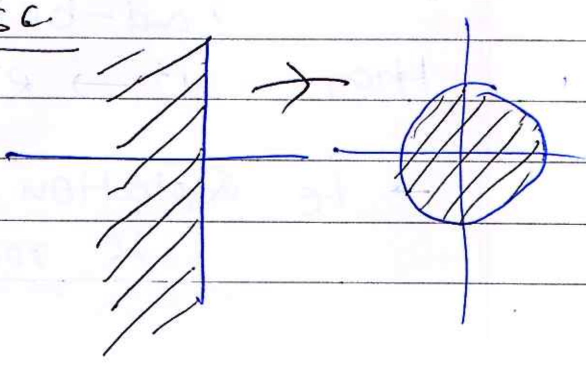


$R^+ \to D(0,1)$

(8) Left half to Unit disc

$\to$ $f(z) = e^{i\theta}\left(\dfrac{z-\beta}{z+\bar{\beta}}\right)$,

& $Re(\beta) < 0$



$(R^- \to D(0,1))$

Question Based on ~~Aa~~ Cauchy Principal Value, GATE-2009

Consider the f^n $f(z) = \dfrac{e^{iz}}{z(z^2+1)}$.

(2M)
① The residue of f at the isolated singular point in the upper half plane $\{z = x+iy \in \mathbb{C} : y>0\}$ is ——

ⓐ $\dfrac{-1}{2e}$ ⓑ $\dfrac{-1}{e}$ ⓒ $\dfrac{e}{2}$ ⓓ. 1

Sol^n:-

$$f(z) = \frac{e^{iz}}{z(z^2+1)} = \frac{e^{iz}}{z(z+i)(z-i)}, \quad \{z = x+iy;\ y>0\}$$

∴ Residue at $z = i$

$$= \lim_{z\to i}\Big((z-i)\,f(z)\Big)_{(z=i)}$$

$$= \frac{e^{iz}}{z(z+i)}\Big|_{(z=i)} = \frac{e^{i\cdot i}}{i(i+i)} = \frac{e^{-1}}{-2}$$

$$= \frac{-1}{2e} \quad \text{— ⓐ}$$

(2M)
② The Cauchy Principal Value of the integral

$$\int_{-\infty}^{\infty} \frac{\sin x}{x(x^2+1)}\,dx.$$

ⓐ. $\dfrac{-2\pi}{(1+2e^{-1})}$ ✓ⓑ. $\pi(1-e^{-1})$ ⓒ. $2\pi(1+e)$

ⓓ. $-\pi(1+e^{-1})$.

Theory (p-106)

Sol^n:- $f(x) = \dfrac{\sin x}{x(x+i)(x-i)}$

Cauchy Principal Value $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx$

$$= 2\pi i \sum_{\mathrm{Img}\,z>0} \mathrm{Res}\,f(z) + \pi i \sum_{\mathrm{Img}\,z=0} \mathrm{Res}(z).$$

Milne-Thomson's Method

(To construct a Regular f^n & Analytic f^n in terms of z)

Case-I, When u is given (put $x=z$, $y=0$ in u_x & u_y)

$$f(z) = \int u_x(z,0)\,dz - i\int u_y(z,0)\,dz + C$$

Case-II, When v is given (put $x=z$, $y=0$ in v_x, v_y)

$$f(z) = i\int v_x(z,0)\,dz + \int v_y(z,0)\,dz + C$$

(You may use the Rel^n $u_x = v_y$, $u_y = -v_x$)

Example ① If $u = e^x(x\cos y - y\sin y)$, find $f(z)$.

$\because u_x = e^x(x\cos y - y\sin y) + e^x(\cos y) = u_x(x,y)$

$\therefore u_x(z,0) = e^z(z) + e^z = e^z(z+1)$

& $u_y(x,y) = e^x\left(x(-\sin y) - \sin y - y\cos y\right)$

$\therefore u_y(z,0) = e^z(0) = 0$

$\therefore$ By Milne's method —

$$f(z) = \int (e^z + ze^z)\,dz - i\int 0\,dz + C$$

$$= e^z + ze^z - \int e^z\,dz + C$$

$$f(z) = e^z + ze^z - e^z + C = ze^z + C$$