Partial Differential Equation

[For CSIR-NET/SET/GATE/PSC/BHU/CUCET/MSc...etc.]

PDE

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Partial Differential Equations

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some definitions

Let f be f^n of x & y and $\frac{\partial f}{\partial y}$ is defined as

$$\frac{\partial f}{\partial y}(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0, y_0+h) - f(x_0, y_0)}{h}.$$

Similarly $\frac{\partial f}{\partial x}(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h}$

P.D.E : A diff^n. Eq^n (DE) which contain partial derivatives of one dependent variable w.r.t. two or more than two independent variable is called Partial Differential Equation (PDE).

EXAMPLE: (a) $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = 0$

(b). $\frac{\partial^2 z}{\partial x^2} + \frac{\partial z}{\partial y} + e^{x-y} = 0$

Degree of PDE; The highest power of the highest order derivative occuring in the PDE, is called degree of that PDE. (Provided given PDE be free from radical fractions & transcendental f^ns)

Order of PDE: The order of the highest order partial derivative occuring in the given PDE is called order of that PDE.

EXAMPLE: order of PDE (a) & (b) is 2.

Classification of first Order PDE

(1). Linear PDE of first Order

A partial diffⁿ of the form

$$\boxed{a(x,y)p + b(x,y)q = c(x,y)z + d(x,y)}$$

is called Linear PDE of first order

EXAMPLE: ⓐ $xyp + q = xyz$ ← linear

ⓑ $x^2p + y^2q = z^2 + yz$ ← Not-linear

Notations:

$$\frac{\partial z}{\partial x} = z_x = p \qquad \frac{\partial^2 z}{\partial x^2} = r \qquad \frac{\partial^2 z}{\partial x \partial y} = s$$

$$\frac{\partial z}{\partial y} = z_y = q \qquad \frac{\partial^2 z}{\partial y^2} = t$$

(2). Semi Linear PDE of first Order

A PDE of the form

$$\boxed{a(x,y)p + b(x,y)q = c(x,y,z)}$$

is called Semi-Linear PDE of 1^{st} order.

EXAMPLE: ⓐ $x^2p + y^2q = z^2 + xy$ ← semi-linear but not linear

ⓑ. $zp + xyq = z$ ← Not semi-Linear.

(3). Quasi Linear PDE of first Order

A PDE of the form

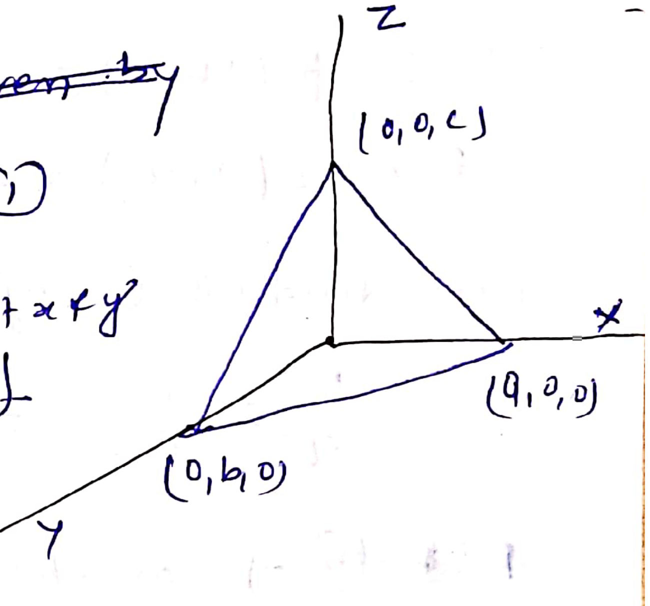
$$\boxed{a(x,y,z)p + b(x,y,z)q = c(x,y,z)}$$

Eg: ⓐ $x^2p = z$ ⓑ. $xp + zq = x^2 + y^2$

<u>Que(7)</u> Form a PDE for all planes having y & z intercepts same.

