Numerical Analysis

[Handwritten Study Material & Solutions]

[For NET/GATE/SET/NBHM/BHU/CUCET/MSc...etc.]

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P. Kalika & K. Munesh
(NET(JRF), GATE, SET)
Email: maths.whisperer@gmail.com

No. of Pages: 102

- The eqn $f(x)=0$ is called an algebraic eqn if it is purely a poly. in x.

  eg: $5x^5+x^3+2x^2-3=0$

- And ~~for~~ If $f(x)$ contains exponential, trigonometric or logarithmic f^n also, then it is called a transcendental equation. eg.

  $\cos x - x^2e^x = 0$, $4x^2+e^x\sin x+\log(x-7)+5=0$

vvI

Solution of Algebraic / transcendental Eqn

If $f(x)=0$ be algebraic / transcendental eqn, $f(x)$ is cts, & diff^n. s.t $f(a).f(b)<0$ for some $a,b\in\mathbb{R}$

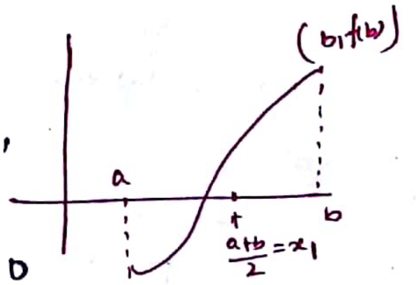
$f:[a,b]\to\mathbb{R}$ (IMVT)

$\Rightarrow \exists\ \alpha\in[a,b]$ s.t $f(\alpha)=0$ (simple root)

$\Rightarrow f'(\alpha)\neq 0$

Then $x_1=\dfrac{a+b}{2}$ is first approximation.

1. BISECTION METHOD



On interval $[a,b]$, we find $x_1=\dfrac{a+b}{2}$,

Next $[a,x_1]$ or $[x_1,b]$

ie either $f(a).f(x_1)<0$ or $f(b)f(x_1)<0$

We ~~Choose~~ continue by choosing interval $[x_i,x_{i+1}]$ s.t $f(x_i).f(x_{i+1})<0$.

Thus $x_n\to\alpha$

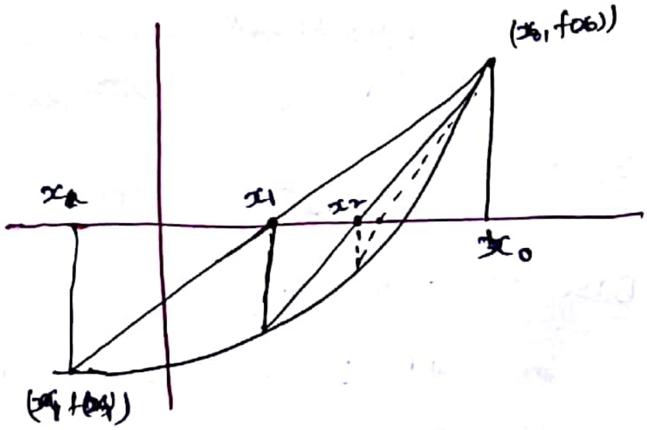
$x_2=\dfrac{x_0+x_1}{2}$, $x_3=\dfrac{x_0+x_2}{2}$ or $\dfrac{x_1+x_2}{2}$ ... $\boxed{x_n=\dfrac{a+b}{2^n}}$ —(i)

- Bisection Method <u>always converges</u>.
- If ϵ_n is error at n^{th} iteration $\Rightarrow \epsilon_n = x_n - \alpha$
  where α is exact root (i.e $f(\alpha)=0$)
- Let ϵ is permissible error then $\boxed{\epsilon \geq \left| \frac{b-a}{2^n} = \epsilon_n \right|}$
  on solving $\boxed{n \geq \left[\frac{\log\left(\frac{b-a}{\epsilon}\right)}{\log 2} \right]}$. n = no. of iteration

- Rate of convergence of Bisection method is linear.
  i.e. $\boxed{\text{Rate of convergence} = 1}$

2. REGULA FALSE METHOD (False position method)

- Two Initial approximation are required.



