

Partial Differential Equation

[For CSIR-NET/SET/GATE/PSC/BHU/CUCET/MSc...etc.]

PDE



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Partial Differential Equations

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some definitions

(3)

Let f be f^n of x & y and $\frac{\partial f}{\partial y}$ is defined

$$\text{as } \frac{\partial f}{\partial y}(x_0, y_0) = \lim_{h \rightarrow 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}$$

Similarly - $\frac{\partial f}{\partial x}(x_0, y_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$

P.D.E : A diffⁿ. Eqⁿ (DE) which contain partial derivatives of one dependent variable w.r.t. two or more than two independent variable is called Partial Differential Equation (PDE).

EXAMPLE : (a) $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = 0$

(b), $\frac{\partial^2 z}{\partial x^2} + \frac{\partial z}{\partial y} + e^{x-y} = 0$

Degree of PDE : The highest power of the highest order derivative occurring in the PDE, is called degree of that PDE. (Provided given PDE be free from radical fractions & transcendental f^n)

Order of PDE : The order of the highest order partial derivative occurring in the given PDE is called order of that PDE.

EXAMPLE : order of PDE (a) & (b) is 2.

Classification of first Order PDE

(4)

(1). Linear PDE of first Order

A partial diff. of the form

$$a(x,y)p + b(x,y)q = c(x,y)z + d(x,y)$$

is called Linear PDE of first order

EXAMPLE:

① $xyzp + q = xyz$ ← linear

② $x^2p + y^2q = z^2 + yz$ ← not-linear

Notations:

$$\frac{\partial z}{\partial x} = z_x = p$$

$$\frac{\partial^2 z}{\partial x^2} = r$$

$$\frac{\partial^2 z}{\partial x \partial y} = s$$

$$\frac{\partial z}{\partial y} = z_y = q$$

$$\frac{\partial^2 z}{\partial y^2} = t$$

(2). Semi-Linear PDE of first Order

A PDE of the form

$$a(x,y)p + b(x,y)q = c(x,y,z)$$

is called Semi-Linear PDE of 1st. order.

EXAMPLE: ① $x^2p + y^2q = z^2 + xy$ ← semi-linear but not linear

② $zp + xyq = z$ ← Not semi-linear

(3). Quasi Linear PDE of first order

A PDE of the form

$$a(x,y,z)p + b(x,y,z)q = c(x,y,z)$$

Ex: ① $x^2p = z$ ② $xp + yq = x^2 + y^2$

Que (7) Form a PDE for all planes having x & z intercepts same. (10)

Solⁿ $\therefore$ Eqⁿ of plane is ~~given by~~

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \quad \text{--- (1)}$$

Now, diffⁿ (1) partially w.r.t x & y
($\&$ try to eliminate constants.)

$$\frac{1}{a} + \frac{z_x}{c} = 0 \Rightarrow a = -\frac{c}{p}$$

$$\frac{1}{b} + \frac{z_y}{c} = 0 \Rightarrow b = -\frac{c}{q}$$

$\Rightarrow \frac{p}{b} = \frac{q}{a} \therefore$ plane having same intercept on x & z axis $\Rightarrow b = c$

Hence

$$c = -\frac{c}{q} \Rightarrow \boxed{q = -1} \leftarrow \text{Required PDE}$$

vvt

[B] Formation of PDE by Eliminating arbitrary functions

For a given relation, which contains n -arbitrary functions, then elimination of these arb. fⁿs gives PDE of order n . Illustrate by Examples

Que (7) Eliminate arbitrary fⁿ from -

$z = f(x+ay) + g(x-ay)$, where $a \neq 0$
and find corresponding PDE. (i)

Solⁿ Diff (i) partially w.r.t x & y , gives -

$$z_x = p = f'(x+ay) + g'(x-ay) \quad \text{--- (ii)}$$

and RHL, $\lim_{h \rightarrow 0^+} \frac{2h}{h} = 2$

(17)

$\Rightarrow \frac{\partial^2 z}{\partial y^2}$ doesn't exist on $\mathbb{R}^2$ (~~here~~)

(as $z_{yy} = \begin{cases} 2 & \geq 0 \\ -2 & \leq 0 \end{cases}$)

Based on Lagrange's Equation winif

[Quasi Linear PDE of first Order]

Consider $P(x,y,z)p + Q(x,y,z)q = R(x,y,z)$

Then general solⁿ of (1) is

$\phi(u,v) = 0 \quad \text{or} \quad u = \phi(v) \quad \text{or} \quad v = \phi(u)$

Where, ϕ is arb. fⁿ & $u(x,y,z) = C_1$ and $v(x,y,z) = C_2$ are two independent solⁿ of Lagrange's Auxillary Eqⁿ given below

$\frac{dx}{P(x,y,z)} = \frac{dy}{Q(x,y,z)} = \frac{dz}{R(x,y,z)} \quad (*)$

Result / Property

* $f(x,y)$ & $g(x,y)$ are s.i.d.b independent iff ~~if~~ any relation b/w f & g .

