# Integral Equation

[For NET/SET/PSC/BHU/CUCET/MSc ...etc.]

(Handwritten Study Material & NET GATE Solutions)

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# Integral Equation (IE)

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| Index | Abbrevation |
|---|---|
|  | ✓ I.E = Integral Equation |
|  | ✓ D.E. = Differential Equation |
|  | ✓ w.r.t = with respect to |
|  | ✓ eqn. = equation. |
|  | ✓ $f^n$ = function |
|  | ✓ w.o.t.f = which of the following |
|  | ✓ $sol^n$ = solution |
|  | ✓ diff = differentiation |
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Def$^n$: INTEGRAL EQUATION (IE)

An I.E. is an eq$^n$ in which an unknown function appears under one or more integral signs.
i.e of the form

$$y(x)\cdot g(x) = f(x) + \lambda \int_{a(x)}^{b(x)} K(x,t)\cdot y(t)\, dt \quad —(*)$$

Where $a(x)$, $b(x)$ are f$^n$ of $x$; $f(x)$, $g(x)$, and $K(x,t)$ are known functions and $y(t)$ is unknown f$^n$ and $\lambda$ is a n-z real / complex parameter. $K(x,t)$ is known as the Kernel of Integral Equation (I.E.)

# Types of Integral Equation

1. Fredholm Integral Eqn.
2. Volterra Integral Eqn.

(A). Fredholm Integral Equation

If $a(x)$ & $b(x)$ are constant in the I.E (*), then I.E. (*) is called FREDHOLM INTEG. EQn
It is of the form

$$\boxed{y(x)g(x) = f(x) + \lambda \int_a^b K(x,t)\, y(t)\, dt} \quad —(a)$$

(I) Fredholm IE of first Kind

If $g(x)=0$ in (a), then
i.e

$$\boxed{f(x) = \lambda \int_a^b K(x,t)\cdot y(t)\, dt}$$

is known as Fredholm Integral Eqn. of first Kind.

(II). Fredholm I.E. of $2^{nd}$ Kind

If $g(x) = 1$ in eq$^n$ (a). i.e a linear integral eqn. of the form

$$\boxed{y(x) = f(x) + \lambda \int_a^b K(x,t)\, y(t)\, dt}$$

is known as Fredholm Integral Eq$^n$ of second Kind.

(III). Homogeneous Fredholm Integ. Eqn. of $2^{nd}$ Kind:

If $f(x) = 0$ in eqn (II) i.e a linear I.E. of the form

$$\boxed{y(x) = \lambda \int_a^b K(x,t)\, y(t)\, dt}$$

is known as the "Homo. Fredholm I.E." of second Kind.

(B). VOLTERRA INTEGRAL EQUATION

An linear integral eqn. of the form

$$\boxed{g(x) \cdot y(x) = f(x) + \lambda \int_{a(x)}^{b(x)} K(x,t)\, y(t)\, dt} \quad \text{—(b)}$$

is known as VOLTERRA I.E if either $a(x)$ or $b(x)$ or both are function of $x$.

(I). Volterra I.E. of first Kind

If $g(x) = 0$ in (b), i.e. a linear IE of the form

$$\boxed{f(x) \cancel{+} \lambda \int_{a(x)}^{b(x)} K(x,t) \cdot y(t)\, dt = 0}$$

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# Volterra Integral Equation of $2^{nd}$ Kind

Given eqn. type

$$u(x) = f(x) + \lambda \int_a^x K(x,t)\, u(t)\, dt \quad \text{—— (2)}$$

There are two method to solve I.E (2).
(1). Approximation method.
(2). Resolvent Kernel Method.

## (1) Approximation method

take $u_0(x) = f(x)$ as the initial/first approximation soln. of (2).

$\therefore u_0(t) = f(t)$.