<u>Sol^n</u> $\because$ Eq^n of plane is ~~given by~~

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \quad \text{—— (1)}$$



Now, diff^n (1) partially w.r.t x & y
(& try to eliminate constants.)

$$\frac{1}{a} + \frac{z_x}{c} = 0 \Rightarrow a = -\frac{c}{p}$$

$$\frac{1}{b} + \frac{z_y}{c} = 0 \Rightarrow b = -\frac{c}{q}$$

$\Rightarrow$ $\cancel{\frac{p}{b} + \frac{1}{q}}$ & $\because$ plane having same intercept on ~~y~~ y & z axis $\Rightarrow$ $b = c$

Hence $c = -\frac{c}{q} \Rightarrow \boxed{q = -1}$ $\leftarrow$ Required PDE

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[B] Formation of PDE by Eliminating arbitrary functions

For a given relation, which contains n-arbitrary functions, then elimination of these arb. f^{n}s gives PDE of order n. <u>Illustrate by Examples</u>

<u>Que(7)</u> Eliminate arbitrary f^n from –

$$z = f(x+ay) + g(x-ay), \quad \text{where } a \neq 0 \quad \text{—— (i)}$$

and find corresponding PDE.

Sol^n Diff (i) partially w.r.t x & y, gives –

$$z_x = p = f'(x+ay) + g'(x-ay) \quad \text{—— (ii)}$$

and RHL, $\lim_{h \to 0^+} \frac{2h}{h} = 2$

$\Rightarrow \frac{\partial^2 z}{\partial y^2}$ doesn't exists on $\mathbb{R}^2$

$\left(\text{as } z_{yy} = \begin{cases} 2 & \geq 0 \\ -2 & < 0 \end{cases} \right)$

Based on Lagrange's Equation

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[Quasi Linear PDE of first Order]

Consider $\boxed{P(x,y,z).p + Q(x,y,z)q = R(x,y,z)}$ —(1)

then general solⁿ of (1) is

$$\boxed{\phi(u,v) = 0 \quad \text{or} \quad u = \phi(v) \quad \text{or} \quad v = \phi(u)}$$

Where, ϕ is arb. f^n & $u(x,y,z) = c_1$ and $v(x,y,z) = c_2$ are two independent solⁿ of Lagrange's Auxillary Eqⁿ given below

$$\boxed{\frac{dx}{P(x,y,z)} = \frac{dy}{Q(x,y,z)} = \frac{dz}{R(x,y,z)}} \quad —(*)$$

Result / Property

* $f(x,y)$ & $g(x,y)$ are s.t.b independent iff $\nexists$ any relation b/w f & g.

* $\frac{\partial(f,g)}{\partial(x,y)} \neq 0$ iff f & g are independent

Type-I: Given that $P(x,y,z)\,p + Q(x,y,z)\,q = R(x,y,z)$
& Lagrange A.E is $\boxed{\dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}}$ —(*)

then we get
two solⁿ, first $u(x,y,z) = c_1$ & second $v(x,y,z) = c_2$ by taking two suitable fractions or by choosing different combination from(*).
And, thus General solⁿ is $\phi(c_1, c_2) = 0$.

Type-II: One solⁿ of Lagrange A.E (*) is obtained by taking two suitable choice/fractions from(*) and other solⁿ is obtained by using first solⁿ.

Note:
(i). We have only general solⁿ for Quasi-Linear PDE of first order.
(ii). Particular solⁿ of given Quasi-LPDE is given by taking partiular choice of fⁿ ϕ, where $\phi(u,v) = 0$ is General solution.