* $\frac{\partial(f,g)}{\partial(x,y)} \neq 0$ iff f & g are independent

Type-I: Given that $P(x,y,z)p + Q(x,y,z)q = R(x,y,z)$

4 Lagrange A.E is $\boxed{\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}} \quad (*)$

then, we get

two solⁿ, first $u(x,y,z) = c_1$ & second $v(x,y,z) = c_2$ by taking two suitable fractions or by choosing different combination from (*). And, thus, general solⁿ is $\phi(c_1, c_2) = 0$.

Type-II: One solⁿ of Lagrange A.E (*) is obtained by taking two suitable choice/fractions from (*) and other solⁿ is obtained by using first solⁿ.

Note: (i). We have only general solⁿ for Quasi-Linear PDE of first order.
(ii). Particular solⁿ of given Quasi-LPDE is given by taking particular choice of f & ϕ , where $\phi(u,v) = 0$ is general solution,

Type-III: Consider $P(x,y,z)p + Q(x,y,z)q = R(x,y,z)$

4 Lagrange A.E is $\boxed{\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}} \quad (*)$

Then, we choose S, T, U fⁿ of x, y, z (called multipliers) s.t. —

$$\boxed{PS + QT + RU = 0}$$

thus, by (*), $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{Sdx + Tdy + Udz}{SP + TQ + UR}$



Second Order Partial differential Equation

(Classification of 2nd order linear PDE)

Consider the general form of 2nd order linear PDE (of two independent variable x, y) as—

$$A u_{xx} + B u_{xy} + C u_{yy} + D u_x + E u_y + F u = G(x, y)$$

①

It can be also given as—

$$A u_{xx} + B u_{xy} + C u_{yy} + f(x, y, u, p, q) = 0$$

$$\text{or } A\delta + B\beta + C\gamma + f(x, y, u, p, q) = 0$$

$$\text{with } A^2 + B^2 + C^2 \neq 0$$

Where A, B, C, D, E, F, G are either constants or function of x and y only (continuous f^n .)

*** The classification of eqn ①, based on A, B, C ~~is given at~~ at point (x, y) is given as—

① If $B^2 - 4AC > 0$ then eqn ① is Hyperbolic

② If $B^2 - 4AC = 0$, then eqn ① is parabolic

③ If $B^2 - 4AC < 0$, then eqn ① is Elliptic

Remarks: If All Constant A, B, C are constant then eqn ① have the same nature throughout.

Practice Que

① Find the characteristic curve of the PDE —

$$y^2 u_{xx} - x^2 u_{yy} = 0 \quad , \quad x \neq 0, y \neq 0$$

②. The PDE $x + 2y + t = 0$ is —

(a). Parabolic & has Ch. curve $\tau = x + 2y$, $\eta = x - 2y$

(b). Reducible to canonical form $u_{\tau\tau} = 0$ & $\tau = x + y$

(c). Parabolic & has general solⁿ: $u = (x-y)f(x+y) + g(x-y)$

(d). Reducible to canonical form $u_{\eta\eta} = 0$ & $\eta = x + 2y$

③. Reduce the following PDE to canonical form —

(I). $u_{xx} + 2u_{xy} + u_{yy} = 0$

(II). $u_{xx} + 2xu_{xy} + x^2 u_{yy} = 0$

④. Find the characteristics of the following PDEs —

(I). $x^2 u_{xx} + 2xy u_{xy} + y^2 u_{yy} = 0$

(II). $x^2 u_{yy} = y^2 u_{xx}$

(III). $(1+x^2)u_{xx} + (1+y^2)u_{yy} + x u_x + y u_y = 0$

(IV). $2u_{xx} + 5u_{yy} - 2x u_{xy} = 0$

(V). $u_{xx} + 4u_{xy} + 4u_{yy} = 0$

Answers

①. Circle & Hyperbola.

②. (b) & (d)

③(I). $u_{\eta\eta} = 0$

(II). $u_{\eta\eta} = u_{\tau\tau}$

(4)(I). Parabolic, family of st. line.

(II). Circle & Hyperbola

(III). Elliptic, no real characteristic

(IV) No real characteristic.

(V). A st. line.

