P KALIKA NOTES

[For NET/GATE/SET/NBHM/JAM/CUCET/MSc/PhD Exams…etc.]

Functional Analysis

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Your Note/Remarks

Some Inequalities

Theorem ①. Let $1 < p, q < \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1$ then for any non-negative real no.s a, b we have

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

Hölder's Inequality (Finite form)

Let $1 \leq p < \infty$ and q be conjugates exponent of p (i.e q is s.t $\frac{1}{p} + \frac{1}{q} = 1$). Let a_i & b_i be complex no.s (in particular they can be real no.s) then —

$$\boxed{\sum_{i=1}^{n} |a_i b_i| \leq \left(\sum_{i=1}^{n} |a_i|^p \right)^{1/p} \left| \sum_{i=1}^{n} |b_i|^q \right|^{1/q}}$$

(Pf: Hint: Take $\alpha = \left(\sum_{i=1}^{n} |a_i|^p \right)^{1/p}$, $\beta = \left| \sum_{i=1}^{n} |b_i|^q \right|^{1/q}$ then take $a = \frac{|a_i|}{\alpha}$, $b = \frac{|b_i|}{\beta}$ in prev. inequality)

Minkowski's Inequality

Let $1 < p < \infty$ and let a_i, b_i are complex no.s then —

$$\boxed{\left(\sum_{i=1}^{n} |a_i + b_i|^p \right)^{1/p} \leq \left(\sum_{i=1}^{n} |a_i|^p \right)^{1/p} + \left(\sum_{i=1}^{n} |b_i|^p \right)^{1/p}}$$

Def^n (NORM): Let X be a vector space over field $\mathbb{F}$ (where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$). A Norm on X is a real valued f^n whose value at x is denoted by $\|x\|$ satisfies the following conditions:

(i). $\|x\| > 0$ if $x \neq 0$

(ii). $\|\alpha x\| = |\alpha| \, \|x\|$

(iii). $\|x+y\| \leq \|x\| + \|y\|$

Eg. W.o.t.f is true ?

× (1). $\|\alpha x\| = |\alpha| |x|$

✓ (2). $|\alpha x| = |\alpha| \, |x|$

× (3). $\|\alpha x\| = \|\alpha\| \, |x|$

× (4) $|\alpha x| = \|\alpha\| \, |x|$

NB: $\underset{\text{vector}}{\|0\|} = \|\underset{\text{scalar}}{0} \cdot \underset{\text{vector}}{x}\| = \underset{\text{scalar}}{|0|} \, \|x\| = \underset{\text{scalar}}{0}$

NLS → Thus $(X, \mathbb{F}, +, \cdot, \|\cdot\|)$ is called a normed linear space (NLS).

Example: (5) Let $(X_1, \|\cdot\|_1)$ and $(X_2, \|\cdot\|_2)$ be two n.l.s. over $\mathbb{F}$. On $X_1 \times X_2$ define add^n and scalar multiplication as follow —

$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, \, y_1 + y_2)$

$\alpha (x, y) = (\alpha x, \alpha y)$

Clearly $X_1 \times X_2$ is a vector space (v.s)

(i) ~~define~~ for $v = (v_1, v_2) \in X_1 \times X_2$

define $\|v\| = \|v_1\|_1 + \|v_2\|_2$ then

$(X_1 \times X_2, \|\cdot\|)$ is a n.l.s.

(ii). define $\|v\| = \max.\{\|v_1\|_1, \|v_2\|_2\}$ then

$(X_1 \times X_2, \|\cdot\|)$ is a n.l.s

Note: We can have more than one norm on a given vector space.

Defn: A sequence $\{x_n\}$ in a metric space is s.t.b 'Cauchy sequence' if $\forall \varepsilon > 0$, $\exists\, n_0(\varepsilon)$ s.t –

$$d(x_n, x_m) < \varepsilon \quad \forall\, n, m \geq n_0(\varepsilon)$$

Defn: A metric space (X, d) is s.t.b complete iff every Cauchy seq. in X converges in X.

Defn: A complete normed linear space is called a 'Banach space'.

Example: (6) The space $\mathbb{R}^n$ and $\mathbb{C}^n$ are Banach spaces with the norm defined by

$$\|x\| = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2} \quad \text{where } x = (x_1, x_2, \ldots, x_n)$$

this norm is denoted by $\|\cdot\|_2$.