- $f(x_n) \cdot f(x_0) < 0$

Since iterative formula for this method is –

$$\boxed{x_{n+1} = x_n - \left(\frac{x_n - x_0}{f(x_n) - f(x_0)}\right) f(x_n) = \frac{x_0 f(x_n) - x_n f(x_0)}{f(x_n) - f(x_0)}} \quad \text{— (ii)}$$

- Rate of convergence of RF-Method is '1'.
- This method <u>always converge</u>.

(3) (3M) Consider solving the following system by Jacobi iteration scheme

$$x + 2my - 2mz = 1$$
$$nx + y + nz = 2$$
$$2mx + 2my + z = 1 \quad \text{where } m, n \in \mathbb{Z}.$$

With initial vector, the scheme converges provided m, n satisfy —

(1) $m+n=3$ (2). $m>n$
(3) $m<n$ (4). $m=n$

Ans: 4

Sol^n Consider $[A:b] = \left[\begin{array}{ccc|c} 1 & 2m & -2m & 1 \\ n & 1 & n & 2 \\ 2m & 2m & 1 & 1 \end{array}\right]$

Trick: we know that, $x_{n+1} = \phi(x_n)$ cgs if $|\phi'(x_n)| < 1$

thus here let $x = \phi_1(x,y,z)$, $y = \phi_2(x,y,z)$, $z = \phi_3(x,y,z)$

Consider z

$$z = \phi_3(x,y,z) = 1 - 2mx - 2my \qquad \text{(by } 3^{rd} \text{ eq}^n\text{)}$$

$$\therefore |\phi_3'| = \left|\frac{\partial\phi_3}{\partial x}\right| + \left|\frac{\partial\phi_3}{\partial y}\right| + \left|\frac{\partial\phi_3}{\partial z}\right| = |-2m| + |-2m| + |0| < 1$$

$$\Rightarrow 2m + 2m < 1 \Rightarrow |m| < 1/4$$

Similarly By 2^{nd} eq^n, $y = 2 - nx - nz = \phi_2(x,y,z)$

$$\Rightarrow |\phi_2'| = \left|\frac{\partial\phi_2}{\partial x}\right| + \left|\frac{\partial\phi_2}{\partial y}\right| + \left|\frac{\partial\phi_2}{\partial z}\right| = |-n| + 0 + |-n| = 2|n| < 1$$

$$\Rightarrow |n| < 1/2$$

$\Rightarrow |m| < 1/4$ & $|n| < 1/2$ & $m, n \in \mathbb{Z}$

$\Rightarrow m = 0 = n \Rightarrow$ op(4). TRUE.

Note: Question based on convergence is VVI.

For that, you need to establish $x_{n+1} = \phi(x_n)$ & then check $|\phi'(x_n)| < 1$ or NOT.

P. Kalika

Problems Based on NUMERICAL INTEGRATION

GATE 2016

(1) Given $I = \int_0^4 f(x)\,dx$, $f(x) = \begin{cases} x & : [0,2] \\ 4-x & : [2,4] \end{cases}$

P: I_2 is the value of I using Trapezoidal Rule with two equal subintervals. Then I_2 is exact value

Q: I_3 is the value of I by using Trapezoidal Rule with three subintervals. Then I_3 is exact value.