Type-III: Consider $P(x,y,z)\,p + Q(x,y,z)\,q = R(x,y,z)$
& Lagrange A.E is $\boxed{\dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}}$ —(*)

Then, we choose S, T, U fⁿs of x, y & z (called multipliers) s.t —

$$\boxed{PS + QT + RU = 0}$$

thus, by (*), $\dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R} = \dfrac{S\,dx + T\,dy + U\,dz}{SP + TQ + UR}$

Second Order Partial differential Equation

(Classification of 2^{nd} order linear PDE)

Consider the general form of 2^{nd} order linear PDE (of two independent variable x & y) as —

$$\boxed{A u_{xx} + B u_{xy} + C u_{yy} + D u_x + E u_y + F u = G(x,y)} \quad \text{—— (1)}$$

It can be also given as —

$$A u_{xx} + B u_{xy} + C u_{yy} + f(x, y, u, p, q) = 0$$

or $$Ar + Bs + Ct + f(x, y, u, p, q) = 0$$

with $A^2 + B^2 + C^2 \neq 0$

where A, B, C, D, E, F, G are either constants or function of x and y only (continuous f^n.)

*** The classification of eqn (1) based on A, B & C ~~is given as~~ at point (x, y) is given as —

(1) If $B^2 - 4AC > 0$ then eqn (1) is Hyperbolic

(2). If $B^2 - 4AC = 0$, then eqn (1) is parabolic

(3). If $B^2 - 4AC < 0$, then eqn (1) is Elliptic

Remarks : If All constant A, B, C are constant then eqn (1) have the same nature throughout.

Practice Que

(1) Find the Characteristic curve of the PDE—

$$y^2 u_{xx} - x^2 u_{yy} = 0 \quad , x \neq 0, y \neq 0$$

(2). The PDE $r + 2s + t = 0$ is —

(a). Parabolic & has Ch. curve $\xi = x+2y$, $\eta = x-2y$

(b). Reducible to canonical form $u_{\xi\xi} = 0$ & $\xi = x+y$

(c). Parabolic & has general Solⁿ. $u = (x-y) f(x+y) + g(x-y)$

(d). Reducible to canonical form $u_{\eta\eta} = 0$ & $\eta = x+y$

(3). Reduce the following PDE to canonical form —

(I). $u_{xx} + 2u_{xy} + u_{yy} = 0$

(II). $u_{xx} + 2x u_{xy} + x^2 u_{yy} = 0$

(4). Find the Characteristics of the following PDEs —

(I). $x^2 u_{xx} + 2xy u_{xy} + y^2 u_{yy} = 0$

(II). $x^2 u_{yy} = y^2 u_{xx}$

(III). $(1+x^2) u_{xx} + (1+y^2) u_{yy} + x u_x + y u_y = 0$

(IV). $2u_{xx} + 5u_{yy} - 2u_{xy} = 0$

(V). $u_{xx} + 4u_{xy} + 4u_{yy} = 0$

— x —

Answers

(1). Circle & Hyperbola.

(2). (b) & (d)

(3)(I). $u_{\eta\eta} = 0$

(II). $u_{\eta\eta} = u_{\xi}$

(4)(I). Parabolic, family of st. line.

(II). Circle & Hyperbola

(III). Elliptic, No real Characteristic

(IV). No real Characteristic.

(V). A st. line.

**

① Two dimensional Eigen function in a Rectangle

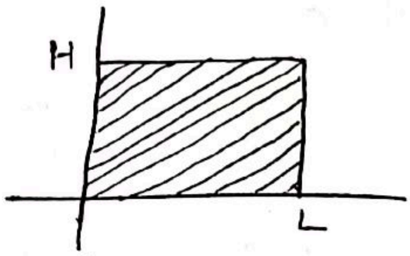
Consider $\phi_{xx} + \phi_{yy} + \lambda\phi = 0$ — ①

$0 < x < L,\ 0 < y < H$

with B.C $\left.\begin{aligned} \phi(x,0) &= \phi(x,H) = 0, & 0 \le x \le L \\ \phi(0,y) &= \phi(L,y) = 0, & 0 \le y \le H \end{aligned}\right\}$

Drichlet B.V Problem.