(7) (In general, $\|x\| = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$, $\|\cdot\|_p$)

Example of Banach space

①. $(\mathbb{R}^n, \|\cdot\|)$

$$\|x\|_2 = \sqrt{x_1^2+x_2^2+\dots+x_n^2}, \quad x=(x_1,\dots x_n)$$

(2). $(\mathbb{C}^n, \|\cdot\|_2)$

$$\|x\|_2 = \sqrt{|z_1|^2+\dots+|z_n|^2}, \quad x=(z_1,\dots,z_n)$$

(3). $(C[0,1], \|\cdot\|_\infty)$.

$$C[0,1] = \{f:[0,1]\to\mathbb{R} \text{ s.t } f \text{ is cts.}\}$$

(4). $\ell_p : 1\le p<\infty$, $\|x\|_p = \left(\sum |x_n|^p\right)^{1/p}$ $\quad 1\le p<\infty$

$$\|x\|_\infty = \sup_n |x_n| \qquad p=\infty$$

(5). $L^p(x) : 1\le p\le\infty$

$1\le p<\infty$, $\|f\|_p = \left(\int |f|^p d\mu\right)^{1/p}$

$p=\infty$, $\|f\|_\infty = \sup_{x\in X} |f(x)|$

(6). $B[0,1] = \{f:[0,1]\to\mathbb{R} \mid f \text{ is b.dd}\}$

$\|f\|_\infty = \sup |f(x)|$

$d(f,g) := \|f-g\|_\infty$

(7) ⓐ If $k_1<k_2$ then $\|\bar{x}\|_{k_2} \le \|\bar{x}\|_{k_1}$

ⓑ If $k_1<k_2$ then $\ell^{k_1} < \ell^{k_2}$

ⓒ. $\ell^1 \subset \ell^2 \subset \dots \subset \ell^\infty$ but NOT equal.

(8). $(C[0,1], \sup\|\cdot\|)$ is Banach space

FUNCTIONAL ANALYSIS

GATE-2022

(15) Let X be a real normed linear space. Let $X_0 = \{x \in X : \|x\| = 1\}$. If X_0 contains two distinct points x & y and the line segment joining them, then w.o.t.f statements is TRUE?

(A). $\|x+y\| = \|x\| + \|y\|$ and x, y are L.I.

(B). $\|x+y\| = \|x\| + \|y\|$ and x, y are L.D.

(C). $\|x+y\|^2 = \|x\|^2 + \|y\|^2$ and x, y are LI.

(D). $\|x+y\| = 2\|x\|\|y\|$ and x, y are LD.

Sol^n

It can be easily solved by choosing a particular eg.

Let $X = \mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ then $x \in X \Rightarrow x = (x_1, x_2)$

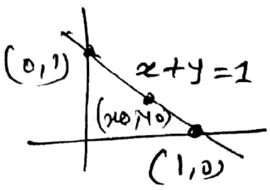
and define norm $\|\cdot\|$ as

$$\| x \| = \| (x_1, x_2) \| = |x_1| + |x_2| \quad \text{——} (1)$$

Now, let two points from X_0 say $(1, 0) = x$, $(0, 1) = y$

then $\because \|x\| = |1| + |0| = 1$ & $\|y\| = |0| + |1| = 1$

$\Rightarrow x, y \in X_0$



and their line segment is $x + y = 1$

Consider any point on the line, say (x_0, y_0)

then, $\because \|(x_0, y_0)\| = |x_0| + |y_0| = x_0 + y_0 = 1 \quad (\because x_0, y_0 \geq 0)$

so clearly (line segment belongs to X_0.

Here, our assumption satisfy the hypothesis.

Isomorphism : If $T \in L(X,Y)$ then T is s.t.b an isomorphism if T is bijective and T^{-1} is bdd (cts.)

THM: Let $T \in L(X,Y)$. Then T is an isomorphism iff $\exists\ c > 0$ s.t $\|T(x)\| \geq c\|x\|$ $\forall x \in X$.

Pf: $T^{-1} : Y \to X$ is cts. (bdd) (Assume that T is isomorphism)

$\therefore \exists\ c > 0$ s.t $\|T^{-1}y\| \leq c\|y\|$ (By def^n of b.dd)

$\Rightarrow \|x\| \leq c\|T(x)\|$

$\Rightarrow \|T(x)\| \geq \frac{1}{c}\|x\|$

$\Leftarrow$ $\because \|T(x)\| \geq c\|x\|$ (let $Tx = y$, $x = T^{-1}y$)

$\Rightarrow \|y\| \geq c\|T^{-1}y\|$

Let $T \circ T^{-1} = I_y$

* **Isometry :** $\|Tx\| = \|x\|$ $\forall x \in X$ where $T \in L(X,Y)$

THM: If T is isometry then T is $1-1$.

NET-16D

Que: Let $T : l_\infty \to l_2$ by $T(a_1, \ldots a_n) = (a_1, \frac{a_2}{2}, \ldots, \frac{a_n}{n}, \ldots)$

Then—

ⓐ. T is cts. linear Map ⓑ T is surjective

ⓒ. T^{-1} exists & cts. ⓓ. T is unif. cts.

Sol^n: $\because l_2 : \|x_2\| = \left(\sum |x|^2\right)^{1/2}$

$\therefore \|T(a_1 \ldots a_n)\|_2^2 = \sum_{n=1}^{\infty} \left|\frac{a_n}{n}\right|^2$