Sol^n

$$I = \int_0^2 f(x)\,dx + \int_2^4 f(x)\,dx$$

$$= \int_0^2 x\,dx + \int_2^4 (4-x)\,dx = \frac{x^2}{2}\Big|_0^2 + \frac{(4-x)^2}{-2}\Big|_2^4$$

$$= 2+2 = 4$$

i.e exact value of $I = 4$

Now for I_2: $x_0 = a$, $x_1 = \frac{a+b}{2}$, $x_2 = b$

$$\therefore I_2 = \frac{h}{2}[f(x_0) + 2f(x_1) + f(x_2)]$$

& $h = \frac{b-a}{2} = \frac{4-0}{2} = 2$

$$\Rightarrow I_2 = \frac{2}{2}[0 + 2.2 + 0] = 4$$

Now for I_3 $x_0 = 0$, $x_1 = 4/3$, $x_2 = 8/3$, $x_3 = 4$

$(x_0 = 0,\ x_1 = x_0 + h,\ x_2 = x_0 + 2h,\ x_3 = x_0 + 3h)$

$$\Rightarrow I_3 = \frac{h}{2}[f(x_0) + 2(f(x_1) + f(x_2)) + f(x_3)]$$

$$= \frac{h}{2}[0 + 2(4/3 + 4/3) + 0] = \frac{4}{3\times\cancel{2}} \times \cancel{2}\times\frac{8}{3} = \frac{32}{9}$$

$$= 3.55$$

here $I_2 = I$ but $I_3 \neq I$

Hence, statement P. is true & Q is false.

GATE-15

(2) If the trapezoidal rule with single interval $[0,1]$ is exact for approximate value of $\int_0^1 (x^3 - cx^2)\,dx$ Then the value of c.

Q.16 Let f be any Polynomial f^n of degree at most 2 over $\mathbb{R}$. If the constants a and b are such that

$$\frac{df}{dx} = a f(x) + 2f(x+1) + b\, f(x+2),\ \forall\ x \in \mathbb{R},$$

then $4a+3b$ $\underline{-7.5}$.

Sol^n $\frac{df}{dx} = a f(x) + 2 f(x+1) + b \underbrace{f(x+2)}_{(i)}$ is for any polynomial f^n of degree at most 2 over $\mathbb{R}$.

So, Let $\underline{f(x) = 1}$

then $(i) \Rightarrow\quad 0 = a + 2 + b \quad \Rightarrow \underline{a+b=-2}$

Now, let $f(x) = x$

then $(i) \Rightarrow\quad 1 = ax + 2(x+1) + b(x+2)$

$\Rightarrow (a+2+b)x + (2+2b) = 1$

on comparing we get, $a+b=-2,\ 2b+2=1$

$\Rightarrow b = \frac{1}{2} - 1 = -\frac{1}{2}$

& $a = -2 + \frac{1}{2} = -\frac{3}{2}$

So, $a = -\frac{3}{2}$ & $b = -\frac{1}{2}$

$\therefore\ 4a+3b = 4\times(-\frac{3}{2}) + 3(-\frac{1}{2}) = -6 - \frac{3}{2} = -\frac{15}{2}$

$= -7.5$

Q.51 The maximum value of the error term of the composite Trapezoidal rule when it is used to evaluate the definite integral

$$\int_{0.2}^{1.4} (\sin x - \log_e x)\, dx$$ 0.022 to 0.028

with 12 sub-intervals of equal length, is equal to _____ (round off to 3 places of decimal).

GATE Solutions

Numerical Analysis- 2018

1M

(9) Let $f: \mathbb{R} \to \mathbb{R}$ be a twice cts.ly differentiable f^n. The order of convergence of the secant method for finding root of the eq^n $f(x)=0$ is —

(a) $\frac{1+\sqrt{5}}{2}$ (b) $\frac{2}{1+\sqrt{5}}$ (c). $\frac{1+\sqrt{5}}{3}$ (d). $\frac{3}{1+\sqrt{5}}$

Sol^n

Since we know that Rate of convergence is as follows —

Bisection < Regula falsi < secant < Newton-R. Method

1   1   1.618 $= \frac{1+\sqrt{5}}{2}$ 2

thus o/p(a).

II. Method (If you don't know about $\frac{1+\sqrt{5}}{2}$)

$\because \sqrt{5} > 2$ so op(b) & op(d) discarded. as < 1.

Now check in (a) & (c)

(a). $\frac{1+\sqrt{5}}{2} = \frac{1+2.236}{2} = \frac{3.236}{2} = 1.618$ — (a)