Linear Programming

[For NET/SET/GATE/PSC/MSc, ...etc.]

(Handwritten Study Material for NET,GATE, SET/SLET, PSC, ...etc)

Operation Research

P. Kalika
(NET(JRF), GATE, SET)
Email: maths.whisperer@gmail.com

Ver.: 2020.7-1

No. of Pages: 110

Optimization / LPP

* Find ~~objective~~ Max(min). of given objective function $f(x_1, x_2, \cdots x_n)$

subject to
$$\left.\begin{aligned} g_1(x_1, x_2, \cdots, x_n) &\le 0 \\ g_2(x_1, x_2, \cdots x_n) &\le 0 \\ &\vdots \\ g_m(x_1, x_2, \cdots x_n) &\le 0 \end{aligned}\right\} \text{Constraints.}$$

$x_i \in \mathbb{R}$.

Where $x_1, x_2, \cdots x_n$ are decision Variables.

LPP : f and g_i's are linear in nature

Max./Min. $z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$

s.t
$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \; (\le = \ge) \; b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \; (\le = \ge) \; b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \; (\le = \ge) \; b_m$$

x_i's $\in \mathbb{R}$

** Feasible Solution : Those $x_i \in \mathbb{R}$ ~~for~~ which satisfy objective function and subject to constraints called Feasible solution (FS).

** * Set of all feasible sol^n is called Feasible Region
* Feasible Region of LPP is always Convex.
* Intersection of convex region is convex again.
* But Union of convex set Need NOT be convex set.

P. Kalika

PAGE :
DATE : / /

LPP

Extreme Point [Corner Point]:

Point of intersection of given constraints (on which we try to find sol^n)

Optimal Solution :-

The feasible solution which minimize or maximize the objective function called optimal solution.

Isoprofit line

Objective f^n

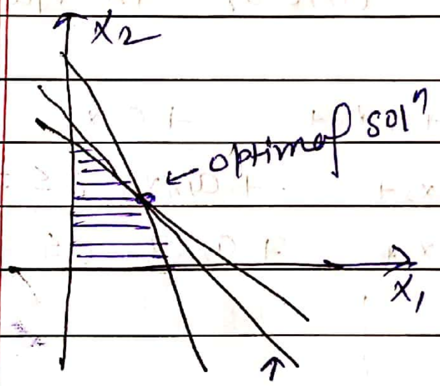
$4x_1 + 3x_2 = 0$

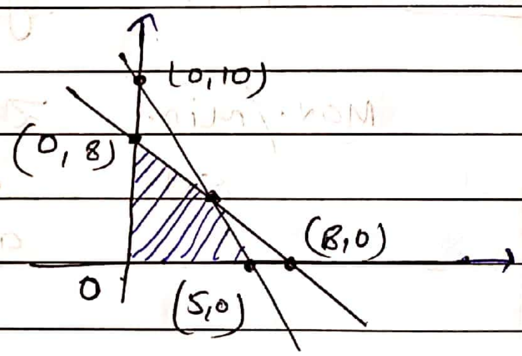
Example:

Max. $z = 4x_1 + 3x_2$

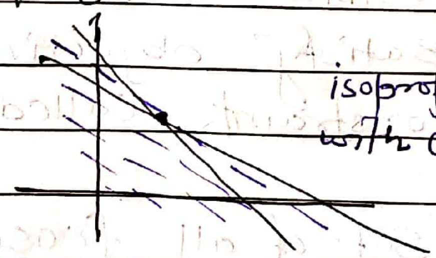
$x_1 + x_2 \leq 18$

$2x_1 + x_2 \leq 10$





→ (1). If infinitely many optimal sol^n exist. then isoprofit line concide (overLap) the constraint line on which optimal sol^n received.