① Two dimensional Eigen function in a Rectangle

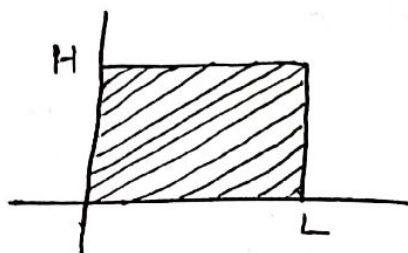
Consider

$$\phi_{xx} + \phi_{yy} + \lambda\phi = 0 \quad \text{--- (1)}$$

$$0 < x < L, \quad 0 < y < H$$

with B.C $\left. \begin{aligned} \phi(x, 0) &= \phi(x, H) = 0, \quad 0 \leq x \leq L \\ \phi(0, y) &= \phi(L, y) = 0, \quad 0 \leq y \leq H \end{aligned} \right\}$

Solution



Dirichlet B.V Problem.

①

* Solution (Eigen fⁿ) will be find using the method of separation of variables.

• Take non-trivial solⁿ as $\phi(x, y) = f(x) \cdot g(y)$ --- (*)
(then $g(0) = g(H) = 0, f(0) = f(L) = 0$)

Now put (*) in (1), then by separable

find $f(x)$ & $g(y)$, finally $\boxed{\phi = f \cdot g}$

∴ Solution will be —

$$\phi_{mn}(x, y) = \sin\left(\frac{m\pi x}{L}\right) \cdot \sin\left(\frac{n\pi y}{H}\right) \leftarrow \text{Eigen f}^n$$

and corresponding Eigen values $\lambda_{mn} = \left(\frac{m\pi}{L}\right)^2 + \left(\frac{n\pi}{H}\right)^2, m, n \in \mathbb{N}$

$$(2) \quad U_{xx} + U_{yy} + dU^2 = 0, \quad 0 < x < L, \quad 0 < y < H$$

$$\text{with } U_x(0, y) = U_x(L, y) = 0$$

$$U_y(x, 0) = U_y(x, H) = 0$$

Newmann
B.V. Problem.

Method: For finding solⁿ, use the method described above (i.e. Method-I)

Thus, solution will be —

$$\text{Eigen-Values: } \lambda_{mn} = \left(\frac{m\pi}{L}\right)^2 + \left(\frac{n\pi}{H}\right)^2, \quad m, n \in \mathbb{N}$$

$$\text{Eigen-function: } U_{mn}(x, y) = \cos\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi y}{H}\right)$$

Note: Eigen-functions corresponding to distinct eigen-values are orthogonal w.r.t. weight f^n (co-eff of. du) in the Rectangle $[0, L] \times [0, H]$.

$$\text{i.e. } \int_{x=0}^L \int_{y=0}^H 1 \cdot U_{kl}(x, y) \cdot U_{mn}(x, y) dy dx = 0$$

$$* \int_0^\pi \sin mx \cdot \sin nx dx = \begin{cases} 0 & , \text{ if } m \neq n \\ \pi/2 & , \text{ if } m = n \end{cases}$$

(vii) ***

* Dirichlet problem on a Rectangle (has unique solⁿ always)

(vii) Given: Laplace Eqⁿ. $u_{xx} + u_{yy} = 0$, $0 < x < a$
 $0 < y < b$

with B.C

(i). $u(x, 0) = f(x)$, $u(x, b) = 0$, $0 < x < a$

$u(0, y) = 0 = u(a, y) = 0$

Then solⁿ is given by

$$u(x, y) = \sum_{n=1}^{\infty} E_n \cdot \sin\left(\frac{n\pi x}{a}\right) \cdot \sinh\left(\frac{n\pi}{a}(b-y)\right)$$

(ii). $u(x, 0) = 0$, $u(x, b) = f(x)$

$u(0, y) = 0 = u(a, y)$

Then solⁿ is —

$$u(x, y) = \sum_{n=1}^{\infty} E_n \cdot \sin\left(\frac{n\pi x}{a}\right) \cdot \sinh\left(\frac{n\pi y}{a}\right)$$

(iii)

$u(x, 0) = 0 = u(x, b)$

$u(0, y) = f(y)$ & $u(a, y) = 0$

Then solⁿ is —

$$u(x, y) = \sum_{n=1}^{\infty} E_n \sinh\left(\frac{n\pi(a-x)}{b}\right) \cdot \sin\left(\frac{n\pi y}{b}\right)$$

(iv)

$u(x, 0) = 0 = u(x, b)$

$u(0, y) = 0$ & $u(a, y) = f(y)$

Then solⁿ is —

$$u(x, y) = \sum_{n=1}^{\infty} E_n \sinh\left(\frac{n\pi(a-x)}{b}\right) \cdot \sin\left(\frac{n\pi y}{b}\right)$$