Linear Algebra

[For NET/GATE/SET/NBHM/JAM/BHU/CUCET/MSc...etc.]

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[Linear Algebra Complete Notes]

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ॐ भूर्भुवः स्वः तत्सवितुर्वरेण्यं भर्गो देवस्य धीमहि धियो यो नः प्रचोदयात्

Kalika

Linear Algebra

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LINEAR ALGEBRA

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LINEAR ALGEBRA (P. Kalika)

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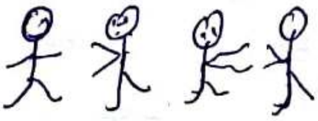


Abbreviation

- S.t.b = said to be
- Id. = identity
- Col.s = Columns
- LD/LI = Linearly dependent / independent
- Coeff = Coefficient.
- arb = arbitrary
- sol^n = solution.
- Hom = Homogeneous.
- eq^n = equation.
- E.value = Eigen value.
- E.vector = Eigen vector
- A.M = Algebraic Multiplicity
- G.M = Geometric Multiplicity
- Ch.poly. = characteristic polynomial
- Tr(A) or tr(A) = trace of matrix A
- det A or |A| = determinant of A
- E.V(A) = Eigenvalues of A.
- Ch. eq^n = characteristic equation

MATRIX

P. Kalika

Note: • For CSIR-NET & GATE (also for PSE SET, NBHM, ...), Matrix topic is very very important.

• From this topic questions are asked like .. finding Linear dependence/independence, solve the system of Linear eqⁿs, finding eigen-value & vector, rank-nullity, orthogonality, ... etc.

• In this section, we study type of matrics & special matrix/special properties, Nature of matrix, finding trace, determinant, orthogonality, Unitary minor/co-factor and checking for bilinear and Quadratic form (vvI for NET).

MATRIX (30~35 marks)

* A matrix $M = [a_{ij}]_{m\times n}$ (defined on next page) is called —

(a). Null Matrix : if $a_{ij}=0$ $\forall i,j$ (denote $O_{m\times n}$)

(b). Row Matrix : if $m=1$ (also called Row vector)

(c). Column Matrix : if $n=1$ (also called Col. vector)

(d). Rectangular Matrix : if $m\neq n$

(e). Square Matrix : if $m=n$

(f). Diagonal Matrix if $m=n$ & $a_{ij}=0$ for $i\neq j$

(g). Scalar Matrix : if $m=n$ & $a_{ij}=0$ for $i\neq j$ & a_{ii} are equal for $\forall i$

→ More details about matrices are discussed in this PDF.

Matrix

Def: A rectangular representation of no.s either complex or real is known as 'Matrix Representation' of numbers.

let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & & & \vdots \\ a_{m1} & a_{m2} & & & a_{mn} \end{bmatrix}_{m \times n}$ ← R_1 ← Row, ← R_2, ← R_m

$\uparrow C_1 \quad \uparrow C_2 \quad - - - \quad \uparrow C_n$ ← col.

then —

(1). Order of matrix is $m \times n$.

(2). $R_1, R_2, \ldots R_m$ represents row & $C_1, C_2, \ldots C_n$ represents Column (col.) & denoted as

$$A = [a_{ij}]_{m \times n}$$

where

a_{ij} = element in i^{th}-row & j^{th} column.

(3). Symbolically represented as || ||, ( ), [ ]

(4). No. of elements in matrix = order of matrix = $m \times n$.

Row Matrix:

A matrix which contains a single row & any no. of col.s. is known as row matrix.

eg. $(1 \ 2 \ 3)_{1 \times 3}$

NB: Row matrix is also known as row-vector.
$(1\hat{i} + 2\hat{j} + 3\hat{k})$

Column Matrix :- A Matrix containg a single column and any no. of rows. eg: $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{3\times1}$

NB: Column matrix is also known as col. Vector.

Null Matrix :- A matrix is s.t.b 'Null Matrix' or 'Zero matrix' if each entry of matrix is '0'.

Eg: $A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{3\times4}$

NB: ✓ Null matrix is denoted as '$O_{m\times n}$'

✓ Null matrix commute as well as anti-commute with each other.