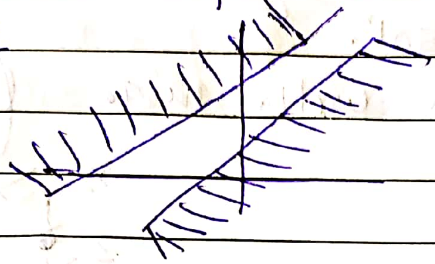
(2). **Unbounded Solution:**

Given LPP has infinitely many sol^n, But optimal sol^n (name) lies / exist at $z=\infty$.

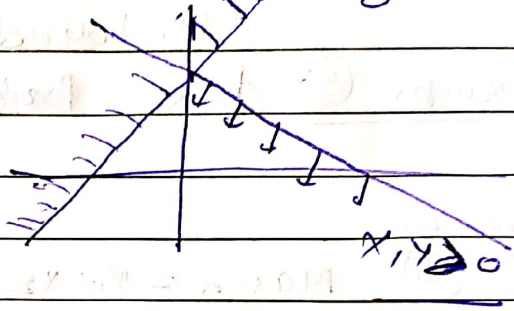
(3). Infeasible sol^n : When there is no feasible Region.

(4). No solution : If there is no common region.

Example ⓐ



ⓑ



Trick Hint : Just plot the ~~objective~~ constraints & look at intersection points with other cond^n/ line. Then check value of objective f^n at those corner points.

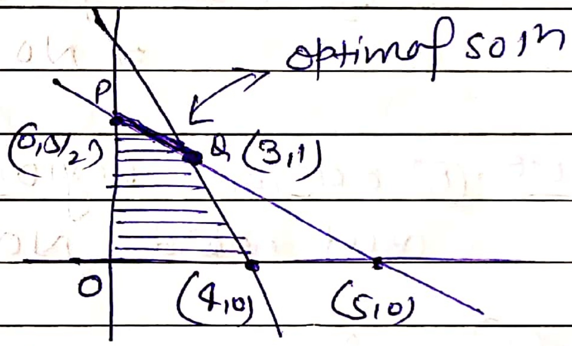
Examples

① Max. $Z = 2x_1 + 4x_2$

s.t $\quad x_1 + 2x_2 \leq 5$

$\quad x_1 + x_2 \leq 4$

$\quad x_1, x_2 \geq 0$



∵ Point	(0,0)	(4,0)	(0, 5/2)	(3,1)	PQ-line
Z	0	8	10	10	10

ie exists more than one optimal sol^n for max Z.

But for min. Z, unique sol^n exists.

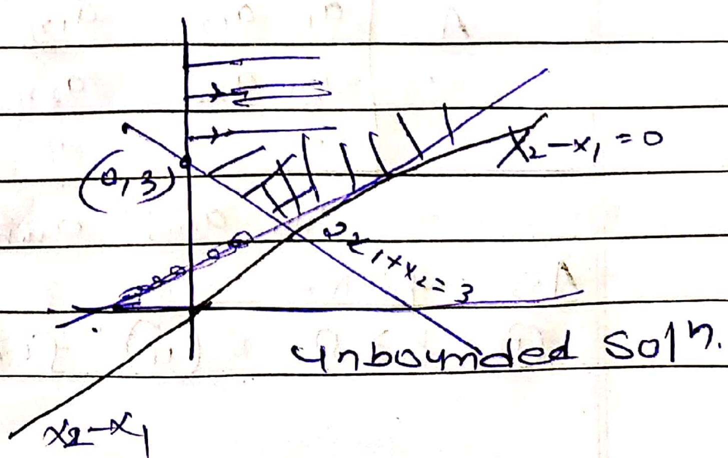
Example

② Max. $Z = 6x_1 + x_2$

s.t $\quad 2x_1 + x_2 \geq 3$

$\quad x_2 - x_1 \geq 0$

$\quad x_1, x_2 \geq 0$



Unbounded sol^n.