$$u_1(t) = f(x) + \lambda \int_a^x K(x,t)\, u_0(t)\, dt$$

$$u_2(t) = f(x) + \lambda \int_a^x K(x,t)\, u_1(t)\, dt$$

$\vdots$

$$u_n(t) = f(x) + \lambda \int_a^x K(x,t)\, u_{n-1}(t)\, dt$$

Thus, soln of I.E (2) is given by

$$\boxed{u(x) = \lim_{n\to\infty} u_n(x)}$$

**Que (12)** Find soln of $u(x) = 1 - \int_0^x (x-t)\, u(t)\, dt$ using approximation method—

**Soln** We have $u(x) = 1 - \int_0^x (x-t)\, u(t)\, dt$

$K(x,t) = x-t$, $\lambda = -1$, $f(x) = 1$

$\therefore u_0(x) = f(x) = 1$

$$u_1(x) = 1 - \int_0^x (x-t)\,dt = 1 - x(t)\Big|_0^x + \frac{t^2}{2}\Big|_0^x$$

$$\frac{(x-t)^2}{-2}\Big|_0^x = +\frac{x^2}{2}$$

$$= 1 - x(x-0) + (x^2/2 - 0)$$

$$= 1 - x^2/2$$

$$u_2(x) = 1 - \int_0^x (x-t)\cdot(1-t^2/2)\,dt$$

$$= 1 - \int_0^x \left[x(1-t^2/2) - t + t^3/2\right] dt$$

$$= 1 - \left[x^2 - x^4/6 - x^2/2 + x^4/8\right]$$

$$u_2(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \cancel{\cos x}$$

$$u_3(x) = 1 - \int_0^x (x-t)\,u_2(t)\,dt$$

$$= 1 - \int_0^x (x-t)\left(1 - x^2/2! + x^4/4! - \cdots\right) dt$$

$$= 1 - \left[xt - \frac{t^3}{3!}x + \frac{t^5}{5!}x - \frac{t^2}{2} + \frac{t^4}{8} - \frac{t^6}{146}\right]_0^x$$

$$= 1 - x^2/2! + x^4/4! - x^6/6! + \cdots$$

$$\therefore u_{n+1}(x) = \sum_{m=1}^{m} (-1)^m \frac{x^{2m}}{(2m)!}$$

$$\therefore u(x) = \lim_{n\to\infty} u_{n+1}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$= \cos x$$

$$\Rightarrow \boxed{u(x) = \cos x}$$

→ If options are given (a) $\cos x$ (b) $e^x$ (c) $-\cos x$ (d) None

Hint: Check for $u(0) = 1 \Rightarrow$ op (d) discarded

Next check for $u'(0) = 0 \Rightarrow$ op(a) only true.

If required check $u''(0) =$ ___

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(2) [Integral Eqn.] : If $y$ is a solⁿ of

$$y(x) - \int_0^x (x-t)\,y(t)\,dt = 1,$$

then w.o.t.f is true?

c/45

Repeated pattern

(a). $y$ is b.dd but NOT periodic in $\mathbb{R}$

(b). $y$ is periodic in $\mathbb{R}$.

(c). $\int_{\mathbb{R}} y(x)\,dx < \infty$

(d). $\int_{\mathbb{R}} \frac{dx}{y(x)} < \infty$

Ans: 4

Solⁿ Given: $y(x) = \int_0^x (x-t)\,y(t)\,dt \cancel{+}\, 1$ — (*).

We solve it by Laplace transformation.

Convolution: $f(x) * g(x) = \int_0^x f(x-t)\,g(t)\,dt$

& $L\left(f(x) * g(x)\right) = L\left(f(x)\right) \cdot L\left(g(x)\right)$

Now applying Laplace on (*).

$$L(y(x)) = L(1) + L\left[\int_0^x f(x-t)\,y(t)\,dt\right]$$

$$= \frac{1}{s} + L\left(x * y(x)\right) \qquad \text{(: Hint check Result)}$$

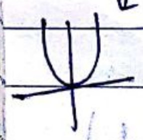
$$= \frac{1}{s} + L(x) \cdot L[y(x)]$$

$$= \frac{1}{s} + \frac{1}{s^2} L\{y(x)\}$$

$$\Rightarrow \left(1 - \frac{1}{s^2}\right) L(y(x)) = \frac{1}{s}$$

$$\Rightarrow L(y(x)) = \frac{s}{s^2-1}$$

cosh(x)



$$\Rightarrow y(x) = L^{-1}\left[\frac{s}{s^2-1}\right] = \cosh(x) = \frac{e^x + e^{-x}}{2}$$

∴ $\cosh(x)$ is b.dd below only & not periodic ⇒ op(1). False and also op(2). False.

op(3). $\int_{\mathbb{R}} y(x)\,dx = \int \cosh x\,dx = \sinh(x)\Big|_{-\infty}^{\infty} = \infty$, so False

op(4). only this left ⇒ op(4) True.

Hint: $\int \frac{1}{\cosh x} = \int \frac{2}{e^x + e^{-x}}$ on solving $4 \cdot \pi/2 = 2\pi < \infty$

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NET 2019 June

(3) [Integral Eqⁿ]: Consider the Integral Eqⁿ (4.75M)

$$\phi(x) - \frac{e}{2}\int_{-1}^{1} x e^{t}\phi(t)\,dt = f(x).$$ Then —

(1) ∃ a cts fⁿ $f:[-1,1]\to(0,\infty)$ for which solⁿ exists
(2) ∃ a cts. fⁿ $f:[-1,1]\to(-\infty,0)$ for which solⁿ exists.
(3) ✓ for $f(x)=e^{-x}(1-3x^2)$, a solⁿ exists.
(4) ✓ for $f(x)=e^{-x}(x+x^3+x^5)$, a solⁿ exists.