Matrix over field :-

If all the entries of a matrix are taken from a field 'F', then it is s.t.b a matrix over field 'F' or simply 'Field Matrix'.

Eg:

$F = \mathbb{R}$, then Real Matrix

$F = \mathbb{C}$, then Complex Matrix

NB ✓ If no field is given/mentioned then by default it is treated as Complex matrix.

✓ Two matrices are comparable for being equal if they are of same order as well as same elts.

Examples: –

(1). $A = \begin{pmatrix} 1 & 2 \\ a & b \end{pmatrix}$, $B = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$ $\Rightarrow$ $x = 1$, $y = 2$, $z = a$, $w = b$

✓ a & b can takes infinite values.

So, Infinite no. of ways for $A = B$

(2). $A = \begin{pmatrix} 1 & 8 \\ a & b \end{pmatrix}$, $B = \begin{pmatrix} 3 & 4 \\ 7 & 8 \end{pmatrix}$. Under what condition $A = B$.

Sol^n :- [Here ∵ $1 \neq 3$, $8 \neq 4$, so we can't take direct comparision]

Consider $\mathbb{Z}_2$ (= the field of integers modulo 2)

$\mathbb{Z}_2 = \{0, 1\}$

then $A = \begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ $\Rightarrow$ $a = 1$, $b = 0$ for $A = B$

(3). How many sol^n of $A = B$ exists if entries are taken from $\mathbb{Z}_p$ (p = prime), $p > 2$. (Que-1).

Sol^n :- ∵ $\mathbb{Z}_p = \{0, 1, 2, \ldots, p-1\}$

$z = a$ (p-options)
$w = b$ (p-options)
$\Big\} \Rightarrow p^2$ – choices or p^2 – solutions.

---

(4). Let $S = \{A = [a_{ij}]_{m \times n} : a_{ij} \in \mathbb{Z}_p, p = \text{prime}\}$

Then how many elements are there in set S. (i.e $|S| = ?$)

Sol^n $A = [a_{ij}]_{m \times n} = \begin{bmatrix} p, & p & \cdots & p \\ p & p & \cdots & p \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}_{m \times n}$ $p \to p$-choices.

$\therefore$ No. of elements $= \underbrace{p \times p \times p \times \dots \times p}_{= m \times n \text{ times}}$

$= p^m p^n = p^{m+n}$

(5). $A = \begin{bmatrix} 5 & 0 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix}$, $A = B$ if —

(a) $F = \mathbb{R}$

(b). $F = \mathbb{C}$

(c) $F = \mathbb{Z}_3$

(d) $F = \mathbb{Z}_5$

Sol^n $\because A = B \Rightarrow \underbrace{5 \equiv 2}_{3 \equiv 0}, \; 0 \equiv 3, \; \underbrace{1 \equiv 4}_{3 \equiv 0}, \; \underbrace{1 \equiv 7}_{3 \times 2 \equiv 0}$

$\because \mathbb{Z}_3 = \{0, 1, 2\}$

$\therefore A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$

Addition of Matrices :→

A & B can be added if they are of same order.

$A = [a_{ij}]_{m \times n}$, $B = [b_{ij}]_{m \times n}$

then $\exists$ C s.t $C = A + B$ of order $m \times n$

s.t $C = [a_{ij} + b_{ij}]_{m \times n} = [c_{ij}]_{m \times n}$

Properties :-

(1). Addition of matrices is commutative.

i.e $A + B = B + A$

(2). If A, B, C are matrices of same order, then

$A + (B + C) = (A + B) + C$ ← Associativity property

Cayley Hamilton theorem

St:- Every sq. matrix satisfies its Ch. poly.

OR

Every Ch. poly. annihilates its matrix.

let $Ch_A(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$

then $Ch_A(A) = A^n + a_{n-1}A^{n-1} + \cdots + aA + a_0 I = 0$

Results

(1). λ is an E. value of A $\Leftrightarrow$ λ is root of $Ch_A(x) = 0$ i.e $|A - \lambda I| = 0$

Pf: let λ is an E. value of A $\Rightarrow \exists\ X (\neq 0)$ s.t

$AX = \lambda X$

$\Leftrightarrow (A - \lambda I)X = 0$

$\Leftrightarrow BX = 0$, where $B = A - \lambda I$ —— (1)

↑ Hom. system of eqn

(1) has n-z soln only when B is singular.

i.e $|B| = 0$.