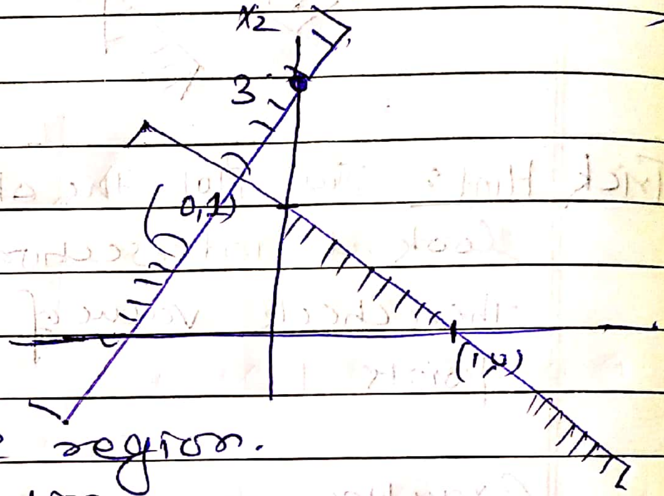
PAGE :
DATE : / /

Note: Unbounded sol^n $\Rightarrow$ Unbounded feasible Region
($\not\Leftarrow$)
i.e Unbdd feasible region need not imply
Unbounded sol^n.

Example (3) take problem with min. (or eg. 2 with min z)

Example (4)

$$\text{Max. } z = x_1 + x_2$$
$$\text{s.t} \quad x_1 + x_2 \le 1$$
$$-3x_1 + x_2 \ge 3$$
$$x_1, x_2 \ge 0$$



sol^n : No feasible region.
$\Rightarrow$ No solution.

Note: (1) Feasible Region is always closed & convex but need NOT be bounded.

(2) A feasible region which is closed, convex & bounded is called "Polytope".

Some

$$\text{Max. } z = c^T x \quad \text{s.t} \quad Ax = b, \; x \ge 0.$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix}, \; x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \; b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \; c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

Assumption
(i) $b \ge 0$. (ii) $\rho(A) = m < n$.

P. Kalika

PAGE :
DATE : / /

(iii). '=' sign

$\exists$ This LPP is called std. form LPP.

* **Redudent Constraints:**

Constraints which doesn't effect the objective functions.

→ * Suppose $C_1, C_2, \ldots C_m \leftarrow L.I$

$C_{m+1}, C_{m+2}, \ldots C_n \leftarrow L.D$

then we construct a $m \times m$, $[\ ]_{m \times n}$ Basis matrix.

and maximum no. of matrix of order $m \times m$ $= {}^nC_m$

→ A variable associated with col. of basis matrix are called Basic Variables.

i.e

$[x_1, x_2, \ldots x_m] \leftarrow$ Basic Variable

$[x_{m+1}, x_{m+2}, \ldots, x_n] \leftarrow$ Non-Basic Variable.

→ A sol^n, obtained by putting non-basic variable equal to '0' & then solve for basic variable is called Basic Solution.

→ No. of max. feasible sol^n = at most ${}^nC_m$.

→ At most m-variable can be non-zero.

* **Degenerate Basic Feasible sol^n:**

If any of the basic variable is '0' in the BFS, then it is called Degenerate BFS.

P. Kalika

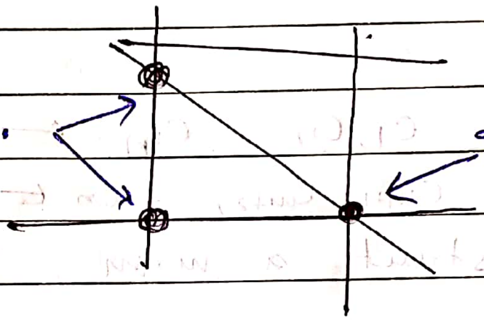
"#" means "no. of"

PAGE :
DATE : / /

* Non-degenrate BFS : If none of variables of BFS is zero.

Trick

(A) If from any corner point, more than two lines passes, then called degenerate F.S.