Solution



method ①

- Solution (Eigen f^n) will be find using the method of separation of variables.
- Take non-trivial sol^n as $\phi(x,y) = f(x) \cdot g(x)$ —(*)
  (then $g(0) = g(H) = 0,\ f(0) = f(L) = 0$)

  Now put (*) in ①, then by separable find $f(x)$ & $g(x)$, finally $\boxed{\phi = f \cdot g}$

& Solution will be —

$$\phi_{mn}(x,y) = \sin\left(\frac{m\pi x}{L}\right) \cdot \sin\left(\frac{n\pi y}{H}\right) \longleftarrow \text{Eigen } f^n$$

and Corresponding Eigen values $\lambda_{mn} = \left(\frac{m\pi}{L}\right)^2 + \left(\frac{n\pi}{H}\right)^2$, $m,n \in \mathbb{N}$

(2) $u_{xx} + u_{yy} + \lambda u = 0$, $0<x<L$, $0<y<H$

with $u_x(0,y) = u_x(L,y) = 0$
$u_y(x,0) = u_y(x,H) = 0$ } ← Newmann B.V. Problem.

Method: For finding sol^n, use the method described above (i.e Method-I)

Thus, solution will be —

Eigen-Values: $\lambda_{mn} = \left(\frac{m\pi}{L}\right)^2 + \left(\frac{n\pi}{H}\right)^2$, $m, n \in \mathbb{N}$
Eigen-function: $u_{mn}(x,y) = \cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi y}{H}\right)$

Note: Eigen.functions corresponding to distinct eigen.values are orthogonal w.r.t. weight f^n (co-eff of. λu) in the Rectangle $[0,L]\times[0,H]$.

i.e
$$\int_{x=0}^{L}\int_{y=0}^{H} 1 \cdot u_{kl}(x,y) \cdot u_{mn}(x,y)\, dy\, dx = 0$$

$$* \quad \int_0^{\pi} \sin mx \cdot \sin nx \, dx = \begin{cases} 0, & \text{if } m \neq n \\ \pi/2, & \text{if } m = n \end{cases}$$

(vvI) ***

* Dirichlet Problem in a Rectangle (has unique soln. always)

(vvi) Given: Laplace Eqn. $u_{xx} + u_{yy} = 0$, $0 < x < a$, $0 < y < b$

with B.C

(i). $u(x,0) = f(x)$, $u(x,b) = 0$, $0 < x < a$

$u(0,y) = 0 = u(a,y) = 0$

Then soln. is given by

$$\boxed{u(x,y) = \sum_{n=1}^{\infty} E_n \cdot \sin\left(\frac{n\pi x}{a}\right) \cdot \sinh\left[\frac{n\pi}{a}(b-y)\right]}$$

(ii). $u(x,0) = 0$, $u(x,b) = f(x)$

$u(0,y) = 0 = u(a,y)$

Then soln. is —

$$\boxed{u(x,y) = \sum_{n=1}^{\infty} E_n \cdot \sin\left(\frac{n\pi x}{a}\right) \cdot \sinh\left(\frac{n\pi y}{a}\right)}$$

(iii). $u(x,0) = 0 = u(x,b)$,

$u(0,y) = f(y)$ & $u(a,y) = 0$

Then soln is —

$$\boxed{u(x,y) = \sum_{n=1}^{\infty} E_n \sinh\left[\frac{n\pi(a-x)}{b}\right] \cdot \sin\left(\frac{n\pi y}{b}\right)}$$

(iv) $u(x,0) = 0 = u(x,b)$

$u(0,y) = 0$ & $u(a,y) = f(y)$

Then soln. is —

$$\boxed{u(x,y) = \sum_{n=1}^{\infty} E_n \sinh\left(\frac{n\pi(a-x)}{b}\right) \cdot \sin\left(\frac{n\pi y}{b}\right)}$$