$\Leftrightarrow (A - \lambda I) = 0$

i.e $ch_A(x) = 0$

$\Rightarrow \lambda$ is an E. value $\Leftrightarrow |A - \lambda I| = 0$

eg. $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $|A - \lambda I| = Ch_A(\lambda) = \lambda^2 + 1 = 0$

$\Rightarrow x = \pm i \in \mathbb{C}$

Hence A has no E. value on $\mathbb{R}$ but 2 E. value in $\mathbb{C}$.

(2). $\lambda = 0$ be an E.value of A then $|\lambda I - A| = 0$

$\Rightarrow (-1)^n |A - \lambda I| = 0$

$\Rightarrow (-1)^n |A| = 0 \quad \{\text{as } \lambda = 0\}$

i.e $\lambda = 0$ is E.V iff $|A| = 0$.

Algebraic Multiplicity

Let λ be E.V of $A_{n\times n}$ s.t $Ch_A(\lambda) = 0$.

$Ch_A'(\lambda) = 0,\ Ch_A''(\lambda) = 0,\ \ldots,\ Ch_A^{m-1}(\lambda) = 0,\ Ch_A^{m}(\lambda) \neq 0$

then we can write it as—

$Ch_A(x) = (x-\lambda)^m P(\lambda) \quad ; \quad \deg(P(x)) = n-m$

Here m is called multiplicity (algebraic) of λ.

Results

(1). The sum of A.M gives the no. of E.values. Counting multiplicity.

eg. $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\quad Ch_A(\lambda) = |A - \lambda I|$

$= -(1-\lambda)^3 \cdot \lambda$

$A.M(\lambda = 0) = 1$

$A.M(\lambda = 1) = 3$

Eigen Space

Let T be L.O on V(F), & λ be E.V of T, then define $\boxed{W_\lambda = \{x \in V(F) : T(x) = \lambda x\}}$

(VI)

POSITIVE DEFINITE MATRIX

let $A = [a_{ij}]_{n \times n}$ be sq. matrix, then —

(1). A is 'Positive definite' if $x^T A x > 0$ for all n.z col. vectors x.

and 'Positive semi-definite' if $x^T A x \geq 0 \quad \forall x$

(2). A is 'Negative definite' if $x^T A x < 0$ $\forall$ non-zero col. vector x.

and 'Negative semi-definite' if $x^T A x \leq 0$

* If A is symmetric matrix, then —

$\because$ i^{th} principal minor of A is the matrix Δ_i formed by the first i-rows & i-col.s of A. (i.e det. of 1x1, 2x2, 3x3 --- along principal diag.)

so first principal minor $\Delta_1 = a_{11}$

2nd ——— " ——— $\Delta_2 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

Results

(1). A is 'Positive definite' if all its principal minors $\Delta_1, \Delta_2, \ldots \Delta_n$ have determinant strictly positive. ie $\boxed{\det(\Delta_i) > 0}$ $\forall \ 1 \leq i \leq n$

Result: A real quadratic form $f = x^T A x$ (or matrix A) in n-variables is —

(1). POSITIVE DEFINITE $\Leftrightarrow$ all the E-Values of A are positive.

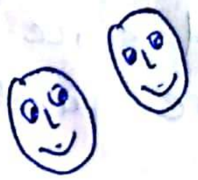
(2). POSITIVE SEMI-DEFINITE $\Leftrightarrow$ if $\lambda_i \geq 0$ & at least one $\lambda_i = 0$

(3). NEGATIVE DEFINITE $\Leftrightarrow$ all $\lambda_i < 0$

(4). NEGATIVE SEMI-DEFINITE $\Leftrightarrow$ all $\lambda_i \leq 0$ & at least one $\lambda_i = 0$

(5). INDEFINITE $\Leftrightarrow$ A has positive & Negative E-Values.

(6) A Pos. Def. Real symm. matrix is non-singular.