Trick

(A) Every corner point corresponds to a basic feasible solution.
But every F.S. corresponds to a corner pt. (need not unique), consider eg. (1)

Example (5) Max. $z = 4x_1 + 3x_2$, s.t $\cancel{x_4} + x_2 \le 8$
$2x_1 + \cancel{2}x_2 \le 10$

Sol^n s.t $x_1 + x_2 \cancel{+} x_3 = 8$
$2x_1 + \cancel{2}x_2 + x_4 = 0$
$\Rightarrow m = 2, n = 4,$
BFS may be $^4C_2 = 6$ (atmost).

For BFS

(1) take $x_1, x_2 \neq 0$ & $x_3 = x_4 = 0$
$x_1 + x_2 = 8$
$2x_1 + \cancel{2}x_2 = \cancel{1}0$ } $\Rightarrow (x_1, x_2, x_3, x_4) = (2, 6, 0, 0)$
BFS

(2) take $x_1, x_3 \neq 0$, $x_2 = x_4 = 0$
$x_1 + x_3 = 8$
$2x_1 + 0 \cdot x_3 = 10$ } $\Rightarrow (x_1, x_2, x_3, x_4) = (5, 0, 3, 0)$
BFS

CSIR-NET Dec2019

CSIR-NET DEC-2019

LPP

DATE: / /
PAGE No.:

Que. (3M) Subject to the conditions

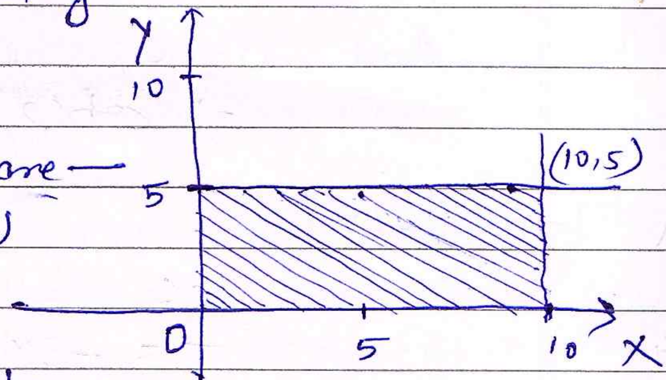
$$0 \le x \le 10$$
$$0 \le y \le 5$$

the minimum value of the f^n $4x-5y+10$ is—

(a) 10 (b) 0 (c). −25 (d). −15

Ans (d).

Sol^n Solve using graphically—



Let $z = 4x - 5y + 10$

and corner points are—

(0,0), (10,0), (0,5), (10,5)

Thus—

Point	(0,0)	(10,0)	(0,5)	(10,5)
z	10	50	−15	25

Thus, minimum value of $z = -15 \Rightarrow$ (d)

(Assam)

Que. (4.75) NET-2019 D

Consider a LP-Problem—

Maximize $z = c^T x$ subject to

$$Ax \le b, \quad x \ge 0.$$

Which of the following are true?

(1). If the primal has an optimal sol^n, then the dual has also an optimal sol^n.

(2). The primal can have an optimal sol^n & the dual may have no feasible sol^n.

(3). The dual can have an optimal sol^n and the primal may have no feasible sol^n.

(4). If the primal has no feasible solⁿ, then the dual also has no feasible solution

(Ans : (1)).

Solⁿ. This question is repeated from GATE.

Recall :

*(a) Primal Unbounded → dual feasible ✗ ; → dual feasible ✓

*(b) primal Infeasible → dual Unbounded ✓ ; or → dual Infeasible ✓

*(c) Primal has optimal solⁿ ⟺ dual has also
⇒ op(1). TRUE. & op(2,3). FALSE

By (b). dual may be infeasible or Unb.dd
⇒ op(4). FALSE.

— P. Kalika
maths.whisperer@gmail.com
All study materials

— * — * —