Ans: 3, 4

Solⁿ Given: $\phi(x)=f(x)+\frac{e}{2}\int_{-1}^{1} x e^{t}\phi(t)\,dt$ — (1)

Let $c=\int_{-1}^{1} e^{t}\phi(t)\,dt$

$$\Rightarrow \boxed{\phi(x)=f(x)+\frac{xe}{2}c} \quad — (2)$$

$$\Rightarrow c=\int_{-1}^{1}\left(e^{t}f(t)+\frac{t}{2}e^{t+1}c\right)dt$$

$$\Rightarrow c=\int_{-1}^{1}e^{t}f(t)\,dt+\int_{-1}^{1}\frac{t}{2}e^{t+1}c\,dt$$

$$\Rightarrow \cancel{c}=\int_{-1}^{1}e^{t}f(t)+\frac{e}{2}\left[te^{t}-\int_{-1}^{1}e^{t}dt\right]_{-1}^{1}c$$

$$\cancel{\Rightarrow \frac{ce}{2}\left[te^{t}-e^{t}\right]_{-1}^{1}}$$

$$\Rightarrow c=\int_{-1}^{1}e^{t}f(t)\,dt+\frac{e}{2}\left[\cancel{e}+e^{-1}-\cancel{e}+e^{-1}\right]c$$

$$\Rightarrow c=\int_{-1}^{1}e^{t}f(t)\,dt+\left(\frac{e}{2}\cdot 2e^{-1}\right)c$$

$$\Rightarrow \boxed{\int_{-1}^{1}e^{t}f(t)\,dt=0} \quad — (3)$$

Now, discarding options

op(1). Here $f(t)>0\ \forall t\in[-1,1]$ & $e^{t}>0\ \forall t$
$\Rightarrow e^{t}f(t)>0 \Rightarrow \int_{-1}^{1}e^{t}f(t)>0$ (by (3))
so op(1). Incorrect.

Op(2) similar to op(1). Integral $<0$, so ~~Incorrect~~
op(2) Incorrect.

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op(2) : $f(t) < 0 \ \forall t\ [-1,1] \Rightarrow e^t f(t) < 0$

$\Rightarrow \int_{-1}^{1} e^t f(t)\,dt < 0 \Rightarrow$ op(2) Incorrect.

op(3). $f(x) = e^{-x}(1-3x^2)$ is sol$^n$ or NOT ? checking

$\int_{-1}^{1} e^t f(t)\,dt = \int_{-1}^{1} e^t \left(e^{-t}(1-3t^2)\right)dt = \int_{-1}^{1} (1-3t^2)\,dt$

$= \left[t - 3\frac{t^3}{3}\right]_{-1}^{1} = 1-1-(-1+1) = 0$

op(4) $f(x) = e^{-x}(x+x^3+x^5)$

$\Rightarrow \int_{-1}^{1} e^t f(t)\,dt = \int_{-1}^{1} \cancel{e^t}\,\cancel{e^{-t}}(t+t^3+t^5)\,dt = \left[\frac{t^2}{2}+\frac{t^4}{4}+\frac{t^6}{6}\right]_{-1}^{1}$

$= \frac{1}{2}+\frac{1}{4}+\frac{1}{6}-\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}\right) = 0$

$\Rightarrow$ op(3) & op(4) TRUE & op(1,2) FALSE.

Que

Que (4.75M) (Repeated)

The integral equation

$$\phi(x) = 1 + \frac{2}{\pi}\int_0^{\pi} (\cos^2 x)\, \phi(t)\, dt \quad \text{has—}$$

(a) No solution (b). Unique solution
(c). More than one but finitely many solⁿ.
(d). Infinitely many solutions.

Ans: (a)

Solⁿ

Let $C = \int_0^{\pi} \phi(t)\, dt$

then $\phi(x) = 1 + \frac{2}{\pi} \cos^2 x . C$

$$\Rightarrow C = \int_0^{\pi} \left(1 + \frac{2C}{\pi} \cos^2 t\right) dt$$

$$= \pi + \frac{2C}{\pi}\left[\int_0^{\pi} \frac{1+\cos 2t}{2}\, dt\right]$$

$$C = \pi + \frac{C}{\pi}\left[t + \frac{\sin 2t}{2}\right]_0^{\pi}$$

$$\Rightarrow C = \pi + \frac{C}{\pi}[\pi] = \pi + C$$

$$\Rightarrow 0 = \pi \rightarrow \leftarrow$$

So, No solution exists. ⇒ (a)

Que. (4.75M)

Assume that $h_1, h_2, g_1$ & $g_2 \in C([a,b])$.
Let $\phi(x) = f(x) + \lambda \int_a^b [h_1(t) g_1(x) + h_2(t) g_2(x)]\, \phi(t)\, dt$

be an integral Equation. Consider the following statements:

$S_1$: If the given I.E. has a solⁿ for some $f \in C([a,b])$
then $\int_a^b f(t) g_1(t)\, dt = 0 = \int_a^b f(t) . g_2(t)\, dt$

$S_2$: Then given I.E. has a unique solⁿ for every