MATRIX SECTION ENDS

(Next Vector space)

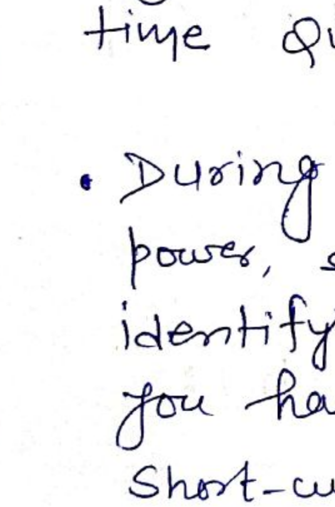
& By: P. Kalika

Vector Space & IPS

This section contains/introduces the basics of V.S, IPS, Basis, dimension, Norm, Orthogonal direct sum, Gram-Schmidt, Diagonalization Block matrix, E.value & Vector, A.M & G.M. Results, Examples, ... etc.

For CSIR-NET/GATE/SET/PSC, ... etc. purpose, this section is also important. From Eigen-value & block, rank-nullity, ... etc, every time question has been asked.

- During prepration, develop your guessing power, so that in exam you can easily identify the correct options, because, in exam you have no more time, you have to adopt short-cut trick to get quick answer.



Love Nature & Live Happy

— P. Kalika

Vector space

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14.03.18

Defn: Let $(V,+)$ be an abelian group and $\mathbb{F}$ is a field s.t $f: \mathbb{F} \times V \to V$. defined as

$$f(\alpha, x) = \alpha \cdot x \quad \leftarrow \text{(external Composition)}$$

Then we say that V form a vector space (V.S) if —

(i). $\forall a, b \in \mathbb{F}$, $x \in V$

$$(a+b)\cdot x = a\cdot x + b\cdot x$$

(ii). $\forall a \in \mathbb{F}$, $x, y \in V$, $\quad a\cdot(x+y) = a\cdot x + b\cdot y$

(iii). $\forall a, b \in \mathbb{F}$, $x \in V$, $\quad a\cdot(b\cdot x) = (a\cdot b)\cdot x$

(iv) $\forall x \in V$, $1 \in \mathbb{F}$ s.t $1\cdot x = x \in V$

then 1 is unit in $\mathbb{F}$.

Example (1) (i). $V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R} \text{ under Component-wise addition} \right\}$

- $(V, +)$ is an abelian group.

Here $\mathbb{F} = \left\{ \begin{bmatrix} a & a \\ a & a \end{bmatrix} : a \in \mathbb{R} \right\}$

Let $x = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ & unit of $\mathbb{F}$ is $\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} = 1$

then

$$1\cdot x = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

i.e $1\cdot x \neq x$

Hence, V is NOT vector space.

Que(46). If $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ then $A^5 = KA + mI$, then

(i) find k & m

(ii) $A^{-3} = pA + qI$, find p & q.

Sol^n

$\because$ Cha eq^n (A) $= x^2 - 2x - 3 = 0$

$\Rightarrow A^2 - 2A - 3I = 0$ —— (*)

$\Rightarrow A^2 = 2A + 3I$

$A^3 = 2A^2 + 3A = 2(2A + 3I) + 3A$

$= 7A + 6I$

$A^4 = 7A^2 + 6A = (14+6)A + 21I = 20A + 21I$

$A^5 = 20A^2 + 21A = (40+21)A + 60I = 61A + 60I$

$\therefore$ $\boxed{k = 61 \text{ \& } m = 60}$

By (*) $I = \frac{1}{3}(A^2 - 2A)$

$\Rightarrow A^{-1} = \frac{1}{3}(A - 2I)$

$A^{-2} = \frac{1}{3}(I - 2A^{-1}) = \frac{1}{3}\left(I - \frac{2}{3}(A - 2I)\right)$

$= \frac{1}{3^2}(7I - 2A)$

$A^{-3} = \frac{1}{9}(7A^{-1} - 2I) = \frac{1}{9}\left(\frac{7}{3}(A - 2I) - 2I\right)$

$= \frac{1}{27}(7A - 20I)$

$\Rightarrow A^{-3} = \frac{7}{27}A - \frac{20}{27}I$

$\Rightarrow$ $\boxed{p = 7/27 \text{ \& } q = -20/27}$

Que(47). Consider $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 4 \end{bmatrix}$. If $A^{-1} = \alpha I + \beta A + \gamma A^2$

then find α, β & γ.